

Dyson Theorem for curves

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ABSTRACT. Let K be a number field and X_1 and X_2 two smooth projective curves defined over it. In this paper we prove an analogue of the Dyson Theorem for the product $X_1 \times X_2$. If $X_i = \mathbb{P}_1$ we find the classical Dyson theorem. In general, it will imply a self contained and easy proof of Siegel theorem on integral points on hyperbolic curves and it will give some insight on effectiveness. This proof is new and avoids the use of Roth and Mordell–Weil theorems.

1 Introduction.

After the proof of the Mordell conjecture by Faltings (the first proof is in [Fa], but [Fa2], [B2] and [Vo2] are nearer to the spirit of this paper), most of the qualitative results in the diophantine approximation of algebraic divisors *by rational points* over curves are solved.

Historically, the first concluding result is the Siegel theorem: An affine hyperbolic curve contains only finitely many S -integral points; we know that we cannot suppose less on the *geometry* of the involved curve: \mathbb{A}^1 and \mathbb{G}_m have, as soon as the field is sufficiently big, infinitely many integral points.

After a long and interesting story of partial results (Liouville, Thue, Siegel, Dyson, Gelfand...), Roth proved that, if α is an algebraic number then, for every $\kappa > 2$, the equation

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{|q|^\kappa}$$

admits only finitely many solutions $\frac{p}{q} \in \mathbb{Q}$. Here again, by Dirichlet theorem, we know that for $\kappa = 2$ the equation may have infinitely many solutions.

Eventually, the already quoted theorem of Faltings close the story: a compact hyperbolic curve contains only finitely many rational points.

It is a fact that, from a *quantitative* point of view, we are still very far from a satisfactory answer (up to the very interesting partial results by Bombieri and his collaborators [B1], [B3], [BVV] and [BC]): In each of the three problems quoted above we are not able to give an upper bound for the heights of the searched solution. And, even worst, we are not able to say if *there is* a solutions to each of these problems.

Let's have a closer look to the Siegel theorem: the modern proof of it rely on the Roth theorem and on the Mordell Weil theorem. Consequently, if one try to find an effective proof by refining the existing proof, one will crash into the problems of effectivity in Roth theorem and in the computation of a basis for the Mordell–Weil group of the Jacobian (problem which seems easier but not yet completely solved). So, at a first

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glance, an effective version of Siegel theorem will be consequence of the solutions of other problems, which seems to be more difficult. This is very unsatisfactory, also because a strong effective version of it will imply a version of the *abc*-conjecture ([Su]).

In this paper we prove a theorem in the spirit of the Dyson Theorem [B1] over the product of two curves. It will easily imply Siegel theorem. Up to standard facts in algebraic geometry and in the theory of heights, the theorem is self contained, and, from our point of view, essentially elementary. Consequently it release Siegel theorem from other big theorems.

We now give a qualitative statement of the main theorem of this paper; for a precise statement, cf. section 2.

Let K be a number field, L be a finite extensions of K and $n := [L : K]$. Let X_1 and X_2 be smooth projective curves over K and $D_i = \text{Spec}(L) \rightarrow X_i$, be effective geometrically reduced divisors of degree $[L : K]$ on X_i . Let H_i be line bundle of degree one over X_i and $h_{H_i}(\cdot)$ height functions associated to H_i . Finally, let S be a finite set of places of K and $\lambda_{D_i, S}(\cdot)$ be Weil functions associated to D_i and S .

1.1 Theorem. *Let ϑ_1, ϑ_2 and ϵ be three rational numbers such that $\vartheta_1 \cdot \vartheta_2 \geq 2n + \epsilon$. Then the set of rational points $(P, Q) \in X_1(K) \times X_2(K)$ such that*

$$\lambda_{D_1, S}(P) > \vartheta_1 \cdot h_{H_1}(P)$$

and

$$\lambda_{D_2, S}(Q) > \vartheta_2 \cdot h_{H_2}(Q)$$

is contained in a proper closed subset whose irreducible components are either fibers or points.

If we apply the theorem to $\mathbb{P}_1 \times \mathbb{P}_1$ and $\vartheta_1 = \vartheta_2 = \sqrt{2n} + \epsilon$ we reobtain the classical theorem of Dyson :

1.2 Corollary. *Let α be an algebraic number of degree n over \mathbb{Q} then there are only finitely many $\frac{p}{q} \in \mathbb{Q}$ such that*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^{\sqrt{2n} + \epsilon}}.$$

If we apply the theorem to $C \times C$ where C is an arbitrary curve, D a reduced divisor such that $\chi(C \setminus D) > 0$ we obtain the following result which implies Siegel theorem on integral points on hyperbolic curves:

1.3 Corollary. *Let C be a smooth projective curve over a number field K ; let D be a reduced divisor over C then for all $p \in C(K)$ we have*

$$\lambda_{D, S}(p) \leq (\sqrt{2n} + \epsilon)h_M(p) + O(1).$$

If $n \geq 3$ the corollary directly imply Siegel theorem; otherwise, in order to obtain the general case of the Siegel theorem, one apply a simple argument a' la Chevalley–Weil.

Using Roth theorem and the weak Mordell–Weil theorem one obtains

$$\lambda_{D,S}(p) \leq \epsilon h_M(p) + O(1);$$

which is much stronger than 1.3 (but, it implies the same qualitative result on integral points). Nevertheless, as already said, the proof we propose here is much simpler and its uneffectiveness is essentially self contained: it does not depend on other theorems.

Let K be a number field and X be a defined over it. Fix an height function h over X and a finite set S of places of K . Fix a positive integer n (if X is rational, we suppose that $n \geq 3$) and denote by $Div^n(X)(K)$ the set of effective divisors of degree n over X defined over K . To our taste, a satisfactory (at least for the moment) effective version of Siegel theorem could be

Conjecture: (*Effective Siegel theorem*) *There is a constant $A := A(K, X, n, S, h)$ depending only on K, X, n, S and h , such that, for every couple $(D, P) \in Div^n(X)(K) \times X(K)$ such that P is a (D, S) -integral point we have*

$$\frac{h(P)}{h(D)} \leq A.$$

It is an amusing exercise to prove that our main theorem 2.1 implies that if the conjecture is false in a *effective way for only one curve* then , for every curve Y and effective divisor D on it, we can explicitly bound the height of the (D, S) -integral points on it:

1.4 Theorem. *Suppose that we can find a curve X for which the following holds: for every constant A we can effectively find a couple (D, P) where D is an effective divisor of degree n on X and $P \in X(K)$ is an (D, S) -integral point such that*

$$h(P) \geq Ah(D).$$

Then, for every curve Y and divisor D' on Y of degree n , one can effectively bound the height of the (S, D') -integral points of Y .

By "effectively find a couple (D, P) ..." we mean that, given A , we can *effectively* compute, as a function of X, K, A etc. the height of D and the height of P .

Consequently, we see that the results of this paper give us the following issue: either we prove the conjecture above, and this will give a quantitative (up to the knowledge of the involved constant A) version of Siegel theorem, or we disprove it in a effective way for a single curve and this will again imply an effective Siegel theorem.

A remark on the language and the methods used: In this paper we decided to use the language of arithmetic geometry a' la Grothendieck and the Arakelov geometry; although this needs a little bit of background, which nowadays is (or should be) standard,

this language allows to better understand and compute the involved constants and the strategy of the proof; this without getting lost in the choice of the constants etc. For a rapid introduction to the Arakelov geometry used in this paper cf. [MB] or the more general [BGS].

2 Statement of the main theorem and notations.

Let K be a number field and O_K be its ring of integers. We will denote by M_K the set of (finite and infinite) places of K . Let S be a finite subset of M_K . We will denote by O_S the ring of S -integers of K . For every $v \in M_K$ let K_v be the completion of K at the place v and k_v be its residue field. For every scheme $X \rightarrow \text{Spec}(O_K)$ we will denote by X_v the base change of it from $\text{Spec}(O_K)$ to $\text{Spec}(K_v)$ and by X_K the base change of it to $\text{Spec}(K)$.

Let L_1, \dots, L_r be finite extensions of K and O_{L_i} be the ring of integers of L_i . We will denote by A the O_K -algebra $\bigoplus O_{L_i}$.

Let $f_1 : \mathcal{X}_1 \rightarrow B := \text{Spec}(O_K)$ and $f_2 : \mathcal{X}_2 \rightarrow B$ be two regular, semistable arithmetic surfaces over O_K . Let $\Delta_i \hookrightarrow \mathcal{X}_1 \times_B \mathcal{X}_2$ ($i = 1, 2$) be the diagonal divisor. We fix a symmetric hermitian structure on the line bundle $\mathcal{O}(\Delta_i)$ ($i = 1, 2$). Let $\sigma \in M_K$ be an infinite place and $P \in (\mathcal{X}_i)_\sigma(\mathbb{C})$; denoting by $\iota_P : (\mathcal{X}_i)_\sigma(\mathbb{C}) \rightarrow (\mathcal{X}_i \times \mathcal{X}_i)_\sigma(\mathbb{C})$ the embedding $\iota_P(x) := (x, P)$, we have a canonical isomorphism $\iota_P^* \mathcal{O}(\Delta) \simeq \mathcal{O}(P)$; as a consequence, for every divisor D of \mathcal{X}_i , the line bundle $\mathcal{O}(D)$ is equipped with a canonical (depending only on the choices made until now) metric and, for every finite set of places $S \subset M_K$ the Weil function $\lambda_{D,S}(\cdot)$ is a well defined function (also this will depend only on the choice of the model and on the metric on the diagonal). The choice of a metric on the diagonal also induces a metric on the relative dualizing sheaf $\omega_{\mathcal{X}_i/B}$; we suppose fixed such a metric; remark that, by construction, the adjunction formula holds: for every section $P : B \rightarrow \mathcal{X}_i$ we have a canonical isomorphism

$$\omega_{\mathcal{X}_i/B}|_P \simeq \mathcal{O}(-P)|_P$$

of *hermitian* line bundles on B . For a general reference on this cf. [MB]. For a reference on Weil functions cf. [HS].

