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Symmetric functions from moduli spaces of curves via vanishing theorems

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Abstract In this paper, we study some families of symmetric functions via geometric properties of the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed genus g stable curves. When $g = 0$, we work out explicit calculations, which extend previous work. In this case, our construction also yields various symmetric polynomials from codimension-one subvarieties of $\overline{\mathcal{M}}_{0,n}$.

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1. Introduction

Let g and $n > 2 - 2g$ be non-negative integers. By $\overline{\mathcal{M}}_{g,n}$ we denote the moduli space of n-pointed genus g stable complex curves. Recall that the points of this space are in one-to-one correspondence with isomorphism classes $[C; p_1, \dots, p_n]$ of (arithmetic) genus g curves with n non-singular marked points and finitely many automorphisms.

The rich and fascinating structure of the space $\overline{\mathcal{M}}_{g,n}$ has been intensively studied in the past decades. Quite often, combinatorial tools have been rather useful in understanding the geometry of this space. Conversely, geometric properties of $\overline{\mathcal{M}}_{g,n}$ have been rarely used as possible tools for combinatorial topics.

The aim of this paper is to introduce some natural symmetric polynomials ξ_n^g and τ_n^g, which are associated with sheaf cohomology groups of $\overline{\mathcal{M}}_{g,n}$. This gives a concrete way of calculating the rank of such groups. For these purposes, we apply standard vanishing results which hold on normal, projective, $\mathbb{Q}$-factorial varieties like $\overline{\mathcal{M}}_{g,n}$ - see Theorems 0.17 and 0.20.

When in particular $g = 0$, we first give recursion relations to carry out an explicit calculation of ξ_n^0 and τ_n^0; furthermore, we determine a natural way of defining symmetric functions on several Cartier divisors of $\overline{\mathcal{M}}_{0,n}$, which are surprisingly defined as differences of symmetric functions on the ambient space $\overline{\mathcal{M}}_{0,n}$ (cf. Claims 0.48 and 0.51).

The paper consists of five sections. In Section 1, we briefly recall basic terminology as well as we introduce notation which will be used in the rest of the paper.

In Section 2, we focus our attention on the space $\overline{\mathcal{M}}_{g,n}$. More precisely, we consider two Weil divisors $A_g(a_1, \dots, a_n)$ and $D_g(a_1, \dots, a_n)$, which are defined for any n-tuple of non-negative integers $(a_1, \dots, a_n) \neq (0, \dots, 0)$ and which are invariant under the action of

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the symmetric group $Sym(n)$ on $\overline{\mathcal{M}}_{g,n}$ - see Formulas (0.10), (0.11), (0.26) and Propositions 0.6, 0.12, respectively.

In Section 3, we apply some vanishing results to these divisors and the Hirzebruch-Riemann-Roch Theorem in order to define symmetric functions on $\overline{\mathcal{M}}_{g,n}$,

$$\xi_n^g(a_1, \ldots, a_n) \text{ and } \tau_n^g(a_1, \ldots, a_n),$$

whose definition is related to the geometry of the space $\overline{\mathcal{M}}_{g,n}$ (cf. Definitions 0.23 and 0.25).

In Section 4, we give examples in the genus zero case. Notably, we deduce recursive relations for the symmetric polynomials $\xi_n^0(a_1, \ldots, a_n)$. In particular, we show that the symmetric functions $\gamma_n(a_1, \ldots, a_n)$ on $\overline{\mathcal{M}}_{0,n}$ in [13] are particular cases of the more general family of symmetric functions $\xi_n^g(a_1, \ldots, a_n)$, which are defined for any genus $g \geq 0$ and not only in the genus-zero case. Furthermore, the functions $\xi_n^0(a_1, \ldots, a_n)$ - when suitably extended - can be used to compute the other symmetric polynomials $\tau_n^0(a_1, \ldots, a_n)$.

In Section 5 we also introduce other natural symmetric functions on suitable effective Cartier divisors of the smooth projective variety $\overline{\mathcal{M}}_{0,n}$. These symmetric functions are defined as a difference of the ξ_n^g's and the τ_n^g's, which exist on the ambient space $\overline{\mathcal{M}}_{0,n}$ (see Claims 0.48 and 0.51).

As for notation and terminology, note that the word *scheme* will throughout refer to an algebraic scheme over $\mathbb{C}$. The term *variety* will be used for an integral scheme. A general convention adopted hereafter is to work additively with divisors and multiplicatively with line bundles.

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2. Preliminaries and Notation

In this section we discuss some fundamental results and some basic facts we shall use throughout. For further details and terminology that is not defined here, the reader is referred to, for example [5].

Let X be a normal, irreducible, projective scheme. Denote by $Z^1(X)$ (resp. $Div(X)$) the group of Weil divisors (resp. Cartier divisors) on X. The symbols $\sim$ and $\equiv$ will denote linear and numerical equivalence on $Div(X)$ and $Z^1(X)$, respectively. As customary, we set

$$A^1(X) = Z^1(X)/\equiv \quad \text{and} \quad Pic(X) = Div(X)/\sim .$$

An element of $Z^1(X) \otimes \mathbb{Q}$ (resp. $Div(X) \otimes \mathbb{Q}$) is called a $\mathbb{Q}$-*divisor* (resp. $\mathbb{Q}$-*Cartier divisor*). By abuse of notation and terminology, K_X denotes the *canonical divisor* of X. By definition, it is the $\mathbb{Q}$-divisor which uniquely extends the canonical divisor on the smooth algebraic scheme $X \setminus Sing(X)$.

