Journal of Algebra **194** (1997), no. 1, 275–298
DOI: 10.1006/jabr.1996.7003

Fabio GAVARINI, Paolo PAPI

"Representations of the Brauer algebra and Littlewood's restriction rules"

INTRODUCTION

Let V be a complex vector space of dimension $2n$, endowed with a symplectic (i.e. non-degenerate bilinear skew-symmetric) form $\langle\ ,\ \rangle$. Consider the symplectic group $Sp(V)$ of linear automorphisms of V preserving the symplectic form $\langle\ ,\ \rangle$. It is well known that all irreducible finite dimensional representations of $Sp(V)$ can be realized as subrepresentations of tensor powers $V^{\otimes m}$ ($m \in \mathbb{N}$). On the other hand, consider the centralizer of the $Sp(V)$–action on $V^{\otimes m}$, which is a quotient of the so-called Brauer algebra $\mathbb{B}_m^{-2n}$: Schur duality tells us that the algebra of operators generated by $Sp(V)$ and the above quotient of the Brauer algebra are mutual centralizer, and establishes a bijective correspondence between the representations of either of these algebras.

The $Sp(V)$–module $V^{\otimes m}$ splits as $V^{\otimes m} = \bigoplus_{k=0}^{[\frac{m}{2}]} T^k\left(V^{\otimes m}\right)$, the subspace $T^k\left(V^{\otimes m}\right)$ being the sum of the $Sp(V)$–isotypic components of $V^{\otimes m}$ which occur for the first time in tensor power $m-2k$; more directly, if $\Psi_{pq} : V^{\otimes m} \longrightarrow V^{\otimes (m+2)}$ is the extension operator which inserts in the positions p and q the canonical element of the skew-form $\langle\ ,\ \rangle$, $T^k\left(V^{\otimes m}\right)$ is the vector space generated by k–fold extensions of the traceless tensors in $V^{\otimes (m-2k)}$ (i.e. tensors killed by any contraction). Note that, if S_m denotes the symmetric group on m letters, then $T^k\left(V^{\otimes m}\right)$ has a natural structure of $Sp(V) \times S_m$–module (even more, of $Sp(V) \times \mathbb{B}_m^{-2n}$–module).

In this paper we show (Theorem 4.1) that, for $n \geq m$ (i.e. in the so-called "stable case"), $T^k\left(V^{\otimes m}\right)$ is obtained by inducing the S_m–module structure from a representation of $S_{m-2k} \times S_{2k}$ built up by taking the tensor product of traceless tensors in $V^{\otimes (m-2k)}$ and $Sp(V)$–invariants in $V^{\otimes (2k)}$. This is proved by considering two actions of the Brauer algebra: the natural action of $\mathbb{B}_m^{-2n}$ on $T^k\left(V^{\otimes m}\right)$ and an action on the induced representation, which we directly define in §3. Relating and comparing these actions we will be able to show that $\mathbb{B}_m^{-2n}$ is the whole centralizer of the $Sp(V)$–action on the induced representation. This fact — whose proof is reduced to a combinatorial calculation — allows us to apply symplectic Schur duality and to get the desired isomorphism using elementary representation theory.

A first application is a proof of Littlewood's restriction rule in the stable case. Namely, let V_λ be an irreducible finite dimensional polynomial $GL(V)$–module indexed by a partition λ of m; its restriction to $Sp(V)$ is no longer irreducible in general. In [L] Littlewood furnished a formula describing the decomposition of V_λ into irreducible $Sp(V)$–modules under the assumption that λ has at most n parts; note that this condition is always satisfied in the stable case. Using the description of $T^k\left(V^{\otimes m}\right)$ we gave, it is not difficult to recover Littlewood's rule using standard techniques of classical invariant theory (cf. §5).

The previous arguments can be repeated almost word-by-word for the orthogonal group; in §6 we point out the few modifications needed.

Finally, in §7, we recover from our main result an explicit realization, inside $V^{\otimes m}$, of the irreducible representations of the Brauer algebra in the stable case, and describe the relation among our results and the combinatorial description of these representations (due to Kerov [K]).

In §2 we introduce the basic definitions and recollect well-known results of representation theory which will be needed in the sequel; almost all the results of this section can be found in Weyl's fundamental book [W].

— — — — —

References

[Br] R. Brauer, *On algebras which are concerned with semisimple continous groups*, Annals of Mathematics **38** (1937), 854–872.

[D-P] C. De Concini, C. Procesi, *A characteristic free approach to invariant theory*, Advances in Mathematics **21** (1976), 330–354.

[D-S] C. De Concini, E. Strickland, *Traceless tensors and the symmetric group*, Journal of Algebra **61** (1979), 112–128.

[K] S. V. Kerov, *Realizations of representations of the Brauer semigroup*, Journal of Soviet Mathematics **47** (1989), 2503–2507.

[L] D. E. Littlewood, *On invariant theory under restricted groups*, Phil. Trans. Roy. Soc. A **239** (1944), 387–417.

[L-P] J-L. Loday, C. Procesi, *Homology of symplectic and orthogonal algebras*, Advances in Mathematics **21** (1976), 93–108.

[W] H. Weyl, *The classical groups*, Princeton University Press, New York, 1946.

[Wz] H. Wenzl, *On the structure of Brauer's centralizer algebras*, Annals of Mathematics **188** (1988), 173–193.