Fix arithmetically ample hermitian line bundles $(M_i, \|\cdot\|_{M_i})$ on \mathcal{X}_i of generic degree one: this means that $\deg(M_i)_K = 1$ and $h_{M_i}(\cdot) > 0$.

We will denote by $(\cdot; \cdot)$ the Arakelov intersection pairing on each of the \mathcal{X}_i .

If D is a effective reduced divisor over \mathcal{X}_i ; write $D := \sum D_j$ where each D_j is an irreducible divisor. Define the following three numbers associated to it: $S(D) := \max\{-(\mathcal{O}(D_j); \mathcal{O}(D_j)); 1\}$; $H(D) := \max\{h_{M_i}(D), 1\}$; and $T(D) := S(D) \cdot H(D)$.

We eventually fix a positive integer and three positive rational numbers ϑ_1, ϑ_2 and ϵ such that

$$\vartheta_1 \cdot \vartheta_2 \geq 2n + \epsilon.$$

The main theorem of this paper is the following generalization of Dyson theorem:

2.1 Theorem. Under the hypotheses above there exist two effectively computable constants R_1 and R_2 , depending only on the \mathcal{X}_i , the hermitian line bundles M_i , the metrics on the diagonals, the ϑ_i and the constant ϵ for which the following holds:

Let L_1, \dots, L_r be finite extensions of K ; denote by n the number $n := \max\{[L_i \cdot L_j : K]\}$, by O_{L_i} the ring of integers of L_i and by A the O_K -algebra $A := \text{Spec}(\oplus O_{L_i})$.

Let

$$D_i: A \rightarrow \mathcal{X}_i$$

be reduced effective divisors over \mathcal{X}_i ($i = 1, 2$).

If $(P, Q) \in \mathcal{X}_1(B) \times \mathcal{X}_2(B)$ is a couple of rational points such that

- (a) $h_{M_1}(P) \geq R_1 \cdot T(D_1) \cdot T(D_2)$
- (b) $\lambda_{D,S}(P) > \vartheta_1 \cdot h_{M_1}(P)$ and $\lambda_{D_2,S}(Q) > \vartheta_2 \cdot h_{M_2}(Q)$;

then

$$h_{M_2}(Q) \leq R_2 \cdot T(D_1) \cdot T(D_2) \cdot h_{M_1}(P).$$

This will easily imply the qualitative theorem and its corollaries.

In the following sections we will introduce the tools we need for the proof of theorem 2.1, we will give the of it in the final section.

3 Small sections.

Let L be a finite extension of K of degree n and O_L its ring of integers.

Let \mathcal{L} be a line bundle over $B := \text{Spec}(O_K)$; we will denote by $\mathcal{O}[\mathcal{L}]$ the O_K -algebra $\text{Sym}(\oplus \mathcal{L}^{\otimes n})$ and by $\mathcal{O}[[\mathcal{L}]]$ the O_K -algebra $\prod \mathcal{L}^{\otimes n}$ with the multiplicative structure given by $(a_n) \cdot (b_n) := (c_n)$ where $c_n := \sum_{i+j=n} a_i \otimes b_j$ (if \mathcal{L} is the trivial line bundle \mathcal{O}_B then $\mathcal{O}[[\mathcal{O}_B]]$ is the usual ring of power series $O_K[[X]]$). If \mathcal{L}_1 and \mathcal{L}_2 are two line bundles we define $\mathcal{O}[\mathcal{L}_1, \mathcal{L}_2]$ and $\mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]]$ in a similar way.

Let $f_{\mathcal{L}}: \mathbb{V}(\mathcal{L}) \rightarrow B$ be the affine B -scheme $\text{Spec}(\mathcal{O}[\mathcal{L}])$ then it is easy to verify that:

- (a) there is a canonical isomorphism $f^*(\mathcal{L}) \simeq \Omega_{(\mathcal{L})/B}^1$;
- (b) if $\mathbf{0}: B \rightarrow \mathbb{V}(\mathcal{L})$ is the canonical section, there is a canonical isomorphism $\widehat{\mathbb{V}(\mathcal{L})}_{\mathbf{0}} \simeq \text{Spf}(\mathcal{O}[[\mathcal{L}]])$.

Let $f: \mathcal{X} \rightarrow \text{Spec}(O_K)$ be an arithmetic surface as in the previous section and let $D: \text{Spec}(O_L) \rightarrow \mathcal{X}$ be a reduced divisor over \mathcal{X} .

Let $f_L: \mathcal{X}_L \rightarrow \text{Spec}(O_L) := B_L$ be a desingularization of the arithmetic surface $\mathcal{X} \times_B \text{Spec}(O_L)$. The morphism D give rise to a section $S_D: B_L \rightarrow \mathcal{X}_L$; moreover, if $p: \mathcal{X}_L \rightarrow \mathcal{X}$ is the natural projection, by construction we have that $p \circ S_D = D$.

We may suppose that $S_D(B_L)$ is contained in the smooth open set of the structural morphism f_L . consequently we can find an open neighborhood \mathcal{U} of $S_D(B_L)$ in \mathcal{X}_L and an étale map $g_D: \mathcal{U} \rightarrow \mathbb{V}(\mathcal{O}(-S_D)|_{S_D})$ sending $S_D(B_L)$ to the zero section. From this we deduce

3.1 Proposition. *Let $(\widehat{\mathcal{X}}_L)_D$ be the completion of \mathcal{X}_L around S_D ; then there is an isomorphism*

$$\Psi_D: (\widehat{\mathcal{X}}_L)_D \longrightarrow \mathrm{Spf}(\mathcal{O}[\mathcal{O}(-S_D)|_{S_D}]).$$

Let $D_1: \mathrm{Spec}(O_{L_1}) \rightarrow \mathcal{X}_1$ and $D_2: \mathrm{Spec}(O_{L_2}) \rightarrow \mathcal{X}_2$ be divisors on \mathcal{X}_1 and \mathcal{X}_2 respectively. As before they define two sections $S_i: \mathrm{Spec}(O_L) \rightarrow \mathcal{X}_L$ ($i = 1, 2$), $L := L_1 \cdot L_2$ being the composite of L_1 and L_2 over K . Let $\xi_{D_1, D_2}: B_L \rightarrow (\mathcal{X}_1 \times \mathcal{X}_2)_L$ be the point obtained from S_1 and S_2 and denote by $(\widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2)_{\xi_{D_1, D_2}}$ the completion of $(\mathcal{X}_1 \times \mathcal{X}_2)_L$ around ξ_{D_1, D_2} ; by the discussion above we obtain an isomorphism

$$\Psi_{D_1, D_2}: (\widehat{\mathcal{X}}_1 \times \widehat{\mathcal{X}}_2)_{\xi_{D_1, D_2}} \longrightarrow \mathrm{Spf}(\mathcal{O}[\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2}]).$$

Let M_L be the set of places of L ; and $\sigma \in M_L$ be an infinite place. The O_L -algebra $(\mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2})]_{\sigma})$ is naturally equipped with the structure of *hermitian algebra* because of the choice of the metrics as in the first section: the hermitian structure being the direct sum structure. If $J \subset \mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2})]$ is an ideal supported on the zero section, the O_L module $\mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2})]/J$ is naturally equipped with the structure of *hermitian module* over O_L ; moreover, if $J \subset J'$ are two ideals as above the natural surjection $\mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2})]/J \rightarrow \mathcal{O}[(\mathcal{O}(-S_1)|_{S_1}; \mathcal{O}(-S_2)|_{S_2})]/J'$ is an isometry: the metric induced by the surjection is the given metric.

If $p_i: (\mathcal{X}_1)_L \times (\mathcal{X}_2)_L \rightarrow (\mathcal{X}_i)_L$ is the natural projection, and N is a line bundle on $(\mathcal{X}_i)_L$, by abuse of notation, we will denote again by N the line bundle $p_i^*(N)$ on $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$.