We recall that a normal, irreducible, projective scheme X is said to be $\mathbb{Q}$-*factorial* if $Z^1(X) \otimes \mathbb{Q} = Div(X) \otimes \mathbb{Q}$, that is, any $\mathbb{Q}$-divisor is a $\mathbb{Q}$-Cartier divisor. In such a case, given $D \in Z^1(X) \otimes \mathbb{Q}$, the least positive integer m_D such that $m_D D$ is Cartier is called the *index of* D. In particular, the smallest positive integer j_X such that $j_X K_X$ is Cartier is called *the index of X*.

We recall that a $\mathbb{Q}$-divisor is called *nef* if its intersection with any irreducible curve in X is non-negative. Furthermore, a nef $\mathbb{Q}$-divisor D is said to be *big* if the highest self-intersection number D^n is positive (we refer the reader to, for example, [10] for equivalent definitions).

Finally, if D is a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier and if μ is a log-resolution of the pair (X, D), then consider

$$K_{X'/X} - \mu^*(D) := K_{X'} - \mu^*(K_X + D) \equiv \sum_i a_i E_i,$$

where the E_i's are distinct irreducible divisors (not necessarily all μ-exceptionals); we recall that the pair (X, D) is called

(i) *canonical* if $a_i \geq 0$, for each E_i μ-exceptional;
(ii) *Kawamata log-terminal* (*k.l.t.*, for short) if $a_i > -1$ for each E_i.

Since $K_{X'/X}$ is always μ-exceptional, then X is said to have only *canonical singularities* if the pair $(X, 0)$ is canonical.

3. Some Fundamental Vanishing Theorems

Among the various results on $\mathbb{Q}$-divisors of normal varieties, we shall apply the machinery of vanishing theorems as a general fundamental tool to define new symmetric functions. In the sequel, we shall be mainly concerned with the following result.

Theorem 0.1. *(see* [10]*, page 73) (General Kodaira Vanishing Theorem) Let (X, Δ) be a k.l.t. pair, where X is a proper, algebraic scheme. Let N be a $\mathbb{Q}$-Cartier Weil divisor on X such that $N \equiv M + \Delta$, where M is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then*

$$H^i(X, \mathcal{O}_X(-N)) = (0), \text{ for each } i < \, dim(X).$$

Note that, when X is smooth, we will also use the standard Kodaira Vanishing Theorem (see, for example, [10], page 62).

4. Some Geometric Properties of Moduli Spaces of n-pointed Stable Curves

In this section we review some properties of moduli spaces of genus g curves with n marked points, which will be used in the next sections.

For any pair of non-negative integers g and n, $n > 2 - 2g$, let $(C; p_1, \ldots, p_n)$ be a reduced, connected, (at worst) nodal curve of arithmetic genus g with n non-singular marked points. Recall that $(C; p_1, \ldots, p_n)$ is *Deligne-Mumford stable* if $\omega_C(\sum_{i=1}^n p_i)$ is ample, where ω_C denotes the dualising sheaf of C. As customary, denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of *stable curves* of arithmetic genus g with n marked points. It is well known that $\overline{\mathcal{M}}_{g,n}$ is projective and has complex dimension $3g - 3 + n$. Moreover, it is normal, $\mathbb{Q}$-factorial and Cohen-Macaulay (see, for instance, [11]). Furthermore, when $g = 0$ and $n \geq 3$, $\overline{\mathcal{M}}_{0,n}$ is a smooth variety.

Theorem 0.2. *(see* [11]*, Theorem 2.5) Let $g \geq 4$ and $n \geq 0$ be integers. Then $\overline{\mathcal{M}}_{g,n}$ has only canonical singularities.*

By Remark 7.25 in [5] and by what we recalled in Section , we have the following result.

Corollary 0.3. *Let $g \geq 4$ and $n \geq 0$ be integers. Then the pair $(\overline{\mathcal{M}}_{g,n}, 0)$ is k.l.t.*

For the purpose of what follows, we need to recall the description of $Pic(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$: see, for instance, [1], [2] for further details. Here, by $Pic(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ we mean the abelian group $Pic(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}$.

Notation 0.4. Recall the following standard definitions.

- Let $\mathcal{L}_i$, $1 \leq i \leq n$, be the line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fibre at the point $[C; p_1, \dots, p_n]$ is the cotangent space at the smooth point p_i to the curve C. Its first Chern class is usually denoted by ψ_i.
- As customary, denote by λ the first Chern class of the vector bundle whose fibre at an n-pointed curve C is the space of holomorphic differentials $H^0(C; \omega_C)$, where ω_C is the dualising sheaf on C.
- Fix, now, $0 \leq i \leq \lfloor g/2 \rfloor$, $S \subset \{1, \dots, n\}$, where $|S| \geq 2$, $n - |S| \geq 2$, if $i = 0$. Moreover, if g is even and $i = \lfloor g/2 \rfloor$, we assume that $n \in S$. Following standard notation, let $\delta_{i,S}$ be the divisor class on $\overline{\mathcal{M}}_{g,n}$ whose generic point is given by a reducible curve with a single node. The removal of such a node yields two irreducible curves C_1 and C_2 of genus i and $g-i$, with $|S|+1$ and $n-|S|+1$ marked points, respectively. The assumptions for $i = 0$ on $|S|$ guarantee that C_1 is a stable curve; the requirement on S for $i = \lfloor g/2 \rfloor$, g even, avoids enumerating the same class twice.
- Next, denote by δ_{irr} the divisor class whose generic point is an irreducible n-pointed genus g curve with just one node.

