arXiv:0811.3838v3 [math.AG] 16 Jun 2010

$\mathbb{P}^r$-SCROLLS ARISING FROM BRILL-NOETHER THEORY AND $K3$-SURFACES

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Abstract. In this paper we study examples of $\mathbb{P}^r$-scrolls defined over primitively polarized $K3$ surfaces S of genus g, which arise from Brill-Noether theory of the general curve in the primitive linear system on S and from Lazarsfeld's results in [25]. We show that such scrolls form an open dense subset of a component $\mathcal{H}$ of their Hilbert scheme; moreover, we study some properties of $\mathcal{H}$ (e.g. smoothness, dimensional computation, etc.) just in terms of $\mathcal{B}_g$, the moduli space of such $K3$'s, and $M_v(S)$, the moduli space of semistable torsion-free sheaves of a given rank on S.

One of the motivation of this analysis is to try to introducing the use of projective geometry and degeneration techniques in order to studying possible limits of semistable vector-bundles of any rank on a general $K3$ as well as Brill-Noether theory of vector-bundles on suitable degenerations of projective curves.

We conclude the paper by discussing some applications to the Hilbert schemes of geometrically ruled surfaces introduced and studied in [9] and [10].

1. Introduction

Smooth curves on $K3$ surfaces, and in particular their Brill-Noether theory, have played a fundamental role in algebraic geometry in the past decades (see e.g. [27], [25], [19], [32], [12], [15], [40], [4], [2], [16] and [41], just to mention a few). The Brill-Noether theory of these curves is both an important subject in its own right, especially because it is connected to the geometry of the surface and, at the same time, it is an important tool to prove results about smooth curves with general moduli with no use of degeneration techniques.

Recall indeed the following fundamental result of R. Lazarsfeld (we will report a slightly weaker version):

Theorem 1.1. *(cf.* [25, Theorem]*) Let S be a $K3$ surface with* $\mathrm{Pic}(S) = \mathbb{Z}[L]$, *with* $L^2 = 2g-2 > 2$. *Let* $\rho(g,r,d) := g-(r+1)(g-d+r)$ *be the* Brill-Noether number.

(i) *If $C \in |L|$ is smooth, and $\rho(g,r,d) < 0$, then*
$$W^r_d(C) := \{\mathcal{A} \in \mathrm{Pic}^d(C) \mid h^0(C,\mathcal{A}) \geq r+1\} = \emptyset,$$
(ii) *If $C \in |L|$ is a general element and $\rho(g,r,d) \geq 0$, then $W^r_d(C)$ is smooth outside $W^{r+1}_d(C)$, and of the expected dimension $\rho(g,r,d)$. In other words, C satisfies Petri's condition.*

An interesting, independent proof of the previous result is contained in [37].

Lazarsfeld's approach for a proof of Theorem 1.1 uses vector-bundle techniques on S. For reader's convenience, we briefly recall Lazarsfeld's construction and set-up in § 4.

Roughly speaking, given a smooth, primitively polarized $K3$ surface (S,L) of genus $g \geq 3$ (i.e. $L^2 = 2g-2$), a general (smooth) curve $C \in |L|$ and a complete linear series $|\mathcal{A}| = \mathfrak{g}^r_d$

Mathematics Subject Classification (2000): 14J28, 14J60, 14C05, 14D20; (Secondary) 14J26, 14H10.
Keywords: Hilbert schemes of high-dimensional scrolls; $K3$ surfaces; Brill-Noether theory; Moduli of curves;
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on C, with suitable properties of global generations (cf. § 4), one can associate to the triple $(S, C, |\mathcal{A}|)$ a rank-$(r+1)$ vector bundle $\mathcal{E}$ on S.

This vector bundle is globally generated; it is *simple* if $|L|$ does not contain either reducible or non-reduced elements (cf. [25]). $\mathcal{E}$ encodes several properties of the Brill-Noether' and Petri's theory of the space $W^r_d(C)$. If, moreover, (S, L) is the general algebraic $K3$ surface of the given polarization g, then $\mathcal{E}$ is also *stable* on S (see e.g. Proposition 4.5).

The main result of Lazarsfeld's paper (a weaker form is Theorem 1.1 recalled above) states that such a C behaves generically from the Petri's theoretical point of view, i.e. C is a *Petri general curve*.

On the other hand, despite the fact that such a C behaves (from the Brill-Noether-Petri's theory point of view) as a curve with *general moduli*, for large g the curve C has *special moduli* (cf. e.g. Theorem 3.9 later on). However, smooth curves $C \in |L|$ of low genus have interesting modular properties, related to the existence of Fano 3-folds of index one of the corresponding sectional genus. These properties have been investigated by Mukai who settled, in particular, a problem raised by Mayer in [26]. He showed that a general curve of genus $g \leq 9$ or $g = 11$ can be embedded as a nonsingular curve in a K3 surface, and that this is not possible for curves of genus $g = 10$, despite an obvious count of parameters indicating the opposite (cf. Theorem 3.9). These facts have been also observed by Beauville in the last section of [7] by means of a local deformation-theoretic analysis.

These are some of the main motivations which explain the deep interest in this subject. One of the aim of this paper is to study some projective geometry which is behind Lazarsfeld's construction. Indeed, it would be interesting to introduce the use of projective geometry and degeneration techniques in order to studying possible limits of semistable vector-bundles of any rank on a general $K3$ as well as Brill-Noether theory of vector-bundles on suitable degenerations of projective curves, which are the hyperplane sections of the $K3$'s.

In more details, for $(S, L, \mathcal{E})$ as above, one can consider the rank-$(r+1)$ vector-bundle on S given by $\mathcal{F} = \mathcal{E} \otimes L$, (more generally, $\mathcal{F}_n := \mathcal{E} \otimes L^{\otimes n}$ for any integer $n \geq 1$). It turns out that $\mathcal{F}$ (equiv. $\mathcal{F}_n$) is very-ample on S, giving rise to a smooth, irreducible $\mathbb{P}^r$-scroll over S (simply called *r-scroll*) which is linearly normal in its projective span $\mathbb{P}^R$, of degree δ, both depending on r, d and g (cf. Proposition 5.2 and Lemma 5.6).

We prove that such r-scrolls have maximal dimensional orbits under the action of the projective transformation group $PGL(R+1, \mathbb{C})$ (cf. Proposition 5.11). This allows us to explicitly compute the dimension of the component $\mathcal{H}_{r+2,\delta}$ of the Hilbert scheme containing such scrolls and to show that this dimension equals the sum of the quantities $\dim(\mathcal{B}_g)$, g and $\dim(M_v(S))$, with $v = v(\mathcal{F})$ the Mukai vector of $\mathcal{F}$ (cf. Theorem 6.1). We also show that $\mathcal{H}_{r+2,\delta}$ is generically smooth and dominates $\mathcal{B}_g$.

As a consequence of our approach, one has also existence results of smooth $\mathbb{P}^r$-scrolls over $K3$ surfaces in projective spaces. Existence of $\mathbb{P}^r$-scrolls is an interesting problem for several reasons.

E.g. it is well-known that there are only finitely many families of smooth 3-folds in $\mathbb{P}^5$ which are not of general type (cf. e.g. [36] and references therein). In [36], the author classifies all smooth 3-folds in $\mathbb{P}^5$ which are scrolls over a surface: these scrolls are only of 4 types. One of these types is a 3-fold of degree 9 which is a $\mathbb{P}^1$-scroll over a $K3$ surface of genus 8 (cf. [36, Example (d)]). However, such a scroll does not arise from Brill-Noether theory (cf. Remark 6.7). At the same time, $\mathbb{P}^r$-scrolls in general occur as special fundamental cases of varieties in *adjunction theory* (cf. e.g. [8]).

We conclude the paper by discussing some applications of our construction to Hilbert schemes of linearly normal, non-special scrolls, which have been studied in [9] and [10]. In particular, moduli behaviour of suitable sub-loci of such Hilbert schemes is considered (cf. Proposition 7.10).

We work over $\mathbb{C}$, the field of complex numbers. For any smooth, projective variety X, the symbol $A(X)$ will denote the Chow-ring of X with intersection product $\cdot$, whereas the symbol $\sim$ will denote the linear equivalence of divisors on X.

Acknowledgments: I wish to warmly thank C. Ciliberto, for having pointed out the content of Remark 6.8, and M.L. Fania, for suggestions concerning Lemma 5.6. Very special thanks to the first referee of this paper, for his/her extremely careful job and for comments and suggestions (expecially in Claim 6.3) which improve the contents and the exposition of the paper.

2. Preliminaries on $K3$ surfaces

In this section we briefly recall some useful results on $K3$ surfaces and moduli spaces of semistable torsion-free sheaves of a given rank on them.

Recall first the following standard definition.

Definition 2.1. A line bundle L on a surface S is called *primitive* if $L \not\sim nL'$ for some $n > 1$ and $L' \in \mathrm{Pic}(S)$.

A *marked surface* (resp. *primitively marked surface*) is a pair (S, L), where S is a surface and $L \in \mathrm{Pic}(S)$ is globally generated (resp. primitive and globally generated).

