A first approach to Hadamard product of varieties

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Motivations (from Algebraic Statistics)

"Statistical Models are Algebraic Varieties'

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Our state space is finite and denoted by $[m] = \{1, 2, \ldots, m\}$

A probability distribution on $[m]$ is a point of the probability simplex

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Definition

Let $\Theta \subset \mathbb{R}^d$ (Θ is called Parameter Space) and

$$\begin{array}{rcl} f : \Theta & \rightarrow & \Delta_{m-1} \\ \theta = (\theta_1, \ldots, \theta_d) & \rightsquigarrow & (p_1(\theta), p_1(\theta), \ldots, p_m(\theta)) \end{array}$$

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Definition

Let $p_1 = \frac{g_0}{h_0}, \dots, p_m = \frac{g_m}{h_m}$, where $g_i, h_i \in \mathbb{R}[\theta_1,\dots,\theta_d]$. Then $\mathcal{M} := f(\Theta) \subset \Delta_{m-1}$ is a parametric algebraic statistical model.

Let $\mathcal{M}$ be a parametric algebraic statistical models with map

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We can extend f to

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The projective algebraic variety associated to $\mathcal{M}$ is

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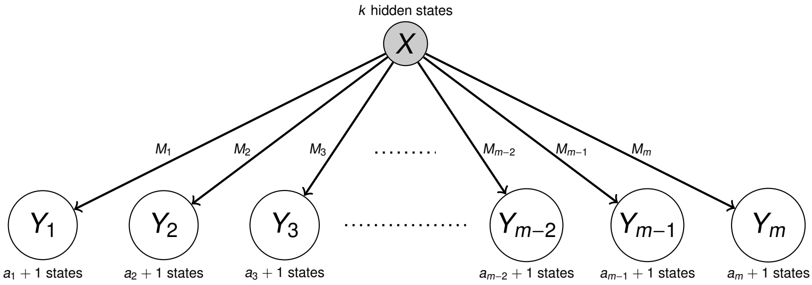
where $\hat{f} : \mathbb{P}^{d-1} \quad \rightarrow \quad \mathbb{P}^{m-1}$

Consider the following graphical model



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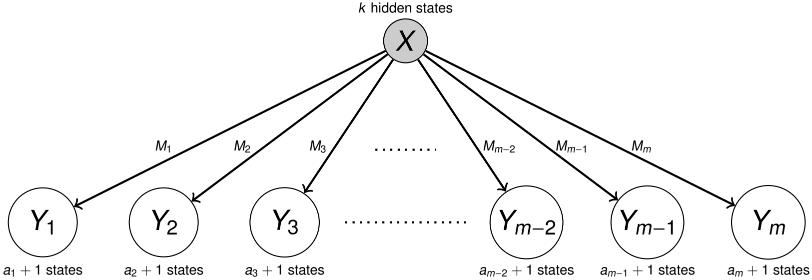

where M_t si a $k \times a_t$ transition matrix with

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The algebraic variety associated to this model is

$$S^k(\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_m})$$

Suppose that the state at X is momentarily fixed as $\tilde{k}$.

For each edge, we have a point

$$\overline{m}_{\tilde{k}Y_t} = \left[(M_t)_{\tilde{k}1} : \cdots : (M_t)_{\tilde{k}a_t}\right] \in \mathbb{P}^{a_t}$$

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Thus, if we define

$$P^{\tilde{k}} := \overline{m}_{\tilde{k}Y_1} \otimes \overline{m}_{\tilde{k}Y_2} \otimes \cdots \times \overline{m}_{\tilde{k}Y_m} \in \mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_m}$$

then $P^{\tilde{k}}$ is a point in the Segre product whose entries (up to scaling) are the expected frequencies of observing patterns conditioned by the state at the root being $\tilde{k}$.

