Algebraic dynamics of polynomial maps: degree growth

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The general setup

X is an algebraic variety defined over a field k

- $f : X \to X$ is a regular (dominant) map
- $f^n = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$

Ask questions of algebraic nature on this dynamical system. Recent sport motivated by:

- the study of holomorphic dynamical systems in arbitrary dimensions;
- the arithmetic of torsion points on abelian varieties (these are preperiodic points for the doubling map).

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X is an algebraic variety defined over $\mathbb{C}$

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The restricted setup

Focus on (dominant) polynomial maps

$$f(x, y) = (P(x, y), Q(x, y)) : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}} \; .$$

- This is a non-trivial class of examples: Hénon maps

$$(x, y) \mapsto (ay, x + P(y))$$

  have been studied in depth (over $\mathbb{C}$ and $\mathbb{R}$), and their dynamics is complicated (positive entropy).
- It is easier to deal with than arbitrary maps: small dimension, simple geometry.

The program

1. Construction of projective compactifications adapted to the dynamics (Favre-Jonsson).
2. The dynamical Mordell-Lang conjecture (Xie).
3. The dynamical Manin-Mumford problem (Dujardin-Favre).

Degree growth

- $\deg(f) = \max\{\deg(P), \deg(Q)\} \in \mathbb{N}^*$;

Problem

Describe the sequence $\deg(f^n)$*:*

- *give an asymptotic;*
- *compute all degrees.*

Motivation: in $(\mathbb{P}^2, \omega_{FS})$ the entropy is bounded by

$$h_{top}(f) \overset{\text{Gromov}}{\leq} \sup_C \limsup_n \frac{1}{n} \log \mathrm{vol}(f^{-n}(C)) = \max\left\{e(f), \limsup_n \frac{1}{n} \log \deg(f^n)\right\}.$$

$e(f) = \#f^{-1}\{p\} =$ topological degree of f.

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Basics on degrees

- $\deg(f \circ g) \le \deg(f) \times \deg(g)$;

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Proof.

If $f = (P, Q)$, $g = (R, S)$, then we have $f \circ g = (P(R, S), Q(R, S))$. $\square$

Basics on degrees

- $\deg(f \circ g) \leq \deg(f) \times \deg(g)$;

Invariance under conjugacy

- if $g = h^{-1} \circ f \circ h$, for some $h \in \mathrm{Aut}[\mathbb{A}^2_k]$ then

$$0 < \frac{1}{C} \leq \frac{\deg(g^n)}{\deg(f^n)} \leq C < \infty \; .$$

Dynamical degree

- The limit $\lambda(f) := \lim_n \deg(f^n)^{1/n}$ exists.

Upper bound

- By Bezout $e(f) \leq \lambda(f)^2$.

Some examples: automorphisms

By Jung and Friedland-Milnor any $f \in \mathrm{Aut}[\mathbb{A}^2_{\mathbb{C}}]$ is conjugated to

- affine map or elementary map

$$f(x,y) = (ax+b, cy+P(x))$$

  in which case $\deg(f^n) \leq \deg(f)$ for all n.
- Hénon-like map $f = h_1 \circ \cdots \circ h_k$ with

$$h_i = (a_i y, x + P_i(y))$$

  $d_i := \deg(P_i) \geq 2$, in which case $\deg(f^n) = \deg(f)^n = (\prod_i d_i)^n$ for all n.

hence $\lambda(f)$ is an integer.

Some examples: monomial maps

- if $f(z) = f(x, y) = (x^a y^b, x^c y^d) = z^M$ with

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix},\ ad \neq bc,\ a, b, c, d \in \mathbb{N}$$

  then $f^n(z) = z^{M^n}$, and $\lambda(f)$ is the spectral radius of M.

hence $\lambda(f)$ is a quadratic integer.

Algebraic stability

- There is a simple geometric condition under which $\deg(f^n)$ can be controlled (Fornaess-Sibony).

Definition

A rational map $f : X \dashrightarrow X$ is algebraically stable iff for any irreducible curve $E \subset X$, the image variety $\check{f}^n(E)$ is not a point of indeterminacy for any $n \geq 1$.