In this section we will construct sections of small norm of suitable line bundles with high order of vanishing along $\xi_{1,2}$. As usual the key lemma is the Siegel Lemma. Before we give the statement (and the proof) of the Siegel Lemma we need, we recall without proof all the tools we need; for the proofs we refer to [Bo] §4.1 and [Sz]:

- a) If E is an hermitian vector bundle over O_K , then we call the real number $\mu_n(E) := \frac{1}{[K;\mathbb{Q}]} \cdot \frac{\widehat{\mathrm{deg}}(E)}{rk(E)}$, *the slope of E* ;
- b) within all the sub bundles of a given hermitian vector bundle E , there is one having maximal slope; we call its slope *the maximal slope of E* and denote it by $\mu_{\max}(E)$;
- c) if E_1 and E_2 are two hermitian vector bundles, we have that $\mu_{\max}(E_1 \oplus E_2) = \max\{\mu_{\max}(E_1); \mu_{\max}(E_2)\}$;
- d) let $f: E \rightarrow F$ be an *injective* morphism between hermitian vector bundles; then $\widehat{\mathrm{deg}}(E) \leq rk(E)(\mu_{\max}(F) + \log\|f\|)$;
- e) There is a constant $\chi(K)$ depending only on K (for the precise value we refer to [Sz]) such that, if E is an hermitian vector bundle on K with $\widehat{\mathrm{deg}}(E) > -rk(E)\chi(K)$, then there is a non torsion element $v \in E$ such that, for every infinite place σ we have $\|v\|_{\sigma} \leq 1$; we define $\|\cdot\|_{\sup}$ to be $\sup\{\|\cdot\|_{\sigma}\}_{\sigma \in M_{\infty}}$;
- f) Let M_{∞} be the set of infinite places of K and $\lambda := (\lambda_{\sigma})_{\sigma \in M_{\infty}}$ be an element of $\mathbb{R}^{[K;\mathbb{Q}]}$ with $\lambda_{\sigma} = \lambda_{\bar{\sigma}}$; we denote by $\mathcal{O}(\lambda)$ the hermitian line bundle $(O_K, \|1\|_{\sigma} = \exp(-\lambda_{\sigma}))$. If E is an hermitian vector bundle over O_K then we denote by $E(\lambda)$ the hermitian vector bundle $E \otimes \mathcal{O}(\lambda)$.

g) (Hilbert–Samuel Formula) There is a constant C , depending on the choices made (but not on the d_i 's), such that, if d_1 and d_2 are sufficiently big, the Hermitian \mathcal{O}_K -module $H^0 = (\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{L}_1^{d_1} \otimes \mathcal{L}_2^{d_2})$ is generated by elements of sup-norm, less or equal then $C^{d_1+d_2}$.

We will also need the following

3.2 Lemma. *Let*

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$$

be an exact sequence of hermitian vector bundles; then

$$\mu_{\max}(E) \leq \max\{\mu_{\max}(E_1), \mu_{\max}(E_2)\}.$$

The proof is straightforward and left to the reader.

The Siegel Lemma we need is the following:

3.3 Lemma. (*Siegel Lemma*) *Let V and W be hermitian vector bundles over \mathcal{O}_K ; Let $\gamma: V \rightarrow W$ be a non injective morphism. Suppose that there is a constant $C > 1$ such that:*

i) V is generated by elements with sup norm at most C ;

ii) for every $\sigma \in M_\infty$ the norm of γ at the place σ is at most C ;

then, if $m = rk(V)$ and $n := rk(Ker(\gamma))$, there is a non zero element $x \in Ker(\gamma)$ with

$$\sup_{\sigma \in M_\infty} \{\log(\|x\|_\sigma)\} \leq \frac{m}{n} \cdot \log(C^2) + \left(\frac{m}{n} - 1\right) \mu_{\max}(W) - \frac{\chi(K)}{[K : \mathbb{Q}]}.$$

Proof: Denote by U the hermitian vector bundle $Ker(\gamma)$ with the induced metric. Observe that, by property (e) above, if $\widehat{\deg}(U(\lambda)) > -n\chi(K)$, then there is a non torsion element $x \in U$ such that

$$\sup_{\sigma \in M_\infty} \{\log(\|x\|_\sigma)\} \leq \sup_{\sigma \in M_\infty} \{\lambda_\sigma\}.$$

An easy computation gives $\widehat{\deg}(U(\lambda)) = \widehat{\deg}(U) + n \cdot \sum_\sigma \lambda_\sigma$. Let W' be the image of γ with the metric induced by the surjection. Thus we have

$$\widehat{\deg}(U(\lambda)) = \widehat{\deg}(V) - \widehat{\deg}(W') + n \cdot \sum_\sigma \lambda_\sigma.$$

By property (d) we have $\frac{\widehat{\deg}(W')}{[K:\mathbb{Q}]} \leq (m-n)(\mu_{\max}(W) + \log(C))$ and by the very definition of Arakelov degree we have $\widehat{\deg}(V) \geq -m[K : \mathbb{Q}] \log(C)$. Consequently

$$\begin{aligned} \widehat{\deg}(U(\lambda)) &= \widehat{\deg}(V) - \widehat{\deg}(W') + n \cdot \sum_{\sigma \in M_\infty} \lambda_\sigma \\ &\geq -2m[K : \mathbb{Q}] \log(C) - (m-n)[K : \mathbb{Q}] \mu_{\max}(W) + n \cdot \sum_{\sigma \in M_\infty} \lambda_\sigma; \end{aligned}$$

thus if we take $\lambda_\sigma = \frac{m}{n} \cdot \log(C^2) + (\frac{m}{n} - 1)\mu_{\max}(W) - \frac{\chi(K)}{[K:\mathbb{Q}]} + \epsilon$ and applying the observation above we conclude.

Let ϑ_1, ϑ_2 and δ be three positive rational numbers. For every couple of positive integers (d_1, d_2) we denote by $\mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}}$ the ideal sheaf of $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$ defined by

$$\sum_{\substack{\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \geq \delta \\ i \leq d_1, j \leq d_2}} \mathcal{O}(-iS_1) \otimes \mathcal{O}(-jS_2) \quad (3.4.1).$$

In the same way, we will denote by $I_{\underline{\vartheta}, \delta, \underline{d}}$ the ideal of $\mathcal{O}[\mathcal{O}(-S_1)|_{S_1}, \mathcal{O}(-S_2)|_{S_2}]$ given by the analogue condition of 3.4.1.

We denote by $\mathcal{A}_{\underline{\vartheta}, \delta, \underline{d}}$ the subscheme of $(\mathcal{X}_1)_L \times (\mathcal{X}_2)_L$ defined by the ideal $\mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}}$ and by $W_{\underline{\vartheta}, \delta, \underline{d}}$ the O_L algebra $\mathcal{O}[\mathcal{O}(-S_1)|_{S_1}, \mathcal{O}(-S_2)|_{S_2}]/I_{\underline{\vartheta}, \delta, \underline{d}}$. Then

- (i) the isomorphism $\Psi_{D_1^{h_1}, D_2^{h_2}}$ induces an isomorphism $\Psi_{1,2}: \mathcal{A}_{\underline{\vartheta}, \delta, \underline{d}} \rightarrow \text{Spec}(W_{\underline{\vartheta}, \delta, \underline{d}})$;
- (ii) The O_L module $W_{\underline{\vartheta}, \delta, \underline{d}}$ has a natural structure of hermitian O_L -module.

Moreover the O_L -module $W_{\underline{\vartheta}, \delta, \underline{d}}$ has a filtration by O_L -submodules F^μ such that $F^\mu/F^{\mu+1} \simeq \mathcal{O}(-iS_1)|_{S_1} \otimes \mathcal{O}(-jS_2)|_{S_2}$ with $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \leq \delta$; this filtration is isometric.

3.4 Proposition. *Let $\epsilon > 0$ be a rational number and $\delta := 2 + \epsilon$; suppose that $\vartheta_1 \cdot \vartheta_2 > (2 + \epsilon)[L : K] + \epsilon$, then there exists a constant A depending only on $\mathcal{X}_i, M_i, [L, K], \vartheta_i$ and ϵ such that the following holds:*

For every couple of irreducible divisor $D_1 \hookrightarrow \mathcal{X}_1$, and $D_2 \hookrightarrow \mathcal{X}_2$ as above and every couple of sufficiently big integers (d_1, d_2) , there is a non zero section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2})$ vanishing along $\mathcal{A}_{\underline{\vartheta}, \delta, \underline{d}}$ and such that, for every infinite place $\sigma \in M_K$ we have

$$\log(\|f\|_\sigma) \leq \frac{A}{\epsilon} \cdot T(D_1) \cdot T(D_2) \cdot (d_1 + d_2);$$

where the $T(D_i)$ are defined as in §2.

Proof: Let $\gamma_{d_1, d_2}: H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2}) \rightarrow W_{\underline{\vartheta}, \delta, \underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2}$ be the composite of the inclusion map $i: H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2}) \hookrightarrow H^0((\mathcal{X}_1)_L \times (\mathcal{X}_2)_L, M_1^{d_1} \times M_2^{d_2})$ and the restriction $H^0((\mathcal{X}_1)_L \times (\mathcal{X}_2)_L, M_1^{d_1} \times M_2^{d_2}) \rightarrow W_{\underline{\vartheta}, \delta, \underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2}$ and let $K(d_1, d_2)$ be its kernel. We have to prove that there exists an element in $K(d_1, d_2)$ having bounded norm.

By 3.2 and (ii) above we have that $\mu_{\max}(W_{\underline{\vartheta}, \delta, \underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2}) \leq A \cdot S(D_1) \cdot S(D_2) \cdot H(D_1) \cdot H(D_2)$ where A is some absolute constant. By (g) above, $H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2})$ is generated by elements with norm bounded by $A^{d_1+d_2}$ where A is again a suitable constant (depending only on the hypotheses).