A *polarized surface* (resp. *primitively polarized surface*) is a pair (S, L), where S is a surface and $L \in \mathrm{Pic}(S)$ is globally generated and ample (resp. primitive, globally generated and ample).

From now on, unless otherwise stated, S will denote a smooth, algebraic $K3$ surface and L a globally generated line bundle with $L^2 > 2$.

As a direct consequence of the analysis contained in the classical paper [38] of Saint Donat, one has:

Proposition 2.2. *Let S be a $K3$ surface such that $\mathrm{Pic}(S) = \mathbb{Z}[L]$ for a globally generated line bundle L with $L^2 > 2$. Then $|L|$ is very ample on S and there exists a positive integer $g \geq 3$ such that $L^2 = 2g - 2$.*

In particular, the general element $C \in |L|$ is a smooth, irreducible curve, of geometric genus g.

For brevity, the integer g will be called the *genus* of S.

Let $g \geq 3$ be any integer. From now on, we will denote by $\mathcal{B}_g$ the moduli space of smooth primitively polarized $K3$ surfaces of genus g. It is well-known that $\mathcal{B}_g$ is smooth, irreducible and of dimension 19 (cf. e.g. [6, Thm.VIII 7.3 and p. 366]). In particular, for $(S, L) \in \mathcal{B}_g$ general, one has $\mathrm{Pic}(S) = \mathbb{Z}[L]$. From Proposition 2.2, $(S, L) \in \mathcal{B}_g$ general determines a smooth, irreducible, projective, primitively polarized $K3$ surface $\Phi_L(S) \subset \mathbb{P}^g$ of degree $2g - 2$, whose sectional genus is g.

Since S is regular, with trivial canonical bundle, and since $\mathcal{T}_S \cong \Omega^1_S$, one has

$$h^0(\mathcal{T}_S) = h^2(\mathcal{T}_S) = 0, \ h^1(\mathcal{T}_S) = 20 \tag{2.3}$$

(cf. [6, Thm.VIII 7.3]).

We now recall the definition of *Mumford-Takemoto* stability of torsion-free sheaves on a smooth, projective $K3$ surface S.

Definition 2.4. (cf. [17, e.g. Definitions 1 and 4, p.85-86]) Let S be a smooth, $K3$ surface and let L be an ample line bundle on S. For $\mathcal{F}$ a torsion-free coherent sheaf on S, the *normalized L-degree* (or simply the *L-slope*) of $\mathcal{F}$ is the rational number

$$\mu_L(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot L}{\mathrm{rk}(\mathcal{F})}.$$

Then, $\mathcal{F}$ is said to be *L-stable* (resp. *L-semistable*) if, for all coherent subsheaves $\mathcal{G} \subset \mathcal{F}$ with $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$, we have $\mu_L(\mathcal{G}) < \mu_L(\mathcal{F})$ (resp., $\mu_L(\mathcal{G}) \leq \mu_L(\mathcal{F})$).

Remark 2.5. For $\mathrm{rk}(\mathrm{Num}(S)) \geq 2$, the definition of L-stability depends on the choice of the numerical equivalence of the class L. On the other hand, if e.g. $\mathrm{Pic}(S) = \mathbb{Z}[L]$, the notion of L-stability (resp. L-semistability) will be simply called *stability* (resp. *semistability*), since it is clear from the context that it is respect to the generator L.

Moreover, recall that a stable torsion-free sheaf $\mathcal{F}$ is *simple*, i.e. $\mathrm{End}(\mathcal{F}) \cong \mathbb{C}$ (cf. e.g. [17, Corollary 8, p. 88]).

Let $\mathcal{F}$ be a torsion-free sheaf on a $K3$ surface S. By results of Mukai in [29, 30, 31], one can consider the *Mukai vector*

$$v = v(\mathcal{F}) \in H^*(S) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

defined as

$$\begin{aligned} v(\mathcal{F}) := ch(\mathcal{F})(1+\omega) &= \mathrm{rk}\,\mathcal{F} + c_1(\mathcal{F}) + (\chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F}))\omega \\ &= \left(\mathrm{rk}\,\mathcal{F}, c_1(\mathcal{F}), \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) + \mathrm{rk}(\mathcal{F})\right) = \\ &= \left(r, c_1, \frac{c_1^2}{2} - c_2 + r\right) \end{aligned} \tag{2.6}$$

where $\omega \in H^4(S, \mathbb{Z})$ is the fundamental class (see e.g. [22, p. 142-143]).

Let L be an ample divisor on S. We denote by

$$\mathcal{M}_v(S, L)$$

the moduli space of *Gieseker-Maruyama* L-semistable torsion-free sheaves $\mathcal{F}$ on S with $v(\mathcal{F}) = v$ (cf. e.g. [17, p. 153-154]).

Remark 2.7. When, in particular, $\mathrm{Pic}(S) = \mathbb{Z}[L]$ then $\mathcal{M}_v(S, L)$ will be simply denoted by $\mathcal{M}_v(S)$ (cf. Remark 2.5).

One denotes by $\mathcal{M}_v(S, L)^{stable}$ the open subset parametrizing stable sheaves. The *expected dimension* of $\mathcal{M}_v(S, L)^{stable}$ is

$$\epsilon := \min\{-1, 2\,r\,c_2 - (r-1)\,c_1^2 - 2\,(r^2 - 1)\} \tag{2.8}$$

(see [22, p. 143]

Remark 2.9. By the Gieseker-Maruyama's construction, when $\epsilon > 0$, the general element of $\mathcal{M}_v(S, L)$ parametrizes a vector bundle on S (cf. e.g. [17, p. 154]). On the other hand, when $\epsilon = 0$, if $\mathcal{M}_v(S, L)^{stable} \neq \emptyset$ then $\mathcal{M}_v(S, L) = \mathcal{M}_v(S, L)^{stable}$ consists of a single, reduced point which represents a stable vector-bundle (cf. e.g. [22, Theorem 6.1.6, p. 143]).

We conclude by recalling the following fundamental result.

Proposition 2.10. *(cf.* [22, Corollary 4.5.2, p. 101]*) Let $\mathcal{F}$ be a torsion-free sheaf corresponding to a stable point $[\mathcal{F}] \in \mathcal{M}_v(S, L)$. Then:*

(i) *the Zariski tangent space of* $\mathcal{M}_v(S,L)$ *at* $[\mathcal{F}]$ *is isomorphic to* $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$*;*
(ii) *if* $\mathrm{Ext}^2(\mathcal{F},\mathcal{F})=0$*, then* $\mathcal{M}_v(S,L)$ *is smooth at* $[\mathcal{F}]$*;*
(iii) *there are bounds*

$$\dim(\mathrm{Ext}^1(\mathcal{F},\mathcal{F})) \geq \dim_{[\mathcal{F}]}(\mathcal{M}_v(S,L)) \geq \dim(\mathrm{Ext}^1(\mathcal{F},\mathcal{F})) - \dim(\mathrm{Ext}^2(\mathcal{F},\mathcal{F})).$$

(iv) *if moreover* $\mathcal{F}$ *is a vector bundle, then* $\mathrm{Ext}^i(\mathcal{F},\mathcal{F}) \cong H^i(\mathcal{F}\otimes\mathcal{F}^\vee)$*, for any* $i\geq 0$*.*

3. Deformations and the Beauville space of pairs

In this section, we review some results on deformation theory that are needed for our aims (for complete details, the reader is referred to e.g. [34, § 3.4.4]) and we recall some fundamental results of Mukai [32, 33] as well as the infinitesimal approach considered by Beauville [7, § 5].

Let Y be a smooth variety and let $X\subset Y$ be a smooth, Cartier divisor. Let $\mathcal{N}_{X/Y}$ be the normal bundle of X in Y. One can define a coherent sheaf $\mathcal{T}_Y\langle X\rangle$ of rank $\dim(Y)$ on Y via the exact sequence :

$$0 \longrightarrow \mathcal{T}_Y\langle X\rangle \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{N}_{X/Y} \longrightarrow 0, \tag{3.1}$$

which is called the *sheaf of germs of tangent vectors to* Y *that are tangent to* X (cf. [34, § 3.4.4]). One has a natural surjective restriction map

$$r:\mathcal{T}_Y\langle X\rangle \longrightarrow \mathcal{T}_X, \tag{3.2}$$

giving the exact sequence

$$0 \longrightarrow \mathcal{T}_Y(-X) \longrightarrow \mathcal{T}_Y\langle X\rangle \longrightarrow \mathcal{T}_X \longrightarrow 0, \tag{3.3}$$

where $\mathcal{T}_Y(-X)$ is the vector bundle of tangent vectors of Y vanishing along X. Since X is smooth, $\mathcal{T}_Y\langle X\rangle$ is a locally free subsheaf of the holomorphic tangent bundle $\mathcal{T}_Y$. More precisely, $\mathcal{T}_Y\langle X\rangle = (\Omega^1_Y(\log X))^\vee$, where $\Omega^1_Y(\log X)$ denotes the sheaf of meromorphic 1-forms on Y that have at most logarithmic poles along X (see e.g. [23]).