Definition

A projective surface $X \supset \mathbb{A}^2_{\mathbb{C}}$ is a good dynamical compactification for f if the (rational) extension of f to X is algebraically stable.

Algebraic stability: examples and consequences

- Affine map and Hénon-like maps are alg. stable in $\mathbb{P}^2$;
- an elementary map $(x, y + P(x))$ is alg. stable in a suitable Hirzebruch surface;
- a monomial map is alg. stable in a suitable product of weighted projective lines.

Fact

When f is alg. stable in X, then $(f^{n+m})^* = (f^n)^* \circ (f^m)^*$ for the natural actions of f^n on the (real) Neron-Severi space of X.

- *$\lambda(f)$ is an algebraic integer;*
- *$\sum_{n\geq 0} \deg(f^n) T^n \in \mathbb{Z}(T)$ (if X dominates $\mathbb{P}^2$)*

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- $\lambda(f)$ *is an algebraic integer;*
- $\sum_{n\geq 0} \deg(f^n) T^n \in \mathbb{Z}(T)$ *(if X dominates $\mathbb{P}^2$)*

Main results

Theorem

Any polynomial map of $\mathbb{A}_k^2$ admits an iterate for which there exists a good dynamical compactification $X \supset \mathbb{A}_k^2$.

Main results: precise form

Theorem

When $e(f) < \lambda(f)^2$, one can choose X s.t.

1. $H_\infty := X \setminus \mathbb{A}^2_k$ *is irreducible and not contracted by* f*;*
2. H_∞ *is irreducible and contracted to a smooth point of* X *that is fixed by* f^N*,* $N \gg 1$*;*
3. H_∞ *has two components intersecting transversally at a fixed point that are contracted to that point by* f^N*.*

Corollary

For any polynomial map of $\mathbb{A}^2_k$*, the real number* $\lambda(f)$ *is a quadratic integer.*

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Corollary

For any polynomial map of $\mathbb{A}^2_k$, *the real number* $\lambda(f)$ *is a quadratic integer.*

First tentative

Optimistic hope:

- find $X = \mathbb{A}^2_{\mathbb{C}} \sqcup E$ with E irreducible and $\check{f}(E) = E$;
- if E exists, the divisorial valuation $\mathrm{ord}_E : \mathbb{C}[x, y] \to \mathbb{Z}$ is f_*-invariant in the sense

$$f_*(\mathrm{ord}_E)(P) := \mathrm{ord}_E(P \circ f) = \lambda(f)\,\mathrm{ord}_E(P) .$$

Difficulties.

- How to find a fixed point for the projective action of f_* on divisorial valuations?
- If a divisorial valuation ν is fixed, is it possible to compactify $\mathbb{A}^2_{\mathbb{C}}$ by adding one irreducible component E at infinity such that $\nu = \mathrm{ord}_E$?

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Identify good valuations

Definition

A good divisorial valuation is a one proportional to ord_E *where* $\mathbb{A}^2_k \sqcup E$ *is a compactification.*

- $X = \mathbb{A}^2_k \sqcup D$, with $D = E_1 \cup \cdots \cup E_r$, and $\nu_i = \mathrm{ord}_{E_i}$.
- Dual divisor: $\check{E}_i \cdot E_j := \delta_{ij}$

Fact

ν_i *is good iff* $\check{E}_i \cdot \check{E}_i > 0$.

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- Dual divisor: $\check{E}_i \cdot E_j := \delta_{ij}$

Theorem

ν_i *is good* $\Leftrightarrow \check{E}_i \cdot \check{E}_i > 0 \Leftrightarrow \check{E}_i$ *is nef and big*

Remark

$\check{E}_i \cdot \check{E}_i$ *only depends on* ν_i *not on the choice of a model*

The space of good valuations I

Definition

Let $\mathcal{V}_1$ be the space of good divisorial valuations on $\mathbb{C}[x, y]$, i.e. of the form $t\,\mathrm{ord}_E$ with $t > 0$ and E is a component at infinity in some compactification such that $\check{E} \cdot \check{E} > 0$.

