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"Quantization of Poisson groups"

INTRODUCTION

"Dualitas dualitatum
et omnia dualitas"
N. Barbecue, "Scholia"

Let G be a semisimple, connected, and simply connected affine algebraic group over an algebraically closed field k of characteristic zero; we consider a family of structures of Poisson group on G, indexed by a multiparameter τ, which generalize the Sklyanin-Drinfel'd one. Then every such Poisson group G^τ has a dual Poisson group H^τ, and $\mathfrak{g}^\tau := Lie(G^\tau)$ and $\mathfrak{h}^\tau := Lie(H^\tau)$ are Lie bialgebras dual to each other.

In 1985 Drinfel'd and Jimbo provided a quantization of $U(\mathfrak{g}) = U(\mathfrak{g}^0)$, namely a Hopf algebra $U_q^Q(\mathfrak{g})$ over $k(q)$, presented by generators and relations, with a $k\big[q,q^{-1}\big]$–form $\mathfrak{U}_Q(\mathfrak{g})$ which for $q \to 1$ specializes to $U(\mathfrak{g})$ as a Poisson Hopf coalgebra. This has been extended to general parameter τ introducing multiparameter quantum groups $U_{q,\varphi}^Q(\mathfrak{g})$ (cf. [R], [CV-1], [CV-2]). Dually, one constructs a Hopf algebra $F_q^P[G]$ of matrix coefficients of $U_q^Q(\mathfrak{g})$ with a $k\big[q,q^{-1}\big]$–form $\mathfrak{F}_P[G]$ which specializes to $F[G]$, as a Poisson Hopf algebra, for $q \to 1$. In particular, $\mathfrak{F}_P[G]$ is nothing but the Hopf subalgebra of "functions" in $F_q^P[G]$ which take values in $k\big[q,q^{-1}\big]$ when "evaluated" on $\mathfrak{U}_Q(\mathfrak{g})$: in a word, the $k\big[q,q^{-1}\big]$–integer valued functions on $\mathfrak{U}_Q(\mathfrak{g})$. This again extends to general τ (cf. [CV-2]).

So far the quantization only dealt with the Poisson group G (or G^τ); the dual group H is involved defining a different $k\big[q,q^{-1}\big]$–form $\mathcal{U}_P(\mathfrak{g})$ (of a quantum group $U_q^P(\mathfrak{g})$) which specializes to $F[H]$ (as a Poisson Hopf algebra) for $q \to 1$ (cf. [DP]), with generalization to the multiparameter case possible again. Here sort of a "mixing dualities" (Hopf duality — among enveloping and function algebra — and Poisson duality — among dual Poisson groups) occurs, which was described (in a formal setting) by Drinfel'd (cf. [Dr], §7), and by Etingof and Kazhdan (cf. [EK-1], [EK-2]). This leads to consider the following: let $F_q^Q[G]$ be the quantum function algebra dual to $U_q^P(\mathfrak{g})$, and look at the "dual" to $\mathcal{U}_P(\mathfrak{g})$ within $F_q^Q[G]$, call it $\mathcal{F}_Q[G]$, namely the Hopf algebra of $k\big[q,q^{-1}\big]$–integer valued functions on $\mathcal{U}_P(\mathfrak{g})$; then this should specialize to $U(\mathfrak{h})$, as a Poisson Hopf coalgebra, for $q \to 1$. The same conjecture can be formulated in the multiparameter case too.