Now we come to the main part of the proof: the O_K -module has rank which is bounded below by $C \cdot d_1 \cdot d_2$. The rank of the O_L -module $W_{\underline{\vartheta}, \delta, \underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2}$ can be bounded from above as follows: the number of the terms of the filtration described in (ii) is the number of couples of positive integers (i, j) with $i \leq d_1, j \leq d_2$ and

$\frac{i}{d_1} + \frac{j}{d_2} \leq \delta$; as soon as d_1 and d_2 are sufficiently big, this number is bounded above by $d_1 \cdot d_2$ multiplied by the area of the triangle with vertices $(0, 0)$, $(\vartheta_1 \cdot \delta, 0)$, and $(0, \vartheta_2 \cdot \delta)$ plus a very small error term, consequently

$$rk_{O_K} \left(W_{\underline{\vartheta}, \delta, \underline{d}} \otimes (M_1^{d_1})|_{S_1} \otimes (M_2^{d_2})|_{S_2} \right) \leq d_1 \cdot d_2 \cdot \frac{(2 + \epsilon)^2}{2\vartheta_1 \cdot \vartheta_2} [L : K] + \epsilon'.$$

From this we obtain that there is a constant A , independent on d_1 and d_2 , such that

$$\frac{rk_{O_K} \left(h^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \times M_2^{d_2}) \right)}{rk_{O_K} (K(d_1, d_2))} \leq \frac{A}{\epsilon}.$$

For every infinite place σ of K , we cover the Riemann surface $\mathcal{X}_{i,\sigma}$ with a finite number of disks over which the line bundle M_i trivializes; inside each disk we take a disk with same center and radius one half of the radius of it; we suppose that also these smaller disks cover the Riemann surface. From the lemma 3.5 below we deduce that we can find a constant A , independent on the D_i 's, such that for every infinite place σ we have $\|\gamma_{d_1, d_2}\|_\sigma \leq A^{d_1 + d_2}$. We apply now 3.3 to this situation and conclude the proof of the proposition.

3.5 Lemma. *Let Δ_R be the disk of radius R . Let $f(x, y)$ be an holomorphic function on $\Delta_R \times \Delta_R$ and $(z_1, z_2) \in \Delta_{R/2} \times \Delta_{R/2}$ then for every (i, j)*

$$\left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} (z_1, z_2) \right| \leq \frac{2^{i+j} i! j!}{R^{i+j}} \cdot \max_{\|x\|=\|y\|=R} \{|f(x, y)|\}.$$

The proof of the Lemma is a straightforward application of the maximum modulus principle and the Cauchy inequality.

4 Index Theorem.

In this section we prove that the section we constructed has a small order of vanishing along a point verifying the inequality of the main theorem. The theorem we need is an analogue of the "Roth index theorem" in this contest. In a first version of the paper we deduced the index theorem from a generalization of the Vojta version of Dyson lemma for curves [Vo]; but, due to the "admissibility hypothesis" in this kind of theorems, this could be applied only in the case when *both* the D_i 's are irreducible.

Let \mathcal{X}_1 and \mathcal{X}_2 be the arithmetic surfaces. Let M be a line bundle over $(\mathcal{X}_1 \times \mathcal{X}_2)_K$ and $f \in H^0((\mathcal{X}_1 \times \mathcal{X}_2)_K, M)$. We fix two positive rational numbers $\vartheta_i \geq 1$.

Let d_1 and d_2 be two positive integers such that $d_i/\vartheta_i \in \mathbb{Z}$.

Let $P := (P_1, P_2) \in (\mathcal{X}_1 \times \mathcal{X}_2)_K(K)$ be a point and z_i be local coordinate around P_i in X_i ($i = 1, 2$). Let e be a local generator of M around P ; consequently, near P , we

can write $f = g \cdot e$ where g is a regular function around P . We will say that f has index at least δ in P with respect to d_1 and d_2 and we will write $\text{ind}_P(f, d_1, d_2) \geq \delta$ if, near P , we have $g = \sum_{i,j} a_{i,j} z_1^i \cdot z_2^j$ with $a_{i,j} = 0$ whenever

$$\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \leq \delta.$$

The subset of $(\mathcal{X}_1 \times \mathcal{X}_2)_K$ of points having index at least δ is a (possibly empty) closed set which will be denoted $Z_\delta(f)$ (if the dependence on the d_i 's is clear from the context).

Let M_i be the line bundles of generic degree one on \mathcal{X}_i ($i = 1, 2$) fixed in the previous section. As in the previous section we will denote by M_i the line bundle $pr_i^*(M_i)$ on $X_1 \times X_2$ ($pr_i: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{X}_i$ being the natural projection).

4.1 Theorem. *Let C and ϵ be positive real numbers. Then we can find constants $B_j = B_j(C, \epsilon)$ depending only on C , the ϑ_1 , and ϵ (and on the other choices made until now, but independent on the d_i 's, having the following property:*

Suppose that:

- (a) $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2; M_1^{d_1} \otimes M_2^{d_2})$ is a global section with $\sup_{\sigma \in M_\infty} \{\|f\|_\sigma\} \leq C^{(d_1+d_2)}$;
- (b) the d_i 's are sufficiently big and divisible and $d_1/d_2 \geq B_1$;
- (c) $P := (P_1, P_2) \in \mathcal{X}_1 \times \mathcal{X}_2(K)$ is a rational point such that

$$B_2 \leq h_{M_1}(P_1) \quad \text{and} \quad \frac{h_{M_2}(P_2)}{h_{M_1}(P_1)} \geq \frac{d_1}{d_2};$$

then

$$\text{ind}_P(f, d_1, d_2) \leq \epsilon.$$

4.2 Remark. The proof of the statement above is directly inspired by the Faltings product theorem [Fa] and can be deduced from it; we propose here a self contained proof (which, to our taste, is simpler than the proof of the product theorem in this situation).

We will use the following standard facts from the height theory of subvarieties, one can find the proofs on [Fa2] or on [Ev]; if Z is a closed subscheme of $\mathcal{X}_1 \times \mathcal{X}_2$ and M is an hermitian line bundle, then we denote by $h_M(Z)$ the height of Z with respect to M as defined in [BGS]:

- (a) Suppose that Z_i are closed irreducible reduced subschemes of \mathcal{X}_i of relative dimension δ_i (over \mathbb{Z}) then

$$h_{M_1^{d_1} + M_2^{d_2}}(Z_1 \times Z_2) = (\delta_1 + \delta_2 + 1)! \cdot d_1^{\delta_1} \cdot d_2^{\delta_2} \cdot \left(\frac{d_1 \cdot h_{M_1}(Z_1)}{(\delta_1 + 1)!} + \frac{d_2 \cdot h_{M_2}(Z_2)}{(\delta_2 + 1)!} \right);$$

this is proved in [Ev Lemma 8].

- (b) Suppose that $\mathcal{X}_i = \mathbb{P}^1$ and $M_i = \mathcal{O}(1)$ and C is a real constant then we can find a constant R depending only on C and the chosen metrics such that the following

holds: suppose that $f_1, \dots, f_r \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{O}(d_1) + \mathcal{O}(d_2))$ are integral global sections such that $\sup_{\sigma \in M_\infty} \{\|f_i\|_\sigma\} \leq C$ and Y is the subscheme of $\mathcal{X}_1 \times \mathcal{X}_2$ defined as the zero set of $\{f_1, \dots, f_r\}$; Let X be an irreducible component of Y with multiplicity m_X then

$$m_X \cdot h_{\mathcal{O}(d_1) + \mathcal{O}(d_2)}(X) \leq R \cdot d_1 \cdot d_2 \cdot (d_1 + d_2);$$

this is proved in [Fa2 Prop.2.17] or [Ev Lemma 9].

(c) If $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} + M_2^{d_2})$ then

$$\begin{aligned} & h_{M_1^{d_1} + M_2^{d_2}}(\operatorname{div}(f)) = \\ & h_{M_1^{d_1} + M_2^{d_2}}(\mathcal{X}_1 \times \mathcal{X}_2) + \sum_{\sigma \in M_\infty} \int_{(\mathcal{X}_1 \times \mathcal{X}_2)_\sigma} \log \|f\|_\sigma (c_1(M_1^{d_1} + M_2^{d_2})_\sigma)^2; \end{aligned}$$

this is a direct consequence of the definition of height (cf. [BGS]); consequently (using point (a)), given a positive constant C we can find a positive constant R , depending only on C , such that the following holds: if $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} + M_2^{d_2})$ is such that $\sup_{\sigma \in M_\infty} \{\|f\|_\sigma\} \leq C$ then

$$h_{M_1^{d_1} + M_2^{d_2}}(\operatorname{div}(f)) \leq R \cdot d_1 \cdot d_2 \cdot (d_1 + d_2 + \log(C)).$$

Proof of 4.1: Let f be the given section and Z be a geometrically irreducible reduced component of $Z_\epsilon(f)$. Extending K if necessary, we may suppose that Z is defined over K . It suffices to prove that, under the hypotheses of the theorem (with explicit and suitable B_i 's) the point P do not belong to Z . There are two cases, depending on the dimension of Z .

Case 1: Dimension of Z equal to one: Let $Y := \operatorname{div}(f)$; since Z is a divisor contained in Y we have

$$Y = m_Z \cdot Z + D;$$

where D is an effective divisor on $(\mathcal{X}_1 \times \mathcal{X}_2)_K$. We claim that, if $d_1/d_2 \geq \vartheta_1/(\epsilon \cdot \vartheta_2)$ then either there is a point $A \in \mathcal{X}_2(K)$ such that $Z = (\mathcal{X}_1)_K \times \{A\}$ or there is a point $B \in \mathcal{X}_1(K)$ such that $Z = \{B\} \times (\mathcal{X}_2)_K$:

4.3 Lemma. *Suppose that Z is not as claimed and $\frac{d_1}{\vartheta_1} \geq \frac{d_2}{\vartheta_2}$, then $m_Z \geq \epsilon \cdot \frac{d_1}{\vartheta_1}$.*

Let's show how the lemma implies the claim: Suppose that Z is not as claimed, then $(Z; M_1) > 0$; consequently

$$d_2 = (Y; M_1) \geq \epsilon d_1 (Z; M_1) > \epsilon \cdot \frac{d_1}{\vartheta_1},$$

so $d_1/d_2 \leq \vartheta_1/\epsilon$; contradiction.

