ON HYPERSURFACES WITH VANISHING HESSIAN AFTER P. GORDAN, M. NOETHER AND U. PERAZZO

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$f(x_0, \ldots, x_N) \in \mathbb{C}[x_0, \ldots, x_N]_d$, $d \geq 1$, $f = f_{\mathrm{red}}$.

- $X = V(f) \subset \mathbb{P}^N$ associated degree d projective hypersurface.

$$\mathrm{Hess}_X = H(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{0 \leq i,j \leq N},$$

hessian matrix of X (or of f, in this case indicated by $H(f)$).
Clearly

$$H(f) = \mathbf{0}_{(N+1)\times(N+1)} \iff d = 1.$$

From now on $d \geq 2$.

$$\mathrm{hess}_X = \det(\mathrm{Hess}_X)$$

hessian of X (or of f, in this case indicated by $h(f)$). There are two possibilities:

1. either $\mathrm{hess}_X \equiv 0$
2. or $\mathrm{hess}_X \in \mathbb{C}[x_0, \ldots, x_N]_{(N+1)(d-2)}$.

We shall be interested in case 1), that is in **hypersurfaces with vanishing hessian**.

Trivial Examples

1. If $\frac{\partial f}{\partial x_i} = 0$ for some $i \Rightarrow h(f) = 0$.
2. Let $\widehat{A} \in GL_{N+1}(\mathbb{C})$, let $\widehat{\mathbf{x}} = \widehat{A}\mathbf{x}$ and let

$$\widehat{f}(\mathbf{x}) = f(\widehat{A}\mathbf{x}).$$

We shall say that $\widehat{f}$ **is linearly equivalent to** f, indicated by

$$\widehat{f} \sim f.$$

Then

$$h(\widehat{f}) = 0 \iff h(f) = 0.$$

3. If $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}$ are **linearly dependent** $\Rightarrow h(f) = 0$.

Cones

Proposition

Let $X = V(f) \subset \mathbb{P}^N$ hypersurface, $d = \deg(X) \geq 2$. Then the following conditions are equivalent:

(i) *X is a cone;*

(ii) *There exists a point $p \in X$ of multiplicity d;*

(iii) *The partial derivatives $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}$ of f are* **linearly dependent***;*

(iv) *Up to a projective transformation, f depends on at most N variables.*

(v) *The dual variety of X, $X^* \subset (\mathbb{P}^N)^*$, is degenerated, i.e. $< X^* > \subsetneq \mathbb{P}^{N*}$.*

Hesse's Claim

$\mathrm{hess}_X \equiv 0 \Rightarrow X = V(f) \subset \mathbb{P}^N$ is a cone.

Or equivalently

$\mathrm{hess}_X \equiv 0 \Rightarrow \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}$ are linearly dependent.

Low dimension & low degree

- For $N = 1$ Calculus I yields Hesse's Claim.
  Suppose from now on $N \geq 2$.
- $N \geq 2$ & $d = 2$:

$$f(\mathbf{x}) = \mathbf{x}^t \cdot B \cdot \mathbf{x} \sim x_0^2 + \ldots + x_r^2$$

with $r + 1 = \mathrm{rk}(B)$ and with $h(f) = \det(B)$. Thus for $d = 2$,

$$X = V(f) \subset \mathbb{P}^N \text{ is a cone} \iff \mathrm{hess}_X \equiv 0.$$

Gordan-Noether Theorem

From now on $N \geq 2$ and $d \geq 3$.

Theorem (Gordan-Noether, 1876)

Let $X = V(f) \subset \mathbb{P}^N$ *hypersurface,* $d = \deg(X) \geq 3$. *Then:*

1. *If* $N \leq 3$, *Hesse's Claim is true.*
2. $\forall N \geq 4$ *&* $\forall d \geq 3$ *there exist counterexamples to Hesse's Claim.*

The counterexamples are
..... *a rare enough phenomenon to merit special study*,
J. G. Semple & J. A. Tyrrell.

"Un esempio semplicissimo" by U. Perazzo

$$f(x_0, x_1, x_2, x_3, x_4) = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2.$$

PERAZZO CUBIC HYPERURFACE$\sim$ 1900**=**
BOURGAIN (SACKSTEDER) TWISTED PLANE$\sim$ 1990

$$\frac{\partial f}{\partial x_0} = x_3^2, \ \frac{\partial f}{\partial x_1} = x_3x_4, \frac{\partial f}{\partial x_2} = x_4^2,$$

$$\frac{\partial f}{\partial x_3} = 2x_0x_3 + x_1x_4, \frac{\partial f}{\partial x_4} = x_1x_3 + 2x_2x_4$$

are **linearly independent** but **ALGEBRAICALLY DEPENDENT**:

$$\frac{\partial f}{\partial x_0}\frac{\partial f}{\partial x_2} - (\frac{\partial f}{\partial x_1})^2 = x_3^2x_4^2 - (x_3x_4)^2 = 0.$$

$\Rightarrow h(f) = 0$ (as we shall see in a moment).

Perazzo's First Classification result

Series of examples for $N \geq 4$ and $d = 3$:

$$g(x_0, x_1, x_2, x_3, x_4, \ldots, x_N) = x_0 x_3^2 + x_1 x_3 x_4 + x_2 x_4^2 + x_5^3 + \ldots + x_N^3.$$

Then $h(g) = 0$ and $X = V(g) \subset \mathbb{P}^N$ is not a cone.

Theorem (Perazzo, 1901)

Let $X = V(f) \subset \mathbb{P}^4$ be a cubic hypersurface, not a cone. Then

$$f \sim x_0 x_3^2 + x_1 x_3 x_4 + x_2 x_4^2.$$

Indeed, not too many examples, at least in degree 3 and for not too many variables.

Gordan-Noether Cremona equivalence Theorem

Theorem (Gordan-Noether, 1876)

Let $X = V(f) \subset \mathbb{P}^N$ *hypersurface,* $d = \deg(X) \geq 2$ *with* $h_X \equiv 0$*. Then there exists a Cremona transformation*

$$\Phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$

such that $\Phi(X)$ *is a cone.*
Moreover the Cremona transformation can be explicitly constructed from f *in such a way that the equation of* $\phi(X)$ *depends on at most* N *variables.*

For example the Perazzo hypersurface

$$X = V(x_0x_3^2 + x_1x_3x_4 + x_2x_4^2) \subset \mathbb{P}^4$$

is Cremona equivalent to

$$V(x_1x_3 + x_2x_4) \subset \mathbb{P}^4 \text{ or to } V(x_0x_3^2 + x_2x_4^2) \subset \mathbb{P}^4$$

Algebro-geometric translation: the Polar Map

$$\begin{array}{rccc} \nabla_f = \nabla_X : & \mathbb{P}^N & \dashrightarrow & \mathbb{P}^{N*} \\ & p & \rightarrow & \left(\frac{\partial f}{\partial x_0}(p) : \ldots : \frac{\partial f}{\partial x_N}(p)\right) \end{array}$$

is **the polar (or gradient) map of** $X = V(f) \subset \mathbb{P}^N$.

- $\mathrm{Sing}(X) := V(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_N}) \subset \mathbb{P}^N$, **base locus scheme of** ∇_X.
- $Z := \overline{\nabla_f(\mathbb{P}^N)} \subseteq (\mathbb{P}^N)^*$ **polar image of** $\mathbb{P}^N$

The restriction of ∇_f to X is the **Gauss map of** X:

$$\begin{array}{rccc} \mathcal{G}_X = \nabla_{f|X} : & X & \dashrightarrow & \mathbb{P}^{N*} \\ & p \in X_{\mathrm{reg}} & \rightarrow & \mathcal{G}_X(p) = [T_pX]. \end{array}$$

$$X^* := \overline{\mathcal{G}_X(X)} \subsetneq Z,$$

is the **dual variety of** X, parametrizing **singular hyperplane sections of** X.