Remark

A valuation $\nu \in \mathcal{V}_1$ is close to $-\deg$ since $\nu(P) < 0$ for all non constant polynomials.

The space of good valuations II

To get a space amenable to a fixed point theorem:

Definition

Let $\mathcal{V}_2$ be the closure of $\mathcal{V}_1$ in the space of all (non-trivial) valuations $\nu : \mathbb{C}[x, y] \to \mathbb{R}_-$.

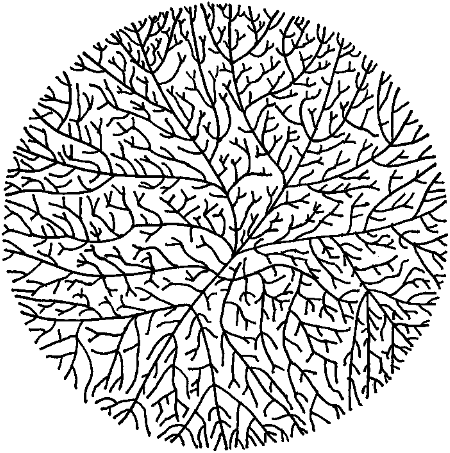
Theorem

The space $\mathcal{V}_2$ is a cone over

$$\mathcal{V}_2' := \{\nu \in \mathcal{V}_2, \min\{\nu(x), \nu(y)\} = -1\} ,$$

and $\mathcal{V}_2'$ is a compact $\mathbb{R}$-tree.

A tree dream



The space of good valuations III

For technical reason, and get a better description of the end points of the tree:

Definition

Let $\mathcal{V}_3$ be the closure of the set of good divisorial valuations $t\,\mathrm{ord}_E$ such that

$$A(t\,\mathrm{ord}_E) := t\,(1 + \mathrm{ord}_E(dx \wedge dy)) < 0 \ .$$

Theorem

The space $\mathcal{V}_3$ is a cone over

$$\mathcal{V}_3' := \{\, \nu \in \mathcal{V}_3,\ \min\{\nu(x), \nu(y)\} = -1\} \ ,$$

and $\mathcal{V}_3'$ is an $\mathbb{R}$-tree whose divisorial end points are either good or associated to a rational pencil.

Existence of the fixed point

Theorem

A polynomial map induces a natural continuous map $f_\bullet$ on the $\mathbb{R}$-tree $\mathcal{V}_3'$.
This map admits a fixed point which attracts all good divisorial valuations when $e(f) < \lambda(f)^2$.

- Invariance of $\mathcal{V}_3'$ is by invariance of nef divisors and the jacobian formula.
- Existence of the fixed point follows from a tracking argument.
- The attraction property is deeper: $\frac{1}{\sqrt{e(f)}} f^*$ is an isometry on the hyperbolic space $\varinjlim_X \mathrm{NS}_{\mathbb{R}}(X)$.

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Construction of the compactification

If the invariant valuation ν is

- divisorial ord_E: either it is good (pick $\mathbb{A}^2_k \sqcup E$) or associated to an rational invariant fibration (pick a suitable Hirzebruch surface);
- not divisorial: allows to construct by induction a sequence of blow ups $X_{n+1} \to X_n \to \mathbb{P}^2$, and f^N is alg. stable in X_n for some $n, N \gg 1$.

Algebraic dynamics of polynomial maps: the dynamical Mordell-Lang conjecture

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The dynamical Mordell-Lang conjecture

- $f : X \to X$ regular dominant map of an algebraic variety defined over $\mathbb{C}$;
- $V \subset X$ a subvariety, and $x \in X$ a point;

Conjecture (Denis, Bell-Ghioca-Tucker)

The set of hitting times $\{n \in \mathbb{N},\ f^n(x) \in V\}$ is a finite union of arithmetic sequences.

An arithmetic sequence is a set $\{an + b, n \in \mathbb{N}\}$ for some integers a, b (possibly zero)

Origin of the conjecture

Theorem (Skolem-Mahler-Lech's theorem)

Suppose $u_n \in \mathbb{C}$ is defined by a recurrence relation $u_{n+k+1} = a_k u_{n+k} + \cdots + a_0 u_n$, $a_i \in \mathbb{C}$. Then the set $\{n \in \mathbb{N},\ u_n = 0\}$ is a finite union of arithmetic sequences.