Proof (of the lemma): Let η be a generic point of Z not contained in D ; we may suppose that the restriction of both projections are étale in a neighborhood of η . Let z_1 and z_2 be local coordinates about the projections of η . In a formal neighborhood of η , the

divisor Z is defined by a irreducible element $h \in K[[z_1, z_2]]$ and Y is defined by the ideal (h^{m_Z}) ; because of our choice of η , we have $h(z_1, z_2) = a_{10}z_1 + a_{01}z_2 + O((z_1 + z_2)^2)$ with $a_{01} \cdot a_{10} \neq 0$, moreover, by definition of $Z_\epsilon(f)$,

$$(h(z_1, z_2))^{m_Z} = \sum_{i,j} a_{ij} \cdot z_1^i \cdot z_2^j$$

with $a_{ij} = 0$ whenever $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \leq \epsilon$.

Consider the covering

$$\begin{aligned} \varphi: K[[z_1, z_2]] &\longrightarrow K[[w_1, w_2]] \\ z_1 &\longmapsto w_1^{\frac{d_2}{\vartheta_2}} \\ z_2 &\longmapsto w_2^{\frac{d_1}{\vartheta_1}}; \end{aligned}$$

by construction $\varphi(h(z_1, z_2)) = a_{10} \cdot w_1^{d_2/\vartheta_2} + a_{01} \cdot w_2^{d_1/\vartheta_1} + O((w_1^{d_2/\vartheta_2} + w_2^{d_1/\vartheta_1})^2)$ and $\varphi(h(z_1, z_2)^{m_Z}) = \sum_{h,k} b_{hk} \cdot w_1^h \cdot w_2^k$ with $\frac{h \cdot \vartheta_1 \cdot \vartheta_2}{d_1 d_2} + \frac{k \cdot \vartheta_1 \cdot \vartheta_2}{d_1 d_2} \geq \epsilon$. On the other side, $\varphi(h(z_1, z_2)^{m_Z}) = a_{10}^{m_Z} \cdot w_1^{\frac{d_2}{\vartheta_2} \cdot m_Z} + O(w_1 + w_2)^{m_Z \cdot d_2/\vartheta_2 + 1}$ (here we use the fact that $d_1/\vartheta_1 > d_2/\vartheta_2$) consequently

$$m_Z \cdot \frac{d_2}{\vartheta_2} \geq \epsilon \cdot \frac{d_1 \cdot d_2}{\vartheta_1 \cdot \vartheta_2};$$

the lemma follows.

We now come to the arithmetic part of the proof, in this case: Z is either $(\mathcal{X}_1)_K \times \{A\}$ or $\{B\} \times (\mathcal{X}_2)_K$ for suitable A and B . It is easy to see that m_Z is $\epsilon \cdot \frac{d_2}{\vartheta_2}$ in the first case and $\epsilon \cdot \frac{d_1}{\vartheta_1}$ in the second (simply compute it on a smooth point of the support of Z and Y). In the first case, by applying point (a) and (c) above, the additive property of the heights, and the hypotheses, we obtain:

$$\begin{aligned} m_Z \cdot h_{M_1^{d_1} + M_2^{d_2}}(Z) &= m_Z \cdot d_1(d_1 \cdot h_{M_1}(\mathcal{X}_1) + 2d_2 \cdot h_{M_2}(A)) \\ &\leq h_{M_1^{d_1} + M_2^{d_2}}(\text{div}(f)) \leq R_1 \cdot d_1 \cdot d_2(d_1 + d_2) \end{aligned}$$

where R_1 is an explicit constant depending only on C ; consequently

$$\frac{\epsilon \cdot d_1 \cdot d_2}{\vartheta_2} \cdot (d_1 \cdot h_{M_1}(\mathcal{X}_1) + 2d_2 \cdot h_{M_2}(A)) \leq R_1 \cdot d_1 \cdot d_2(d_1 + d_2)$$

because $m_Z = \epsilon \cdot \frac{d_2}{\vartheta_2}$; this implies that

$$d_2 \cdot h_{M_2}(A) \leq \frac{R_2 \cdot \vartheta_2}{\epsilon}(d_1 + d_2);$$

similarly, in the second case we obtain

$$\frac{\epsilon \cdot d_1 \cdot d_2}{\vartheta_1} \cdot (2 \cdot h_{M_1}(B) + d_2 \cdot h_{M_2}(\mathcal{X}_2)) \leq R_1 \cdot d_1 \cdot d_2 \cdot (d_1 + d_2),$$

thus

$$h_{M_1}(B) \leq \frac{R_2}{\epsilon} \cdot \left(1 + \frac{d_2}{d_1}\right).$$

This implies that, if $d_1/d_2 \geq 1$, the point (P_1, P_2) cannot be on Z as soon as $\frac{d_2}{d_1} \cdot h_{M_2}(P_2) \geq h_{M_1}(P_1) \geq \frac{2R_2}{\epsilon} \cdot \max\{\vartheta_1, \vartheta_2\}$.

Case 2: Dimension of Z equal to zero : Denote by $(P, Q) \in (\mathcal{X}_1 \times \mathcal{X}_2)_K(K)$ the support of Z . In this case we need to project on $\mathbb{P}^1 \times \mathbb{P}^1$. We fix once for all a finite set of coverings $\gamma_{i,j}: (\mathcal{X}_i)_K \rightarrow \mathbb{P}^1$ with the property that, if $U_{i,j} \subseteq (\mathcal{X}_i)_K$ is the open set over which $\gamma_{i,j}$ is étale, then $\bigcup_j U_{i,j} = (\mathcal{X}_i)_K$ and $\gamma_{i,j}^*(\mathcal{O}(1)) \simeq (M_i)_K^{t_i}$ for suitable t_i (we fix such isomorphisms). We also suppose that each $\gamma_{i,j}$ extends to a generically finite morphism $\gamma_{i,j}: \mathcal{X}_i \rightarrow \mathbb{P}_{O_K}^1$ (this can always be obtained after a suitable blow up of \mathcal{X}_i). We equip the line bundle $\mathcal{O}(1)$ on \mathbb{P}^1 with the Fubini-Study metric $\|\cdot\|_{FS}$ and we fix a constant A such that

$$A^{-1}\gamma_{i,j}(\|\cdot\|_{FS}) \leq \|\cdot\|_{M_i}^{t_i} \leq A\gamma_{i,j}(\|\cdot\|_{FS})$$

We may suppose that $(P, Q) \in (\mathcal{X}_1 \times \mathcal{X}_2)(K)$ is contained in $U_{1,1} \times U_{2,1}$. Denote by Γ the morphism $\gamma_{1,1} \times \gamma_{1,2}: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Put $d_i = t_i \cdot a_i$; then $\Gamma^*(\mathcal{O}(a_1, a_2)) = M_1^{d_1} + M_2^{d_2}$ and $g := \Gamma_*(f) \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1 \cdot t_2, d_2 \cdot t_1))$. It is easy to verify that there exists an absolute constant A_1 such that

$$\|g\|_{FS} \leq A_1^{(d_1+d_2)} \|f\|$$

and that $(P', Q') := \Gamma(P, Q)$ is contained in $Z_\epsilon(g)$. Consequently it suffices to prove the theorem with $\mathcal{X}_1 = \mathcal{X}_2 = \mathbb{P}^1$ and $M_1 = M_2 = \mathcal{O}(1)$ equipped with the Fubini-Study metric.

We first look to the irreducible components Z' of $Z_{\epsilon/2}$ containing (P', Q') . If there is such a Z' of dimension one, then we are reduced to the previous case and we are done. We may then suppose that the support of Z' is (P', Q') too. Let I_ϵ and $I_{\epsilon/2}$ be the ideal of Z and Z' in the completion $K[[z_1, z_2]]$ of the local ring of $\mathbb{P}^1 \times \mathbb{P}^1$ in (P', Q') ; let $h = \alpha \cdot z_1^{r_1} z_2^{r_2} + \dots$ be an element of $I_{\epsilon/2}$ then

$$\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \cdot \partial z_2^{i_2}} h = \alpha_1 z_1^{(r_1-i_1)'} z_2^{(r_2-i_2)'} + \dots$$

(where $(a)' := \sup\{a, 0\}$) for a suitable α_1 ; and α_1 can be zero only if α is zero or $\alpha \neq 0$ and one of the $(r_j - i_j)'$ is zero. If $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \frac{\epsilon}{2}$ and $h \in I_{\epsilon/2}$ then $\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \cdot \partial z_2^{i_2}} h \in I_\epsilon \subseteq (z_1, z_2)$. This implies that h , and consequently $I_{\epsilon/2}$ is contained in $(z_1^{\epsilon d_1/(4 \cdot \vartheta_1)}, z_2^{\epsilon d_2/(4 \cdot \vartheta_2)})$. Thus the multiplicity of Z' in $Z_{\epsilon/2}(g)$ is at least $\frac{1}{\vartheta_1 \cdot \vartheta_2} \cdot \left(\frac{\epsilon}{4}\right)^2 \cdot d_1 d_2$.