When for example $g = 0$, the class λ equals 0 and, following [1], we have

$$\psi_i = \sum_{\substack{i \in S \\ j \neq k \notin S}} \delta_{0,S}, \tag{0.5}$$

where $S \subset \{1, \dots, n\}$, $|S|, |S^c| \geq 2$. Note that each $\delta_{0,S}$ is an irreducible subvariety of $\overline{\mathcal{M}}_{0,n}$, since it is isomorphic to a product of moduli spaces of rational curves. Therefore the divisors ψ_i are Cartier divisors. This depends on the more general fact that $\overline{\mathcal{M}}_{0,n}$ is smooth, for each $n \geq 3$, so each Weil divisor is Cartier.

As proved in [3], $Pic(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}}$ is an abelian group generated by λ, δ_{irr}, ψ_i $(1 \leq i \leq n)$, and $\delta_{i,S}$ $(0 \leq i \leq \lfloor g/2 \rfloor,\ S \subset \{1, 2, \dots, n\})$. When $g = 0$, the generators reduce to the classes $\delta_{0,S}$, $|S| \geq 2$ (see [9]).

We now consider some results on divisors of $\overline{\mathcal{M}}_{g,n}$ which will be fundamental to applying Theorem 0.1. In this way, we will be able to define various symmetric functions on $\overline{\mathcal{M}}_{g,n}$.

Proposition 0.6. *Consider $\overline{\mathcal{M}}_{g,n}$, with $g \geq 0$ and $n > 2 - 2g$.*

(i) *If $g = 0$ and $n \geq 4$, the Cartier divisor*

$$D_1(a_1, \dots, a_n) := \sum_{i=1}^{n} a_i \psi_i - \sum_{|S| \geq 2} \delta_{0,S} \tag{0.7}$$

is ample on $\overline{\mathcal{M}}_{0,n}$ for $a_i \in \mathbb{N}$, $1 \leq i \leq n$.

(ii) *Let $g \geq 1$ and $n \geq 0$ be integers such that $2g - 2 + n > 0$. Let $a_1, \dots, a_n, b$ be positive integers such that $b > 11$. Then the $\mathbb{Q}$-divisor*

$$D_2(a_1, \dots, a_n, b) = \sum_{i=1}^{n} a_i \psi_i + b\lambda - \delta_{irr} - \sum_{i \geq 0} \sum_{S} \delta_{i,S} \tag{0.8}$$

is ample. Note that the last sum in (0.8) ranges over all pairs (i, S) such that the corresponding divisor $\delta_{i,S}$ is well defined.

Proof. (i) Clearly, for any reduced and irreducible subvariety $Y \subset \overline{\mathcal{M}}_{0,n}$, we have

$$\left(\sum_{i=1}^{n} a_i\psi_i - \sum_{|S|\geq 2} \delta_{0,S}\right)^{dim(Y)} \cdot Y \geq \left(\sum_{i=1}^{n} \psi_i - \sum_{|S|\geq 2} \delta_{0,S}\right)^{dim(Y)} \cdot Y.$$

By the Grothendieck-Riemann-Roch Theorem, it follows that $\sum_{i=1}^{n} \psi_i - \sum_{|S|\geq 2} \delta_{0,S} = \kappa_1$, which is well known to be ample on $\overline{\mathcal{M}}_{0,n}$. Thus $D_1^{dim(Y)} \cdot Y > 0$, for each $Y \subset \overline{\mathcal{M}}_{0,n}$; hence D_1 is ample by the Nakai-Moishezon criterion for ampleness.

(ii) By what was proved in [4], the divisor

$$\sum_{i=1}^{n} \psi_i + b\lambda - \delta_{irr} - \sum_{i\geq 0}\sum_{S} \delta_{i,S}$$

is ample on $\overline{\mathcal{M}}_{g,n}$ for $g \geq 1$, $n \geq 0$, $2g-2+n > 0$. Since $a_i > 0$, the claim easily follows by applying Nakai-Moishezon's criterion as in (i). □

Since $\overline{\mathcal{M}}_{g,n}$ is $\mathbb{Q}$-factorial, denote by m_i and m_λ the indices of ψ_i, $1 \leq i \leq n$, and of λ, respectively. We set

(0.9)
$$b_i := m_i a_i,\ 1 \leq i \leq n,\ \text{and}\ B := m_\lambda b.$$

For the sake of simplicity, put $\delta := \delta_{irr} + \sum_{i\geq 0}\sum_S \delta_{i,S}$. We thus define the (Weil) divisor

(0.10)
$$A_g(a_1, \dots, a_n) := \begin{cases} D_1(a_1, \dots, a_n) & \text{for } g = 0, n \geq 4, \\ \sum_{i=1}^{n} b_i\psi_i + B\lambda - 2\delta & \text{for } g \geq 1,\ n \geq 0,\ 2g-2+n > 0, \end{cases}$$

which is ample by Proposition (0.6).