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then $P^{\tilde{k}}$ is a point in the Segre product whose entries (up to scaling) are the expected frequencies of observing patterns conditioned by the state at the root being $\tilde{k}$.
Summing over all possible states at X, we obtain the joint distribution

$$P = P^1 + P^2 + \cdots + P^k.$$

Since we are summing k points on the variety $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_m}$, we obtain $P \in S^k(\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_m})$

M.A. Cueto, E.A. Tobis and J. Yu, *An implicitization challenge for binary factor analysis*, J. Symbolic Comput. **45** (2010), no. 12, 1296–1315.
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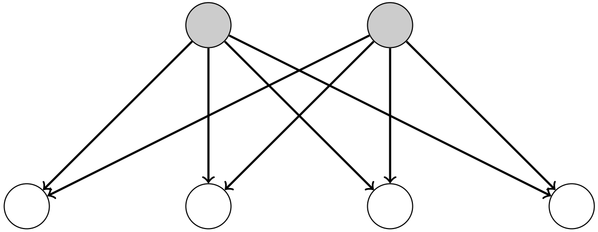

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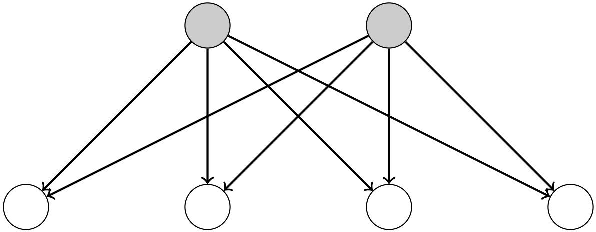

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$$V_M = S^1((\mathbb{P}^1)^4) \star S^1((\mathbb{P}^1)^4)$$

Definition

Let $p, q \in \mathbb{P}^n$ be two points of coordinates respectively $[a_0 : a_1 : \ldots : a_n]$ and $[b_0 : b_1 : \ldots : b_n]$. If $a_i b_i \neq 0$ for some i, their Hadamard product $p \star q$ of p and q, is defined as

$$p \star q = [a_0 b_0 : a_1 b_1 : \ldots : a_n b_n].$$

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The Hadamard product of two varieties $X, Y \in \mathbb{P}^n$ is

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What are the properties of $X \star Y$ w.r.t the properties of X and Y ?

Our definiton(s)

Definition

Given varieties $X, Y \subset \mathbb{P}^n$ *we consider the usual Segre product*

$$X \times Y \subset \mathbb{P}^N$$

$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0 b_0 : a_0 b_1 : \ldots : a_n b_n]$$

and we denote with z_{ij} *the coordinates in* $\mathbb{P}^N$*. Let* $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ *be the projection map from the linear space defined by equations* $z_{ii} = 0, i = 0, \ldots, n$*. The* Hadamard product *of* X *and* Y *is*

$$X \star Y = \overline{\pi(X \times Y)},$$

where the closure is taken in the Zariski topology.

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In the ring $K[x_0, \ldots, x_n, y_0, \ldots, y_n, z_0, \ldots, z_n]$ consider the ideals

$$I(\mathbf{y}) = \text{ image of } I \text{ under the map } x_i \to y_i,\ i = 0, \ldots, n$$
$$J(\mathbf{z}) = \text{ image of } J \text{ under the map } x_i \to z_i,\ i = 0, \ldots, n$$
$$L_{I,J} = I(\mathbf{y}) + J(\mathbf{z}) + \langle x_i - y_i z_i,\ i = 0, \ldots, n \rangle.$$

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One has

$$I(X \star Y) = I(X) \star_R I(Y).$$

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Moreover, given a projective transformation $f : \mathbb{P}^n \to \mathbb{P}^n$ we can have

$$f(X \star Y) \neq f(X) \star f(Y)$$

Basic results

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Definition

Given a positive integer r and a variety $X \subset \mathbb{P}^n$, the r-th Hadamard power *of X is*

$$X^{\star r} = X \star X^{\star (r-1)},$$

where $X^{\star 0} = [1 : \cdots : 1]$.

$\dim(X^{\star r}) \leq r \dim(X)$ and $X^{\star r}$ cannot be empty if X is not empty.