Algebro-geometric translation: the Polar map

KEY FORMULA: $H(f) = Jac(\nabla_f : \mathbb{C}^{N+1} \to \mathbb{C}^{N+1})$.

$$T_{\nabla_f(p)} Z = \mathbb{P}(\mathrm{Im}(H(f)(p))) \subseteq (\mathbb{P}^N)^* \ \ \forall p \in \mathbb{P}^N \text{ general.}$$

$$\dim Z = \mathrm{rk}(H(f)) - 1$$

$$h(f) \equiv 0 \iff Z \subsetneq \mathbb{P}^{N*}$$

$$\iff \exists g \in \mathbb{C}[y_0, \dots, y_N] \ : \ g(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N}) \equiv 0.$$

$$\iff \frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N} \text{are } \textbf{algebraically} \text{ dependent}$$

$$\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N} \textbf{algebraically} \text{ dependent} \overset{Hesse's Claim}{\Rightarrow}$$

$$\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_N} \ \textbf{linearly} \text{ dependent?}$$

Curvature and $h(f)$

Lemma

$$(d-1)x_0^2 h(f) = \begin{vmatrix} \frac{d}{d-1}f & \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_N} \\ \frac{\partial f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_N} & \frac{\partial^2 f}{\partial x_1 \partial x_N} & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_N} \end{vmatrix}$$

$$K(p) = -\frac{\begin{vmatrix} 0 & \frac{\partial f}{\partial x_1}(p) & \cdots & \frac{\partial f}{\partial x_N}(p) \\ \frac{\partial f}{\partial x_1}(p) & \frac{\partial^2 f}{\partial x_1^2}(p) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_N}(p) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_N}(p) & \frac{\partial^2 f}{\partial x_1 \partial x_N}(p) & \cdots & \frac{\partial^2 f}{\partial x_N \partial x_N}(p) \end{vmatrix}}{\left(\left(\frac{\partial f}{\partial x_1}(p)\right)^2 + \ldots \left(\frac{\partial f}{\partial x_N}(p)\right)^2 \right)^{\frac{N+1}{2}}}$$

is the **Gaussian Curvature of** $X = V(f) \subset \mathbb{P}^N$ **at** $p \in (X \setminus \mathrm{Sing}(X)) \cap (\mathbb{P}^N \setminus V(x_0))$.

Curvature and $h(f)$

$$h(f) \equiv 0 \Rightarrow h(f) \equiv 0 \ (\text{mod. } f) \iff K \equiv 0.$$

- $N = 2$, $C = V(f) \subset \mathbb{P}^2$ reducible curve of degree d
- $H(C) = V(h(f)) \subset \mathbb{P}^2$ **Hessian curve of C**
  $(H(C) = \mathbb{P}^2 \iff h(f) \equiv 0)$
- $f / h(f) \iff C \subseteq H(C) \iff$ every $p \in C \setminus \mathrm{Sing}(C)$ is a **flex** $\iff$ C is a union of lines.
- $f(x_0, x_1, x_2) = x_0x_1x_2 \Rightarrow h(f) = 2x_0x_1x_2 \not\equiv 0$ but $h(f) = 0$ (mod. f).
- $h(f) = 0 \stackrel{?}{\Rightarrow}$ the lines pass through a fixed point, that is C is a cone?
- Gordan-Noether Theorem is true for $N = 2$:

$$C \text{ is a cone} \iff h(f) \equiv 0.$$

Gordan-Noether Theorem in $\mathbb{P}^3$

Let us recall a more or less well known result about the characterization of developable surfaces in $\mathbb{R}^3$.

Theorem

Let $S = V(f) \subset \mathbb{P}^3$ be an irreducible surface. Then the following conditions are equivalent:

1. $K(p) = 0\ \forall p \in S_{\text{reg}}$ *(that is f divides $h(f)$);*
2. *S is a* **cone** *or S is the* **developable of tangent lines to** *$C \subset \mathbb{P}^3$ (equivalently S is a* **developable surface***).*

Let us state Gordan-Noether's refinement:

Theorem (Gordan-Noether, 1876)

Let $S = V(f) \subset \mathbb{P}^3$. Then

$$S \text{ is a cone} \iff h(f) \equiv 0.$$

What measures f divides $h(f)$?

Let

$$\mathcal{G}_X : X \dashrightarrow X^* \subseteq Z \subseteq (\mathbb{P}^N)^*.$$

Lemma (B. Segre, 1951)

Let $X = V(f) = X_1 \cup \ldots \cup X_r \subset \mathbb{P}^N$, with $X_i = V(f_i)$ and $f = f_1 \cdots f_r$, $p \in X_i$ general. Then

$$\mathrm{rk}(d\mathcal{G}_X)_p = \mathrm{rk}_{(f_i)} H(f) - 2.$$

In particular

$$\dim(X_i^*) = \mathrm{rk}(d\mathcal{G}_X)_p = \mathrm{rk}_{(f_i)} H(f) - 2 \leq \mathrm{rk}(H(f)) - 2 \leq \dim(Z) - 1.$$

$$h(f) \equiv 0 \Rightarrow X^* \subsetneq Z \subsetneq (\mathbb{P}^N)^*.$$

Proof of the Lemma (vector=column vector)

$$\widehat{X} = V(f) \subset \mathbb{C}^{N+1}, \;\; \mathbf{v} : \; [\mathbf{v}] = p \in X_{=} V(f_i),$$

$$\mathbf{w} \in T_{\mathbf{v}}\widehat{X} = T_{\mathbf{v}}\widehat{X}_i \iff \nabla_{f_i}(\mathbf{v})^t \cdot \mathbf{w} = 0 \iff \nabla_f(\mathbf{v})^t \cdot \mathbf{w} = 0.$$

$$\mathbf{v}^t \cdot (H(f)(\mathbf{v}) = (d-1)\nabla_f(\mathbf{v}) \Longrightarrow$$

$$\mathbf{v}^t \cdot (H(f)(\mathbf{v})) \cdot \mathbf{w} = (d-1)\nabla_f(\mathbf{v})^t \cdot \mathbf{w} = 0.$$

$$(d\mathcal{G}_X)_{\mathbf{v}} = d\mathcal{G}_{X_i} : T_{\mathbf{v}}\widehat{X} \to \mathbb{C}^{N+1}, \;\; (d\mathcal{G}_X)_{\mathbf{v}}(T_{\mathbf{v}}\widehat{X}) \subseteq \nabla_f(\mathbf{v})^t \cdot \mathbf{x} = 0.$$

$$L : V \to W \text{ linear}, \; V_1 \subset V, W_1 \subset W, \mathrm{codim}(V_1) = \mathrm{codim}(W_1) = 1,$$
$$L(V) \not\subset W_1, L(V_1) \subseteq W_1 \Rightarrow \mathrm{rk}(L_{|V_1}) = \mathrm{rk}(L) - 1.$$

Gordan-Noether Identity

$f \in \mathbb{C}[x_0, \dots, x_n]_d$, $f = f_{\text{red}}$

$$h(f) \equiv 0$$

$$\Rightarrow \exists g \in \mathbb{C}[y_0, \dots, y_n] \ : \ g(\nabla_f(\mathbf{x})) = g(\frac{\partial f}{\partial x_0}(\mathbf{x}), \dots, \frac{\partial f}{\partial x_n}(\mathbf{x})) \equiv 0.$$

- $Z \subseteq T = V(g) \subset \mathbb{P}^N$ (we can assume equality if $\text{codim}(Z) = 1$);
- we can also assume $\frac{\partial g}{\partial y_i}(\nabla_f(\mathbf{x})) \not\equiv 0$ for some i by taking g a generator of minimal degree in $I(Z)$;
- Well defined
$$\psi_g = \nabla_g \circ \nabla_f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$
(first instance of Gordan-Noether-Perazzo map).