On the other hand, we can also consider the Weil divisor

(0.11)
$$D_g(a_1, \dots, a_n) := \sum_{i=1}^{n} a_i(m_i\psi_i) = \sum_{i=1}^{n} b_i\psi_i$$

(note that $m_i = 1$, $1 \leq i \leq n$, when $g = 0$).

We now prove the following result (see for example [7] for an alternative approach).

Proposition 0.12. *Let $g \geq 0$ and $n > 2-2g$. Let $(a_1, \dots, a_n) \neq (0, \dots, 0)$ be non-negative integers. The divisor $D_g(a_1, \dots, a_n)$ in* (0.11) *is nef and big on $\overline{\mathcal{M}}_{g,n}$.*

Proof. Each $\mathbb{Q}$-divisor ψ_i, $1 \leq i \leq n$, is nef, as follows from [8]. Since the integers $b_i = m_i a_i$ are non-negative, $\sum_{i=1}^{n} b_i\psi_i$ is nef too. To prove that $\sum_{i=1}^{n} b_i\psi_i$ is big, we first show that each of the ψ_i's has positive top self-intersection. This is equivalent to showing that

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_i^{3g-3+n} > 0.$$

As proved by purely geometric arguments in [6], we have

(0.13)
$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_i^{3g-3+n} = \int_{\overline{\mathcal{M}}_{g,1}} \psi_i^{3g-2} = \frac{1}{g!(24)^g}$$

and

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_i^{n-3} = 1. \tag{0.14}$$

Since each ψ_i is nef, this proves that it is also big. Furthermore, we also have

$$\psi_1^{i_1} \cdot \psi_2^{i_2} \cdot \ldots \cdot \psi_n^{i_n} \geq 0 \tag{0.15}$$

when $\sum_{j=1}^n i_j = 3g - 3 + n$. Indeed, when $g = 0$ and $n \geq 3$,

$$\psi_1^{i_1} \ldots \psi_n^{i_n} = \binom{n-3}{i_1 \ldots i_n} > 0. \tag{0.16}$$

On the other hand, when $g = 1$, denote by $\pi_k : \overline{\mathcal{M}}_{1,k+1} \to \overline{\mathcal{M}}_{1,k}$, $k \geq 1$, the map which forgets the last point and contracts unstable components. By applying $\pi_{k,*}$ for $1 \leq k \leq n-1$, the push-forward of the class $\psi_1^{i_1} \ldots \psi_n^{i_n}$ is a positive multiple of the Mumford class κ_1, which is known to be ample on $\overline{\mathcal{M}}_{1,1}$. Analogously, for $g \geq 2$, by a formula due to C. Faber, (0.15) can be pushed forward to obtain a polynomial in the Mumford classes over $\overline{\mathcal{M}}_g$ with positive rational coefficients. Since Mumford classes are effective, then Formula (0.15) holds. Therefore we have

$$(\sum_{i=1}^n b_i \psi_i)^{3g-3+n} > 0.$$

Since $\sum_{i=1}^n b_i \psi_i$ is nef, it is also big. Hence the claim follows. $\square$

5. Vanishing theorems on $\overline{\mathcal{M}}_{g,n}$ and related symmetric functions

The aim of this section is to determine various symmetric functions on $\overline{\mathcal{M}}_{g,n}$. When, in particular, $g = 0$ and $n \geq 3$ we shall relate our symmetric functions with those studied in [13] (see, for example, Section).

Theorem 0.17. *Let $g \geq 0$ and $n > 2 - 2g$ be non-negative integers such that $(\overline{\mathcal{M}}_{g,n}, 0)$ is k.l.t. Let $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ be non-negative integers. Then*

$$H^q(\overline{\mathcal{M}}_{g,n}, \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(-D_g(a_1, \ldots, a_n)) = (0), \text{ for each } q < 3g - 3 + n.$$

In particular, the vanishings hold when $g = 0$ and $n \geq 4$ or when $g \geq 4$ and $n \geq 0$ (see Theorem 0.2 and Corollary 0.3).

Proof. By Proposition 0.12, the divisor $D_g(a_1, \ldots, a_n)$ is big and nef. By Theorem 0.2 and Corollary 0.3 applied to $\overline{\mathcal{M}}_{g,n}$, we can apply Theorem 0.1. $\square$

By Proposition 5.75 in [10], the dualising sheaf $\omega_{\overline{\mathcal{M}}_{g,n}}$ is isomorphic to $\mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(K_{\overline{\mathcal{M}}_{g,n}})$. Thus by Serre's duality we have the following result.

Corollary 0.18. *Under the assumptions of Theorem 0.17,*

$$\begin{aligned} \chi(K_{\overline{\mathcal{M}}_{g,n}} + D_g(a_1, \ldots, a_n)) &= h^0(K_{\overline{\mathcal{M}}_{g,n}} + D_g(a_1, \ldots, a_n)) = \\ h^{3g-3+n}(-D_g(a_1, \ldots, a_n)) &= (-1)^{3g-3+n}\chi(-D_g(a_1, \ldots, a_n)), \end{aligned} \tag{0.19}$$

where $\chi(-)$ denotes the Euler characteristic of the sheaf $-$, and $h^j(\overline{\mathcal{M}}_{g,n}, -)$ the dimensions of the cohomology vector spaces.