Conjecture $\Rightarrow$ Theorem.

Take $X = \mathbb{A}^{k+1}_{\mathbb{C}}$, f linear, $x = (u_0, \cdots, u_k)$, and V a hyperplane. $\square$

Theorem (Falting-Vojta)

Let G be a (semi)-abelian variety over $\mathbb{C}$, let V be a subvariety, and let Γ be a finitely generated subgroup of $G(\mathbb{C})$. Then $V(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.

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The polynomial case

- $f : \mathbb{A}^2 \to \mathbb{A}^2$ polynomial dominant map;
- V an irreducible curve, and $x \in X$ a point.

Conjecture

When $\{n \in \mathbb{N},\ f^n(x) \in V\}$ is infinite, then either x or V is pre-periodic.

Theorem (J. Xie)

For any polynomial map $f : \mathbb{A}^2_{\bar{\mathbb{Q}}} \to \mathbb{A}^2_{\bar{\mathbb{Q}}}$ the previous conjecture is true.

Automorphisms

Theorem (Bell-Ghioca-Tucker)

For any polynomial automorphism $f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ *the set* $\{n \in \mathbb{N},\ f^n(x) \in V\}$ *is infinite iff* x *or* V *is periodic.*

- Their method applies to any étale maps in any dimension.
- Elaboration of the original method of Skolem based on *p*-adic methods.

First reduction

- Use a specialization argument to reduce to the case f, x, V have coefficients in $\mathbb{Q}$.
- Pick a large prime number p not dividing denominators in the coef. of f, x, V, and such that f mod p remains an automorphism.

Work in $\mathbb{Q}_p$: completion of $\mathbb{Q}$ w.r.t the p-adic norm $|p| = \frac{1}{p}$.

$$\mathbb{Z}_p := \{ x \in \mathbb{Q}_p, |x|_p \leq 1 \} = \text{closure of } \mathbb{Z} .$$

Skeleton of the argument

1. Replace f by f^N to get $\bar{f}(\bar{x}) = \bar{x}$ in $\mathbb{A}^2_{\mathbb{F}_p}$;

- The map f is then an analytic automorphism of the open ball $B(x, 1)$;

2. Extend the map $n \mapsto f^n(x)$ to an analytic map $\Phi : \mathbb{Z}_p \to \mathbb{A}^2_{\mathbb{Q}_p}$ s.t. $\Phi(n) = f^n(x)$ for all n;

- For an equation $V = \{h = 0\}$ we have

$$\{n \in \mathbb{N},\ f^n(x) \in V\} \subset \{t \in \mathbb{Z}_p,\ h \circ \Phi(t) = 0\}$$

which is finite or equal to $\mathbb{Z}_p$.

p-adic parameterization

Theorem (Poonen)

Let $f(x) = \sum_I a_I x^I$, $|a_I| \to 0$, $a_I \in \mathbb{Z}_p$ be an analytic automorphism of the closed unit polydisk $\overline{B(0,1)}^d$ such that

$$f \equiv id \mod p^c \text{ with } c > \frac{1}{p-1} .$$

Then there exists an analytic map Φ on $\mathbb{Z}_p \times \overline{B(0,1)}^d$ s.t. $\Phi(n,x) = f^n(x)$ for all n.

- Any point belongs to a one dimensional disk on which f is conjugated to a translation by 1.
- In the complex domain, an analog statement holds in 1d, but not in 2d!
- One line proof but the margin is too small!!!

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Ingredients in Xie's approach

Theorem (J. Xie)

Pick any polynomial map $f : \mathbb{A}^2_{\bar{\mathbb{Q}}} \to \mathbb{A}^2_{\bar{\mathbb{Q}}}$*, any irred. curve* V *and any point* x*. If the set* $\{n \in \mathbb{N},\ f^n(x) \in V\}$ *is infinite, then either* x *or* V *are pre-periodic.*

1. A local analog of DML for special maps.
2. Arithmetical arguments : Siegel's theorem, height argument.
3. Affine geometry: existence of good compactifications, a special device to construct auxiliary polynomials.