Functions h_i

$e = \deg(g(\mathbf{y})) \geq 1$ (if $e = 1$, then X is a cone)

$$g(\nabla_f(\mathbf{x})) \equiv 0 \overset{\text{Euler's Formula}}{\Longrightarrow} 0 = e \cdot g(\nabla_f(\mathbf{x})) = \sum_{i=0}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}) \frac{\partial g}{\partial y_i}(\mathbf{x})(\nabla_f(\mathbf{x})).$$

Let

$$\rho(\mathbf{x}) = m.c.d(\frac{\partial g}{\partial y_0}(\nabla_f(\mathbf{x})), \ldots, \frac{\partial g}{\partial y_N}(\nabla_f(\mathbf{x})))$$

so that

$$\rho(\mathbf{x}) \cdot h_i(\mathbf{x}) = \frac{\partial g}{\partial y_i}(\nabla_f(\mathbf{x}))$$

and

$$\psi_g = (h_0 : \ldots : h_N) : \mathbb{P}^N \to \mathbb{P}^N,$$

with $m.c.d.(h_0, \ldots, h_N) = 1$.

Gordan-Noether Identity

Theorem (Gordan-Noether, 1876)

(**Gordan–Noether Identity**) *Let notation be as above and let* $F \in \mathbb{C}[x_0, \ldots, x_N]_m$. *Then:*

$$\sum_{i=0}^{N} \frac{\partial F}{\partial x_i}(\mathbf{x}) h_i(\mathbf{x}) = 0 \iff F(\mathbf{x}) = F(\mathbf{x} + \lambda \psi_g(\mathbf{x})) \quad \forall \lambda \in \mathbb{K} \supseteq \mathbb{C}, \forall \mathbf{x} \in \mathbb{C}^{N+1}.$$

(Idea of proof) The implication $\Leftarrow$ is trivial. Consider $\Rightarrow$.

$$F(\mathbf{x}) - F(\mathbf{x} + \lambda \psi_g(\mathbf{x})) = \sum_{k=1}^{m} \Phi_k \lambda^k.$$

$$\Phi_1 = \sum_{i=0}^{N} \frac{\partial F}{\partial x_i} h_i = 0$$

by hypothesis. First year Calculus + some identities of $\psi_g(\mathbf{x})$ yields

$$\Phi_k = 0 \Rightarrow \Phi_{k+1} = 0 \text{ (proof postponed)}.$$

Gordan -Noether Identity for f

$$\text{Euler Formula: } \sum_{i=0}^{N} y_i \frac{\partial g}{\partial y_i}(\mathbf{y}) = e \cdot g(\mathbf{y}) \Longrightarrow$$

$$0 = e \cdot g(\nabla_f(\mathbf{x})) = \sum_{i=0}^{N} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i}(\nabla_f(\mathbf{x})) = \rho(\mathbf{x})(\sum_{i=0}^{N} \frac{\partial f}{\partial x_i}(\mathbf{x}) h_i(\mathbf{x})),$$

$\Longrightarrow$

$$f(\mathbf{x}) = f(\mathbf{x} + \lambda \psi_g(\mathbf{x})), \ \ \forall \mathbf{x} \in \mathbb{C}^{N+1} \ \forall \lambda \in \mathbb{K} \supseteq \mathbb{C}.$$

In particular if $p \in X = V(f)$ and if Ψ_g is defined at p we have

$$0 = f(p) = f(p + \lambda \psi_g(p)), \ \forall \lambda \in \mathbb{K} \supseteq \mathbb{C}$$

so that the line $< p, \psi_g(p) >$ is contained in X. We shall see in a moment that X is **developable** at least along $< p, \psi_g(p) >$.

Gordan-Noether Identity for the Polar Map

$$\frac{\partial}{\partial x_j} g(\nabla_f) \equiv 0$$

Chain Rule

$$\sum_{i=0}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i} h_i = 0$$

by Gordan-Noether Identitites

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) = \frac{\partial f}{\partial x_j}(\mathbf{x}+\lambda\psi_g(\mathbf{x})) \ \ \forall j = 0, \ldots, N, \ \ \forall \mathbf{x} \in \mathbb{C}^{N+1}, \ \forall \lambda \in \mathbb{K} \supseteq \mathbb{C},$$

so that

$$\nabla_f(\mathbf{x}) = \nabla_f(\mathbf{x} + \lambda\psi_g(\mathbf{x})) \ \ ; \ \forall \mathbf{x} \in \mathbb{C}^{N+1}.$$

In particular

$$\frac{\partial f}{\partial x_j}(\psi_g(\mathbf{x})) = 0 \ \ \forall j = 0, \ldots, N \ \ \forall \mathbf{x} \in \mathbb{C}^{N+1} \Longrightarrow \psi_g(\mathbb{P}^N) \subseteq \mathrm{Sing}(X).$$

Hesse Claim for $N = 2$

Let us recall that $h(f) \equiv 0$ yields

$$X^* \subsetneq Z \subsetneq \mathbb{P}^2.$$

Then $Z = V(g) \subset \mathbb{P}^2$ and by Gordan-Noether Identity

$$f(\mathbf{x}) = f(\mathbf{x} + \lambda \psi_g(\mathbf{x})) \ \ \forall \lambda \in \mathbb{C} \ \ \forall \mathbf{x} \in \mathbb{C}^3.$$

Thus

$$Z^* \subseteq \mathrm{Sing}(X) \subset \mathbb{P}^2 \Rightarrow \psi_g(\mathbb{P}^2) = Z^* = \{point\}$$

$$0 = f(p) = f(p + \lambda \psi_g(p)) \ \ \forall p \in X \ \ \forall \lambda \in \mathbb{C},$$

i.e.

$$X = \{ \text{ union of lines through } Z^* \}.$$

Gordan-Noether Identity for $\psi_g(\mathbf{x})$

Lemma

Let $F = F_1 \cdot F_2 \in \mathbb{C}[x_0, \ldots, x_N]_m$ *with* $F_i \in \mathbb{C}[x_0, \ldots, x_N]_{m_i}$, $i = 1, 2$. *Then*

$$\begin{array}{c} F(\mathbf{x}) = F(\mathbf{x} + \lambda\psi_g(\mathbf{x})) \\ \forall\lambda \in \mathbb{K} \supseteq \mathbb{C}, \forall \mathbf{x} \in \mathbb{C}^{N+1} \end{array} \iff \begin{array}{c} F_i(\mathbf{x}) = F_i(\mathbf{x} + \lambda\psi_g(\mathbf{x})) \\ i = 1, 2, \forall\lambda \in \mathbb{K} \supseteq \mathbb{C}, \forall \mathbf{x} \in \mathbb{C}^{N+1} \end{array}$$

The implication $\Leftarrow$ is obvious. Consider $\Rightarrow$. Write

$$F_i(\mathbf{x} + \lambda\psi_g(\mathbf{x})) = \sum_{j=0}^{d_i} A^i_j(\mathbf{x})\lambda^j \in (\mathbb{C}[\mathbf{x}])[\lambda].$$