In particular, (0.19) *holds when $g = 0$ and $n \geq 4$ or when $g \geq 4$ and $n \geq 0$.*

Proof. Since we have

$$\chi(\mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(K_{\overline{\mathcal{M}}_{g,n}} + D_g(a_1,\ldots,a_n))) = (-1)^{3g-3+n}\chi(\mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(-D_g(a_1,\ldots,a_n))),$$

the claim follows from Theorem 3.1 and Serre's duality. □

Since the divisor $A_g(a_1,\ldots,a_n)$ in (0.10) is ample, then it is in particular nef and big. Thus, the same proof of Theorem 0.17 yields the following result.

Theorem 0.20. *Let $g \geq 0$ and $n > 2-2g$ be non negative integers such that $(\overline{\mathcal{M}}_{g,n}, 0)$ is k.l.t. Then, with the same notation adopted in* (0.10)*, we have*

$$H^q(\overline{\mathcal{M}}_{g,n}, \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(-A_g(a_1,\ldots,a_n))) = (0), \text{ for each } q < 3g-3+n.$$

In particular, the vanishings above hold when $g = 0$ and $n \geq 4$ or when $g = 0$ and $n \geq 4$.

Therefore we have the following result.

Corollary 0.21. *Under the same assumptions of Theorem 0.20,*

$$\begin{array}{ll} \chi(K_{\overline{\mathcal{M}}_{g,n}} + A_g(a_1,\ldots,a_n)) & = h^0(K_{\overline{\mathcal{M}}_{g,n}} + A_g(a_1,\ldots,a_n)) = \\ h^{3g-3+n}(-A_g(a_1,\ldots,a_n)) & = (-1)^{3g-3+n}\chi(-A_g(a_1,\ldots,a_n)), \end{array} \tag{0.22}$$

where $\chi(-)$ is the Euler characteristic of the sheaf $-$, and $h^j(\overline{\mathcal{M}}_{g,n}, -)$ the dimensions of the cohomology spaces.

In order to define some natural symmetric functions on $\overline{\mathcal{M}}_{g,n}$, one can use Corollary 0.21. Indeed, observe that the symmetric group $Sym(n)$ (or Σ_n) acts in a natural way on $\overline{\mathcal{M}}_{g,n}$ by permuting the markings.

Recall Notation 0.4. The action of $Sym(n)$ permutes the isomorphism classes of the line bundles $\mathcal{L}_i$ in the obvious manner. The divisor λ is $Sym(n)$-invariant, since it does not depend on the marked points. By definition, the divisor δ_{irr} is $Sym(n)$-invariant. Finally, even if $\delta_{i,S}$ is not $Sym(n)$-invariant, the divisor $\sum_{i>0,|S|\geq 2}\delta_{i,S}$ is. Therefore we have the following definition.

Definition 0.23. *For any n-tuple $(a_1,\ldots,a_n)$ of non-negative integers and either for $b > 11$, $g \geq 4$ and $n \geq 1$ or for $g = 0$ and $n \geq 4$, one has the symmetric function*

$$\begin{array}{rcl} \tau_n^g(a_1,\ldots,a_n) & := & h^0(\overline{\mathcal{M}}_{g,n}, \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(K_{\overline{\mathcal{M}}_{g,n}} + A_g(a_1,\ldots,a_n))) \\ & = & \chi(K_{\overline{\mathcal{M}}_{g,n}} + A_g(a_1,\ldots,a_n)). \end{array} \tag{0.24}$$

Other natural symmetric functions on $\overline{\mathcal{M}}_{g,n}$ are determined by Corollary 0.18. Indeed, we have the following.

Definition 0.25. *For any n-tuple $(a_1,\ldots,a_n) \neq (0,\ldots,0)$ of non-negative integers, and either for $g = 0$ and $n \geq 4$ or for $g \geq 4$ and $n \geq 1$, one has the symmetric function*

$$\begin{array}{rcl} \xi_n^g(a_1,\ldots,a_n) & := & h^0(\overline{\mathcal{M}}_{g,n}, \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}(K_{\overline{\mathcal{M}}_{g,n}} + D_g(a_1,\ldots,a_n))) \\ & = & \chi(K_{\overline{\mathcal{M}}_{g,n}} + D_g(a_1,\ldots,a_n)). \end{array} \tag{0.26}$$

Clearly, if σ_i, $1 \leq i \leq n$, denotes the i^{th}-elementary symmetric function, then

$$\xi_n^g(a_1,\ldots,a_n) \in \mathbb{Q}[\sigma_1,\ldots,\sigma_n], \text{ and } \tau_n^g(a_1,\ldots,a_n) \in \mathbb{Q}[\sigma_1,\ldots,\sigma_n]$$

whenever they are defined.

Remark 0.27. Let $R := \mathbb{Q}[\sigma_1, \ldots]$ be the infinite polynomial ring in the indeterminates $\{\sigma_d\}_{d\in\mathbb{N}}$. Define a grading on R by assigning to the variable σ_d the weight d. For $f \in R$, the degree of f is the highest weight of the monomials which appears in f. By direct inspection, the degree of τ_n^g and ξ_n^g is $3g - 3 + n$, since they both contain monomials of total degree $3g - 3 + n$.