Recall the following basic result:

Proposition 3.4. *(see* [34, Proposition 3.4.17]*) The locally trivial deformations of the pair* (Y,X) *are controlled by the sheaf* $\mathcal{T}_Y\langle X\rangle$*; namely,*

- *the obstructions lie in* $H^2(Y,\mathcal{T}_Y\langle X\rangle)$*;*
- *first-order, locally trivial deformations are parametrized by* $H^1(Y,\mathcal{T}_Y\langle X\rangle)$*;*
- *infinitesimal automorphisms are parametrized by* $H^0(Y,\mathcal{T}_Y\langle X\rangle)$*.*

The map which associates to a first-order, locally trivial deformation of (Y,X) *the corresponding first-order deformation of* X *is the map*

$$H^1(r): H^1(Y,\mathcal{T}_Y\langle X\rangle) \longrightarrow H^1(X,\mathcal{T}_X), \tag{3.5}$$

induced in cohomology by (3.2).

Definition 3.6. (cf. [7, § 5]) Let $\mathcal{KC}_g$ be the moduli space parametrizing pairs (S,C), where $C\subset S$ is a smooth curve such that $(S,\mathcal{O}_S(C))\in\mathcal{B}_g$, $g\geq 3$, where $\mathcal{B}_g$ is as in § 2.

In [7, § 5], the author more precisely considers $\mathcal{KC}_g$ with its algebraic stack structure; we will not need this generality in what follows.

There is an induced, dominant morphism

$$\pi:\mathcal{KC}_g \longrightarrow \mathcal{B}_g \tag{3.7}$$

given by the natural projection. From [7, § (5.2)], for any $(S,C)\in\mathcal{KC}_g$, by Serre duality one has $H^2(S,\mathcal{T}_S\langle C\rangle) = H^0(S,\Omega^1_S(\log C))^\vee = (0)$. Furthermore, since C is a smooth curve of genus $g \geq 3$ and since $\mathcal{T}_S \cong \Omega^1_S$ (being S a $K3$ surface and $\mathcal{T}_S$ a rank-two vector bundle on it) by (3.3) we have $H^0(S, \mathcal{T}_S\langle C\rangle) = (0)$. From Proposition 3.4, $\mathcal{K}C_g$ is smooth, of dimension

$$\dim(\mathcal{K}C_g) = h^1(S, \mathcal{T}_S\langle C\rangle) = 19 + g.$$

Since the fibers of π are connected, $\mathcal{K}C_g$ is also irreducible.

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g, which is irreducible and of dimension $3g-3$, since $g \geq 3$ by assumption. One has a natural morphism

$$c_g : \mathcal{K}C_g \longrightarrow \mathcal{M}_g \tag{3.8}$$

defined as

$$c_g((S,C)) = [C] \in \mathcal{M}_g,$$

where $[C]$ denotes the isomorphism class of $C \subset S$.

The main results concerning the morphism c_g are contained in the following:

Theorem 3.9 (Mukai). *With notation as above:*

(i) c_g *is dominant for* $g \leq 9$ *and* $g = 11$ *(cf.* [32]*);*
(ii) c_g *is not dominant for* $g = 10$ *(cf.* [32]*). More precisely,* $\mathrm{Im}(c_{10})$ *is a hypersurface in* $\mathcal{M}_{10}$ *(cf.* [13]*);*
(iii) c_g *is generically finite onto its image, for* $g = 11$ *and for* $g \geq 13$*, but not for* $g = 12$ *(cf.* [33]*).*

For the infinitesimal counterpart of the above results, see [7, § (5.2)]

4. Brill-Noether theory and the Mukai-Lazarsfeld vector bundle

We will briefly recall the vector bundle techniques used in Lazarsfeld's approach for the proof of Theorem 1.1. These are contained in [25, § 1].

Let S be a smooth, polarized, projective $K3$ surface and let $C_0 \subset S$ be a smooth, irreducible curve of genus g. Given a curve C and integers d and r, one can consider

$$V^r_d(C) \subset \mathrm{Pic}^d(C)$$

the non-empty, open subset of $W^r_d(C)$ consisting of line bundles $\mathcal{A}$ on C such that:

(i) $h^0(\mathcal{A}) = r+1$ and $\deg(\mathcal{A}) = d$; and
(ii) both $\mathcal{A}$ and $\omega_C \otimes \mathcal{A}^\vee$ are globally generated (where ω_C denotes the canonical bundle of C).

If we fix a smooth curve $C \in |\mathcal{O}_S(C_0)|$ and a line bundle $\mathcal{A} \in V^r_d(C)$ - equivalently $|\mathcal{A}| = \mathfrak{g}^r_d$ on C - one can associate to the pair $(C, \mathcal{A})$ a rank-$(r+1)$ vector bundle $\mathcal{E} = \mathcal{E}_{C,\mathcal{A}}$ on S as follows: since $\mathcal{A}$ is globally generated, we have a canonical surjective map

$$ev_{C,\mathcal{A}} : H^0(\mathcal{A}) \otimes \mathcal{O}_S \twoheadrightarrow \mathcal{A}$$

of $\mathcal{O}_S$-modules (thinking $\mathcal{A}$ as a sheaf on S); thus, $ker(ev_{C,\mathcal{A}})$ is a vector-bundle on S, therefore, we set $\mathcal{E} = \mathcal{E}_{C,\mathcal{A}} := ker(ev_{C,\mathcal{A}})^\vee$ (for details, cf. [25, § 1]). One has the exact sequence on S:

$$0 \to \mathcal{E}^\vee \to H^0(\mathcal{A}) \otimes \mathcal{O}_S \to \mathcal{A} \to 0. \tag{4.1}$$

Dualizing (4.1), we get

$$0 \to H^0(\mathcal{A})^\vee \otimes \mathcal{O}_S \to \mathcal{E} \to \omega_C \otimes \mathcal{A}^\vee \to 0, \tag{4.2}$$

since $\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{A}, \mathcal{O}_S) \cong \omega_C \otimes \mathcal{A}^\vee$ (cf. [21, Lemma 7.4, p. 242]). The vector bundle $\mathcal{E}$ will be called the *Mukai-Lazarsfeld* vector bundle.

If, as is costumary, one considers the *Brill-Noether number*:

$$\rho(\mathcal{A}) := g - h^0(\mathcal{A})h^1(\mathcal{A}) = g - (r+1)(r+g-d), \tag{4.3}$$

from (4.1), (4.2) and the fact that S is regular with $\omega_S \cong \mathcal{O}_S$, it is easy to observe the following facts ([25, § 1]):

(E1) $\mathcal{E}$ is globally generated,
(E2) $c_1(\mathcal{E}) = [C_0]$, $c_2(\mathcal{E}) = \deg(\mathcal{A}) = d$,
(E3) $h^0(\mathcal{E}^\vee) = h^2(\mathcal{E}) = 0$, $h^1(\mathcal{E}^\vee) = h^1(\mathcal{E}) = 0$,
(E4) $h^0(\mathcal{E}) = h^0(C, \mathcal{A}) + h^1(C, \mathcal{A}) = 2r + g - d + 1$;
(E5) $\chi(\mathcal{E} \otimes \mathcal{E}^\vee) = 2 - 2\rho(\mathcal{A})$. More precisely, $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = h^2(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$ and $h^1(\mathcal{E} \otimes \mathcal{E}^\vee) = 2\rho(\mathcal{A})$.

Another fundamental property of $\mathcal{E}$ is the following:

Lemma 4.4. *(cf.* [25, Lemma 1.3]*) If $\mathcal{E}^\vee$ has non-trivial endomorphisms, i.e. if $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) \geq 2$, then $|\mathcal{O}_S(C_0)|$ contains a reducible (or multiple) curve.*

We have the following:

Proposition 4.5. *Let (S, L) be a primitively polarized $K3$ surface, such that $L^2 > 2$ and $|L|$ contains neither reducible nor non-reduced curves. Let $C \in |L|$ be any smooth curve and let $\mathcal{A} \in V^r_d(C)$. Then:*
(i) $\mathcal{E}$ and $\mathcal{E}^\vee$ are simple bundles on S.
(ii) Assume further that $(S, L) \in \mathcal{B}_g$ is general, then both $\mathcal{E}$ and $\mathcal{E}^\vee$ are stable bundles on S. In particular, $\dim(M_v(S)) = 2\rho(\mathcal{A})$, where $v = v(\mathcal{E})$ (cf. Remark 2.7).

Proof. (i) This is a direct consequence of Lemma 4.4.
(ii) We prove it here for reader's convenience. Assume there exists a destibilizing sequence of the form

$$0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{F}_2 \to 0 \tag{4.6}$$

where $\mathcal{F}_i$ are sheaves on S, $1 \leq i \leq 2$. Since by assumption $(S, L) \in \mathcal{B}_g$ is general, then $\mathrm{Pic}(S) = \mathbb{Z}[L]$; therefore, there exist integers $a_i \in \mathbb{Z}$ such that $c_1(\mathcal{F}_i) \sim a_i L$. Since $\mathcal{E}$ is globally generated then so is $\mathcal{F}_2$ and so $a_2 \geq 0$.