From

$$F_1(\mathbf{x} + \lambda\psi_g(\mathbf{x})) \cdot F_2(\mathbf{x} + \lambda\psi_g(\mathbf{x})) = F(\mathbf{x} + \lambda\psi_g(\mathbf{x})) = F(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$$

we deduce $d_1 = d_2 = 0$ so that

$$F_i(\mathbf{x} + \lambda\psi_g(\mathbf{x})) = A^i_0(\mathbf{x}) = F_i(\mathbf{x}).$$

Gordan-Noether Identity for $\psi_g(\mathbf{x})$

Let $g_j = \frac{\partial g}{\partial y_j}(\nabla_f(\mathbf{x}))$ and recall that

$$\sum_{i=0}^{N} \frac{\partial^2 f}{\partial x_k \partial x_i} h_i = 0 \;\; \forall k = 0, \ldots, N.$$

Then

$$\sum_{i=0}^{N} \frac{\partial g_j}{\partial x_i} h_i = \sum_{i=0}^{N} \left(\sum_{k=0}^{N} \frac{\partial g_j}{\partial (\frac{\partial f}{\partial x_k})} \frac{\partial^2 f}{\partial x_k \partial x_i} \right) h_i$$

$$= \sum_{k=0}^{N} \frac{\partial g_j}{\partial (\frac{\partial f}{\partial x_k})} \left(\sum_{i=0}^{N} \frac{\partial^2 f}{\partial x_k \partial x_i} h_i \right) = 0.$$

Since $g_j(\mathbf{x})$ satisfy the Gordan-Noether Identity also $h_j(\mathbf{x})$ satisfy the Identity.

Gordan-Noether Identity for $\psi_g(\mathbf{x})$

$$h_j(\mathbf{x}) = h_j(\mathbf{x} + \lambda\psi_g(\mathbf{x}))$$
$$\forall\lambda \in \mathbb{K} \supseteq \mathbb{C}, \;\; \forall\mathbf{x} \in \mathbb{C}^{N+1} \;\; \forall j = 0, \ldots, N,$$

implies

$$\psi_g(\mathbf{x}) = \psi_g(\mathbf{x} + \lambda\psi_g(\mathbf{x})) \;\; \forall\lambda \in \mathbb{K} \supseteq \mathbb{C}, \forall\mathbf{x} \in \mathbb{C}^{N+1}.$$

In particular

$$\psi_g(\psi_g(\mathbf{x})) = 0$$

and

$$\psi_g(\mathbb{P}^N) \subseteq V(h_0, \ldots, h_N) = \mathrm{Bs}(\psi_g) \subset \mathbb{P}^N,$$

yielding

$$\mathrm{codim}(\overline{\psi_g(\mathbb{P}^N)}) \leq 2.$$

Another proof of Hesse Claim for $N = 2$

$$\psi_g(\mathbb{P}^2) \subseteq \mathrm{Bs}(\psi_g) = V(h_0, h_1, h_2) \Longrightarrow \psi_g(\mathbb{P}^2) = q \in \mathrm{Bs}(\psi_g(\mathbb{P}^2))$$

Then for every $p \in X \setminus (X \cap \mathrm{Bs}(\psi_g))$ and for every $\lambda \in \mathbb{C}$ we have

$$0 = f(p) = f(p + \lambda\psi_g(p)) = f(p + \lambda q)$$

$$\Longrightarrow \langle p, q \rangle \subseteq X \Longrightarrow X \text{ union of lines through } q,$$

that is a cone with vertex $q = \psi_g(\mathbb{P}^2)$.

An useful Remark

The following useful Remark is contained in Lemma 3.10 of Ciliberto-Russo-Simis:

Lemma

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface. Let $H = \mathbb{P}^{n-1}$ be a hyperplane not contained in X, let $h = H^*$ be the corresponding point in $(\mathbb{P}^N)^*$ and let π_h denote the projection from the point h. Then:

$$\nabla_{X \cap H} = \pi_h \circ (\nabla_{X|H}).$$

In particular, $\overline{\nabla_{V(f) \cap H)}(H)} \subset \pi_h(Z)$.

Hesse Claim for $N = 3$

$$h(f) \equiv 0 \Longrightarrow X^* \subsetneq Z \subsetneq (\mathbb{P}^3)^*,$$

and

$$\psi_g(\mathbb{P}^3) \subseteq V(h_0, h_1, h_2, h_3) \subset \mathbb{P}^3$$

Two cases:

1. $\dim(\psi(\mathbb{P}^3)) = 0 \Rightarrow X$ cone;
2. or $\dim(\psi_g(\mathbb{P}^3)) = 1$ and $\overline{\psi_g(\mathbb{P}^3)}$ is an irreducible component of $\mathrm{Bs}(\psi_g)$.

By the Lemma and the $N = 2$ case we can suppose
$Z = V(g) \subset \mathbb{P}^3$
(otherwise a general plane section of X consists of d lines through a point, X is a union of d planes through a line and the claim is true)
so that

$$\psi_g(p) = \psi_g(p + \lambda\psi_g(p)) \ \ \lambda \in \mathbb{C} \ \ \forall p \in \mathbb{P}^3$$

Hesse Claim for $N = 3$

Let

$$q_1, q_2 \in \psi_g(\mathbb{P}^3) \text{ general points}$$

$$\psi_g^{-1}(q_i) \text{ union of cones with vertex } q_i$$

$$\Rightarrow \overline{\psi_g^{-1}(q_i)} \text{ union of cones of vertex } q_i.$$

$$\overline{\psi_g^{-1}(q_1)} \cap \overline{\psi_g^{-1}(q_2)} \subseteq V(h_0, h_1, h_2, h_3) = \mathrm{Bs}(\psi_g) \text{ is a union of curves}$$

$$r \in \overline{\psi_g^{-1}(q_1)} \cap \overline{\psi_g^{-1}(q_2)}, \quad r \neq q_i, \quad i = 1, 2.$$

$$\langle r, q_i \rangle \subseteq \overline{\psi_g^{-1}(q_i)} \cap \mathrm{Bs}(\psi_g).$$

Hesse Claim for $N = 3$

By contradiction let

$$q \in < r, q_i > \cap \psi_g^{-1}(q_i).$$

From

$$\psi_g(q) = \psi_g(q + \lambda q_i) \Rightarrow \psi_g(r) = \psi_g(q) \neq \mathbf{0} \Rightarrow r \notin \mathrm{Bs}(\psi_g).$$

$$\Longrightarrow < r, q_i > = \overline{\psi_g(\mathbb{P}^3)} = \mathrm{Bs}(\psi_g)$$

$$\Longrightarrow \mathrm{Bs}(\psi_g) = \overline{\psi_g(\mathbb{P}^3)} = Z^* = < q_1, q_2 >$$

CONTRADICTION (no surface has a line as dual variety!).

$\dim(\nabla_f(\mathbb{P}^N)) \leq 2$ implies X a cone

The previous results and the Useful Remark/Lemma yield:

Corollary

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface with vanishing hessian. If $\dim(\nabla_f(\mathbb{P}^N)) \leq 2$, then X is a cone with positive dimensional vertex.

How small can be Z without being X being a cone? This is a very intriguing and quite subtle question.