5. The Symmetric Functions τ and ξ in Genus Zero

In this section, we focus on the case $g = 0$. Since $\overline{\mathcal{M}}_{0,n}$ is smooth, all the integers m_i in (0.9) are equal to 1. Thus, with the same notation adopted in (0.9), $b_i = a_i$ for each $1 \leq i \leq n$. Here we show how to compute recursive formulas for the symmetric functions $\tau_n^0(a_1, \ldots, a_n)$ in (0.23) and $\xi_n^0(a_1, \ldots, a_n)$ in (0.26). Furthermore, this will also enable us to define some symmetric functions on several effective Cartier divisors of $\overline{\mathcal{M}}_{0,n}$.

In this section, we shall work under the assumption $g = 0$ and $n \geq 3$. Since $\overline{\mathcal{M}}_{0,n}$ is smooth for each $n \geq 3$, $K_{\overline{\mathcal{M}}_{0,n}}$ is of index one. Additionally, it is possible to give a closed expression for $K_{\overline{\mathcal{M}}_{0,n}}$.

Proposition 0.28. *Let $n \geq 3$ be an integer. Then*

$$K_{\overline{\mathcal{M}}_{0,n}} = \sum_{i=1}^{n} \psi_i - 2 \sum_{|S|\geq 2} \delta_{0,S}. \tag{0.29}$$

Proof. We use induction on n. For $n = 3$, both sides of (0.29) are zero since the moduli space $\overline{\mathcal{M}}_{0,3}$ is a point. Denote by $\pi_n : \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n-1}$, $n \geq 4$, the map which forgets the last point and passes to the 'stable' model. Since π_n is a smooth morphism, then

$$K_{\overline{\mathcal{M}}_{0,n}} = \pi_n^*(K_{\overline{\mathcal{M}}_{0,n-1}}) + \psi_n - \sum_{\substack{i,n\in S\\ |S|=2}} \delta_{0,S}.$$

The result thus follows from the relation

$$\pi_n^*(\psi_i) = \psi_i - \delta_{0,\{i,n\}}.$$

□

Remark 0.30. By (0.5) and suitable relations in [9], Formula (0.29) can be further simplified (see, for instance, [14]). This expression shows more clearly the invariance of $K_{\overline{\mathcal{M}}_{0,n}}$ with respect to the action of the symmetric group, as explained in Section .

Let us now consider the ample divisor $A_0(a_1, \ldots, a_n)$ in (0.7). By (0.29), we get

$$K_{\overline{\mathcal{M}}_{0,n}} + A_0(a_1, \ldots, a_n) \sim \sum_{i=1}^{n} (a_i + 1)\psi_i - 3 \sum_{|S|\geq 2} \delta_{0,S}. \tag{0.31}$$

Therefore from Definition (0.23) we get the symmetric function

$$\tau_n^0(a_1, \ldots, a_n) = h^0(\overline{\mathcal{M}}_{0,n}, \sum_{i=1}^{n} (a_i + 1)\psi_i - 3 \sum_{|S|\geq 2} \delta_{0,S}), \tag{0.32}$$

which is defined for each $a_i > 0$, $1 \leq i \leq n$.

On the other hand, by (0.11), (0.29) and by the fact that $m_i = 1$, $1 \leq i \leq n$, we get

$$K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1, \ldots, a_n) \sim \sum_{i=1}^{n} (a_i + 1)\psi_i - 2 \sum_{|S| \geq 2} \delta_{0,S}. \tag{0.33}$$

Accordingly, from Definition (0.25) we obtain the symmetric function

$$\xi_n^0(a_1, \ldots, a_n) = h^0(\overline{\mathcal{M}}_{0,n}, \sum_{i=1}^{n} (a_i + 1)\psi_i - 2 \sum_{|S| \geq 2} \delta_{0,S}) \tag{0.34}$$

for any n-tuple $(a_i, \ldots, a_n) \neq (0, \ldots, 0)$ of integers.

It is possible to compute the ξ_n^0's via simple recursion relations, which can be derived from the geometry of $\overline{\mathcal{M}}_{0,n}$. More precisely, we have the following result.

Theorem 0.35. *The function $\xi_n^0(a_1, \ldots, a_n) \in \mathbb{Q}[\sigma_1, \ldots, \sigma_n]$ is determined by the recursive relations*

(0.36)

$$\xi_{n+1}^0(a_1, \ldots, a_n, 0) = -\xi_n^0(a_1, \ldots, a_n) + \sum_{i=1}^{n} \sum_{j=0}^{a_i - 1} \xi_n^0(a_1, \ldots, a_{i-1}, j, a_{i+1}, \ldots, a_n, a_3)$$

and the initial condition

$$\xi_3^0(a_1, a_2, a_3) = 1. \tag{0.37}$$

Proof. First of all, note that the initial condition (0.37) holds since $\overline{\mathcal{M}}_{0,3}$ is a point. Next, observe that, by Serre's duality,

$$\begin{aligned} \xi_n^0(a_1, \ldots, a_n) &= h^0(\overline{\mathcal{M}}_{0,n}, K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1, \ldots, a_n)) \\ &= h^{n-3}(\overline{\mathcal{M}}_{0,n}, -D_0(a_1, \ldots, a_n)) \\ &= (-1)^{n-3} \chi_n(-D_0(a_1, \ldots, a_n)), \end{aligned}$$

where $\chi_n(-D_0(a_1, \ldots, a_n))$ denotes the Euler characteristic of the sheaf associated with $-D_0(a_1, \ldots, a_n)$. Our strategy is similar to that in [13]. Indeed, to obtain (0.36), we compare the symmetric functions ξ_n^0 and ξ_{n+1}^0. By Remark 0.27, ξ_n^0 is a symmetric function of degree at most $n-3$. The map from the ring of symmetric functions in $a_1, \ldots, a_n$ to the ring of symmetric functions in $a_1, \ldots, a_{n-1}$ is bijective for symmetric functions of degree at most $n-1$. Hence $\xi_n^0(a_1, \ldots, a_n)$ is completely determined by $\xi_n^0(a_1, \ldots, a_{n-1}, 0)$.