Every differential operator $\frac{\partial^{i_1+i_2}}{\partial z_1^{i_1} \partial z_2^{i_2}}$ with $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \frac{\epsilon}{2}$ can be seen as a linear endomorphism $D^{(i_1, i_2)}$ of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d_1, d_2))$. For every infinite place $\sigma \in M_\infty$ the norm of the operator $D^{(i_1, i_2)}$ (with i_1 and i_2 bounded as above) is bounded from above

by $2^{\max\{\vartheta_1, \vartheta_2\} \cdot (d_1 + d_2)}$; thus applying point (b) above and the hypotheses we obtain

$$m_{Z'} \cdot (d_1 \cdot h_{\mathcal{O}(1)}(P') + d_2 \cdot h_{\mathcal{O}(1)}(Q')) \leq R' \cdot d_1 \cdot d_2 \cdot (d_1 + d_2)$$

where R' depends only on A ; consequently, since $m_{Z'} \geq \frac{1}{\vartheta_1 \cdot \vartheta_2} \cdot \left(\frac{\epsilon}{4}\right)^2 \cdot d_1 \cdot d_2$,

$$d_1 \cdot h_{\mathcal{O}(1)}(P') + d_2 \cdot h_{\mathcal{O}(1)}(Q') \leq \vartheta_1 \cdot \vartheta_2 \cdot \left(\frac{4}{\epsilon}\right)^2 \cdot R'(d_1 + d_2).$$

If we suppose that $\frac{1}{\vartheta_1 \cdot \vartheta_2} \cdot \left(\frac{4}{\epsilon}\right)^2 \cdot R' \leq h_{\mathcal{O}(1)}(P) \leq h_{\mathcal{O}(1)}(Q)$ the point (P, Q) cannot belong to Z' and this concludes the proof of the theorem.

5 Generalized Cauchy inequalities.

Fix $\vartheta_i \in \mathbb{Q}_{\geq 1}$ and the divisors $D_i: \text{Spec}(O_L) \rightarrow \mathcal{X}_i$ as in the previous sections. For every rational positive δ and couple of positive integers (d_1, d_2) , let $\mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}}$ be the ideal sheaf of $\mathcal{X}_1 \times \mathcal{X}_2$ defined in section 3 and let $p: \tilde{\mathcal{X}}_\delta \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ be the blow up along it and let E_δ be the corresponding exceptional divisor on it. Since, for every infinite place σ , we fixed a (symmetric) metric on $\mathcal{O}_{(\mathcal{X}_i \times \mathcal{X}_i)_\sigma}(\Delta_i)$, the line bundle $\mathcal{O}_{\tilde{\mathcal{X}}_\sigma}(E_\delta)$ is naturally equipped with the structure of *hermitian line bundle*.

Let M be an hermitian line bundle on $\mathcal{X}_1 \times \mathcal{X}_2$; by abuse of notation, we will denote again by M the pull back of M to $\tilde{\mathcal{X}}_\delta$.

If $P_i \in \mathcal{X}_i(K)$ are K -rational points of \mathcal{X}_i , they extend to sections $P_i: \text{Spec}(O_K) \rightarrow \mathcal{X}_i$. We will denote by $P: \text{Spec}(O_K) \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ the section $P_1 \times P_2$ and by $\tilde{P}: \text{Spec}(O_K) \rightarrow \tilde{\mathcal{X}}_\delta$ the strict transform of P .

The theorem we want to prove in this section is the following:

5.1 Theorem. *There is a constant C depending only on the models, the metrics, the ϑ_i 's etc. but independent on the d_i 's for which the following holds:*

Suppose that there exists a global section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M \otimes \mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}})$ such that $\sup_{\sigma \in M_\infty} \{\|f\|_\sigma\} \leq A$ and a rational point $P := P_1 \times P_2: \text{Spec}(O_K) \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ such that $\text{ind}_P(f, d_1, d_2) \leq \epsilon$; then there exists $\epsilon' \leq \epsilon$, two positive integers i_1 and i_2 and a non zero global section $\tilde{f} \in H^0(\tilde{P}, M \otimes \omega_{\mathcal{X}_1/B}^{i_1} \otimes \omega_{\mathcal{X}_2/B}^{i_2} \otimes \mathcal{O}(-E_{\delta-\epsilon'}))$ such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \epsilon$ and

$$\sup_{\sigma \in M_\infty} \{\|\tilde{f}\|_\sigma\} \leq A \cdot C^{(d_1 + d_2)}.$$

Before we start the proof of the theorem, we need to introduce some notations and some tools.

Let \mathcal{L}_1 and \mathcal{L}_2 be two line bundles on $\text{Spec}(O_K)$. For every couple of positive integers (i_1, i_2) we define the differential operator

$$D^{(i_1, i_2)}: \mathcal{O}[\mathcal{L}_1, \mathcal{L}_2] \longrightarrow \mathcal{O}[\mathcal{L}_1, \mathcal{L}_2] \otimes \mathcal{L}_1^{i_1} \otimes \mathcal{L}_2^{i_2}$$

in the following way: let e_1 (resp. e_2) be a local generator of \mathcal{L}_1 (resp. of \mathcal{L}_2) then we define

$$D^{(i_1, i_2)}(e_1^a \otimes e_2^b) := \binom{a}{i_1} \cdot \binom{b}{i_2} \cdot e_1^{a-i_1} \otimes e_2^{b-i_2} \otimes (e_1^{i_1} \otimes e_2^{i_2})$$

and extend it linearly to $\mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]]$; one easily verify that this definition do not depends on the choice of the local generators. The module $\mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]] \otimes \mathcal{L}_1^{i_1} \otimes \mathcal{L}_2^{i_2}$ has a natural structure of $\mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]]$ -module (multiplication on the right) and one easily verify that $D^{(i_1, i_2)}$ is a differential operator: it is O_K -linear (by definition) and it satisfy the (iterated) Leibnitz-rule; for instance $D^{n,0}(f \cdot g) = \sum \binom{n}{i} \cdot D^{(i,0)}(f) \cdot D^{(n-i,0)}(g)$ ($D^{(i,0)}(f) \in \mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]] \otimes \mathcal{L}_1^i$ and $D^{(n-i,0)}(g) \in \mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]] \otimes \mathcal{L}_1^{n-i}$, consequently $D^{(i,0)}(f) \cdot D^{(n-i,0)}(g) \in \mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]] \otimes \mathcal{L}_1^n$).

If $\sigma \in M_\infty$ is an infinite place, then $\mathcal{O}[[\mathcal{L}_1, \mathcal{L}_2]]_\sigma$ is (non canonically) isomorphic to the ring of formal power series in two variables and the operators $D^{(a,b)}$ are the usual iterated derivatives.

Although it is not necessary, we will tacitly authorize ourself to pass to the Hilbert class field extension: consequently we will suppose that every line bundle on B is trivial; this is not necessary, but highly simplify the notations.

In the same way as in 3.1, denoting by $(\mathcal{X}_1 \widehat{\times} \mathcal{X}_2)_P$ the formal completion of $\mathcal{X}_1 \times \mathcal{X}_2$ around P , we have a canonical isomorphism

$$\Psi_P: (\mathcal{X}_1 \widehat{\times} \mathcal{X}_2)_P \xrightarrow{\sim} \mathrm{Spf}(\mathcal{O}[[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]]).$$

We will denote by $\widehat{I_P} \subset \mathcal{O}[[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]]$ the ideal corresponding to the ideal of definition of $(\mathcal{X}_1 \times \mathcal{X}_2)_P$ defining the point section P (with the reduced structure).

Proof: (Of theorem 5.1) Let $p_i: \mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathcal{X}_i$ the projection; the restriction of the ideal sheaf $p_i^*(\mathcal{O}(-D_i))$ to $(\mathcal{X}_1 \widehat{\times} \mathcal{X}_2)_P$ will correspond, via Ψ , to a principal ideal of $\mathcal{O}[[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]]$ generated by an element G_i . We denote then by $I_{\delta, \underline{d}} \subset \mathcal{O}[[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]]$ the ideal generated by the elements $G_1^i \cdot G_2^j$ with $\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \geq \delta$; it corresponds to the restriction to $(\mathcal{X}_1 \widehat{\times} \mathcal{X}_2)_P$ of the ideal sheaf $\mathcal{I}_{\vartheta, \delta, \underline{d}}$. Consequently, a global section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2; M \otimes \mathcal{I}_{\vartheta, \delta, \underline{d}})$ restricted to $(\mathcal{X}_1 \widehat{\times} \mathcal{X}_2)_P$ will determine an element

$$F = \sum_{\frac{i}{d_1} \cdot \vartheta_1 + \frac{j}{d_2} \cdot \vartheta_2 \geq \delta} a_{ij} \cdot G_1^i \cdot G_2^j.$$

If (i_1, i_2) is a couple of indices such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \epsilon$ then a direct computation using the iterated Leibnitz rule gives $D^{(i_1, i_2)}(F) \in I_{\delta-\epsilon, \underline{d}} \otimes M|_P \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2}$.

If, as we suppose the index $\mathrm{ind}_P(f, d_1, d_2)$ of f at P is less or equal then ϵ , then we can find a couple of positive integers (i_1, i_2) such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \epsilon$ and the class of $D^{(i_1, i_2)}(f)$ in $(\mathcal{O}[[\mathcal{O}(-P_1)|_{P_1}, \mathcal{O}(-P_2)|_{P_2}]] \otimes M|_P \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2}) / \widehat{I_P} \simeq H^0(P, M \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2})$ defines a *non zero* section in $\tilde{f} \in H^0(P, M \otimes \mathcal{O}(-i_1 P_1)|_{P_1} \otimes \mathcal{O}(-i_2 P_2)|_{P_2})$.