Proof $\Phi_k = 0 \Rightarrow \Phi_{k+1} = 0$

$$F(\mathbf{x} + \lambda\psi_g(\mathbf{x})) - F(\mathbf{x}) = \sum_{k=0}^{\deg(F)} \Phi_k \lambda^k$$

$\Phi_1 = 0$ by hip. Suppose $\Phi_k = \sum_{i_1,\ldots,i_k} \frac{\partial^k F}{\partial x_{i_1} \ldots \partial x_{i_k}} \frac{h_{i_1} \cdots h_{i_k}}{k!} = 0$

$(*) = \sum_{j=0}^{N} h_j \cdot \frac{\partial}{\partial x_j}(h_{i_1} \cdots h_{i_k}) = 0$ (iterate the identity for $h_i(\mathbf{x})$).

$$\Phi_{k+1} = \frac{1}{k+1} \sum_{j=0}^{N} h_j [\sum_{i_1,\ldots,i_k} \frac{\partial}{\partial x_j}(\frac{\partial^k F}{\partial x_{i_1} \ldots \partial x_{i_k}}) \cdot \frac{h_{i_1} \cdots h_{i_k}}{k!}] + (*) =$$

$$= \frac{1}{k+1} \sum_{j=0}^{N} h_j \frac{\partial \Phi_k}{\partial x_j} = 0.$$

Cremona equivalence with a cone

$$(h_0(\mathbf{x}), \ldots, h_N(\mathbf{x})) \neq \mathbf{0}.$$

We can suppose $h_N(\mathbf{x}) \neq 0$ and take

$$\lambda = -\frac{x_N}{h_N(\mathbf{x})} \in \mathbb{C}(x_0, \ldots, x_N) \supseteq \mathbb{C}.$$

Thus by Gordan-Noether Identity

$$h_i(\mathbf{x}) = h_i(\mathbf{x} - \frac{x_N}{h_N(\mathbf{x})}\psi_g(\mathbf{x})) = h_i(x_0 - \frac{x_N}{h_N}h_0, \ldots, x_{N-1} - \frac{x_N}{h_N}h_{N-1}, 0).$$

Cremona equivalence with a cone

$$\begin{cases} x'_0 &= & x_0 & - & \dfrac{h_0(\mathbf{x})}{h_N(\mathbf{x})} x_N \\ \vdots & & & & \\ x'_i &= & x_i & - & \dfrac{h_i(\mathbf{x})}{h_N(\mathbf{x})} x_N \\ \vdots & & & & \\ x'_N &= & x_N & & \end{cases}$$

$$\begin{cases} x_0 &= & x'_0 & + & \dfrac{h_0(\mathbf{x})}{h_N(\mathbf{x})} x'_N & = & x'_0 & + & \dfrac{h_0(x'_0,\dots,x'_{N-1},0)}{h_N(x'_0,\dots,x'_{N-1},0)} x'_N \\ \vdots & & & & & & & & \\ x_i &= & x'_i & + & \dfrac{h_i(\mathbf{x})}{h_N(\mathbf{x})} x'_N & = & x'_i & + & \dfrac{h_i(x'_0,\dots,x'_{N-1},0)}{h_N(x'_0,\dots,x'_{N-1},0)} x'_N \\ \vdots & & & & & & & & \\ x_N &= & x'_N & & & & & & \end{cases}$$

Cremona equivalence to a cone

$$f(\mathbf{x}) = f(\mathbf{x} - \frac{x_N}{h_N(\mathbf{x})}\psi_g(\mathbf{x})) = f(x'_0, \ldots, x'_{N-1}, 0).$$

We have thus proved

Theorem (Gordan-Noether, 1876)

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface, $d = \deg(X) \geq 2$, with $h(f) \equiv 0$. Then there exists a Cremona transformation

$$\Phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$

such that $\Phi(X)$ is a cone.
Moreover the Cremona transformation can be explicitly constructed from f in such a way that the equation of $\Phi(X)$ depends on at most N variables.

Example Perazzo hypersurface in $\mathbb{P}^4$

$$f(\mathbf{x}) = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$$
$$g(\mathbf{y}) = y_0y_2 - y_1^2$$
$$(h_0, \ldots, h_4) = (x_4^2, -2x_3x_4, x_3^2, 0, 0), h_0 \neq 0$$

$$\left\{\begin{array}{ccccc} x_0' & = & x_0 & & \\ x_1' & = & x_1 & + & 2\frac{x_3}{x_4}x_0 \\ x_2' & = & x_2 & - & \frac{x_3^2}{x_4^2}x_0 \\ x_3' & = & x_3 & & \\ x_4' & = & x_4 & & \end{array}\right. \quad \left\{\begin{array}{ccccc} x_0 & = & x_0' & & \\ x_1 & = & x_1' & - & 2\frac{x_3'}{x_4'}x_0' \\ x_2 & = & x_2' & + & \frac{x_3'^2}{x_4'^2}x_0' \\ x_3 & = & x_3' & & \\ x_4 & = & x_4' & & \end{array}\right.$$

$$f(0, x_1', x_2', x_3', x_4') = x_1'x_3'x_4' + x_2'x_4'^2 = x_4'(x_1'x_3' + x_2'x_4')$$

$$\Phi(V(x_0x_3^2 + x_1x_3x_4 + x_2x_4^2)) = V(x_1x_3 + x_2x_4).$$

Another Cremona equivalence

$$f(\mathbf{x}) = x_0x_3^2 + x_1x_3x_4 + x_2x_4^2$$
$$g(\mathbf{y}) = y_0y_2 - y_1^2$$
$$(h_0, \ldots, h_4) = (x_4^2, -2x_3x_4, x_3^2, 0, 0)$$

Since $h_1 \neq 0$

$$\left\{\begin{array}{lllll} x_0' & = & x_0 & + & \frac{x_4^2}{2x_3x_4}x_1 \\ x_1' & = & x_1 & & \\ x_2' & = & x_2 & + & \frac{x_3^2}{2x_3x_4}x_1 \\ x_3' & = & x_3 & & \\ x_4' & = & x_4 & & \end{array}\right. \quad \left\{\begin{array}{lllll} x_0 & = & x_0' & - & \frac{x_4'^2}{2x_3'x_4'}x_1' \\ x_1 & = & x_1' & & \\ x_2 & = & x_2' & - & \frac{x_3'^2}{2x_3'x_4'}x_1' \\ x_3 & = & x_3' & & \\ x_4 & = & x_4' & & \end{array}\right.$$

$$f(x_0', 0, x_2', x_3', x_4') = x_0'x_3'^2 + x_2'x_4'^2$$

$$\Phi(V(x_0x_3^2 + x_1x_3x_4 + x_2x_4^2)) = V(x_0x_3^2 + x_2x_4^2).$$

Cremona Linearization of the Veronese Surface in $\mathbb{P}^5$

$$f(\mathbf{x}) = \begin{vmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{0,1} & x_{1,1} & x_{1,2} \\ x_{0,2} & x_{1,2} & x_{2,2} \end{vmatrix}, \quad h(f) = -16f^2 \neq 0$$

$X = V(f) \subset \mathbb{P}^5$ secant variety Veronese surface $S \subset \mathbb{P}^5$

$$\left\{\begin{array}{rclcc} x'_{0,0} & = & x_{0,0} & & \\ x'_{0,1} & = & x_{0,1} & & \\ x'_{0,2} & = & x_{0,2} & & \\ x'_{1,1} & = & x_{1,1} & - & x_{0,1}^2 \\ x'_{1,2} & = & x_{1,2} & - & x_{0,1}x_{0,2} \\ x'_{2,2} & = & x_{2,2} & - & x_{0,2}^2 \end{array}\right. \qquad \left\{\begin{array}{rclcc} x_{0,0} & = & x'_{0,0} & & \\ x_{0,1} & = & x'_{0,1} & & \\ x_{0,2} & = & x'_{0,2} & & \\ x_{1,1} & = & x'_{1,1} & + & {x'_{0,1}}^2 \\ x_{1,2} & = & x'_{1,2} & + & x'_{0,1}x'_{0,2} \\ x_{2,2} & = & x'_{2,2} & + & {x'_{0,2}}^2 \end{array}\right.$$

$$\Phi(S) = V(x_{1,1}, x_{1,2}, x_{2,2}) \subset \mathbb{P}^5$$

$\Phi(V(f)) = V(x_{1,1}x_{2,2} - x_{1,2}^2) \subset \mathbb{P}^5$ is a cone with vertex $\Phi(S)$.