Let us denote by $\mathcal{L}_{i,n}$ the line bundle corresponding to the divisors ψ_i on $\overline{\mathcal{M}}_{0,n}$. By $\pi_{n+1} : \overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ we denote the map which forgets the last marked point and passes to the 'stable' model. We recall that

$$\pi_{n+1}^*(\mathcal{L}_{i,n}) \otimes \mathcal{O}\left(\delta_{0,\{i,n+1\}}\right) \cong \mathcal{L}_{i,n+1}, \ i = 1, \ldots, n. \tag{0.38}$$

In particular, since $\delta_{0,\{i,n+1\}}\delta_{0,\{j,n+1\}} = 0$, $i \neq j$, the restrictions of $\mathcal{L}_{i,n+1}$ and $\pi_{n+1}^*(\mathcal{L}_{i,n})$ to $\delta_{0,\{i,n+1\}}$ are isomorphic. On the other hand, note that $\mathcal{L}_{i,n+1}$ is trivial when restricted to $\delta_{0,\{i,n+1\}}$, whereas $\pi_{n+1}^*(\mathcal{L}_{i,n})$ is not. By what we recalled above, if we tensor the exact sequence

$$0 \to \mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(-\delta_{0,\{i,n+1\}}) \to \mathcal{O}_{\overline{\mathcal{M}}_{0,n}} \to \mathcal{O}_{\delta_{0,\{i,n+1\}}} \to 0$$

by

$$\pi_{n+1}^*(\mathcal{L}_{i,n}^\vee)^{b+1} \otimes (\mathcal{L}_{i,n+1}^\vee)^a$$

and we take the duals in (0.38), we obtain the exact sequence:

$$0 \to \pi_{n+1}^*(\mathcal{L}^{\vee b}{}_{i,n}) \otimes \mathcal{L}^{\vee a+1}{}_{i,n+1} \to \pi_{n+1}^*(\mathcal{L}^{\vee b+1}{}_{i,n+1}) \otimes \mathcal{L}^{\vee a}{}_{i,n+1} \to \pi_{n+1}^*(\mathcal{L}^{\vee a}{}_{i,n}) \otimes \mathcal{O}_{\delta_{0,\{i,n+1\}}} \to 0,$$

where a, b are non-negative integers. By repeated application of the last exact sequence, we have (for the sake of simplicity we switch to divisor notation):

$$\chi_{n+1}(-D_0(a_1,\ldots,a_n)) = \chi_{n+1}(-\pi^*_{n+1}(D_0(a_1,\ldots,a_n))) \tag{0.39}$$

$$-\sum_i \sum_{j=0}^{a_i-1} \chi_{n+1}\left(-\pi^*_{n+1}(a_1\psi_1+\ldots a_{i-1}\psi_{i-1}+j\psi_i+\ldots+a_n\psi_n)\otimes\mathcal{O}_{\delta_{0,\{i,n+1\}}}\right).$$

Since $\pi^*_{n+1}(\mathcal{O}_{\overline{\mathcal{M}}_{0,n-1}}) \cong \mathcal{O}_{\overline{\mathcal{M}}_{0,n}}$, Leray's isomorphism and Theorem 0.1 yield

$$\chi_{n+1}(-\pi^*_{n+1}(\sum_{i=1}^n a_i\psi_i)) = \chi_n(-(\sum_{i=1}^n a_i\psi_i)) = (-1)^{n-3}\xi^0_n(a_1,\ldots,a_n).$$

Analogously,

$$\chi_{n+1}\left(-\pi^*_{n+1}(a_1\psi_1+\ldots a_{i-1}\psi_{i-1}+j\psi_i+\ldots+a_n\psi_n)\otimes\mathcal{O}_{\delta_{0,\{i,n+1\}}}\right)$$
$$= (-1)^{n-3}\xi^0_n(a_i,\ldots,a_{i-1},j,\ldots,a_n).$$

Thus the claim follows since

$$\chi_{n+1}(-\sum_{i=1}^n a_i\psi_i) = (-1)^{n-2}\xi^0_n(a_1,\ldots,a_n,0).$$

□

Remark 0.40. So far, the functions $\xi^0_n(a_1,\ldots,a_n)$ have been defined for non-negative integers. However, by the Grothendieck-Riemann-Roch Theorem, we remark that

$$\begin{aligned} h^0(\overline{\mathcal{M}}_{0,n}, K_{\overline{\mathcal{M}}_{0,n}} + \textstyle\sum_{i=1}^n a_i\psi_i) &= \int_{\overline{\mathcal{M}}_{0,n}} e^{K_{\overline{\mathcal{M}}_{0,n}}+\sum_{i=1}^n a_i\psi_i} Td(\overline{\mathcal{M}}_{0,n}) \\ &= \xi^0_n(a_1,\ldots,a_n). \end{aligned} \tag{0.41}$$

By the second equality in (0.41), the functions ξ^0_n can be extended to any n-tuple $a_1,\ldots,a_n$ of real numbers.

