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"Quantum function algebras as quantum enveloping algebras"

INTRODUCTION

"Nel mezzo a una q–algebra di funzioni
io ci ritrovo una tal forma intera
che l'algebra di Lie dual mi doni"
N. Barbecue, "Scholia"

Let G be a connected, simply connected, semisimple algebraic group over an algebraically closed field k of characteristic zero, and consider on it the Sklyanin-Drinfel'd structure of Poisson group (cf. for instance [DP] §11 or [Ga] §1, or even [Dr]). Then $\mathfrak{g} := Lie(G)$ is a Lie bialgebra, $F[G]$ is a Poisson Hopf algebra, and $U(\mathfrak{g})$ is a Poisson Hopf coalgebra. Let H be the corresponding dual Poisson (algebraic) group of G, whose tangent Lie bialgebra $\mathfrak{h} := Lie(H)$ is the (linear) dual of $\mathfrak{g}$: then again $F[H]$ is a Poisson Hopf algebra, and $U(\mathfrak{h})$ is a Poisson Hopf coalgebra.

The quantum group $U_q^Q(\mathfrak{g})$ of Drinfel'd and Jimbo provides a quantization of $U(\mathfrak{g})$: namely, $U_q^Q(\mathfrak{g})$ is a Hopf algebra over $k(q)$ which has a $k\big[q,q^{-1}\big]$–form $\mathfrak{U}^Q(\mathfrak{g})$ which for $q \rightarrow 1$ specializes to $U(\mathfrak{g})$ as a Poisson Hopf coalgebra. Dually, by means of a Peter-Weyl type axiomatic trick one constructs a Hopf algebra $F_q^P[G]$ of matrix coefficients of $U_q^Q(\mathfrak{g})$ with a $k\big[q,q^{-1}\big]$–form $\mathfrak{F}^P[G]$ which specializes to $F[G]$, as a Poisson Hopf algebra, for $q \rightarrow 1$. So far the quantization only dealt with the Poisson group G; the dual group H is involved defining a different $k\big[q,q^{-1}\big]$–form $\mathcal{U}^P(\mathfrak{g})$ (of a quantum group $U_q^P(\mathfrak{g})$) which specializes to $F[H]$ (as a Poisson Hopf algebra) for $q \rightarrow 1$ (cf. [DP] or [DKP]). In a dual fashion, it is proved in [Ga] — in a wider context — that the dual (in the Hopf sense) quantum function algebra $F_q^Q[G]$ has a $k\big[q,q^{-1}\big]$–integer form $\mathcal{F}^Q[G]$ which for $q \rightarrow 1$ specializes to $U(\mathfrak{h})$, as a Poisson Hopf coalgebra. Therefore quantum function algebras can also be thought of as quantum enveloping algebras, whence the title of the paper.

In this paper we stick to the case of the group $G = SL(n+1)$.

Our first goal is to relate the latter result above with the well-known presentation of $F_q^P[SL(n+1)]$ by generators and relations (cf. [FRT]). Namely, inspired by the definition of $\mathcal{F}^Q[G]$ and $\mathcal{F}^P[G]$, we define two $k\big[q,q^{-1}\big]$–integer forms $\widetilde{F}_q^Q[SL(n+1)]$ and $\widetilde{F}_q^P[SL(n+1)]$ (along with a third one, $\widetilde{F}_q[SL(n+1)]$) of $F_q^P[SL(n+1)]$. These inherit a presentation by generators and relations, which enables us to prove that they specialize to $U(\mathfrak{h})$ (as a Poisson Hopf coalgebra) for $q \rightarrow 1$. As a second step, since for $U(\mathfrak{h})$ one has the Poincaré-Birkhof-Witt (PBW in short) theorem which provides "monomial" basis, because of the previous result we are led to look for PBW-like theorems for $F_q^P[SL(n+1)]$: we provide two of them, both closely related with the classical PBW theorem for $U(\mathfrak{h})$.

The paper is organized as follows. Sections 1 and 2 are introductory. Sections 3 and 4 are devoted to integer forms of $F_q^P[SL(n+1)]$ and their specialization. Section 5 is an excursus, where we explain the relation among the constructions and results in this paper and those in [Ga]: this is one of the main motivation of this work; on the other hand, this section can be skipped without affecting the comprehension of the rest of the paper, which is completely self-contained. In section 6 we briefly outline the extension of the previous results to the quantum function algebra $F_q[GL(n+1)]$. Finally, section 7 deals with PBW theorems.

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