The Sarkisov Program for Mori fibred Calabi–Yau pairs

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Birational maps of $\mathbb{P}^2$

Second half of 19th century: developments of a general theory of birational maps of $\mathbb{P}^2$–Cremona, Enriques, Noether, de Jonquières, Castelnuovo..

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Theorem (Noether–Castelnuovo)

Cr_2 is generated by $\mathrm{PGl}_3(\mathbb{C})$ and a standard quadratic transformation

$$C\colon (x, y) \dashrightarrow \left(\frac{1}{x}, \frac{1}{y}\right)$$

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- to higher dimensions:
  - Study $\mathrm{Cr}_n = \mathrm{Bir}\,\mathbb{P}^n$ for $n \geq 3$
  - So far, very little– no meaningful way to probe even Cr_3
- to open varieties:
  - consider proper birational maps between open algebraic surfaces

$$\varphi\colon U \dashrightarrow V$$

  with $U, V \subset \mathbb{P}^2$ or U, V subsets of rational surfaces
  - Iitaka's philospohy: formalism of pairs

Symplectic transformations of the plane

Subgroup of birational maps of $\mathbb{P}^2$ preserving standard volume form of $(\mathbb{C}^*)^2$

$$\mathrm{SCr}_2 = \left\{ f\colon\ \mathbb{C}^2 \dashrightarrow \mathbb{C}^2, f \in \mathrm{Cr}_2 \,|\, f^*\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = \frac{dx}{x} \wedge \frac{dy}{y} \right\}$$

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Theorem (Usnich–Blanc)

SCr_2 is generated by $(\mathbb{C}^*)^2$, $\mathrm{Sl}_2(\mathbb{Z})$ and the birational map

$$P\colon (x, y) \dashrightarrow \left(y, \frac{1+y}{x}\right)$$

Symplectic Cremona group

Possible generalisations?
Subgroup of Cr_n of maps preserving standard volume form of $(\mathbb{C}^*)^n$

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Examples of symplectic Cremona transformations

(i) $(\mathbb{C}^*)^n$, $\mathrm{Sl}_n(\mathbb{Z})$ are subgroups of SCr_n

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Examples of symplectic Cremona transformations

(i) $(\mathbb{C}^*)^n$, $\mathrm{Sl}_n(\mathbb{Z})$ are subgroups of SCr_n

(ii) (mutations) Let $\mathbb{T} = \operatorname{Spec} \mathbb{C}[N]$ be an n-dimensional torus, $h \in M = \mathrm{Hom}(N, \mathbb{Z})$, and $f \in \mathbb{C}[h^\perp] \subset \mathbb{C}[N]$, then

$$\phi\colon x^\gamma \mapsto x^\gamma \cdot f^{<h,\gamma>}$$

is volume preserving.

Another generalisation of results on Cr_2 and SCr_2

Generalisation of Noether-Castelnuovo's theorem on Cr_2:

Theorem (Sarkisov, Reid, Corti, Hacon-McKernan)

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Generalisation of Usnich-Blanc's theorem on SCr_2:

Theorem (Corti–K.)

A volume preserving birational map between Mori fibred Calabi–Yau pairs is a composition of volume preserving Sarkisov links.

Sarkisov program

Proof of Noether-Castelnuovo:
(Noether) if $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ not biregular, then $\exists\, \mathbb{P}^2 \overset{c}{\dashrightarrow} \mathbb{P}^2$ quadratic with $\deg(c \circ \varphi) < \deg \varphi$

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(Castelnuovo) proof relies on:

Theorem

Any birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a chain of the following elementary maps

$$\begin{tikzcd} \mathbb{P}^2 \arrow[d] & \mathbb{F}_1 \arrow[l, "\varepsilon"'] \arrow[d] \\ \mathrm{pt} & \mathbb{P}^1 \arrow[l] \end{tikzcd}$$

Type I

$$\begin{tikzcd} \mathbb{F}_k \arrow[r, dashed] \arrow[d] & \mathbb{F}_{k\pm1} \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & \mathbb{P}^1 \end{tikzcd}$$