Taking first Chern classes in (4.6), we get $L = c_1(\mathcal{E}) = c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2) = (a_1 + a_2)L$. Hence $a_1 + a_2 = 1$. If $a_2 \geq 1$, then $a_2 \leq 0$ and hence we would have

$$\frac{a_1}{\mathrm{rk}(\mathcal{F}_1)} \leq 0 < \frac{1}{\mathrm{rk}(\mathcal{E})},$$

i.e. the sequence would not be destibilizing. Thus, we must have $a_2 = 0$ and $a_1 = 1$.

However, if $a_2 = 0$, since $\mathcal{F}_2$ is globally generated, it follows that the torsion-free part of $\mathcal{F}_2$ is a trivial bundle. Since (4.6) is destibilizing, then $\mathrm{rk}(\mathcal{F}_1) < \mathrm{rk}(\mathcal{E})$ and hence the torsion-free part of $\mathcal{F}_2$ is non-zero. Therefore $\mathcal{E}$ has a trivial quotient; this quotient can be combined with a global section of $\mathcal{E}$ to give a non-trivial endomorphism of $\mathcal{E}$, contradicting $h^0(\mathcal{E} \otimes \mathcal{E}^\vee) = 1$ as it follows from (i) and Remark 2.5. This implies $\mathcal{E}$ is stable and so $\mathcal{E}^\vee$ is.

The dimension count follows from the fact that, being stable, $[\mathcal{E}] \in M_v(S)$ is a smooth point; thus $h^1(\mathcal{E} \otimes \mathcal{E}^\vee) = \dim(T_{[\mathcal{E}]}(M_v(S))$. This equals $2\rho(\mathcal{A})$ as it follows from (E5). □

5. High-dimensional scrolls arising from Brill-Noether theory and $K3$'s

Let (S, L) be a primitively, polarized $K3$ surface of genus $g \geq 3$, and let $\mathcal{A} \in V^r_d(C)$, for $C \in |L|$ any smooth curve. Let $\mathcal{E} = \mathcal{E}_{C,A}$ be the Mukai-Lazarsfeld vector bundle as in § 4.

Consider

$$\mathcal{F} := \mathcal{E} \otimes L. \tag{5.1}$$

In what follows, we will compute some cohomological properties of the vector bundle $\mathcal{F}$ as well as of the ruled projective variety defined by the pair $(F, \mathcal{O}_F(1))$, where $F = \mathbb{P}(\mathcal{F})$ (cf. Propositions 5.2, 5.11, Lemma 5.6 and Definition 5.10). It is clear from the computations that one could more generally consider $\mathcal{F}_n := \mathcal{E} \otimes L^{\otimes n}$, for any $n \geq 1$: all the results naturally extend to this more general case, with only tedious and longer computations but with no change with respect to the the geometric strategies. Therefore, for the reader convenience and for a hopefully clearer presentation, we prefer to focus on the case $\mathcal{F}_1 = \mathcal{F}$ as in (5.1) and leave to the reader as an exercise the more general case $\mathcal{F}_n$.

Proposition 5.2. *Let (S, L) be a primitively, polarized $K3$ surface of genus $g \geq 3$. With notation as above, we have:*

- (F1) *$\mathcal{F}$ is a vector bundle of rank $r+1$, which is globally generated and very-ample;*
- (F2) *$c_1(\mathcal{F}) = (r+2)[L]$;*
- (F3) *$c_2(\mathcal{F}) = r(r+3)(g-1) + d$;*
- (F4) *$h^0(\mathcal{F}) = 3g - 3 - d + (r+1)(g+1)$;*
- (F5) *$h^1(\mathcal{F}) = h^2(\mathcal{F}) = 0$;*
- (F6) *$\chi(\mathcal{F} \otimes \mathcal{F}^\vee) = 2 - 2\rho(\mathcal{A})$. Precisely, $h^0(\mathcal{F} \otimes \mathcal{F}^\vee) = h^2(\mathcal{F} \otimes \mathcal{F}^\vee) = 1$ and $h^1(\mathcal{F} \otimes \mathcal{F}^\vee) = 2\rho(\mathcal{A})$; in particular, if $|L|$ contains neither reducible nor non-reduced curves, then $\mathcal{F}$ is simple.*

If, moreover, $(S, L) \in \mathcal{B}_g$ is general, then $\mathcal{F}$ is stable on S.

Proof. (F1): By construction, $\mathcal{F}$ has the same rank of $\mathcal{E}$. Moreover, $\mathcal{F}$ is globally generated, since $\mathcal{E}$ is (cf. (E1)). Furthermore, since L is very-ample from Proposition 2.2, then $\mathcal{F}$ is.

(F2): $c_1(\mathcal{F}) = c_1(\mathcal{E}) + (r+1)c_1(L)$; therefore one can finish the argument by using (E2).

(F3): By standard computations on vector bundles, we have

$$c_2(\mathcal{E} \otimes L) = \binom{r+1}{2} L^2 + r c_1(\mathcal{E}) \cdot L + c_2(\mathcal{E}) \in A^2(S)$$

Since $c_1(\mathcal{E}) = L$ and $c_2(\mathcal{E}) = d$, we have

$$c_2(\mathcal{F}) = (\frac{r(r+1)}{2} + r)L^2 = r(r+3)(g-1) + d.$$

(F4) and (F5): Tensoring (4.2) by L, we get

$$0 \to H^0(\mathcal{A})^\vee \otimes L \to \mathcal{F} \to \omega_C^{\otimes 2} \otimes \mathcal{A}^\vee \to 0. \tag{5.3}$$

Therefore, (F4) and (F5) follow from the facts $h^1(S, L) = h^2(S, L) = h^1(C, \omega_C^{\otimes 2} \otimes \mathcal{A}^\vee) = 0$, $h^0(\mathcal{A}) h^0(\mathcal{O}_S(L)) = (r+1)(g+1)$ and $h^0(C, \omega_C^{\otimes 2} \otimes \mathcal{A}^\vee) = 3(g-1) - d$.

(F6): This follows from (E5) and from the obvious fact $\mathcal{F} \otimes \mathcal{F}^\vee = \mathcal{E} \otimes \mathcal{E}^\vee$.

The last assertion follows from the fact that $\mathcal{E}$ is stable for $(S, L) \in \mathcal{B}_g$ general (cf. Proposition 4.5 (ii)). □

Observe that the map $\mathcal{F} \to \mathcal{F} \otimes L^{-1} = \mathcal{E}$ induces an isomorphism

$$\mathcal{M}_{v(\mathcal{F})}(S, L) \cong \mathcal{M}_{v(\mathcal{E})}(S, L).$$

In particular, from Proposition 4.5 (ii), we have

$$\dim(\mathcal{M}_{v(\mathcal{F})}(S, L)) = 2\rho(\mathcal{A}). \tag{5.4}$$

For $(S, L) \in \mathcal{B}_g$ general and $v = v(\mathcal{F})$, with $\mathcal{F}$ as above, we now want to study the projective geometry of the pairs $(S, \mathcal{F})$, with $[\mathcal{F}] \in \mathcal{M}_v(S)$ (cf. Remark 2.7). To this aim, let $\eta : F \to S$ be the *projective space bundle* on S given by

$$F := \mathbb{P}_S(\mathcal{F}) = Proj(Sym(\mathcal{F})).$$

Notice that $\dim(F) = r + 2$.

From Proposition 5.2 (F1), the tautological linear system $|\mathcal{O}_F(1)|$ is base-point-free and very ample; therefore the morphism

$$\Phi : F \to \mathbb{P}^R$$

induced by $|\mathcal{O}_F(1)|$ is an embedding, where

$$R = h^0(\mathcal{F}) - 1 = (r+1)(g+1) + 3(g-1) - (d+1). \tag{5.5}$$

Moreover, from Proposition 5.2 (F5), the structural morphism η and Leray's isomorphisms, we get

$$h^i(F, \mathcal{O}_F(1)) = 0, \quad \text{for any } i \geq 1.$$

Lemma 5.6. *Let $\xi \in Div(F)$ be the class of divisor on F corresponding to the tautological line bundle $\mathcal{O}_F(1)$. Then*

$$\xi^{r+2} = c_1(\mathcal{F})^2 - c_2(\mathcal{F}) = (g-1)(3r^2 + 5r + 8) - d. \tag{5.7}$$

Proof. Since S is a surface, from [21, pag. 429], we have the relation

$$\xi^{r+1} - \eta^*(c_1(\mathcal{F})) \cdot \xi^r + \eta^*(c_2(\mathcal{F})) \cdot \xi^{r-1} = 0 \tag{5.8}$$

in the degree (r+1)-part $A^{r+1}(F)$ of the Chow ring $A(F)$. If we intersect (5.8) with ξ, we get

$$\xi^{r+2} = \eta^*(c_1(\mathcal{F})) \cdot \xi^{r+1} - \eta^*(c_2(\mathcal{F})) \cdot \xi^r.$$