Corollary

Let $X = V(f) \subset \mathbb{P}^N$ be a hypersurface with vanishing hessian and let notation be as above. Then

(i) *for $p \in \mathbb{P}^N$ general,*

$$\overline{\nabla_f^{-1}(\nabla_f(p))} = < p, (T_{\nabla_f(p)} Z)^* > = \mathbb{P}^{codim(Z)}.$$

(ii)

$$Z^* = \overline{\bigcup_{p \in \mathbb{P}^N \text{general}} (T_{\nabla_f(p)} Z)^*} \subseteq \mathrm{Sing}(X).$$

$X = S(1,2)^* \subset \mathbb{P}^4$ has vanishing hessian and it is not a cone

Let $S(1,2) \subset (\mathbb{P}^4)^*$ the cubic rational normal scroll surface generated by a line $L \subset \mathbb{P}^4$ and a conic $C \subset \mathbb{P}^4$: $L \cap \langle C \rangle = \emptyset$.
Then:

- $X = S(1,2)^* = V(x_0x_3^2 + x_1x_3x_4 + x_2x_4^2) \subset \mathbb{P}^4$ is a Perazzo Cubic Hypersurface;
- $\mathrm{Sing}(X)_{\mathrm{red}} = \mathbb{P}^2 = L^\perp \subset X \subset \mathbb{P}^4$ (hyperplanes through the directrix line L);
- $Z = V(y_0y_2 - y_1^2) \subset (\mathbb{P}^4)^*$ is a quadric cone with vertex L;
- $\mathrm{Sing}(X)$ is non reduced along the conic $Z^* \subset \mathrm{Sing}(X)$;
- the hyperplanes $T_r \mathrm{Sing}(X) \supset T_r Z^*$, $r \in Z^*$ foliate $\mathbb{P}^4$ and $T_r \mathrm{Sing}(X) \cap X$ are planes foliating X, i.e X is a **scroll in planes tangent to** Z^*.
- $\mathrm{hess}_X \equiv 0$.

Hypersurfaces with vanishing hessian in $\mathbb{P}^4$

Theorem (Gordan-Noether, 1876; Franchetta, 1954)

Let $X \subset \mathbb{P}^4$ be an irreducible hypersurface, not a cone, and let $d = \deg(X)$. The following conditions are equivalent:

1. *X has vanishing hessian;*
2. *X is a scroll in $\mathbb{P}^2$ tangent to a plane rational curve induced by a foliation of hyperplanes on X;*
3. *$X \simeq S^*$ with $S \subset \mathbb{P}^4$ a degree d scroll surface with directrix a line over a rational plane curve.*

Also in this case we have that Z^* is a plane rational curve with $< Z^* >= \mathbb{P}^2 \subset \mathrm{Sing}(X)$.
Moreover Z is a cone over a plane rational curve.

Key question

$$\mathrm{hess}_X \equiv 0 \Rightarrow Z^* \subset \mathrm{Sing}(X).$$

The key question:

$$< Z^* >= \mathbb{P}^\tau \overset{?}{\subseteq} \mathrm{Sing}(X).$$

Also for Cubics a priori we only have:

$$Z^* \subset \mathrm{Sing}(X) \Rightarrow SZ^* \subseteq X,$$

where SZ^* is **secant variety to** $Z^* \subset \mathbb{P}^N$. One would desire

$$SZ^* \subseteq \mathrm{Sing}(X).$$

$N = 4$: $\mathrm{hess}_X \equiv 0 \Rightarrow < Z^* > = \mathbb{P}^2 \subseteq \mathrm{Sing}(X)$

ACCORDING to Garbagnati-Repetto, PRAGMATIC 2006:

$$Z^* \subseteq \mathrm{Bs}(\psi_g) \subset \mathbb{P}^4 \Longrightarrow 1 \leq \dim(Z^*) \leq \dim(\mathrm{Bs}(\psi_g)) \leq 2.$$

$$\dim(Z^*) = 2, \quad r_1, r_2 \in Z^* \text{ general} \Longrightarrow \exists t \in \overline{\psi_g^{-1}(r_1)} \cap \overline{\psi_g^{-1}(r_2)},$$

$$\Longrightarrow < r_i, t > \subseteq \mathrm{Bs}(\psi_g) = Z^* \Longrightarrow Z^* \text{ cone}$$

$$\Longrightarrow Z \subset (\mathbb{P}^4)^* \mathit{degenerated} \Longrightarrow X \subset \mathbb{P}^4 \text{ cone.}$$

CONTRADICTION. Thus $\dim(Z^*) = 1$ and $\dim(\mathrm{Bs}(\psi_g)) = 2$.

$N = 4$: $\mathrm{hess}_X \equiv 0 \Rightarrow < Z^* > = \mathbb{P}^2 \subseteq \mathrm{Sing}(X)$

$$\dim(Z^*) = 1, \quad r_1, r_2 \in Z^* \text{ general} \Longrightarrow \dim(\overline{\psi_g^{-1}(r_1)} \cap \overline{\psi_g^{-1}(r_2)}) = 2,$$

$$+ \ \overline{\psi_g^{-1}(r_1)} \cap \overline{\psi_g^{-1}(r_2)} \subseteq \mathrm{Bs}(\psi_g) \Longrightarrow$$

$\exists$ an irreducible surface P, irreducible component of $\mathrm{Bs}(\psi_g)$:

$$Z^* \subseteq P, \quad P \subseteq \overline{\psi_g^{-1}(r_1)} \cap \overline{\psi_g^{-1}(r_2)} \ \forall r_1, r_2 \in Z^* \text{ general} \ .$$

$$\Longrightarrow < r_i, p > \subseteq P \ \ \forall p \in P \text{ general}$$

$$\Longrightarrow Z^* \subseteq \mathrm{Vert}(R) \Longrightarrow P = \mathbb{P}^2 = \langle Z^* \rangle.$$

Classification for $N \geq 5$

1. for every $d \geq 3$ and for every $N \geq 5$ there exist many different classes of examples (Ciliberto, —, Simis; 2008);
2. for all them $\langle Z^* \rangle \subset \mathrm{Sing}(X)$.
3. the first challenging case is $N = 5$ to try to show $\langle Z^* \rangle \subset \mathrm{Sing}(X)$.
4. ONE MORE GORDAN-NOETHER IDENTITY IS MISSING:

$$T_r Z^* \subset \mathrm{Sing}(X) \Longleftrightarrow \sum_{i=0}^{N} \frac{\partial^2 h_j}{\partial x_k \partial x_i} h_i = 0$$

$$\Longrightarrow TZ^* \subseteq \mathrm{Sing}(X).$$

How to generalize the results of Gordan-Noether-Franchetta for $N = 4$?

The Perazzo map of hypersurfaces with vanishing hessian

Let $X = V(f) \subset \mathbb{P}^N$ be a reduced hypersurface with vanishing hessian.

$$\nabla_f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N \text{ polar map, } Z = \nabla_f(\mathbb{P}^N) \subsetneq (\mathbb{P}^N)^*.$$

$$\mathcal{P}_X := \mathcal{G}_Z^* \circ \nabla_f : \mathbb{P}^N \dashrightarrow \mathbb{G}(\mathrm{codim}(Z) - 1, N)$$

$$p \mapsto (T_{\nabla_f(p)} Z)^* = \mathrm{Sing}(Q_p)$$

is **the Perazzo map of** X.