Type II

$$\begin{tikzcd} \mathbb{F}^1 \arrow[r, "\varepsilon"] \arrow[d] & \mathbb{P}^2 \arrow[d] \\ \mathbb{P}^1 \arrow[r] & \mathrm{pt} \end{tikzcd}$$

Type III

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, equal, "\iota"] \arrow[d] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[d] \\ \mathbb{P}^1 & \mathbb{P}^1 \end{tikzcd}$$

Type IV

Sarkisov program

A Mori fibre space (Mfs) X/S

- a $\mathbb{Q}$-factorial terminal variety X
- and a fibration $f\colon X \to S$ such that $f_*\mathcal{O}_X = \mathcal{O}_S$, $-K_X$ is f-ample, and $\rho(X) - \rho(S) = 1$

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Classical Minimal Model Program (MMP): if Z is a uniruled projective manifold, the MMP on Z terminates with a Mfs

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Structure theorem: identifies "generators" of $\mathrm{Bir}\,X$ for any Mfs X

Sarkisov program

Sarkisov link $\varphi\colon X/S \dashrightarrow X'/S'$: a commutative diagram

Type I
inverse Type III

$$\begin{tikzcd}
 & Z \arrow[dl] \arrow[r, dashed] & X' \arrow[d] \\
X \arrow[d] & & S' \arrow[dll] \\
S & &
\end{tikzcd}$$

$Z \to X$ divisorial contraction; $Z \dashrightarrow X'$ sequence of flips, flops and inverse flips

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$X \dashrightarrow X'$ sequence of flips, flops and inverse flips

All intermediate varieties terminal $\mathbb{Q}$-factorial

Calabi–Yau pairs

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A Mori fibred CY pair $(X/S, D)$ is the end product of a classical MMP and the end product of a log-MMP for (X, D)

Examples of Calabi-Yau pairs

Mori fibred rational Calabi-Yau surface pairs

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2. $(\mathbb{F}_N, D)$ where D is
   - $C_1 + C_2 + C_3 + C_4$, $(C_1, C_2, C_3, C_4) = (\sigma, f, \sigma + kf, f)$
   - $C_1 + C_2 + C_3$, $(C_1, C_2, C_3) = (\sigma, f, \sigma + (1+k)f)$
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3. a few extra cases $(\mathbb{F}_N, D)$ where $N \leq 2$ and σ not a component of D

Examples of Calabi-Yau pairs

4. (S, D) a Mf CY surface pair and $\sigma\colon \tilde{S} \to S$ the blowup of $P_1, \cdots, P_k$ distinct points on $D - \operatorname{Sing}(D)$. The pair $(\tilde{S}, \sigma_*^{-1}D)$ is a (t,dlt) CY pair if (S, D) is dlt, and a (t, lc) pair otherwise

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7. X a smooth weak Fano 3-fold and $D \in |-K_X|$ a general anticanonical section; then (X, D) is a (t, dlt) CY pair

Volume preserving birational maps

Let (X, D) and (X', D') be CY pairs. A birational map $\varphi\colon X \dashrightarrow X'$ is volume preserving if there exists a common log resolution

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\varphi"] & & X' \end{tikzcd}$$

such that $p^*(K_X + D) = q^*(K_{X'} + D')$.

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Equivalently: φ is volume preserving if for all geometric valuations E

$$a(E, K_X + D) = a(E, K_{X'} + D')$$

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Factorisation of volume preserving maps of lc CY pairs

Theorem (CK)

Let (X, D) and (X', D') be lc CY pairs and $\varphi \colon X \dashrightarrow X'$ a volume preserving birational map. Then there are $\mathbb{Q}$-factorial (t, dlt) CY pairs (Y, D_Y), $(Y', D_{Y'})$ and a commutative diagram of birational maps:

$$\begin{tikzcd} Y \arrow[r, dashed, "\chi"] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\ X \arrow[r, dashed, "\varphi"] & X' \end{tikzcd}$$

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where:

(1) the morphisms $g \colon Y \to X$, $g' \colon Y' \to X'$ are volume preserving;

(2) $\chi \colon Y \dashrightarrow Y'$ is a volume preserving isomorphism in codimension 1 which is a composition of volume preserving flips, flops and inverse flips between terminal $\mathbb{Q}$-factorial varieties

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- limiting case of the Sarkisov program for pairs
- Sarkisov program for varieties

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Working with pairs usually spoils the singularities of the underlying varieties

Working with varieties does not preserve singularities of pairs

Factorisation of volume preserving maps of lc CY pairs

A non-example.. or how it could go wrong!