Since, by Proposition 5.2 (F3), $c_2(\mathcal{F})$ consists of $r(r+3)(g-1) + d$ points on S, then $\eta^*(c_2(\mathcal{F})) \cdot \xi^r = r(r+3)(g-1) + d$ by definition of tautological line-bundle. Therefore, we have

$$\xi^{r+2} = \eta^*(c_1(\mathcal{F})) \cdot \xi^{r+1} - c_2(\mathcal{F}).$$

If we now intersect (5.8) with $\eta^*(c_1(\mathcal{F}))$, we get

$$\xi^{r+1} \cdot \eta^*(c_1(\mathcal{F})) - \eta^*(c_1(\mathcal{F})^2) \cdot \xi^r = 0,$$

since $\eta^*(c_1(\mathcal{F})) \cdot \eta^*(c_2(\mathcal{F})) \cdot \xi^{r-1} = 0$ by dimensional reasons. In particular, $\xi^{r+1} \cdot \eta^*(c_1(\mathcal{F})) = \eta^*(c_1(\mathcal{F})^2) \cdot \xi^r$. From Proposition 5.2 (F2), we have that $c_1(\mathcal{F})^2 = 2(r+2)^2(g-1)$. Thus, reasoning as above we have $\xi^{r+1} \cdot \eta^*(c_1(\mathcal{F})) = c_1(\mathcal{F})^2 = 2(r+2)^2(g-1)$, which completely proves (5.7)

□

Remark 5.9. Observe that $R = R(r, d)$ and ξ^{r+2} depend both on the integers r and d, i.e. on the numerical data of $|\mathcal{A}| = \mathfrak{g}^r_d$ on C, as $\mathcal{F}$ does.

Definition 5.10. The $(r+2)$-dimensional, smooth projective variety

$$\Phi(F) := \mathcal{P} \subset \mathbb{P}^R$$

is said to be the $\mathbb{P}^r$-*scroll* (or simply, the r-*scroll*) determined by the pair $(\mathcal{F}, S)$. We will denote by $\delta := \deg(\mathcal{P}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F}) = (g-1)(3r^2 + 5r + 8) - d$ (cf. Lemma 5.6).

For any $x \in S$, let $F_x := \eta^{-1}(x)$. The r-dimensional linear space $\mathcal{P}_x := \Phi(F_x) \cong \mathbb{P}^r$ is called a *ruling* of $\mathcal{P}$. Since $h^1(S, \mathcal{F}) = h^1(\mathcal{P}, \mathcal{O}_{\mathcal{P}}(1)) = 0$, we will say that the pair $(\mathcal{F}, S)$ determines $\mathcal{P} \subset \mathbb{P}^R$ as a *linearly normal, non-special r-scroll.*

If moreover $\mathcal{F}$ is stable, then $\mathcal{P}$ will be called also *stable.*

In order to study the Hilbert schemes parametrizing such scrolls, we need to compute some cohomological properties.

Proposition 5.11. *Let $g \geq 3$ be a postive integer. Let $(S, L) \in \mathcal{B}_g$ and let $[\mathcal{F}] \in \mathcal{M}_{v(\mathcal{F})}(S, L)$ be as in* (5.1)*. Let $\mathcal{P}$ be the r-scroll determined by the pair $(\mathcal{F}, S)$.*
Denote by $G_{\mathcal{P}} \subset PGL(R+1)$ the subgroup of projective transformations fixing $\mathcal{P}$. Then, $\dim(G_{\mathcal{P}}) = 0$.

Proof. There is an obvious inclusion of algebraic groups $G_{\mathcal{P}} \hookrightarrow \mathrm{Aut}(\mathcal{P})$. We will show that $\dim(\mathrm{Aut}(\mathcal{P})) = 0$. Since $\mathrm{Aut}(\mathcal{P})$ is an algebraic group, its dimension equals $h^0(\mathcal{P}, \mathcal{T}_{\mathcal{P}})$, where $\mathcal{T}_{\mathcal{P}}$ denotes the tangent bundle of $\mathcal{P}$.

Consider the exact sequence

$$0 \to \mathcal{T}_{rel} \to \mathcal{T}_{\mathcal{P}} \to \eta^*(\mathcal{T}_S) \to 0 \tag{5.12}$$

arising from the structure morphism $\mathcal{P} \cong F \stackrel{\eta}{\to} S$.

As it follows from our definition of $\mathcal{P}$ and from [34, (4.33), p. 244], we have the exact sequence:

$$0 \to \mathcal{O}_{\mathcal{P}} \to \eta^*(\mathcal{F}^{\vee}) \otimes \mathcal{O}_{\mathcal{P}}(1) \to \mathcal{T}_{rel} \to 0. \tag{5.13}$$

If we apply η_* to (5.13), since $R^1\eta_*(\mathcal{O}_{\mathcal{P}}) = 0$, we get

$$0 \to \mathcal{O}_S \to \mathcal{F}^{\vee} \otimes \mathcal{F} \to \eta_*(\mathcal{T}_{rel}) \to 0. \tag{5.14}$$

Since $h^0(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 1$ and $h^1(\mathcal{O}_S) = 0$, from Proposition 5.2 (F6) and (5.14), we get

$$h^0(\eta_*(\mathcal{T}_{rel})) = h^2(\eta_*(\mathcal{T}_{rel})) = 0, \;\; h^1(\eta_*(\mathcal{T}_{rel})) = 2\rho(\mathcal{A}).$$

Since $R^i\eta_*(\mathcal{T}_{rel}) = 0$, $i \geq 1$, by Leray's isomorphisms we get

$$h^i(\mathcal{T}_{rel}) = \left\{ \begin{array}{ll} 0 & \text{if} \;\; 0 \leq i \leq r+2, \, i \neq 1 \\ 2\rho(\mathcal{A}) & \text{if} \;\; i = 1. \end{array} \right. \tag{5.15}$$

Thus, by considering (2.3) and Leray's isomorphisms, from (5.12), we obtain

$$h^i(\mathcal{T}_{\mathcal{P}}) = \left\{ \begin{array}{ll} 0 & \text{if} \;\; 0 \leq i \leq r+2, \, i \neq 1 \\ 2\rho(\mathcal{A}) + 20 & \text{if} \;\; i = 1. \end{array} \right. \tag{5.16}$$

□

Remark 5.17. Observe that from the proof of Proposition 5.11, one more precisely has:

$$H^1(\mathcal{P}, \mathcal{T}_{rel}) \cong H^1(S, \mathcal{F}^{\vee} \otimes \mathcal{F}) = H^1(\mathcal{E}^{\vee} \otimes \mathcal{E}),$$

i.e.

$$H^1(\mathcal{P}, \mathcal{T}_{rel}) \cong T_{[\mathcal{F}]}(\mathcal{M}_{v(\mathcal{F})}(S, L)) \cong T_{[\mathcal{E}]}(\mathcal{M}_{v(\mathcal{E})}(S, L)) \tag{5.18}$$

(cf. Proposition 2.10).

Another fundamental step is the following computation.

Proposition 5.19. *Assumptions as in Proposition 5.11. If $\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}$ denotes the normal bundle of $\mathcal{P}$ in $\mathbb{P}^R$, then:*

(i) $h^0(\mathcal{P}, \mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 18 + 2g - 2(r+1)(r+g-d) + (R+1)^2$;
(ii) $h^i(\mathcal{P}, \mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$, *if* $i \geq 1$.

Proof. From the Euler sequence

$$0 \to \mathcal{O}_{\mathcal{P}} \to H^0(\mathcal{O}_{\mathcal{P}}(1))^{\vee} \otimes \mathcal{O}_{\mathcal{P}}(1) \to \mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}} \to 0$$

and from the fact that $\mathcal{P}$ is linearly normal, non-special and is a scroll over a $K3$ surface, we get

$$h^0(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) = (R+1)^2 - 1, \ h^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) = 1, \ h^i(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) = 0, \text{ for any } i \geq 2.$$

Consider the tangent sequence

$$0 \to \mathcal{T}_{\mathcal{P}} \to \mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}} \to N_{\mathcal{P}/\mathbb{P}^R} \to 0. \tag{5.20}$$

The above computations on the cohomology of $\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}$, together with Proposition 5.11, show that $h^i(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$, for any $i \geq 2$. Moreover, from (5.16) and (5.20), we have

$$\chi(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = h^0(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) - h^1(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 18 + 2\rho(\mathcal{A}) + (R+1)^2. \tag{5.21}$$

The rest of the proof will be concentrated on showing that $h^1(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$.

Since $h^2(\mathcal{T}_{\mathcal{P}}) = 0$ (cf. (5.16)) then, from (5.20), we have that $h^1(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$ iff the map $H^1(\mathcal{T}_{\mathcal{P}}) \to H^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}})$ is surjective, where $h^1(\mathcal{T}_{\mathcal{P}}) = 20 + 2\rho(\mathcal{A})$ and $h^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) = 1$.

Claim 5.22. *The map $H^1(\mathcal{T}_{\mathcal{P}}) \to H^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}})$ arising from* (5.20) *is surjective.*

Proof of Claim 5.22. From the Euler sequence on $\mathcal{P}$, we get

$$H^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) \cong H^2(\mathcal{O}_{\mathcal{P}}). \tag{5.23}$$

By Leray's isomorphism, the latter is isomorphic to $H^2(\mathcal{O}_S)$.