$\omega_{\mathcal{X}_1/B}^{i_1}|_{P_1} \otimes \omega_{\mathcal{X}_2/B}^{i_2}|_{P_2} \otimes \mathcal{I}_{\vartheta, \delta - \epsilon, \underline{d}}$ (we used adjunction formula).

Let $\sigma \in M_\infty$ be an infinite place; we can fix once for all a covering of $(\mathcal{X}_i)_\sigma$ by open sets U_j analytically equivalent to a disk (with coordinate z); the metric on the restriction of $\mathcal{O}(P_i)$ to U_i is given by a \mathcal{C}^∞ function $\rho_j(z)$: the norm of the section $\mathbb{1}_{P_i}$ will be $|z - z(P_i)| \cdot \rho(z)$. Due to the way we constructed the metric on $\mathcal{O}(P_i)$, we can find constants A_ℓ , *independent on P_i* , such that $A_1 \leq \rho_j(z) \leq A_2$: indeed it suffices to take a covering of $(\mathcal{X}_i \times \mathcal{X}_i)_\sigma$ made by product of disks where the line bundle $\mathcal{O}(\Delta_i)$ trivializes. A direct application of 3.5 gives a constant C_1 independent on P and on the d_i 's such that

$$\sup\{\|\tilde{f}\|_\sigma\} \leq A \cdot C_1^{(d_1+d_2)}.$$

The section \tilde{f} extends to a section, denote it again by \tilde{f} , of $(M \otimes \omega_{\mathcal{X}_1}^{i_1} \otimes \omega_{\mathcal{X}_2}^{i_2})_\sigma$ on a neighborhood of P , which we may suppose to be one a product of the U_i above; a similar argument shows that $\sup\{\|\tilde{f}\|\} \leq A \cdot C_1^{(d_1+d_2)}$.

Let $\tilde{X} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ be the blow up along the ideal $\mathcal{I}_{\vartheta, \delta - \epsilon, \underline{d}}$ and $E_{\delta - \epsilon}$ be the exceptional divisor; let $\tilde{P}: \text{Spec}(O_K) \rightarrow \tilde{X}$ be the strict transform of P . By definition \tilde{f} corresponds to a non zero section $\tilde{f} \in H^0(\tilde{P}, M \otimes \omega_{\mathcal{X}_1/B}^{i_1} \otimes \omega_{\mathcal{X}_2/B}^{i_2}(-E_{\delta - \epsilon}))$. We will now give an upper bound for the norm of \tilde{f} . As before, once we take a suitably chosen (once for all) open covering of $(\mathcal{X}_i)_\sigma$, in the analytic topology, the existence of the upper bound as in the statement of the theorem is consequence of 5.2 below.

Let \mathbb{D} be an open disk, $0 \in \mathbb{D}$ be a point on it and z be a coordinate with a simple zero on 0. Suppose that $\rho_i(z)$ ($i = 1, 2$) are two \mathcal{C}^∞ functions on \mathbb{D} ; suppose that we can find two positive constants B_1 and B_2 such that $B_1 \leq \rho_i(z) \leq B_2$. We define two metrics $\|\cdot\|_i$ on $\mathcal{O}(0)$ by the formula $\|\mathbb{1}_0\|_i = |z|\rho_i(z)$.

Let $p_i: \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$ the i -th projection, we will denote by $\mathcal{O}(-0_i)$ the line bundle $p_i^*(\mathcal{O}(-0))$ and by z_i the holomorphic function $p_i(z)$ (it is the canonical section of $\mathcal{O}(0_i)$). We will suppose that $\mathcal{O}(0_i)$ is equipped with the pull-back, via p_i of the metric $\|\cdot\|_i$.

Fix positive rational numbers ϑ_i and δ . For every couple of sufficiently divisible positive integers (d_1, d_2) define $\mathcal{I}_{\vartheta, \delta, \underline{d}}$ to be the ideal sheaf of $\mathcal{O}_{\mathbb{D} \times \mathbb{D}}$ generated by the monomials $z_1^{i_1} \cdot z_2^{i_2}$ with $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \geq \delta$.

Let $b: \tilde{X} \rightarrow \mathbb{D} \times \mathbb{D}$ be the blow up of $\mathcal{I}_{\vartheta, \delta, \underline{d}}$ and $E_\delta \subset \tilde{X}$ be the exceptional divisor. The metrics on $\mathcal{O}(0_i)$ induce a metric on the line bundle $\mathcal{O}(E)$.

5.2 Theorem. *There exists a constant B depending only on ϑ_i , δ and the constants A_i such that if the d_i 's are sufficiently big and divisible, $f \in H^0(\mathbb{D} \times \mathbb{D}, \mathcal{I}_{\vartheta, \delta, \underline{d}})$ and \tilde{f} is the corresponding section in $H^0(\tilde{X}, \mathcal{O}(-E))$ then*

$$\|\tilde{f}\| \leq \|f\| \cdot B^{(d_1+d_2)}.$$

Proof: We can find a very small positive constant α such that, if the d_i are sufficiently

big, we have a surjection

$$\bigoplus_{\delta \leq \frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \delta + \alpha} \mathcal{O}(-i_1 \cdot 0_1) \otimes \mathcal{O}(-i_2 \cdot 0_2) \twoheadrightarrow \mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}}.$$

To simplify notations we will denote by H the set

$$\left\{ (i_1, i_2) \in \mathbb{Z} \times \mathbb{Z} / \delta \leq \frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \delta + \alpha \right\}.$$

Denoting by \mathbf{P} the projective bundle $\mathbf{Proj}(\bigoplus_{(i_1, i_2) \in H} \mathcal{O}(-i_1 \cdot 0_1) \otimes \mathcal{O}(-i_2 \cdot 0_2))$ over $\Delta \times \Delta$ we get a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xhookrightarrow{\iota} & \mathbf{P} \\ & \searrow & \downarrow \\ & & \mathbb{D} \times \mathbb{D}. \end{array}$$

Moreover, by construction we have an *isometry* $\iota^*(\mathcal{O}(1)) \simeq \mathcal{O}(-E)$. Remark that \mathbf{P} is isomorphic to $\mathbb{D} \times \mathbb{D} \times \mathbb{P}^N$ for a suitable N . Denote by u_{i_1, i_2} (with $\delta \leq \frac{i_1}{d_1} + \frac{i_2}{d_2} \leq \delta + \alpha$) the homogeneous coordinates on \mathbb{P}^N ; the blow up \tilde{X} is defined by the equations

$$u_{j_1, j_2} \cdot z_1^{i_1} \cdot z_2^{i_2} = u_{i_1, i_2} \cdot z_1^{j_1} \cdot z_2^{j_2}$$

for all (i_1, i_2) and (j_1, j_2) in H .

Let's work on the local chart $u_{i_1, i_2} \neq 0$; a local computation shows that over this chart

$$\|E\| = \frac{|z_1^{i_1} \cdot z_2^{i_2}|}{|u_{i_1, i_2}|} \cdot \sqrt{\sum_{(j_1, j_2) \in H} \left(|u_{j_1, j_2}| \cdot \rho_1^{j_1} \cdot \rho_2^{j_2} \right)^2}.$$

Let $f \in H^0(\mathbb{D} \times \mathbb{D}, \mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}})$. The pull-back $b^*(f)$ naturally defines a global section $\tilde{f} \in H^0(\tilde{X}, \mathcal{O}(-E))$. Over the chart $u_{i_1, i_2} \neq 0$ we can find a holomorphic function h such that $f = z_1^{i_1} \cdot z_2^{i_2} \cdot h$. In order to conclude the proof of the theorem we have to give an upper bound for

$$|h| \cdot \frac{\sqrt{\sum |u_{j_1, j_2}|^2 \cdot \rho_1^{2j_1} \cdot \rho_2^{2j_2}}}{|u_{i_1, i_2}|}. \quad (5.3.1)$$

Fix a very small positive ϵ ; up to change the local chart we may suppose that we are in the disk

$$\frac{|u_{j_{r_1}, j_{r_2}}|}{|u_{i_1, i_2}|} \leq 1 + \epsilon;$$

consequently, we can find a constant B_1 depending only on the norms, in particular independent on the d_i 's, such that, if the d_i 's are sufficiently big, the expression in 5.3.1 can be upper bounded by

$$|h| \cdot B_1^{(d_1 + d_2)}.$$

Since h is holomorphic, the function $|h|$ will assume its maximum on the border. We

may assume that the d_i are such that $\frac{d_i \cdot \delta}{\vartheta_i} \in \mathbb{N}$. On our chart, $z_1^{\frac{\delta \cdot d_1}{\vartheta_1}} = z_1^{i_1} \cdot z_2^{i_2} \cdot u_{\frac{\delta \cdot d_1}{\vartheta_1}, 0}$ (resp. $z_2^{\frac{\delta \cdot d_2}{\vartheta_2}} = z_1^{i_1} \cdot z_2^{i_2} \cdot u_{0, \frac{\delta \cdot d_2}{\vartheta_2}}$) and $|u_{\frac{\delta \cdot d_2}{\vartheta_2}, 0}|$ (resp. $u_{0, \frac{\delta \cdot d_2}{\vartheta_2}}$) is less or equal to $1 + \epsilon$. consequently, if $|z_1^{\frac{\delta \cdot d_1}{\vartheta_1}}| = 1$ (resp. $|z_2^{\frac{\delta \cdot d_2}{\vartheta_2}}| = 1$) then $1 \leq |z_1^{i_1} \cdot z_2^{i_2}| \cdot (1 + \epsilon)$ thus

$$|h| \leq \frac{\|f\|}{|z_1^{i_1} \cdot z_2^{i_2}|} \leq (1 + \epsilon) \cdot \|f\|;$$

the conclusion of the theorem easily follows.