Let $H_i \subset \mathbb{P}^n, i = 0, \ldots, n$, be the hyperplane $x_i = 0$ and set

$$\Delta_i = \bigcup_{0 \leq j_1 < \ldots < j_{n-i} \leq n} H_{j_1} \cap \ldots \cap H_{j_{n-i}}.$$

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Δ_i = points having *at most* $i + 1$ non-zero coordinates

Δ_0 is the set of coordinates points
Δ_{n-1} is the union of the coordinate hyperplanes.

$$\Delta_0 \subset \Delta_1 \subset \ldots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{P}^n$$

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Lemma

If $X \cap \Delta_{n-i} = \emptyset$, then $X^{\star r} \cap \Delta_{n-ri+r-1} = \emptyset$.

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If $X \cap \Delta_{n-i} = \emptyset$, then $X^{\star r} \cap \Delta_{n-ri+r-1} = \emptyset$.

Example (closure is necessary)

Let $X \subset \mathbb{P}^2$ be the curve $x_0x_1 - x_2^2 = 0$ and $p = [1 : 0 : 2]$.

$$p \star X = \{x_1 = 0\}$$

The point $[0 : 0 : 1]$ cannot be obtained as Hadamard product $p \star x$, with $x \in X$.

Suppose $p \in \mathbb{P}^n \setminus \Delta_{n-1}$, that is, p has no coordinate equal zero.
Let $L \subset \mathbb{P}^n$ be the linear space of equations

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$$D_P = \begin{pmatrix} p_0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p_n \end{pmatrix}$$

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Then $p \star L$ is the linear space of equations

$$M'\mathbf{x} = 0.$$

where $M' = MD_p{}^{-1}$.

Terracini's Lemma Given a general point $P \in S^k(X)$, lying in the subspace $\langle P_1, \ldots, P_k \rangle$ spanned by k general points on X, then the tangent space $T_{S^k(X),P}$ to $S^k(X)$ at P is

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Lemma (Hadamard version of Terracini's Lemma)

Consider varieties $X, Y \subset \mathbb{P}^n$. If $p \in X$ and $q \in Y$ are general points, then

$$T_{p\star q}(X \star Y) = \langle p \star T_q(Y), q \star T_p(X) \rangle.$$

Moreover, if $p_1, \ldots, p_r \in X$ are general points and $p_1 \star \ldots \star p_r$ is a general point, then

$$T_{p_1\star\ldots\star p_r}(X^{\star r}) = \langle p_2 \star \ldots \star p_r \star T_{p_1}(X), \ldots, p_1 \star \ldots \star p_{r-1} \star T_{p_r}(X) \rangle.$$

Hadamard powers of a line

Lemma

Let $n \geq 2$ and $n \geq r$. If $L \subset \mathbb{P}^n$ is a line such that $L \cap \Delta_{n-2} = \emptyset$, then

$$L^{\star r} = \bigcup_{p_i \in L} p_1 \star \ldots \star p_r,$$

that is, the closure operation is not necessary.

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Theorem

Let $L \subset \mathbb{P}^n$, $n > 1$, be a line. If $L \cap \Delta_{n-2} = \emptyset$, then $L^{\star r} \subset \mathbb{P}^n$ is a linear space of dimension $\min\{r, n\}$.

If $L \cap \Delta_{n-2} = \emptyset$ fails in the previous theorem then $L^{\star r}$ is still linear, but possibly of deficient dimension.
Consider, for example, the line $L \subset \mathbb{P}^5$ of equation

$$\begin{cases} 2x_0 - x_1 = 0 \\ x_1 + 3x_2 - x_4 = 0 \\ 3x_2 - x_3 = 0 \\ 16x_3 - 12x_4 - 3x_5 = 0 \end{cases}$$

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$$L^{\star 2} : \begin{cases} 9x_2 - x_3 = 0 \\ 192x_1 + 64x_3 - 48x_4 - 9x_5 = 0 \\ 768x_0 + 64x_3 - 48x_4 - 9x_5 = 0 \end{cases}$$

$$L^{\star 3} : \begin{cases} 27x_2 - x_3 = 0 \\ 8x_0 - x_1 = 0 \end{cases} \qquad L^{\star 4} : \begin{cases} 81x_2 - x_3 = 0 \\ 16x_0 - x_1 = 0 \end{cases}$$

and $\dim(L^{\star r}) = 3$ for all $r \geq 4$.

