

The Sarkisov Program for Mori fibred Calabi–Yau pairs

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Birational maps of \mathbb{P}^2

Second half of 19th century: developments of a general theory of birational maps of \mathbb{P}^2 —Cremona, Enriques, Noether, de Jonquières, Castelnuovo..

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Theorem (Noether–Castelnuovo)

Cr_2 is generated by $\mathrm{PGL}_3(\mathbb{C})$ and a standard quadratic transformation

$$C: (x, y) \dashrightarrow \left(\frac{1}{x}, \frac{1}{y} \right)$$

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- to higher dimensions:

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- to higher dimensions:
 - Study $\text{Cr}_n = \text{Bir } \mathbb{P}^n$ for $n \geq 3$
 - So far, very little—no meaningful way to probe even Cr_3
- to open varieties:
 - consider proper birational maps between open algebraic surfaces

$$\varphi: U \dashrightarrow V$$

with $U, V \subset \mathbb{P}^2$ or U, V subsets of rational surfaces

- litaka's philosophy: formalism of pairs

Symplectic transformations of the plane

Subgroup of birational maps of \mathbb{P}^2 preserving standard volume form of $(\mathbb{C}^*)^2$

$$\mathrm{SCr}_2 = \left\{ f: \mathbb{C}^2 \dashrightarrow \mathbb{C}^2, f \in \mathrm{Cr}_2 \mid f^* \left(\frac{dx}{x} \wedge \frac{dy}{y} \right) = \frac{dx}{x} \wedge \frac{dy}{y} \right\}$$

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Theorem (Usnich–Blanc)

SCr_2 is generated by $(\mathbb{C}^*)^2$, $\mathrm{Sl}_2(\mathbb{Z})$ and the birational map

$$P: (x, y) \dashrightarrow \left(y, \frac{1+y}{x} \right)$$

Symplectic Cremona group

Possible generalisations?

Subgroup of Cr_n of maps preserving standard volume form of $(\mathbb{C}^*)^n$

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Examples of symplectic Cremona transformations

(i) $(\mathbb{C}^*)^n$, $\text{Sl}_n(\mathbb{Z})$ are subgroups of SCr_n

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Examples of symplectic Cremona transformations

- (i) $(\mathbb{C}^*)^n$, $\text{Sl}_n(\mathbb{Z})$ are subgroups of SCr_n
- (ii) (mutations) Let $\mathbb{T} = \text{Spec } \mathbb{C}[N]$ be an n -dimensional torus, $h \in M = \text{Hom}(N, \mathbb{Z})$, and $f \in \mathbb{C}[h^\perp] \subset \mathbb{C}[N]$, then

$$\phi: x^\gamma \mapsto x^\gamma \cdot f^{\langle h, \gamma \rangle}$$

is volume preserving.

Another generalisation of results on Cr_2 and SCr_2

Generalisation of Noether-Castelnuovo's theorem on Cr_2 :

Theorem (Sarkisov, Reid, Corti, Hacon-McKernan)

A birational map between Mori fibre spaces is a composition of Sarkisov links.

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Generalisation of Usnich-Blanc's theorem on SCr_2 :

Theorem (Corti-K.)

A volume preserving birational map between Mori fibred Calabi-Yau pairs is a composition of volume preserving Sarkisov links.

Sarkisov program

Proof of Noether-Castelnuovo:

(Noether) if $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ not biregular, then $\exists \mathbb{P}^2 \xrightarrow{c} \mathbb{P}^2$ quadratic with $\deg(c \circ \varphi) < \deg \varphi$

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(Castelnuovo) proof relies on:

Theorem

Any birational map $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a chain of the following elementary maps

$$\begin{array}{ccc} \mathbb{P}^2 & \xleftarrow{\varepsilon} & \mathbb{F}_1 \\ \downarrow & & \downarrow \\ \text{pt} & \longleftarrow & \mathbb{P}^1 \end{array}$$

Type I

$$\begin{array}{ccc} \mathbb{F}_k & \dashrightarrow & \mathbb{F}_{k\pm 1} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \end{array}$$

Type II

$$\begin{array}{ccc} \mathbb{F}^1 & \xrightarrow{\varepsilon} & \mathbb{P}^2 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \longrightarrow & \text{pt} \end{array}$$

Type III

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$$

Type IV

Sarkisov program

A Mori fibre space (Mfs) X/S

- a \mathbb{Q} -factorial terminal variety X
- and a fibration $f: X \rightarrow S$ such that $f_*\mathcal{O}_X = \mathcal{O}_S$, $-K_X$ is f -ample, and $\rho(X) - \rho(S) = 1$