6 Proof of Theorem 2.1.

In this section we will give the proof of the main theorem of the paper: Theorem 2.1. For simplicity we will assume that the set of places S has cardinality one. The general case is similar and can be obtained *mutatis mutandis* as explained for instance in [HS D.2.2.1].

We recall all the tools and the ingredients: \mathcal{X}_i are two regular arithmetic surfaces projective over $B := \text{Spec}(O_K)$ over which we fixed arithmetically ample hermitian line bundles M_i and symmetric hermitian metrics on $\mathcal{O}(\Delta_i)$ (Δ_i being the diagonal on $\mathcal{X}_i \times \mathcal{X}_i$). Eventually we fix a place $\sigma \in M_K$.

We fix two finite extensions L_i of K and two reduced divisors $D_i: B_{L_i} := \text{Spec}(O_{L_i}) \rightarrow \mathcal{X}_i$. We denote by L the composite field $L_1 \cdot L_2$ and by n the degree of the extension L/K . We fix two positive rational numbers $\vartheta_i \geq 1$ and a positive ϵ such that $\vartheta_1 \cdot \vartheta_2 \geq 2n + \epsilon$. We will denote by $T(D_i)$ the positive real number $\sup\{1, h_{M_i}(D_i)\} \cdot \sup\{1, -(\mathcal{O}(D_i), \mathcal{O}(D_i))\}$ (the pairing being the Arakelov intersection pairing). Theorem 2.1 will be consequence of the following:

6.1 Theorem. *There exists a constant A depending only on the arithmetic surfaces \mathcal{X}_i , the ϑ_i , the ϵ , the hermitian line bundles M_i , the symmetric metrics on the diagonals $\mathcal{O}(\Delta_i)$ and the place σ , for which the following holds:*

- Let $D_i \subset \mathcal{X}_i$ be divisors as above, and $P_i \in \mathcal{X}_i(B)$ be two rational sections such that
- (i) $\lambda_{D_1, S}(P_1) > \vartheta_1 \cdot h_{M_1}(P_1)$ and $\lambda_{D_2, S}(P_2) > \vartheta_2 \cdot h_{M_2}(P_2)$;
 - (ii) $h_{M_1}(P_1) \geq A \cdot T(D_1) \cdot T(D_2)$.

Then

$$h_{M_2}(P_2) \leq A \cdot T(D_1) \cdot T(D_2) \cdot h_{M_1}(P_1).$$

Proof: Suppose that σ is an infinite place, then take a covering of $(\mathcal{X}_i)_\sigma$ by open U_{ij} , sets analytically equivalent to the disk of radius 1 and such that the open subsets analytically equivalent to the disk of radius 1/2 also cover the $(\mathcal{X}_i)_\sigma$. We can then find a constant A_2 such that if U_{ij_k} are the open sets containing the $(D_i)_\sigma$ and $(P_i)_\sigma$ are not contained in the U_{ij_k} then $\lambda_{D_i, S}(P_i) \leq A_2$. Consequently, we see that, taking A much bigger than A_2 (which is independent on the D_i), in this case condition (i) and condition (ii) are in

contradiction. In particular the theorem holds in this case. A similar argument holds if σ is a finite place.

Suppose that $\vartheta_1 \cdot \vartheta_2 = 2n + \epsilon$; define $\epsilon_1 := \frac{\epsilon}{n+1}$ and $\delta := 2 + \epsilon_1$.

For every couple of positive integers d_1 and d_2 , let $\mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}}$ be the ideal sheaf on $\mathcal{X}_1 \times \mathcal{X}_2$ defined in §3 and having support on $D_1 \times D_2 \subset \mathcal{X}_1 \times \mathcal{X}_2$. A direct application of 3.4 gives, as soon as the d_i 's are sufficiently big, the existence of a constant A_3 and a non zero global section $f \in H^0(\mathcal{X}_1 \times \mathcal{X}_2, M_1^{d_1} \otimes M_2^{d_2} \otimes \mathcal{I}_{\underline{\vartheta}, \delta, \underline{d}})$ such that

$$\sup_{\sigma \in M_\infty} \{\log \|f\|_\sigma\} \leq A_3 \cdot T(D_1) \cdot T(D_2)(d_1 + d_2).$$

Fix a constant A_4 such that $h_{\omega_{\mathcal{X}_i/B}}(\cdot) \leq A_4 \cdot h_{M_i}(\cdot)$ and let ϵ_2 such that $\epsilon_2 < \frac{\epsilon_1}{1+2 \cdot A_4}$. One apply Theorem 4.1 with $C = A_3 \cdot T(D_1) \cdot T(D_2)$ and $\epsilon = \epsilon_2$ and deduce the existence of two constants for which that theorem apply; following the proof one can see that the sup of these two constants is again of the form $A_4 \cdot T(D_1) \cdot T(D_2)$ with A_4 independent on the D_i 's.

Suppose that $P_i: B \rightarrow \mathcal{X}_i$ are two sections which satisfy hypothesis (i) and such that

$$h_{M_2}(P_2) > A_4 \cdot T(D_1) \cdot T(D_2) h_{M_1}(P_1)$$

we will prove that there exists a constant A_5 such that $h_{M_1}(P_1) \leq A_5 \cdot T(D_1) \cdot T(D_2)$, and this will be the conclusion of the proof.

In the sequel we will denote by h_i the real numbers $h_{M_i}(P_i)$.

Take d to be a very big ad divisible positive integer; let d_i be integers such that $d_i h_i \sim d$ and such that

$$\frac{h_2}{h_1} > \frac{d_1}{d_2}$$

(in order to keep the proof as readable as possible we avoid to introduce more small constants). The hypotheses of theorem 4.1 are satisfied consequently the index of f at $P_1 \times P_2$ is smaller then ϵ_2 . Let $\tilde{\mathcal{X}} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2$ be the blow up of the ideal $\mathcal{I}_{\underline{\vartheta}, \delta - \epsilon_2, \underline{d}}$ and $E_{\delta - \epsilon_2}$ be the exceptional divisor; let $\tilde{P}: B \rightarrow \tilde{\mathcal{X}}$ be the strict transform of $P := P_1 \times P_2$. We apply theorem 5.1 and deduce the existence of an absolute constant A_6 , a couple of indices (i_1, i_2) and a non zero section $\tilde{f} \in H^0(\tilde{P}, M_1^{d_1} \otimes M_2^{d_2} \otimes \omega_{\mathcal{X}_1/B}^{i_1}|_{P_1} \otimes \omega_{\mathcal{X}_2/B}^{i_2}(-E_{\delta - \epsilon_2}))$ such that $\frac{i_1}{d_1} \cdot \vartheta_1 + \frac{i_2}{d_2} \cdot \vartheta_2 \leq \epsilon_2$ and $\sup_{\sigma \in M_\infty} \{\log \|\tilde{f}\|_\sigma\} \leq A_6 \cdot T(D_1) \cdot T(D_2)(d_1 + d_2)$.

A local computation involving the definition of the blow up implies that there exist two indices j_1 and j_2 such that $\frac{j_1}{d_1} \cdot \vartheta_1 + \frac{j_2}{d_2} \cdot \vartheta_2 \geq \delta - \epsilon_2$ such that

$$\widehat{\deg}(\tilde{P}^*(E_{\delta - \epsilon_2})) \geq j_1 \cdot \lambda_{D_1, S}(P_1) + j_2 \cdot \lambda_{D_2, S}(P_2)$$

(recall that $\widehat{\deg}(\cdot)$ denotes the Arakelov degree). Thus we deduce

$$\begin{aligned}
& -A_6 \cdot T(D_1) \cdot T(D_2)(d_1 + d_2) \\
& \leq d_1 \cdot h_1 + d_2 \cdot h_2 + i_1 \cdot h_{\omega_{x_1/B}}(P_1) + i_2 \cdot h_{\omega_{x_2/B}}(P_2) - \widehat{\deg}(\tilde{P}^*(E_{\delta-\epsilon_2})) \\
& \leq 2d + \frac{i_1}{d_1} \cdot d_1 \cdot h_{\omega_{x_1/B}}(P_1) + \frac{i_2}{d_2} \cdot d_2 \cdot h_{\omega_{x_2/B}}(P_2) - (j_1 \cdot \lambda_{D_1,S}(P_1) + j_2 \cdot \lambda_{D_2,S}(P_2)) \\
& \leq 2d + 2A_4 \cdot \epsilon_2 \cdot d - \left(\frac{j_1}{d_1} \cdot \vartheta_1 \cdot h_1 \cdot d_1 + \frac{j_2}{d_2} \cdot \vartheta_2 \cdot h_2 \cdot d_2 \right) \\
& \leq ((2 + 2\epsilon_2 \cdot A_4) - (2 + \epsilon_1 - \epsilon_2)) \cdot d
\end{aligned}$$

from this and by our choice of the ϵ_i 's, we deduce

$$-A_6 \cdot T(D_1) \cdot T(D_2) \cdot \left(\frac{1}{h_1} + \frac{1}{h_2} \right) \leq -\epsilon_3$$

where $\epsilon_3 = \epsilon_1 - (1 + 2A_4) \cdot \epsilon_2$; thus

$$h_1 \leq \frac{2 \cdot A_6}{\epsilon_3} \cdot T(D_1) \cdot T(D_2)$$

and from this the conclusion follows.

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