Equations of $L^{\star r}$

Proposition

Let L in $\mathbb{P}^n$, $n > 1$, be a line with $L \cap \Delta_{n-2} = \emptyset$, and let $r < n$.

- *If* $L = \text{rowspan}\begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{10} & a_{11} & \dots & a_{1n} \end{pmatrix}$*, then*

$$L^{\star r} = \text{rowspan}\begin{pmatrix} a_{00}^r & a_{01}^r & \dots & a_{0n}^r \\ a_{00}^{r-1}a_{10} & a_{01}^{r-1}a_{11} & \dots & a_{0n}^{r-1}a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{10}^r & a_{11}^r & \dots & a_{1n}^r \end{pmatrix}.$$

- *In terms of Plücker coordinates, if $0 \le i_0 < i_1 < \dots < i_r \le n$, then*

$$[i_0, i_1, \dots, i_r]_{L^{\star r}} = \prod_{0 \le j < k \le r} [i_j, i_k]_L \ .$$

If $L \subset \mathbb{P}^n$ is a line and $r < n$ then

$$\dim(\langle L^{\star r}\rangle) = \dim(L^{\star r}) = r = \binom{r+1}{1} - 1 = \binom{r+\dim(L)}{\dim(L)} - 1$$

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Example

Let P be the 2-plane in $\mathbb{P}^5$ spanned by $[3 : 1 : 4 : 1 : 5 : 9]$, $[2 : 6 : 5 : 3 : 5 : 8]$ *and* $[9 : 7 : 9 : 3 : 2 : 3]$.

- $\dim(P^{\star 2}) = 4$
- $\deg(P^{\star 2}) = 3$

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Let P be the 2-plane in $\mathbb{P}^5$ spanned by $[3:1:4:1:5:9]$, $[2:6:5:3:5:8]$ *and* $[9:7:9:3:2:3]$.

- $\dim(P^{\star 2}) = 4$
- $\deg(P^{\star 2}) = 3$
- $P^{\star 2}$ *is singular in codimension* 2, *with singular locus*

$$\overline{\{p \star p : p \in P\}}.$$

If $L \subset \mathbb{P}^n$ is a line and $r < n$ then

$$\dim(\langle L^{\star r}\rangle) = \dim(L^{\star r}) = r = \binom{r+1}{1} - 1 = \binom{r+\dim(L)}{\dim(L)} - 1$$

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- $\dim(\langle P^{\star 2}\rangle) = 5 = \binom{2+\dim(P)}{\dim(P)} - 1$

Lemma

Let $L \subset \mathbb{P}^n$ be a generic linear space of dimension m. Then the linear span $\langle L^{\star r}\rangle$ has dimension $\min(\binom{m+r}{r} - 1, n)$.

Let $L = \langle p_0, p_1, \ldots, p_m\rangle$ and correspondingly write:

$$L = \text{rowspan}\begin{pmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m0} & a_{m1} & \ldots & a_{mn} \end{pmatrix}.$$

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$$\langle L^{\star r}\rangle = \langle p_0^{\star r_0} \star p_1^{\star r_1} \star \ldots \star p_m^{\star r_m} : r_i \in \mathbb{Z}_{\geq 0} \text{ and } r_0 + r_1 + \ldots + r_m = r\rangle,$$

$$L^{\star} = \text{rowspan}\begin{pmatrix} a_{00}^r & a_{01}^r & \ldots & a_{0n}^r \\ \ldots & \ldots & \ldots & \ldots \\ \prod a_{i0}^{r_i} & \prod a_{i1}^{r_i} & \ldots & \prod a_{im}^{r_i} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m0}^r & a_{m1}^r & \ldots & a_{mn}^r \end{pmatrix}.$$

The tropical approach

Given an irreducible variety $X \subset \mathbb{P}^n$ not contained in Δ_{n-1}, let $I \subset \mathbb{C}[x_0^{\pm}, \ldots, x_n^{\pm}]$ be the defining ideal of X .