Let $E = \mathbb{P}^1 \times \mathbb{P}^1$ and W the total space of the vector bundle $\mathcal{O}_E(-1,-2)$.

Let $D_W \subset W$ be a smooth surface such that $D_W \cap E$ is a ruling in E and a (-2)-curve in D_W.

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Let $f\colon W \to Y$ be the contraction of E along the first ruling and $f'\colon W \to Y'$ the contraction along the second ruling.

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Then (Y, D) and (Y', D') are both dlt, Y' is terminal, Y is canonical but not terminal, and the map $Y \dashrightarrow Y'$ is volume preserving.

Factorisation of volume preserving maps of lc CY pairs
Sketch proof

Step 1. Take a common resolution

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, dashed, "\varphi"] & & X' \end{tikzcd}$$

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$$\begin{aligned} K_W + D_W &= f^*(K_X + D) + F \\ &= f'^*(K_{X'} + D') + F \end{aligned}$$

for $D_W = \sum_{a_E=-1} E, F = \sum_{a_E>0} a_E E$

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for $D_W = \sum_{a_E=-1} E, F = \sum_{a_E>0} a_E E$

Step 2. $(Y, D_Y)/(Y', D_{Y'})$=end product of the $(K_W + D_W)$-MMP over X/X'
Crucial: $W \dashrightarrow Y$ and $W \dashrightarrow Y'$ isomorphisms near nklt loci

Factorisation of volume preserving maps of lc CY pairs
Sketch proof

Step 1. Take a common resolution

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ X \arrow[rr, dashed, "\varphi"] & & X' \end{tikzcd}$$

$$\begin{aligned} K_W + D_W &= f^*(K_X + D) + F \\ &= f'^*(K_{X'} + D') + F \end{aligned}$$

for $D_W = \sum_{a_E = -1} E, F = \sum_{a_E > 0} a_E E$

Step 2. $(Y, D_Y)/(Y', D_{Y'})$=end product of the $(K_W + D_W)$-MMP over X/X'
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Step 3. Induced $Y \overset{\chi}{\dashrightarrow} Y'$ isomorphism in codimension 1
For suitable Θ (perturbation of D_Y), (Y, Θ) is klt and χ is a $(K_Y + \Theta)$-MMP
χ is a composition of $(K_Y + \Theta)$-flips that are also $(K + D)$-flops and all intermediate varieties are terminal

Sketch proof of the main theorem

Setup:

$$\begin{tikzcd}
Y \arrow[r, dashed, "\chi"] \arrow[d, "g"'] & Y' \arrow[d, "g'"] \\
X \arrow[r, dashed, "\varphi"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\
S & S'
\end{tikzcd}$$

(i) Y and Y' have $\mathbb{Q}$-factorial terminal singularities and $g\colon Y \to X$ and $g'\colon Y \to X'$ are birational morphisms

(ii) $\chi\colon Y \dashrightarrow Y'$ is the composition of flips, flops and inverse flips between terminal $\mathbb{Q}$-factorial varieties

(iii) $p\colon X \to S$ and $p'\colon X' \to S'$ are Mfs

Sketch proof of the main theorem

(Hacon–McKernan) Two Mfs that are end products of the classical MMP on the same variety are connected by a chain of Sarkisov links

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$$\chi_i \colon Y_i \dashrightarrow Y_{i+1}$$

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Depending on whether χ_i is a flip, flop or antiflip, X_i/S_i and X_{i+1}/S_{i+1} are both end products of the MMP on Y_i or on Y_{i+1}