A simplified situation

- f, x and V are defined over $\mathbb{Q}$;
- f is alg. stable in $\mathbb{P}^2$ and $\deg(f^n) \to \infty$;
- H_∞ is contracted to a point, say q_∞;
- the invariant valuation in $\mathcal{V}_3'$ is *not divisorial*, e.g. $e(f) < \lambda(f)$;
- the curve V is a line.

Assumption: x is not preperiodic and the set $\mathcal{N} := \{n \in \mathbb{N},\ f^n(x) \in V\}$ is infinite.

Aim: V is preperiodic.

The line contains the super-attracting point I

Step 1: $f^n(x) \to q_\infty \in \mathbb{P}^2_{\mathbb{C}}$ is impossible

- Blow-up at q_∞: f maps again the whole divisor to a fixed point $q_\infty^{<1>}$ that attracts x. Repeat the process until $q_\infty^{<n>}$ is not in the closure of V.
- same argument works when $\mathbb{C}$ is replaced by some $\mathbb{P}^2_{\mathbb{C}_p}$ for some prime p.

The line contains the super-attracting point II

Step 2: the point x is preperiodic

- Uniform upper bound for $|f^n(x)|_p$ for all $n \in \mathcal{N}$ and all place p.
- Height of $f^n(x)$ is bounded for all $n \in \mathcal{N}$.

Remark

This ends the proof when f is a Hénon automorphism. When f is birational, Xie proves that V not periodic implies $f^n(V) \ni q_\infty$ for some n.

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This ends the proof when f is a Hénon automorphism. When f is birational, Xie proves that V not periodic implies $f^n(V) \ni q_\infty$ for some n.

The line does not contain the super-attracting point

- Build a sequence of irrreducible pre-images

$$V_{-k+1} \xrightarrow{f} V_{-k} \xrightarrow{f^k} V$$

with $\mathcal{N}_k := \{n \in \mathbb{N}, f^n(x) \in V_{-k}\}$ infinite.

- Siegel's theorem: V_{-k} has at most two places at infinity

Simplification: V_{-k} has a single place for all k.

$$\nu_{-k}(P) := \mathrm{ord}_\infty(P|_{V_{-k}}) \in \mathbb{Z} \cup \{+\infty\} \text{ associated to } V_{-k} \, .$$

Auxiliary polynomial

Assumption: the map f has at least three points of indeterminacy in a good compactification

Theorem

There exists $P \in \mathbb{C}[x, y]$ s.t. $\nu_{-k}(P) > 0$ for all $k \geq 0$.

Consequence

The function $P|_{V_{-k}}$ is identically zero for all k and V is pre-periodic.

Auxiliary polynomial : existence

- Choose a resolution $f^M : X \to \mathbb{P}^2$
- $X = \mathbb{A}^2 \sqcup \left(\cup_1^s E_i \cup F\right)$ with F (reducible but) connected, and $f^{-M}\{-\deg\} \subset \{\mathrm{ord}_{E_i}\}$
- The curve V_{-k} does not intersect F

Aim: build an ample divisor supported on F so that $X \setminus F$ is affine.

- roughly: start with $\frac{1}{\lambda(f)^M}(f^M)^* H_\infty \in \mathrm{NS}_{\mathbb{R}}(X)$;
- modify it to get zero value on the E_i's.
- Need to contract a couple of E_i's.

$\to$ Look at Xie's paper for detail !!!

The difficulties in the general case

- The curve might have more than one place at infinity (≤ 2 by Siegel's theorem).
- The case the invariant valuation is divisorial is substantially harder.
- Remove the assumption on the existence of sufficiently many indeterminacy points: need to construct suitable height and prove a height bound.
- Need to treat the case $e = \lambda(f)^2$ separately.

Algebraic dynamics of polynomial maps: the dynamical Manin-Mumford conjecture

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Distribution of preperiodic points

The set up.