The *tropicalization* of X is the set

$$\mathrm{trop}(X) = \{\mathbf{w} \in \mathbb{R}^n : \mathrm{in}_{\mathbf{w}}(I) \text{ contains no monomial}\}$$

where

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) \ : f \in I \rangle$$

and $\mathrm{in}_{\mathbf{w}}(f)$ is the sum of all nonzero terms of $f \sum_\alpha c_\alpha x^\alpha$ such that $\alpha \cdot w$ is maximum.

It is the support of a pure polyhedral subfan of the Gröbner fan of I. That subfan has positive integer multiplicities attached to its facets, and these balance along ridges.

Tropical geometry provides powerful tools to study Hadamard products, because of the following connection.

Lemma (Maclagan-Sturmfels, 2015)

The tropicalization of the Hadamard products of two varieties is the Minkowski sum of their tropicalizations. In symbols, if $X, Y \subset \mathbb{P}^n$ are irreducible varieties, then

$$\mathrm{trop}(X \star Y) = \mathrm{trop}(X) + \mathrm{trop}(Y),$$

as weighted balanced fans.

Theorem

Let $L_1, L_2, \ldots, L_r \subset \mathbb{P}^n$ be generic linear spaces of dimensions $m_1, m_2, \ldots, m_r$, respectively. Set

$$m = m_1 + m_2 + ... + m_r$$

and

$$d = \binom{m_1 + m_2 + \ldots + m_r}{m_1,\ m_2,\ \ldots\ ,\ m_r}.$$

Assume $m < n$. Then $L_1 \star L_2 \star \ldots \star L_r$ has dimension m and degree

$$\begin{cases} d & \text{if } L_i \text{ are pairwise distinct} \\ \frac{d}{r!} & \text{if } L_i \text{ are the same.} \end{cases}$$

In full generality, when L_i form a multiset with multiplicites $r_1, \ldots, r_k$, the dimension is m and the degree is $\frac{d}{(r_1!)\ldots(r_k!)}$.

trop(L_i) equals the *standard tropical linear space* Λ_{m_i} of dimension m_i

$$\mathrm{trop}(L_i) = \Lambda_{m_i} = \bigcup_{0 \le j_1 < \ldots < j_{m_i} \le n} \mathrm{pos}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_{m_i}}).$$

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is generically $(r_1!) \cdots (r_k!)$ to 1 $(r_1 + \cdots + r_k = r)$.

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$$\begin{aligned}\deg(L_1 \star \ldots \star L_r) &= \mathrm{mult}_{\mathbf{0}}\Big(\mathrm{trop}(L_1 \star \ldots \star L_r)\ \cap_{\mathrm{st}} \Lambda_{n-m}\Big)\\ &= \mathrm{mult}_{\mathbf{0}}\Big(\frac{d}{(r_1!)\ldots(r_k!)}\ \Lambda_m\ \cap_{\mathrm{st}} \Lambda_{n-m}\Big)\\ &= \frac{d}{(r_1!)\ldots(r_k!)}.\end{aligned}$$

Star Configurations

Definition

A set of $\binom{m}{r}$ points $\mathbb{X} \subset \mathbb{P}^n$ is a star configuration if there exist linear spaces $H_1, \ldots, H_m \subset M \subset \mathbb{P}^n$ such that:

- $r = \dim M = \dim H_i + 1$.
- *H_i are in linear general position in M.*
- $\mathbb{X} = \bigcup_{1 \leq i_1 < \ldots < i_r \leq m} H_{i_1} \cap \ldots \cap H_{i_r}$.