Let $C \in |L|$ be the smooth curve appearing in the definition of $\mathcal{F}$ and so of $\mathcal{P}$. From the exact sequence defining C in S, we get

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{N}_{C/S} \to 0$$

which gives

$$H^2(\mathcal{O}_S) \cong H^1(\mathcal{N}_{C/S}). \tag{5.24}$$

On the other hand, (3.1) becomes

$$0 \longrightarrow \mathcal{T}_S\langle C\rangle \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{N}_{C/S} \longrightarrow 0. \tag{5.25}$$

From (2.3), we get

$$0 \longrightarrow H^0(\mathcal{N}_{C/S}) \longrightarrow H^1(\mathcal{T}_S\langle C\rangle) \xrightarrow{H^1(r)} H^1(\mathcal{T}_S) \xrightarrow{\alpha} H^1(\mathcal{N}_{C/S}) \longrightarrow 0, \tag{5.26}$$

where $H^1(r)$ is as in (3.5) and since $h^2(\mathcal{T}_S\langle C\rangle) = 0$ (cf. after Definition 3.6 and [7]).

Since $h^2(\mathcal{T}_{rel}) = 0$ (cf. (5.15)), from (5.12), Leray's isomorphism and from the natural commutativity of the diagram

$$\begin{array}{ccc} H^1(\mathcal{P}, \mathcal{T}_{\mathcal{P}}) & \to & H^1(S, \mathcal{T}_S) \\ \downarrow & & \downarrow^{\alpha} \\ H^1(\mathcal{P}, \mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}}) & \overset{\cong}{\to} & H^1(S, \mathcal{N}_{C/S}) \end{array}$$

arising from (5.12) and (5.26), we have $H^1(\mathcal{T}_{\mathcal{P}}) \twoheadrightarrow H^1(\mathcal{T}_S)$. Since α is surjective, by the identifications (5.23) and (5.24), the map $H^1(\mathcal{T}_{\mathcal{P}}) \to H^1(\mathcal{T}_{\mathbb{P}^R}|_{\mathcal{P}})$ is also surjective. □

From Claim 5.22, we deduce that also $h^1(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$. □

Remark 5.27. We want to stress the geometric meaning of the cohomological computations in the proof of Proposition 5.19, when $(S, L) \in \mathcal{B}_g$ is general.

Since $\mathcal{N}_{C/S} \cong \omega_C$, then

$$H^1(\mathcal{N}_{C/S}) \cong \mathbb{C} \ \text{ and } \ H^0(\mathcal{N}_{C/S}) = T_{[C]}(|L|) \cong \mathbb{C}^g. \tag{5.28}$$

With the notation introduced after Definition 3.6, the sequence

$$0 \longrightarrow H^0(\mathcal{N}_{C/S}) \longrightarrow H^1(\mathcal{T}_S\langle C\rangle) \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow 0 \tag{5.29}$$

can be read as the natural differential sequence

$$0 \longrightarrow T_{[C]}(|L|) \longrightarrow T_{(S,C)}(\mathcal{K}C_g) \longrightarrow T_{[S]}(\mathcal{B}_g) \longrightarrow 0; \tag{5.30}$$

indeed, $\mathcal{B}_g$ is smooth of dimension 19 at $[S]$, whereas $h^1(\mathcal{T}_S) = 20$ (cf. (2.3)) and $h^1(\mathcal{N}_{C/S}) = 1$ by (5.28); in other words the elements of $\mathrm{Ker}(\alpha)$ can be identified with the first-order deformations of S preserving the genus g polarization.

Since by the Leray spectral sequence $H^1(\eta^*(\mathcal{T}_S)) \cong H^1(\mathcal{T}_S)$, putting together (5.12), (5.18), (5.26), and taking into account the interpretation of (5.29), we have:

$$\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & T_{[\mathcal{F}]}(\mathcal{M}_{v(\mathcal{F})}(S)) & & & \\
 & & & \downarrow & & & \\
 & & & H^1(\mathcal{T}_{\mathcal{P}}) & \to & H^1(\mathcal{T}_{\mathbb{P}^R|_{\mathcal{P}}}) & \to 0 \\
 & & & \downarrow & & || & \\
0 \to & T_{[S]}(\mathcal{B}_g) & \to & H^1(\mathcal{T}_S) & \to & H^1(\mathcal{N}_{C/S}) & \to 0 \\
 & & & \downarrow & & & \\
 & & & 0 & & & ,
\end{array}$$

which gives another interpretation of Claim 5.22.

6. Hilbert schemes of r-scrolls

Basic information about Hilbert schemes parametrizing r-scrolls as in § 5 are essentially given by the following result.

Theorem 6.1. *Let $g \geq 3$ be an integer. For $(S, L) \in \mathcal{B}_g$ general, for any smooth $C \in |L|$ and any $\mathcal{A} \in V^r_d(C)$, let $M_v(S)$ be the moduli space of torsion-free sheaves on S, with $v = v(\mathcal{F})$ the Mukai vector of $\mathcal{F}$ associated to $\mathcal{A}$ as in (5.1). Let $\rho := g - (r+1)(g+r-d)$.*

The r-scrolls $\mathcal{P}$ determined by the pairs $(S, \mathcal{F})$ fill-up an open dense subset of an irreducible component of the Hilbert scheme parametrizing $(r+2)$-dimensional subvarieties of $\mathbb{P}^R$ of degree $\delta = (g-1)(3r^2+5r+8) - d$, which we denote by $\mathcal{H}_{r+2,\delta}$. The general point $[\mathcal{P}] \in \mathcal{H}_{r+2,\delta}$ parametrizes a smooth, non-special, stable r-scroll, which is linearly normal in $\mathbb{P}^R$.

Furthermore:

(i) $\mathcal{H}_{r+2,\delta}$ *is generically smooth;*
(ii) $\dim(\mathcal{H}_{r+2,\delta}) = 18 + 2g - 2(r+1)(g+r-d) + (R+1)^2$*;*
(iii) $\mathcal{H}_{r+2,\delta}$ *dominates* $\mathcal{B}_g$.

Proof. Denote by $\mathcal{M}_v \xrightarrow{\tau} \mathcal{B}_g$ the relative moduli space of rank-$(r+1)$ torsion-free sheaves with given Mukai vector v so that, for $(S, L) \in \mathcal{B}_g$, $\tau^{-1}((S, L)) = M_v((S, L))$ (cf. [1, 24]).

Since, for any $(S, L) \in \mathcal{B}_g$, $M_v((S, L))$ is irreducible of dimension $2\rho = 2g - 2(r+1)(g+r-d)$ (cf. (4.3)), then $\mathcal{M}_v$ is irreducible, of dimension $18 + 2g - 2(r+1)(g+r-d)$.

Up to shrinking to an open, dense subset $\mathcal{B}_g^0 \subset \mathcal{B}_g$, we have that $\mathrm{Pic}(S) = \mathbb{Z}[L]$. Let $\mathcal{M}_v^0$ be the restriction to $\mathcal{B}_g^0$ of $\mathcal{M}_v$.

The universal bundle exists locally in the classical (or étale) topology of $\mathcal{M}_v^0$ (cf. [17, p. 154]). This is enough for our dimensional computations. Indeed, from this and from Proposition 5.2 - (F5), on a non-empty $U \subset \mathcal{M}_v^0$, we have $\mathcal{F}_U \xrightarrow{\pi} U$ the universal bundle and $\pi_*(\mathcal{F}_U)$ is a vector bundle of rank $R+1$, which can be assumed to be trivial on U. In particular, we can choose independent global sections $s_0, \ldots, s_R$ of $\pi_*(\mathcal{F}_U)$.

Consider $\mathcal{G}_{\mathcal{U}} := U \times PGL(R+1)$ which is irreducible, of dimension $18+2g-2(r+1)(g+r-d)+(R+1)^2$. An element of $\mathcal{G}_{\mathcal{U}}$ can be regarded as a triple $((S,L),\mathcal{F},\sigma) := \gamma_\sigma$, where $(S,L) \in \mathcal{B}_g$ is a general, primitively polarized $K3$ surface of genus g, $[\mathcal{F}] \in M_v(S)$ is general, and σ is a projective transformation. Moreover, the sections $s_0, \ldots, s_R$ induce independent sections of $H^0(S,\mathcal{F})$ and therefore determine a morphism $F = \mathbb{P}(\mathcal{F}) \to \mathcal{P} \subset \mathbb{P}^R$.

Let $\mathrm{Hilb}(\delta, r+2, R)$ denote the Hilbert scheme of varieties of degree $\delta = (g-1)(3r^2+5r+8)-d$ and dimension $r+2$ in $\mathbb{P}^R$. Consider the morphism

$$\Psi : \mathcal{G}_{\mathcal{U}} \to \mathrm{Hilb}(\delta, r+2, R)$$

which maps the triple γ_σ to the r-scroll $\sigma(\mathcal{P})$.

We define $\mathcal{H}_{r+2,\delta}$ to be the closure in the Zariski topology of the image of the above map to the Hilbert scheme. By definition of $\mathcal{G}_{\mathcal{U}}$, it follows that $\mathcal{H}_{r+2,\delta}$ is irreducible and dominates $\mathcal{B}_g$ via the forgetful morphism. Its general point represents a smooth, linearly normal and stable r-scroll $\mathcal{P}$ in $\mathbb{P}^R$ of degree δ (cf. Proposition 5.2 (F5)-(F6) and Definition 5.10).