- $f : X \to X$ a regular dominant map on an algebraic variety $/\mathbb{C}$;
- $\mathrm{Per}(f) = \{x \in X,\, f^n(x) = x \text{ for some } n \geq 1\}$;
- $\mathrm{PrePer}(f) = \{x \in X,\, f^n(x) = f^m(x) \text{ for some } n > m \geq 0\}$.

The problem.

- Describe the distribution of Per(f) (and/or PrePer(f)) in X.
- In the euclidean topology: look at the limits of atomic measures equidistributed over $\{f^n = \mathrm{id}\}$;

$$\begin{cases} \text{Bedford-Smillie: automorphisms of } \mathbb{A}^2_{\mathbb{C}}; \\ \text{Lyubich, Briend-Duval: endomorphisms of } \mathbb{P}^d_{\mathbb{C}}; \\ \text{Many other cases: Dinh, Sibony, etc...} \end{cases}$$

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The abelian case (the Manin-Mumford conjecture)

- X abelian variety (compact complex torus that is projective);
- $f(x) = k \cdot x = \underbrace{x + \cdots + x}_{k \text{ times}}$ with $k \geq 2$;
- $\mathrm{PrePer}(f) = \mathrm{Tor}(X)$.

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- $\mathrm{PrePer}(f) = \mathrm{Tor}(X)$.

Theorem (Raynaud)

Pick $V \subset X$ irreducible s.t. $\mathrm{Tor}(f) \cap V$ is Zariski dense. Then V is a translate by a torsion point of an abelian subvariety.

The abelian case (the Manin-Mumford conjecture)

- X abelian variety (compact complex torus that is projective);
- $f(x) = k \cdot x = \underbrace{x + \cdots + x}_{k \text{ times}}$ with $k \geq 2$;
- $\mathrm{PrePer}(f) = \mathrm{Tor}(X)$.

Theorem (Raynaud)

Pick $V \subset X$ irreducible s.t. $\mathrm{PrePer}(f) \cap V$ is Zariski dense. Then V is preperiodic.

Towards a DMM conjecture?

Question (The DMM conjecture)

Pick $V \subset X$ irreducible s.t. $\mathrm{PrePer}(f) \cap V$ *is Zariski dense. Does this imply V to be preperiodic?*

- Wrong!!! Counterexamples for endomorphisms of $\mathbb{P}^2$ (Ghioca-Tucker-Zhang, Pazuki)
- True in some cases: a very general endomorphism of $\mathbb{P}^d_{\mathbb{C}}$ (Fakhruddin)

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Variations on the DMM problem

Question

Given f, describe all irreducible subvarieties $V \subset X$ s.t. $\mathrm{PrePer}(f) \cap V$ is Zariski dense.

Question

Describe the maps f for which the DMM conjecture has a positive/negative answer.

The DMM problem
The case of polynomial automorphisms
Ideas of proof

The DMM problem for polynomial automorphisms

$f : \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ an automorphism.

- When f is affine or elementary $(x, y) \mapsto (ax + b, cy + P(y))$ the DMM problem has a positive answer (exercice).

In the sequel suppose

$$f(x, y) = (ay, x + P(y)),\ \deg(P) \geq 2$$

is of Hénon type.

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First reductions

$f(x, y) = (ay, x + P(y))$, $\deg(P) \geq 2$ and V an irreducible curve

- Assumption:
  $V \cap \mathrm{PrePer}(f)$ is Zariski dense
- Conclusion:

First reductions

$f(x, y) = (ay, x + P(y))$, $\deg(P) \geq 2$ and V an irreducible curve

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Question

Is the set $\mathrm{Per}(f) \cap V$ *is finite for any irreducible curve* V*?*

Main theorem

Theorem (Dujardin-Favre)

Suppose $f(x,y) = (ay, x + P(y))$ with $|\mathrm{Jac}(f)| = |a| \neq 1$. Then the set $\mathrm{Per}(f) \cap V$ is finite for any irreducible curve V.

Counter-examples: reversible maps

- $f(x, y) = (y, -x + y^2)$, $f^{-1} = (-y + x^2, x)$;
- $f^{-1} = \sigma \circ f \circ \sigma$ with $\sigma(x, y) = (y, x)$;
- $\Delta = \{(x, x)\}$, $\Delta \cap f^n(\Delta) \subset \mathrm{Fix}(f^{2n})$;

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Proposition

$$|\Delta \cap f^n(\Delta)| \to \infty.$$

Proof.

