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"A Brauer algebra theoretic proof of Littlewood's restriction rules"

INTRODUCTION

"Non potrai dir che quest' è cosa dura:
usando la dualità di Brauer
dimostrazione dar, novella e pura"

N. Barbecue, "Scholia"

Let U be a complex vector space, endowed with an orthogonal or symplectic form, and let G be either $O(U)$ or $Sp(U)$ respectively. Consider a simple polynomial $GL(U)$–module V_λ (associated in a standard way to a partition λ), and restrict it to G. If $\lambda_1^t + \lambda_2^t \leq dim(U)$, in the orthogonal case (λ^t being the dual partition to λ), or $\lambda_1^t \leq dim(U)/2$, in the symplectic case, then its decomposition into simple G–modules is described by the Littlewood's restriction rule (cf. [L]), which gives a formula for the multiplicity in V_λ of each simple G–module. The main aim in this article is to prove this formula.

It is well known (cf. e.g. [W], [H]) that one can realize a copy of V_λ inside the tensor power $U^{\otimes f}$, where f is the sum of parts of λ (i.e. λ is a partition of f). By the general theory of centralizer algebras, a bijection $V_\lambda \longleftrightarrow M_\lambda$ exists between simple $GL(U)$–modules and simple modules over $End_{GL(U)}\left(U^{\otimes f}\right)$ — the centralizer algebra of the $GL(U)$–action on $U^{\otimes f}$ — occurring in $U^{\otimes f}$, which interchanges dimensions and multiplicities. Similarly, a bijection $W_\mu \longleftrightarrow N_\mu$ exists between simple G–modules and simple modules over $End_G\left(U^{\otimes f}\right)$ — the centralizer algebra of the G–action on $U^{\otimes f}$ — occurring in $U^{\otimes f}$ (which is now thought of as a G–module), which interchanges dimensions and multiplicities. Then we have an identity $\left[V_\lambda : W_\mu\right] = \left[N_\mu : M_\lambda\right]$, thus to get the multiplicity $\left[V_\lambda : W_\mu\right]$ we can compute the above right-hand-side term instead: in other words, instead of studying $V_\lambda\Big|_G^{GL(U)}$ we study $N_\mu\Big|_{End_{GL(U)}(U^{\otimes f})}^{End_G(U^{\otimes f})}$. Therefore, if

$$\left[V_\lambda : W_\mu\right] = C_\mu^\lambda \qquad (\star)$$

is the identity given in Littlewood's restriction formula, our aim is to prove that

$$\left[N_\mu : M_\lambda\right] = C_\mu^\lambda \qquad (\star\star)$$

Now, one has that $End_{GL(U)}\left(U^{\otimes f}\right) = \mathbb{C}\left[S_f\right]$, with S_f acting on $U^{\otimes f}$ by index permutation. On the other hand, $End_G\left(U^{\otimes f}\right)$ is a quotient of the Brauer algebra $\mathcal{B}_f^{(\epsilon N)}$, where $N = dim_{\mathbb{C}}(U)$ and ϵ is the "sign" of the form on U ("+" for orthogonal and "−" for symplectic case); the kernel of $\pi_U : \mathcal{B}_f^{(\epsilon N)} \twoheadrightarrow End_G\big(U^{\otimes f}\big)$ is also known, essentially from the Second Fundamental Theorem of Invariant Theory (for the group G). In the stable case (i.e. when $f \leq N/2$ in the symplectic case and $f \leq N$ in the orthogonal case) π_U is an isomorphism, and Littlewood's formula can be proved as a corollary of a suitable description of $V^{\otimes f}$ (cf. [GP]). In the general case a different approach is necessary.

To describe $\mathcal{B}_f^{(x)}$ we can display an explicit basis D_f — whose elements are certain graphs — and assign the multiplication rules for elements in this basis — based on "composition" of graphs. Then from the previously mentioned description of $Ker\big(\pi_U\big)$ we take out an explicit set of linear generators of this kernel.

In addition, the simple G–modules N_μ are quotients of certain $\mathcal{B}_f^{(\varepsilon N)}$–modules N'_μ which have a nice combinatorial description (in terms of graphs related to those of D_f); moreover, we prove that the kernel of the epimorphism $N'_\mu \twoheadrightarrow N_\mu$ is just $Ker\big(\pi_U\big).N'_\mu$. Now, the multiplicity $\big[N'_\mu : M_\lambda\big]$ is exactly equal to the right-hand-side part of $(\star)$; then it is enough for us to show that in $Ker\big(\pi_U\big).N'_\mu$, as a $\mathbb{C}\big[S_f\big]$–module, there are no components of type M_λ for λ such that $\lambda_1^t + \lambda_2^t \leq dim(U)$ (in the orthogonal case) or $\lambda_1^t \leq dim(U)/2$ (in the symplectic case). We deduce this fact from the previous description of $Ker\big(\pi_U\big)$.

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