Our starting aim was to achieve this goal, i. e. to construct $F_q^Q[G]$ and its $k\big[q,q^{-1}\big]$–form $\mathcal{F}_Q[G]$, and to prove that $\mathcal{F}_Q[G]$ is a deformation of the Poisson Hopf coalgebra $U(\mathfrak{h})$. This goal is succesfully attained by performing a suitable dualization of the quantum double construction. But by the way, this leads to discover a *new quantum group*, which we call

$U_q^M(\mathfrak{h})$, which is for $U(\mathfrak{h})$ what $U_q^M(\mathfrak{g})$ is for $U(\mathfrak{g})$; in particular it has an integer form $\mathfrak{U}_Q(\mathfrak{h})$ which is a quantization of $U(\mathfrak{h})$, and an integer form $\mathcal{U}_P(\mathfrak{h})$ which is a quantization of $F^\infty[G]$, the function algebra of the formal Poisson group associated to G. Furthermore, we exhibit a Hopf pairing between $U_q^{M'}(\mathfrak{g})$ and $U_q^M(\mathfrak{h})$ which gives a quantization of the various pairings occurring among the algebras attached to the pair (G,H). Once again, all this extends to the multiparameter case. Thus in particular we provide a (infinitesimal) quantization for a wide class of Poisson groups (the H^τ's). Now, in the summer of 1995 (when the present work was already accomplished) a quantization of any Poisson group was presented in [EK-1] and [EK-2], but its greatest generality goes along with some lack of concreteness. In contrast, our construction is extremely concrete; moreover, it allows specialization at roots of 1, construction of quantum Frobenius morphisms, and so on (like for $\mathfrak{U}_Q(\mathfrak{g})$ and $\mathcal{U}_P(\mathfrak{g})$), which is not possible in the approach of [EK-1], [EK-2].

Finally, a brief sketch of the main ideas of the paper. Our aim is to study the "dual" of a quantum group $U_{q,\varphi}^M(\mathfrak{g})$, where M is any lattice of weights.

First, we select as operation of "dualization" the most naïve one, namely taking the *full linear dual* (rather than the usual — restricted — Hopf dual), the latter being a *formal* Hopf algebra (rather than a common Hopf algebra). Second, as $U_{q,\varphi}^M(\mathfrak{g})$ is a quotient of a quantum double $D_{q,\varphi}^M(\mathfrak{g}) := D\left(U_{q,\varphi}^M(\mathfrak{b}_-), U_{q,\varphi}^M(\mathfrak{b}_+), \pi_\varphi\right)$, its linear dual $U_{q,\varphi}^M(\mathfrak{g})^*$ embeds into $D_{q,\varphi}^M(\mathfrak{g})^*$. Third, since $D_{q,\varphi}^M(\mathfrak{g}) \cong U_{q,\varphi}^M(\mathfrak{b}_+) \otimes U_{q,\varphi}^M(\mathfrak{b}_-)$ (as coalgebras) we have $D_{q,\varphi}^M(\mathfrak{g})^* \cong U_{q,\varphi}^M(\mathfrak{b}_+)^* \,\widehat{\otimes}\, U_{q,\varphi}^M(\mathfrak{b}_-)^*$ (as algebras), where $\widehat{\otimes}$ denotes topological tensor product. Fourth, since quantum Borel algebras of opposite sign are perfectly paired, their linear duals are suitable completions of quantum Borel algebras again. Thus we find a presentation of $U_{q,\varphi}^M(\mathfrak{g})^*$ by generators and relations: this leads us to *define* $U_{q,\varphi}^M(\mathfrak{h}) := U_{q,\varphi}^{M'}(\mathfrak{g})^*$ (where M' depends on M), and gives all claimed results. Because of their construction, we call the new objects $U_{q,\varphi}^M(\mathfrak{h})$ *(multiparameter) quantum formal groups.*

In contrast, we also present an alternative approach, yielding other *new* objects — denoted by $F_{q,\varphi}^{M,\infty}[G]$ — which we call *(multiparameter) formal quantum groups.* The *similar* but *different* terminology reveals the fact that $U_{q,\varphi}^M(\mathfrak{h})$ and $F_{q,\varphi}^{M,\infty}[G]$ provide two *different* quantizations of the *same* classical objects $U(\mathfrak{h}^\tau)$ and $F^\infty[G^\tau]$, arising from two different ways of realizing $F^\infty[G^\tau]$.

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