$W_X = \overline{\mathcal{P}_X(\mathbb{P}^N)} \subset \mathbb{G}(\mathrm{codim}(Z) - 1, N)$ is the **Perazzo image of** X;

$$\mu = \dim W.$$

If $X = V(f) \subset \mathbb{P}^N$ is not a cone, then $\mu \geq 1$.

$$X = V(f) \subset \mathbb{P}^N \text{ cubic hypersurface}$$

$$r \in \mathbb{P}^N, \quad H_r \ : \ \sum_{i=0}^{N} \frac{\partial f}{\partial x_i}(r)x_i = 0 \text{ polar hyperplane of } r$$

$$s \in \mathbb{P}^N, \quad Q_s \ : \ \sum_{i,j=0}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(s)x_i x_j = 0 \text{ polar hyperquadric of } s$$

Reciprocity Law of Polarity : $r \in H_s \iff s \in Q_r$.

EXAMPLE : $X = S(1,2)^* \subset \mathbb{P}^4, \ \mathrm{Sing}(X) = \mathbb{P}^2$.

$$s \in \mathrm{Sing}(X) \Longrightarrow H_s = \mathbb{P}^4$$

$$\Longrightarrow \mathrm{Sing}(X) = \mathbb{P}^2 \subseteq Q_r, r \in \mathbb{P}^4 \text{ general } \Longrightarrow Q_r \text{ cone, } r \in \mathbb{P}^4 \text{ general}$$

$$\Longrightarrow \det(Q_r) = \mathrm{hess}_X(r) = 0, \ r \in \mathbb{P}^4 \text{ general } \Longrightarrow \mathrm{hess}_X \equiv 0.$$

Perazzo map of cubic hypersurfaces

Theorem

Let $X = V(f) \subset \mathbb{P}^N$ be a cubic hypersurface with vanishing hessian. Let $w = [(T_{\Phi_X(p)}Z)^*] \in W_X \subset \mathbb{G}(\alpha, N)$ and $r \in (T_{\Phi_X(p)}Z)^*$, $p \in \mathbb{P}^N$ general points. Then:

$$\overline{\mathcal{P}_X^{-1}(w)} = \bigcap_{r \in (T_{\Phi_X(p)}Z)^*} \operatorname{Sing}(Q_r) = \mathbb{P}_w^{N-\mu}.$$

$$\operatorname{codim}(Z) = 1 \Rightarrow \overline{\mathcal{P}_X^{-1}(r)} = \mathbb{P}^{N-\mu} = \operatorname{Sing}(Q_r) = T_r \operatorname{Sing}(X) \supset T_r Z^*,$$

with $r \in Z^*$ general.

The general fiber of the Perazzo map of a CUBIC Hypersurface foliates $\mathbb{P}^N$ with linear spaces TANGENT to Z^*

Congruence of the fibers of the Perazzo map

The fibers of the Perazzo map form a **congruence of order one of linear spaces**, that is if :

$$\Theta \subset \mathbb{G}(N-\mu, N), \;\; \dim(\Theta) = \mu, \;\; \pi : \mathcal{U} \to \Theta$$

the restriction of the universal family, then the tautological map

$$p : \mathcal{U} \to \mathbb{P}^N$$

is birational and the $\mathbb{P}^{N-\mu}$'s of the family foliate $\mathbb{P}^N$.

The easiest examples of congruences of linear spaces of dimension β is given by the family of linear spaces of dimension $\beta + 1$ passing through a fixed linear space $L = \mathbb{P}^\beta \subset \mathbb{P}^N$.

The previous examples motivate the following:

Definition

An irreducible cubic hypersurface $X \subset \mathbb{P}^N$ with vanishing hessian, not a cone, will be called a *Special Perazzo Cubic Hypersurface* if the general fibers of its Perazzo map form a congruence of linear spaces of dimension $N - \mu$ passing through a fixed $\mathbb{P}^{N-\mu-1}$.

Example

$$\sigma \geq 2, \quad \sigma \in \mathbb{N}.$$

$$f(\mathbf{x}) = \sum_{i=0}^{\sigma} x_i C^i(x_{N-\sigma+1}, \ldots, x_N) + D(x_{\sigma+1}, \ldots, x_N) \in \mathbb{C}[x_0, \ldots, x_N]_3,$$

$$C^i(x_{N-\sigma+1}, \ldots, x_N) \in \mathbb{C}[x_0, \ldots, x_N]_2,$$

$$D(x_{\sigma+1}, \ldots, x_N) \in \mathbb{C}[x_0, \ldots, x_N]_3.$$

Then the associated hypersurface $X = V(f) \subset \mathbb{P}^N$ is called a **Perazzo Cubic Hypersurface**

1. $h(f) \equiv 0$;
2. $X = V(f)$ is not a cone for general C_i's and D.

Special Cubic Hypersurface=Perazzo Hypersurface

Proposition

Let $X \subset \mathbb{P}^N$ be a cubic hypersurface with vanishing hessian, not a cone. Then

$$X \text{ Special Perazzo} \iff X \text{ Perazzo}$$

Fundamental Inequality for Cubics with vanishing hessian

Theorem

Let $X = V(f) \subset \mathbb{P}^N$ *be a cubic hypersurface with vanishing hessian, not a cone, with* $\mathrm{codim}(Z) = 1$*. Then:*

$$\dim(Z^*) \leq \frac{N-1}{2}.$$

Classification of Cubic Hypersurfaces with vanishing hessian for $N \leq 6$

Theorem (Gondim, ——;20??)

Let $X = V(f) \subset \mathbb{P}^N$ ba a cubic hypersurface with vanishing hessian, not a cone. If $N \leq 6$, then

$$f \sim \textit{ Perazzo Cubic Hypersurface.}$$

For $N \geq 7$ there exists cubic hypersurfaces with $h(f) \equiv 0$, not cones, which are not projectively equivalent to a Perazzo Cubic Hypersurface.

These examples are related to an interesting phenomenon:

$$\operatorname{codim}(X^*, Z) > 1 \iff \operatorname{rk}_{(f)}(H(f)) < \operatorname{rk}(H(f)).$$

Examples with $\mathrm{codim}(X^*, Z) > 1$, $\mathrm{codim}(Z) = 1$

1. Let $R = SW \subset \mathbb{P}^{\frac{3n}{2}+2}$, $n = 4, 8, 16$, be the secant variety of a Severi varieties $W^n \subset \mathbb{P}^{\frac{3n}{2}+2}$;
2. $p \in R$ be a smooth point, $X = R \cap T_pR = V(f) \subset \mathbb{P}^{\frac{3n}{2}+1}$ is a cubic hypersurface;
3. $$codim(Z) = 1, \;\; codim(X^*, Z) = n + 2;$$
4. $$\Rightarrow \mathrm{rk}(\mathrm{Hess}_X) = \frac{3n}{2} + 1, \;\; \mathrm{rk}_{(f)}(\mathrm{Hess}_X) = n + 2.$$
5. NOT PERAZZO HYPERSURFACES

codim(Z) and codim(X^*, Z) arbitrary large as a function of the degree

(——, A. SIMIS; 2014):

$$M = \begin{bmatrix} x_0 & x_1 & \cdots & x_{r-2} & x_{r-1} & x_r \\ x_{r+1} & x_{r+2} & \cdots & x_{2r-1} & x_{2r} & x_{2r+1} \\ x_{2r+2} & x_{2r+3} & \cdots & x_{3r} & x_{3r+1} & 0 \\ x_{3r+2} & x_{3r+3} & \cdots & x_{4r} & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ x_{r(r+5)/2-1} & x_{r(r+5)/2} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$f := \det(M), \quad X = V(f) \subset \mathbb{P}^{\frac{r(r+5)}{2}}, \quad \deg(f) = r+1.$$

$$\operatorname{codim}(Z) = \binom{r}{2}, \quad \operatorname{codim}(X^*, Z) = r.$$

Standard artinian graded algebras

Definition

$$A = \bigoplus_{i=0}^{d} A_i$$

be an artinian associative and commutative graded $\mathbb{K}$*-algebra with* $A_0 = \mathbb{K}$ *and* $A_d \neq 0$. *Let*

$$\begin{array}{cccc} \bullet : & A_i \times A_{d-i} & \to & A_d \\ & (\alpha, \beta) & \to & \alpha \bullet \beta \end{array}$$

be the restriction of the multiplication in A.