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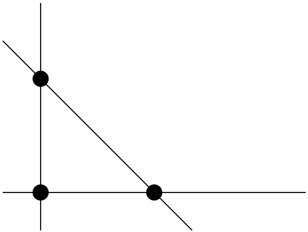

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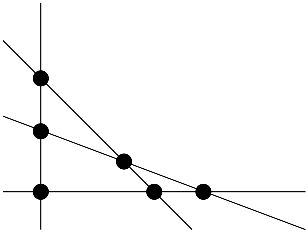

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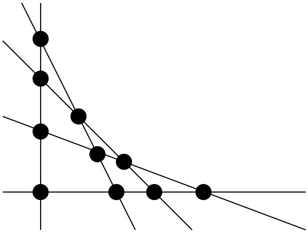

Let $0 \neq I \subset R = k[x_0, \ldots, x_n]$, be a homogeneous ideal. We define the m–th symbolic power of I to be

$$I^{(m)} = R \cap \bigcap_{P \in Ass(I)} I^m R_P.$$

Containment Problem: Given an ideal I, for which pairs (m, r) we have $I^{(m)} \subseteq I^r$?

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Theorem (Ein-Lazarsfeld-Smith, 2001,Hochster-Huneke, 2002)

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Theorem (B, Harbourne, 2010)

The bound of ELS-HH is sharp for every e and every n.

Given $I \subset k[x_0, \ldots x_N]$ we define

$$\alpha(I) = \min\{t \ : \ I_t \neq \emptyset\}$$

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Chudnovsky type conjecture: Let $I \subset k[x_0, \dots x_N]$ be the ideal of a finite set S of points in $\mathbb{P}^N$. Then

$$\frac{\alpha(I^{(m)})}{m} = \frac{\alpha(I) + N - 1}{N}.$$

if and only if S is a star configuration or contained in a hyperplane

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- $N = 2$ B, Chiantini, 2011
- $N = 3$ Bauer, Szemberg, 2013
- $N \geq 4$ still open

Definition

Let $Z \subset \mathbb{P}^n$ be a finite set of points. The $r-th$ square-free *Hadamard power of Z is*

$$Z^{\underline{\star} r} = \{p_1 \star \ldots \star p_r : p_i \in Z \text{ and } p_i \neq p_j \text{ for } i \neq j\}.$$

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Lemma

Let $L \subset \mathbb{P}^n$ be a line, $Z \subset L$ be a set of m points and $r \leq n$. If $L \cap \Delta_{n-2} = \emptyset$ and $Z \cap \Delta_{n-1} = \emptyset$, then $Z^{\underline{\star} r}$ is a set of $\binom{m}{r}$ points.

Lemma

Let $L \subset \mathbb{P}^n$ *be a line, let* $p_i \in L \setminus \Delta_{n-1}$, $1 \leq i \leq m$, *be* m *distinct points, and* $r \leq n$. *Set* $M = L^{\star r}$ *and* $H_i = p_i \star L^{\star(r-1)}$, $1 \leq i \leq m$. *If* $L \cap \Delta_{n-2} = \emptyset$, *then whenever* $i_1, \ldots, i_j$ *are distinct:*

- $H_{i_1} \cap \ldots \cap H_{i_j} = p_{i_1} \star \ldots \star p_{i_j} \star L^{\star(r-j)}$, *for* $j \leq r$.
- $H_{i_1} \cap \ldots \cap H_{i_j} = \emptyset$, *for* $r < j$.

In particular, the linear spaces H_i *are in linear general position in* M.

Lemma

Let $L \subset \mathbb{P}^n$ be a line, let $p_i \in L \setminus \Delta_{n-1}$, $1 \leq i \leq m$, be m distinct points, and $r \leq n$. Set $M = L^{\star r}$ and $H_i = p_i \star L^{\star(r-1)}$,$1 \leq i \leq m$. If $L \cap \Delta_{n-2} = \emptyset$, then whenever $i_1, \ldots, i_j$ are distinct:

- $H_{i_1} \cap \ldots \cap H_{i_j} = p_{i_1} \star \ldots \star p_{i_j} \star L^{\star(r-j)}$, *for* $j \leq r$.
- $H_{i_1} \cap \ldots \cap H_{i_j} = \emptyset$, *for* $r < j$.

In particular, the linear spaces H_i are in linear general position in M.