Use Arnold's result: $\mathrm{mult}_{(x,x)}(f^n(\Delta), \Delta) = O(1)$. $\square$

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Proposition

$$|\Delta \cap f^n(\Delta)| \asymp 2^n.$$

Proof.

The image $f^n(\Delta)$ converges to a laminar current. $\square$

Some conjectures

Conjecture

Suppose $\mathrm{Per}(f) \cap V$ *is infinite. Then* $f^{-n} = \sigma \circ f^n \circ \sigma$ *for some* $n \geq 1$ *and some involution* σ*.*

Conjecture (Weak form)

Suppose $\mathrm{Per}(f) \cap V$ *is infinite. Then* $\mathrm{Jac}(f)$ *is a root of unity.*

Conjecture (Effective bounds)

Fix f *for which the DMM conjecture has a positive answer. Give a bound on* $\mathrm{Per}(f) \cap V$ *in terms of* $\deg(V)$*.*

Szpiro-Ullmo-Zhang' strategy

Reduce to the case $f(x, y) = (ay, x + y^2 + c)$, $V = \{Q = 0\}$ with $a, c \in \mathbb{Q}$, $Q \in \mathbb{Q}[y]$.

Assumption: $V \cap \mathrm{Per}(f)$ is infinite.

Conclusion: $|a| = 1$?

- Step 1: describe the distribution of periodic point on V to get $\mu_V^+ = \mu_V^-$.
- Step 2: exploit the equality of measures and use a renormalization argument to conclude.

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The Green function

$f(x, y) = (x_1, y_1) = (ay, x + y^2 + c)$,
$\|(x, y)\| = \max\{|x|, |y|\}$

- if $|y| \geq |x| \geq R \gg 1$, then $|y_1| = |y|^2 \geq |y| = |x_1| \geq R$.
- $\frac{1}{2^n} \log \max\{ 1, \|f^n(x, y)\|\}$ converges when $n \to +\infty$ uniformly to a Green function G^+

Properties:

- $G^+ \geq 0$, G^+ is continuous;
- $G^+ \circ f = 2G^+$
- $\{G^+ = 0\} = \{ (x, y), \sup_{n \geq 0} \|f^n(x, y)\| < +\infty\}$

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Distribution of periodic points

Theorem

Suppose $z_n \in V$ is a sequence of distinct periodic points. Then

$$\frac{1}{\deg(z_n)} \sum_{w \text{Galois conj. to } z_n} \delta_w \longrightarrow \mu_V^+ := c_+ \, \Delta(G^+|_V) \; .$$

Corollary

$$G^+|_V = c \, G^-|_V$$

Proof

- Build G_p^+ over any $\mathbb{C}_p$ for any prime;
- Sum them up to get a height:

$$h(z) := \frac{1}{\deg(z)} \sum_{w \text{ Galois conj. to } z} \sum G_p^+(z) \ .$$

- $h(z) = 0$ when z is periodic
- The height function $h|_V$ is a good height: one can apply Autissier' result to conclude.

The renormalization argument

Assumptions:

- $z \in V_{\mathrm{reg}}$ hyperbolic fixed point;
- $W^u_{\mathrm{loc}}(z)$ and $W^s_{\mathrm{loc}}(z)$ cut V transversally

$$df(z) = \begin{bmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{bmatrix}, \; |\lambda^+| > 1 > |\lambda^-|$$

Main idea: compute the Hölder exponent $\kappa^\pm$ of $G^\pm|_V$ near z.

- Transversality implies $G^+|_V$ and $G^+|_{W^u_{\mathrm{loc}}}$ have the same exponent
- Linearization: $f|_{W^u_{\mathrm{loc}}(z)}(t) = \lambda^+ t$
- $G^+(t) \asymp |t|^{\kappa^+}$
- $G^+ \circ f(t) = 2G^+ \Longrightarrow 2 = |\lambda^+|^{\kappa^+}$