Since $h^i(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R}) = 0$, for any $i \geq 1$, $[\mathcal{P}]$ is a smooth point of the component of $\mathrm{Hilb}(\delta, r+2, R)$ containing $\mathcal{H}_{r+2,\delta}$.

Next we compute the dimension of $\mathcal{H}_{r+2,\delta}$. Given a general point of $\mathcal{H}_{r+2,\delta}$ corresponding to a r-scroll $\mathcal{P}$, from Proposition 5.11 we know that $\dim(G_{\mathcal{P}}) = 0$; in particular, if one had $\dim \Psi^{-1}([\mathcal{P}]) > 0$, the positive dimension of the general fibre would not be related to projective trasformations of $\mathcal{P}$; in other words, $\Psi^{-1}([\mathcal{P}])$ has to be transverse to the PGL-directions.

In any case, a parameter computation shows that

$$\dim(\mathcal{H}_{r+2,\delta}) \leq 18+2g-2(r+1)(g+r-d)+(R+1)^2. \tag{6.2}$$

Claim 6.3. *For $\gamma_\sigma \in \mathcal{G}_{\mathcal{U}}$ general, the fibre of $\Psi^{-1}(\Psi(\gamma_\sigma))$ is zero-dimensional at γ_σ.*

Proof of Claim 6.3. Suppose that S_1 and S_2 are $K3$'s and $\mathcal{F}_1$ and $\mathcal{F}_2$ are vector bundles as above on S_1 and S_2, respectively. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the resulting scrolls embedded in $\mathbb{P}^R$ via their tautological linear systems. Then, we have to show that:

(i) if F_1 is isomorphic to F_2 (as an abstract variety), then S_1 is isomorphic to S_2;

(ii) if $\mathcal{P}_1 = \mathcal{P}_2$ in $\mathbb{P}^R$ then the isomorphism in (i) sends $\mathcal{F}_1$ to $\mathcal{F}_2$.

To prove (i), let $\pi_i : F_i \to S_i$ be te structural morphism of $\mathbb{P}^r$-bundles, $1 \leq i \leq 2$. Assume there exists an isomorphism $\phi : F_1 \to F_2$. Since for any $x \in S_1$, $\pi_1^{-1}(x)$ is a $\mathbb{P}^r$, the image $\pi_2(\phi(\pi_1^{-1}(x))$ in S_2 is covered by rational curves. Since S_2 is a $K3$ surface, then $\pi_2(\phi(\pi_1^{-1}(x))$ has to be either a rational curve or a point.

As x varies in S_1, the fibres $\pi_1^{-1}(x)$ sweep out F_1 and therefore also F_2, hence their images must cover S_2. Since S_2 cannot be covered by a family of rational curves, we conclude that for a general point $x \in S_1$ then $\pi_2(\phi(\pi_1^{-1}(x))$ has to be a point in S_2. This implies that the isomorphism ϕ preserves the fibration π_1 and so induces a map $\varphi : S_1 \to S_2$ such that

$\pi_2 \circ \phi = \varphi \circ \pi_1$. The commutativity of these maps and the fact that ϕ is an isomorphism imply that also φ is an isomorphism, proving (i).
(ii) Since $\mathcal{P}_1 = \mathcal{P}_2$ in $\mathbb{P}^R$, the isomorphism ϕ is such that $\phi^*(\mathcal{O}_{F_2}(1)) = \mathcal{O}_{F_1}(1)$. Since $(\pi_i)_*(\mathcal{O}_{F_i}(1)) = \mathcal{F}_i$, $1 \leq i \leq 2$, the commutativity of the maps in (i) shows that $\varphi^*(\mathcal{F}_2) = \mathcal{F}_1$. □

From the formula for $h^0(\mathcal{N}_{\mathcal{P}/\mathbb{P}^R})$ in Proposition 5.19 and from Claim 6.3, one has that (6.2) is an equality and that $\mathcal{H}_{r+2,\delta}$ is a component of $\mathrm{Hilb}(\delta, r+2, R)$, which is generically smooth and of that dimension. □

Remark 6.4. An interesting problem is to analyze possible limits in the component $\mathcal{H}_{r+2,\delta}$ of the general element it parametrizes. In the same spirit of [11], subject of a future work will be to bridge the study of this Hilbert scheme, via projective and degenerations techniques, with the one of vector bundles on $K3$'s and Brill-Noether theory of vector bundles on projective curves.

Corollary 6.5. *Assumptions as in Theorem 6.1. Assume further $r = 1$. For any*

$$\gamma(g) \leq d \leq g, \tag{6.6}$$

where

$$\gamma(g) := \begin{cases} \frac{g+2}{2} & \text{if } g \text{ even,} \\ \frac{g+3}{2} & \text{if } g \text{ odd.} \end{cases}$$

is the general gonality*, the general point of $\mathcal{H}_{3,\delta(d)}$ parametrizes a smooth, non-special* 3*-fold scroll over S, with $(S, L) \in \mathcal{B}_g$ general, of degree $\delta(d) = 16(g-1) - d$, which is linearly normal in $\mathbb{P}^{5g-1-d}$.*

Proof. By construction, there exists a $\mathfrak{g}^1_d$ on $C \in |L|$ general iff $\rho(g, 1, d) \geq 0$. The integer which minimizes the non-negative function $\rho(g, 1, d)$ is $\gamma(g)$ as above. Indeed, by definition of gonality and by the results in [25] (cf. also Theorem 1.1)

$$\rho(g, 1, \gamma(g)) = \begin{cases} 0 & \text{if } g \text{ even,} \\ 1 & \text{if } g \text{ odd.} \end{cases}$$

On the other hand, since we are considering $\mathfrak{g}^1_d$'s with $\mathcal{A} \in V^1_d(C)$, in particular $h^1(\mathcal{A}) \geq 1$. By the Riemann Roch theorem we have $d = g + 1 - h^1(\mathcal{A}) \leq g$. This explains the bounds in (6.6). The rest of the statement directly follows from Theorem 6.1. □

Remark 6.7. Observe that e.g. the $K3$-scroll of degree 9 in $\mathbb{P}^5$, which is the only possible smooth scroll in $\mathbb{P}^5$ over a $K3$-surface (cf. [36, Theorem, p. 452]), cannot arise from Brill-Noether theory on such a $K3$. Indeed, in this case S is of genus $g = 8$, being $S = \mathbb{G}(1,5) \cap \mathbb{P}^8 \subset \mathbb{P}^{14}$ (cf. [36, Example (d), p. 452]). If we want $R = 5$, one should have $d = 33$ which is impossible for a $\mathfrak{g}^1_d$ on a smooth curve of genus 8 in $|L|$.

Remark 6.8. Take $r = 1$ and d as in (6.6). Let $\mathcal{E}$ be the vector-bundle associated to a $\mathfrak{g}^1_d$ as in (E1) - (E5). By the Koszul exact sequence we have

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_Z(L) \to 0, \tag{6.9}$$

where Z is a zero-dimensional subscheme of S of length d, which consists of an element of the $\mathfrak{g}^1_d$. Tensoring the Koszul sequence by L one has

$$0 \to L \to \mathcal{F} \to \mathcal{I}_Z(L^{\otimes 2}) \to 0. \tag{6.10}$$

One has a geometric interpretation of (6.10).

The quotient of $\mathcal{F}$ corresponds to a unisecant divisor $\mathcal{S}$ of $\mathcal{P}$, which intesects at just one point the general line of $\mathcal{P}$ and entirely contains the lines of $\mathcal{P}$ which are over the scheme $Z \subset S$.

The dimension of such a familiy of surfaces is $\dim(|\mathcal{I}_Z(L^{\otimes 2}) \otimes L^\vee|)$, i.e. $\dim(|\mathcal{I}_Z(L)|) = g+2-d$, as it follows from (6.9), from $h^1(\mathcal{O}_S) = 0$ and from (E4).

7. Applications to Hilbert schemes of non-special ruled surfaces

Let $(S, L) \in \mathcal{B}_g$ be general. Let $\mathcal{A} \in V^r_d(C)$, for $C \in |L|$ any smooth curve. Let $\mathcal{F}$ be the associated vector bundle as in (5.1). If one tensors the exact sequence defining C in S by $\mathcal{F}$, one gets

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{F}|_C \to 0. \tag{7.1}$$

From (E3), (E4) and Proposition 5.2, one has:

$$h^0(\mathcal{F}|_C) = (R+1) - (2r+g-d+1) = (r+3)(g-1) \text{ and } h^1(\mathcal{F}|_C) = 0. \tag{7.2}$$

In particular, $\mathcal{F}|_C$ is non-special and very-ample on C. Moreover

$$c_1(\mathcal{F}|_C) = c_1(\mathcal{F}) \cdot C = 2(r+2)(g-1).$$

From the surjectivity in (7.1), it is clear that the pair $(C, \mathcal{F}|_C)$ determines a smooth scroll Σ, of dimension $r+1$, which is a $\mathbb{P}^r$-bundle over the curve C and which is contained in the r-scroll $\mathcal{P}$ over S studied in the previous sections. The degree of Σ is $2(r+2)(g-1)$, its sectional genus is g; furthermore, $\Sigma \subset \mathbb{P}^{(r+3)(g-1)-1}$ is non-special and linearly normal.