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Sarkisov program (Hacon-McKernan)

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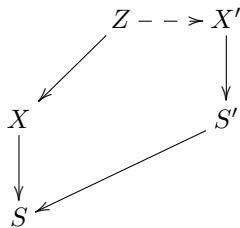
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Structure theorem: identifies “generators” of $\text{Bir } X$ for any Mfs X

Sarkisov program

Sarkisov link $\varphi: X/S \dashrightarrow X'/S'$: a commutative diagram

Type I
inverse Type III

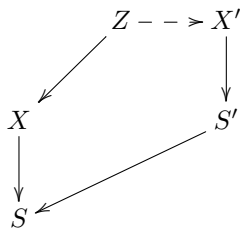


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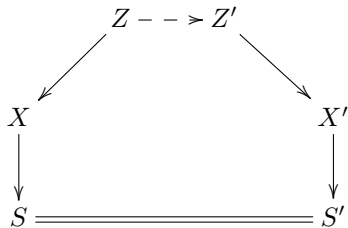
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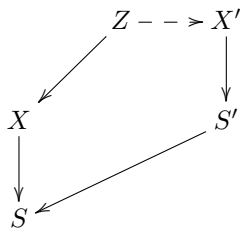


$Z \rightarrow X$ and $Z' \rightarrow X'$ divisorial contractions; $Z \dashrightarrow Z'$ sequence of flips, flops and inverse flips

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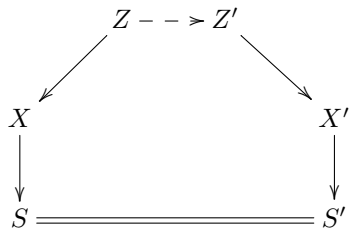
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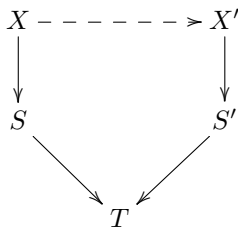
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$Z \rightarrow X$ and $Z' \rightarrow X'$ divisorial contractions; $Z \dashrightarrow Z'$ sequence of flips, flops and inverse flips

Type IV



$X \dashrightarrow X'$ sequence of flips, flops and inverse flips

All intermediate varieties terminal \mathbb{Q} -factorial

Calabi–Yau pairs

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- the pair (X, D) is dlt (resp. lc)

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A Mori fibred CY pair $(X/S, D)$ is the end product of a classical MMP and the end product of a log-MMP for (X, D)

Examples of Calabi-Yau pairs

Mori fibred rational Calabi-Yau surface pairs

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- $C_1 + C_2 + C_3 + C_4, (C_1, C_2, C_3, C_4) = (\sigma, f, \sigma + kf, f)$
- $C_1 + C_2 + C_3, (C_1, C_2, C_3) = (\sigma, f, \sigma + (1+k)f)$
- $C_1 + C_2, (C_1, C_2) = (\sigma, f, \sigma + (2+k)f, f)$

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 - $C_1 + C_2, (C_1, C_2) = (\sigma, f, \sigma + (2+k)f, f)$
3. a few extra cases (\mathbb{F}_N, D) where $N \leq 2$ and σ not a component of D

Examples of Calabi-Yau pairs

4. (S, D) a Mf CY surface pair and $\sigma: \tilde{S} \rightarrow S$ the blowup of P_1, \dots, P_k distinct points on $D - \text{Sing}(D)$. The pair $(\tilde{S}, \sigma_*^{-1}D)$ is a (t,dlt) CY pair if (S, D) is dlt, and a (t, lc) pair otherwise

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7. X a smooth weak Fano 3-fold and $D \in |-K_X|$ a general anticanonical section; then (X, D) is a (t, dlt) CY pair

Volume preserving birational maps

Let (X, D) and (X', D') be CY pairs. A birational map $\varphi: X \dashrightarrow X'$ is **volume preserving** if there exists a common log resolution

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \overset{\varphi}{\dashrightarrow} & X' \end{array}$$

such that $p^*(K_X + D) = q^*(K_{X'} + D')$.