We say that A **satisfies the Poincarè Duality Property** *if:*

(i) $\dim_{\mathbb{K}}(A_d) = 1$;

(ii) $\bullet : A_i \times A_{d-i} \to A_d \simeq \mathbb{K}$ *is non-degenerate for every* $i = 0, \ldots, [\frac{d}{2}]$.

Definition

The algebra A is said to be **standard** *if*

$$A \simeq \frac{\mathbb{K}[x_0, \ldots, x_N]}{I},$$

as graded algebras, with $I \subset \mathbb{K}[x_0, \ldots, x_N]$ a homogeneous ideal.

Let us remark that this implies $\sqrt{I} = (x_0, \ldots, x_N)$ because $(x_0, \ldots, x_N)^m \subseteq I$ for $m \geq d + 1$.

Characterization of artinian Gorenstein algebras

Proposition

Let A be a graded artinian $\mathbb{K}$-algebra. Then A satisfies the Poincarè Duality Property if and only if it is Gorenstein.

Examples

Example

Let

$$Q = \mathbb{K}[\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_N}]$$

$$F(\mathbf{x}) \in \mathbb{K}[x_0, \dots, x_N]_d.$$

For

$$G \in Q \Rightarrow G(F) \in \mathbb{K}[x_0, \dots, x_N]$$

$$\mathrm{Ann}_Q(F) = \{G \in Q \ : \ G(F) = 0\} \subset Q \textit{ homogeneous ideal .}$$

$$A = \frac{Q}{\mathrm{Ann}_Q(F)}$$

is a standard artinian Gorenstein graded $\mathbb{K}$*-algebra with* $A_i = 0$ *for* $i > d$ *and* $A_d \neq 0$*.*

Characterization of standard Gorenstein algebras

The following is one of the main applications of the Theory of Inverse Systems of Macaulay:

Theorem

Let

$$A = \bigoplus_{i=0}^{d} A_i \simeq \frac{\mathbb{K}[x_0, \ldots, x_N]}{I}$$

be an artinian standard graded $\mathbb{K}$*-algebra. Then*

$$A \text{ Gorenstein} \iff A \simeq \frac{Q}{\mathrm{Ann}_Q(F)}$$

for some $F \in \mathbb{K}[x_0, \ldots, x_N]_d$

$$\mathrm{Ann}_Q(F)_1 = 0 \iff \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_N} \text{ are linearly independent.}$$

Suppose

$$\mathrm{Ann}_Q(F)_1 = 0$$

$$A = \frac{Q}{\mathrm{Ann}_Q(F)}, \quad L = a_0x_0 + \ldots + a_Nx_N \in A_1$$

$$\bullet L^{d-2} : A_1 \to A_{d-1}$$

corresponds to

$\phi_L : A_1 \times A_1 \to \mathbb{K}$ symmetric bilinear form by Poincarè Duality

matrix of ϕ_L w. resp. to basis $\{x_0, \ldots, x_N\} = \mathrm{Hess}(F)(a_0, \ldots, a_N)$

$$\bullet L^{d-1} \text{ isomorphism } \iff \mathrm{hess}(F)(a_0, \ldots, a_N) \neq 0$$

Thus

$$\bullet L^{d-1} \text{ isomorphism for } L \in A_1 \text{ general} \iff \mathrm{hess}(F) \neq 0$$

Hence homogeneous polynomial whose derivates are linearly dependent but algebraically dependent are such that $\bullet L^{d-1}$ is not an isomorphism for any $L \in A_1$.
This should be compared with the following fundamental result.

Hard Lefschetz Theorem

The following fundamental result of S. Lefschetz is known as the **Hard Lefschetz Theorem**.

Theorem

Let $X \subset \mathbb{P}^N_{\mathbb{C}}$ be a smooth irreducible complex projective variety of dimension $n \geq 1$ and let $[H] \in H^2(X)$ be the class of a hyperplane section. Then $\forall q = 1, \ldots, n$

$$\bullet [H]^q : \quad H^{n-q}(X) \quad \to \quad H^{n+q}(X)$$

is an isomorphism.

In particular

$$\bullet [H]^{n-2} : H^2(X) \to H^{2n-2}(X)$$

is an isomorphism and

$$\bigoplus_{i=0}^{n} H^{2i}(X)$$

is a Gorenstein $\mathbb{C}$–algebra by Poincarè Duality Theorem.

Corollary

Let notation be as above. Then:

$$\bullet[H]^k : \quad H^i(X) \quad \to \quad H^{i+2k}(X) \tag{1}$$

is injective for $i \leq n-k$ and surjective for $i \geq n-k$.

Proof.

$$\text{For } i \leq n-k \quad H^i(X) \xrightarrow{\bullet[H]^k} H^{i+2k}(X) \xrightarrow{\bullet[H]^{n-k-i}} H^{2n-i}(X)$$

is an isomorphism by Hard Lefschetz Theorem so that the first map is injective.

$$\text{For } i \geq n-k \quad H^{2n-i-2k}(X) \xrightarrow{\bullet[H]^{i-n+k}} H^i(X) \xrightarrow{\bullet[H]^k} H^{i+2k}(X).$$

is an isomorphism by Hard Lefschetz Theorem so that the second map is surjective. □

Definition of Strong Lefschetz Properties

Definition

$A = \bigoplus_{i=0}^{d} A_i$ artinian associative and commutative graded $\mathbb{K}$-algebra with $A_d \neq 0$.
A has **the Strong Lefschetz Property**, briefly *SLP*, if $\exists L \in A_1$:

$$\bullet L^k : A_i \to A_{i+k}$$

is of maximal rank, that is injective or surjective, $\forall\, 0 \leq i \leq d$ and $\forall\, 0 \leq k \leq d-k$.
A has **the Strong Lefschetz Property in the narrow sense** if $\exists L \in A_1$ such that the multiplication map

$$\bullet L^{d-2i} : A_i \to A_{d-i}$$

is an isomorphism $\forall\, i = 0, \ldots, [\frac{d}{2}]$.

Polynomials having linearly independent partial derivatives but vanishing hessian (or having vanishing hessians of *higher order*) naturally produces counterexamples to the previous conditions.

The techniques and ideas behind the construction and classification of polynomials with vanishing hessian but depending on all the variables modulo linear change of coordinates, allowed recently Rodrigo Gondim to produce examples of homogeneous polynomials of any degree $d \geq 3$ whose associated artinian Gorenstein $\mathbb{K}$-algebras violate the Strong Lefschetz Property at the desired i with $1 \leq i \leq [\frac{d}{2}]$.