Theorem

Let $L \subset \mathbb{P}^n$ be a line, $Z \subset L$ be a set of m points and $r \leq \min\{m, n\}$. If $L \cap \Delta_{n-2} = \emptyset$ and $Z \cap \Delta_{n-1} = \emptyset$, then $Z^{\underline{\star} r}$ is a star configuration in $M = L^{\star r}$.

$$L : \begin{cases} x_0 - 2x_2 + x_3 = 0 \\ x_1 - 5x_2 + x_3 = 0 \end{cases}$$

$$Z$$

$$\begin{aligned} p_1 &= [1 : 1 : 2 : 3] \\ p_2 &= [1 : 2 : 1 : 1] \\ p_3 &= [2 : 3 : 3 : 4] \\ p_4 &= [3 : 4 : 5 : 7] \\ p_5 &= [3 : 5 : 4 : 5] \end{aligned}$$

$Z^{\underline{\star}2}$

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$Z^{\underline{\star} 2}$

$$\begin{aligned} p_1 \star p_2 &= [1:2:2:3] \\ p_1 \star p_3 &= [2:3:6:12] \\ p_1 \star p_4 &= [3:4:10:21] \\ p_1 \star p_5 &= [3:5:8:15] \\ p_2 \star p_3 &= [2:6:3:4] \\ p_2 \star p_4 &= [3:8:5:7] \\ p_2 \star p_5 &= [3:10:4:5] \\ p_3 \star p_4 &= [6:12:15:28] \\ p_3 \star p_5 &= [6:15:12:20] \\ p_4 \star p_5 &= [9:20:20:35] \end{aligned}$$

$$L^{\star 2} : 15x_0 - 2x_1 - 10x_2 + 3x_3 = 0$$

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$$p_1 \star L : \begin{cases} x_0 - x_2 + \frac{1}{3}x_3 = 0 \\ x_1 - \frac{5}{2}x_2 + \frac{1}{3}x_3 = 0 \end{cases}$$

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$$p_2 \star L : \begin{cases} x_0 - 2x_2 + x_3 = 0 \\ \frac{1}{2}x_1 - 5x_2 + x_3 = 0 \end{cases}$$

$Z^{\underline{\star}2}$

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$Z^{\underline{\star}3}$

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$Z^{\underline{\star} 3}$

$$\begin{aligned} p_1 \star p_2 \star p_3 &= [1:3:3:6] \\ p_1 \star p_2 \star p_4 &= [3:8:10:21] \\ p_1 \star p_2 \star p_5 &= [3:10:8:15] \\ p_1 \star p_3 \star p_4 &= [1:2:5:24] \\ p_1 \star p_3 \star p_5 &= [2:5:8:20] \\ p_1 \star p_4 \star p_5 &= [9:20:40:105] \\ p_2 \star p_3 \star p_4 &= [6:24:15:28] \\ p_2 \star p_3 \star p_5 &= [3:15:6:10] \\ p_2 \star p_4 \star p_5 &= [9:40:20:35] \\ p_3 \star p_4 \star p_5 &= [9:30:30:70] \end{aligned}$$

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$$L : \begin{cases} x_0 - 2x_2 + x_3 = 0 \\ x_1 - 5x_2 + x_3 = 0 \end{cases}$$

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$$\begin{aligned} p_1 \star L^{\star 2} &: \quad 15x_0 - 2x_1 - 5x_2 + x_3 = 0 \\ p_2 \star L^{\star 2} &: \quad 15x_0 - x_1 - 10x_2 + 3x_3 = 0 \\ p_3 \star L^{\star 2} &: \quad \tfrac{15}{2}x_0 - \tfrac{2}{3}x_1 - \tfrac{10}{3}x_2 + \tfrac{3}{4}x_3 = 0 \\ p_4 \star L^{\star 2} &: \quad 5x_0 - \tfrac{1}{2}x_1 - 2x_2 + \tfrac{3}{7}x_3 = 0 \\ p_5 \star L^{\star 2} &: \quad 5x_0 - \tfrac{2}{5}x_1 - \tfrac{5}{2}x_2 + \tfrac{3}{5}x_3 = 0 \end{aligned}$$

Thank for your attention