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$$a(E, K_X + D) = a(E, K_{X'} + D')$$

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Factorisation of volume preserving maps of lc CY pairs

Theorem (CK)

Let (X, D) and (X', D') be lc CY pairs and $\varphi: X \dashrightarrow X'$ a volume preserving birational map. Then there are \mathbb{Q} -factorial (t, dlt) CY pairs (Y, D_Y) , $(Y', D_{Y'})$ and a commutative diagram of birational maps:

$$\begin{array}{ccc} Y & \xrightarrow{\chi} & Y' \\ g \downarrow & & \downarrow g' \\ X & \xrightarrow{\varphi} & X' \end{array}$$

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where:

- (1) the morphisms $g: Y \rightarrow X$, $g': Y' \rightarrow X'$ are volume preserving;
- (2) $\chi: Y \dashrightarrow Y'$ is a volume preserving isomorphism in codimension 1 which is a composition of volume preserving flips, flops and inverse flips between terminal \mathbb{Q} -factorial varieties

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Main subtlety: Ensure that the decomposition of χ only involves varieties with terminal singularities

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- limiting case of the Sarkisov program for pairs
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Working with pairs usually spoils the singularities of the underlying varieties

Working with varieties does not preserve singularities of pairs

Factorisation of volume preserving maps of lc CY pairs

A non-example.. or how it could go wrong!

Let $E = \mathbb{P}^1 \times \mathbb{P}^1$ and W the total space of the vector bundle $\mathcal{O}_E(-1, -2)$.

Let $D_W \subset W$ be a smooth surface such that $D_W \cap E$ is a ruling in E and a (-2) -curve in D_W .

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Let $f: W \rightarrow Y$ be the contraction of E along the first ruling and $f': W \rightarrow Y'$ the contraction along the second ruling.

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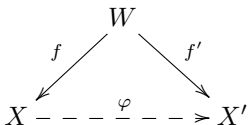
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Then (Y, D) and (Y', D') are both dlt, Y' is terminal, Y is canonical but not terminal, and the map $Y \dashrightarrow Y'$ is volume preserving.

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Sketch proof

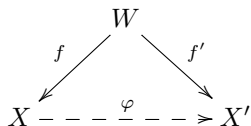
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$$\begin{aligned}K_W + D_W &= f^*(K_X + D) + F \\ &= f'^*(K_{X'} + D') + F\end{aligned}$$

$$\text{for } D_W = \sum_{a_E = -1} E, F = \sum_{a_E > 0} a_E E$$

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 \end{aligned}$$

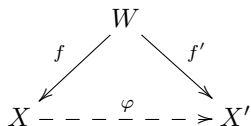
$$\text{for } D_W = \sum_{a_E = -1} E, F = \sum_{a_E > 0} a_E E$$

Step 2. $(Y, D_Y)/(Y', D_{Y'}) = \text{end product of the } (K_W + D_W)\text{-MMP over } X/X'$
 Crucial: $W \dashrightarrow Y$ and $W \dashrightarrow Y'$ isomorphisms near nklc loci

Factorisation of volume preserving maps of lc CY pairs

Sketch proof

Step 1. Take a common resolution



$$\begin{aligned}
 K_W + D_W &= f^*(K_X + D) + F \\
 &= f'^*(K_{X'} + D') + F
 \end{aligned}$$

$$\text{for } D_W = \sum_{a_E = -1} E, F = \sum_{a_E > 0} a_E E$$

Step 2. $(Y, D_Y)/(Y', D_{Y'}) = \text{end product of the } (K_W + D_W)\text{-MMP over } X/X'$
 Crucial: $W \dashrightarrow Y$ and $W \dashrightarrow Y'$ isomorphisms near nkt loci

Step 3. Induced $Y \xrightarrow{\chi} Y'$ isomorphism in codimension 1

For suitable Θ (perturbation of D_Y), (Y, Θ) is klt and χ is a $(K_Y + \Theta)$ -MMP
 χ is a composition of $(K_Y + \Theta)$ -flips that are also $(K + D)$ -flops and all intermediate varieties are terminal

Sketch proof of the main theorem

Setup:

$$\begin{array}{ccc} Y & \xrightarrow{\chi} & Y' \\ g \downarrow & & \downarrow g' \\ X & \xrightarrow{\varphi} & X' \\ p \downarrow & & \downarrow p' \\ S & & S' \end{array}$$

- (i) Y and Y' have \mathbb{Q} -factorial terminal singularities and $g: Y \rightarrow X$ and $g': Y \rightarrow X'$ are birational morphisms
- (ii) $\chi: Y \dashrightarrow Y'$ is the composition of flips, flops and inverse flips between terminal \mathbb{Q} -factorial varieties
- (iii) $p: X \rightarrow S$ and $p': X' \rightarrow S'$ are Mfs

Sketch proof of the main theorem

(Hacon–McKernan) Two Mfs that are end products of the classical MMP on the same variety are connected by a chain of Sarkisov links

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$$\chi_i: Y_i \dashrightarrow Y_{i+1}$$

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Depending on whether χ_i is a flip, flop or antiflip, X_i/S_i and X_{i+1}/S_{i+1} are both end products of the MMP on Y_i or on Y_{i+1}