Remark 7.3. Observe that, in contrast with what occurs for the r-scroll $\mathcal{P}$ over S (cf. Remark 5.9), the degree and the embedding dimension of the scroll Σ over C arising from the above construction are independent from $d = \deg(\mathcal{A})$.

When in particular $r = 1$, Σ is a geometrically ruled surface, of degree $n := 6g-6$, which is ruled by lines, non-special and linearly normal in its projective span $\mathbb{P}^h$, where $h := 4g-5$. From now on, we will call such a surface a *scroll of genus g*.

Basic information about the Hilbert scheme parametrizing scrolls of genus g are contained in [9, Theorem 1.2] and [10, Theorem 2] (these have been also studied in [5]).

Theorem 7.4. *Let $g \geq 0$ be an integer and set $k := \min\{1, g-1\}$. If $n \geq 2g+3+k$, then there exists a unique, irreducible component $\mathcal{H}_{n,g}$ of the Hilbert scheme of scrolls of genus g, of degree n in $\mathbb{P}^{n-2g+1}$, such that the general point $[Y] \in \mathcal{H}_{n,g}$ represents a smooth, non-special and linearly normal scroll Y of genus g. Furthermore,*

(i) *$\mathcal{H}_{n,g}$ is generically reduced;*
(ii) $\dim(\mathcal{H}_{n,g}) = 7(g-1) + (n-2g+2)^2$*;*
(iii) *$\mathcal{H}_{n,g}$ dominates the moduli space $\mathcal{M}_g$ of smooth curves of genus g.*

Moreover, if $g \geq 1$, let $[Y] \in \mathcal{H}_{n,g}$ be a general point and let $(\mathcal{G}, C)$ be a pair which determines Y, where $[C] \in \mathcal{M}_g$ general and $\mathcal{G}$ is a degree n rank-two vector bundle on C. Then $[\mathcal{G}]$ is a general point in $U_C(n)$, the moduli space *of rank-two, semistable vector bundles of degree n on C.*

By the uniqueness of $\mathcal{H}_{6g-6,g}$, for $(S, L) \in \mathcal{B}_g$ general, $C \in |L|$ general and $\mathcal{A} \in V^1_d(C)$, for any admissible d, the scrolls Σ arising from the above construction are all contained in $\mathcal{H}_{6g-6,g}$ (cf. Remark 7.3).

Question 7.5. Does any such Σ correspond to a smooth point of $\mathcal{H}_{6g-6,g}$?

The answer is yes and it is proved in the following:

Proposition 7.6. *Let $g \geq 3$ be an integer. Let $(S, L) \in \mathcal{B}_g$ be general, $C \in |L|$ general and $\mathcal{A} \in V^1_d(C)$, for any admissible d. Let $[\Sigma] \in \mathcal{H}_{6g-6,g}$ be the Hilbert point corresponding to the scroll Σ arising from the pair $(C, \mathcal{F}|_C)$. Then, $[\Sigma]$ is a smooth point of the Hilbert scheme $\mathcal{H}_{6g-6,g}$.*

Proof. We have to show that $[\Sigma] \in \mathcal{H}_{6g-6,g}$ is unobstructed. A sufficient condition is to show that $h^1(\mathcal{N}_{\Sigma/\mathbb{P}^h}) = 0$.

To see this, consider the normal sequence of $\Sigma \subset \mathbb{P}^h$:

$$0 \to \mathcal{T}_\Sigma \to \mathcal{T}_{\mathbb{P}^h|_\Sigma} \to \mathcal{N}_{\Sigma/\mathbb{P}^h} \to 0. \tag{7.7}$$

One wants to compute first the cohomology of $\mathcal{T}_{\mathbb{P}^h|_\Sigma}$.

For the latter, it is sufficient to consider the Euler sequence of $\mathbb{P}^h$ restricted to Σ. If we denote by $\eta : \Sigma \to C$ also the structural morphism of the ruled surface Σ, one has that $\eta_*(\mathcal{O}_\Sigma) = \mathcal{O}_C$ and $\eta_*(\mathcal{O}_\Sigma(H)) = \mathcal{F}|_C$. Therefore, from (7.2) and from the Euler sequence, one easily finds

$$h^0(\mathcal{T}_{\mathbb{P}^h|_\Sigma}) = (h+1)^2 + g - 1 \ \text{ and } \ h^i(\mathcal{T}_{\mathbb{P}^h|_\Sigma}) = 0, \text{ for any } i \geq 1. \tag{7.8}$$

On the other hand, since Σ is a scroll of genus g, it is well-known that $\chi(\mathcal{T}_\Sigma) = h^0(\mathcal{T}_\Sigma) - h^1(\mathcal{T}_\Sigma) = 6 - 6g$.

Therefore, from (7.7) and the above computations , we get

$$h^i(\mathcal{N}_{\Sigma/\mathbb{P}^h}) = 0,$$

for any $i \geq 1$, which proves the assertion. □

In particular, we have

$$h^0(\mathcal{N}_{\Sigma/\mathbb{P}^h}) = h^0(\mathcal{T}_{\mathbb{P}^h|_\Sigma}) - \chi(\mathcal{T}_\Sigma) = 7(g-1) + (h+1)^2 = \dim_{[\Sigma]}(T_{\mathcal{H}_{6g-6,g}})$$

as it has to be.

The next natural question is the following.

Question 7.9. Let $g \geq 3$ be an integer. Let $[Y] \in \mathcal{H}_{6g-6,g}$ be a general point. Is it true that there exists a pair $(C, \mathcal{F}|_C)$ as above which determines Y?

For large values of g, the answer is obviously NO. Indeed, if $g = 10$ and $g \geq 12$, from Theorem 3.9 we know that the pair $(C, \mathcal{F}|_C)$ cannot determine a scroll with general moduli. Therefore, one can finish the argument by using Theorem 7.4 (iii).

For $g \leq 11$, $g \neq 10$, observe first that since we are considering $\mathfrak{g}^1_d$'s on C of genus g, we must consider d as in (6.6). The next proposition shows that even if for $3 \leq g \leq 11$, $g \neq 10$, the curve C has general moduli (cf. Theorem 3.9), the answer to Question 7.9 is negative also for any such g and for any d as in (6.6).

Proposition 7.10. *Let $3 \leq g \leq 11$, $g \neq 10$, be an integer. For any d as in (6.6), scrolls Σ of genus g arising from the above construction fill up an open dense subset of a closed subscheme of $\mathcal{H}_{6g-6,g}$, denoted by $\mathcal{K}_d$, which is irreducible and dominates $\mathcal{M}_g$.*

Moreover, for any $\gamma(g) \leq d \neq d' \leq g$, one has $\mathcal{K}_d \neq \mathcal{K}_{d'}$.

Proof. Fix d an integer as above. By the above considerations, scrolls arising from these constructions depend at most on the following parameters:

- $19 + g = \dim(\mathcal{K}C_g)$, plus
- $2\rho(g, 1, d) = \dim(M_{v(\mathcal{F})}(S))$, where $\mathcal{F}$ associated to the $\mathfrak{g}^1_d$ on C;

- for any $[\mathcal{F}] \in M_{v(\mathcal{F})}(S)$, we consider $\mathcal{F}|_C$ and consequently the embedding dimension $h = 4g-5$ of the scroll Σ determined by the pair $(C, \mathcal{F}|_C)$. Thus, one takes into account all the projective transformations of Σ via $PGL(h+1, \mathcal{C})$.

Therefore, for any d,

$$\dim(\mathcal{K}_d) \leq 18 + g + 2\rho(g,1,d) + (h+1)^2 = 18 - g - 4 + 4d + (h+1)^2.$$

From Theorem 7.4 (ii) we have that $\dim(\mathcal{H}_{6g-6,g}) = 7(g-1) + (h+1)^2$. Observe that the assumptions on d implies that $18 - g - 4 + 4d + (h+1)^2 < 7(g-1) + (h+1)^2$, for any g.

Any $\mathcal{K}_d$ dominates $\mathcal{M}_g$ as it follows from Theorem 3.9.

The fact that any $\mathcal{K}_d$ is irreducible directly follows from the construction, when $\rho(g,1,d) > 0$ by the well-known results of Fulton-Lazarsfeld in [18]. On the other hand, when $\rho(g,1,d) = 0$, one can conclude by using [14, Theorem 1].

For the last assertion, without loss of generality, one can assume $d < d'$. The fact that $\mathcal{K}_d \neq \mathcal{K}_{d'}$ for $d \neq d'$ directly follows from the fact that a general $\mathcal{A} \in W^1_{d'}(C)$ cannot belong to $W^1_d(C)$, otherwise $|\mathcal{A}|$ would have some base point, against the assumption of generality for $\mathcal{A} \in W^1_{d'}(C)$ (cf. also § 4). □

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