This allows us to relate the functions $\xi^0_n(a_1,\ldots,a_n)$ to the symmetric polynomials γ_n in [13], which are defined as follows:

$$\gamma_n(a_1,\ldots,a_n) := \chi(\overline{\mathcal{M}}_{0,n}, \sum_{i=1}^n a_i\psi_i),$$

where $(a_1,\ldots,a_n)$ is an arbitrary n-tuple of non-negative integers $(a_1,\ldots,a_n)$. The next result will show that functions γ_n in the genus-zero case, studied in [13], are particular cases of the more general family of symmetric functions $\xi^g_n(a_1,\ldots,a_n)$, defined for any genus $g \geq 0$.

More precisely, the following result holds.

Theorem 0.42. *For any n-tuple of integers $(a_1,\ldots,a_n) \neq (0,\ldots,0)$, $n \geq 3$, we have*

$$\xi^0_n(a_1,\ldots,a_n) = (-1)^{n-3}\gamma_n(-a_1,\ldots,-a_n).$$

Proof. By Serre's duality and by definition of the Euler characteristic of a sheaf, we have

$$\begin{aligned} \xi^0_n(a_1,\ldots,a_n) = \ & h^0(\overline{\mathcal{M}}_{0,n}, K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1,\ldots,a_n)) \\ &= h^{n-3}(\overline{\mathcal{M}}_{0,n}, -a_1\psi_1-\ldots-a_n\psi_n) \\ &= (-1)^{n-3}\chi(-a_1\psi_1-\ldots-a_n\psi_n) \\ &= (-1)^{n-3}\gamma_n(-a_1,\ldots,-a_n). \end{aligned}$$

□

Theorem 0.42 shows that the symmetric polynomials $\xi_n^0(a_1,\ldots,a_n)$ can be evaluated for any n-tuple of integers, thus extending the definition of the symmetric polynomials studied in [13].

It is also possible to compute the symmetric polynomials $\tau_n^0(a_1,\ldots,a_n)$ in terms of the ξ_n^0's. To this end, we express the boundary δ as a linear combination of the ψ_i's.

Proposition 0.43. *Let $n \geq 5$ be an integer and consider the moduli space $\overline{\mathcal{M}}_{0,n}$. There exist rational numbers c_i, $1 \leq i \leq n$, such that*

$$\delta = \sum_{i=1}^{n} c_i \psi_i, \tag{0.44}$$

where δ is the boundary divisor of $\overline{\mathcal{M}}_{0,n}$.

Proof. We will prove the proposition by showing how to compute the numbers c_i. If we multiply (0.44) by ψ_k^{n-4}, $1 \leq k \leq n$, we obtain a system of n equations, namely

$$\psi_k^{n-4}\delta = c_1\psi_1\psi_k^{n-4} + \ldots c_{k-1}\psi_{k-1}\psi_k^{n-4} + c_k\psi_k^{n-3} + \ldots + c_n\psi_n\psi_k^{n-4},\ 1 \leq k \leq n. \tag{0.45}$$

By (0.16), the intersection numbers ψ_k^{n-3} and $\psi_j\psi_k^{n-4}$ equal 1 and $n-3$ respectively. The coefficient matrix of (0.45) is thus equal to

$$\begin{pmatrix} 1 & n-3 & \ldots & n-3 \\ n-3 & 1 & \ldots & n-3 \\ \ldots & \ldots & \ldots & \ldots \\ n-3 & n-3 & \ldots & 1 \end{pmatrix}.$$

It is easy to check that the determinant of such a matrix is non-singular. By dimensional computations, the intersection number $\psi_k^{n-4}\delta$ is 1 only on the boundary components $\delta_{0,\{i,j\}}$, where i and j are indices in $\{1,\ldots,n\}$ other than k. Since there are $\binom{n-1}{2}$ such components, the system (0.45) is non-homogeneous with non-singular matrix coefficients. Therefore Cramer's rule allows computing the numbers c_i's explicitly. □

We can finally show the relation between the symmetric functions τ_n^0 and ξ_n^0.

Theorem 0.46. *Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of n-pointed genus zero stable curves. Then*
(1) $\tau_3^0(a_1,a_2,a_3) = 1$;
(2) $\tau_4^0(a_1,a_2,a_3,a_4) = \sum_{i=1}^n a_i - 4$;
(3) $\tau_n^0(a_1,\ldots,a_n) = \xi_n^0(a_1-c_1,\ldots,a_n-c_n)$, $n \geq 5$, where c_i, $1 \leq i \leq n$, are the numbers computed in Proposition 0.43.

Proof. (1) and (2) follow from direct inspection, since $\overline{\mathcal{M}}_{0,3}$ is a point and $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$. As for (3), note that

$$\tau_n^0(a_1,\ldots,a_n) = h^0(K_{\overline{\mathcal{M}}_{0,n}} + \sum_{i=1}^n a_i\psi_i - \delta) = \int_{\overline{\mathcal{M}}_{0,n}} e^{K_{\overline{\mathcal{M}}_{0,n}} + \sum_{i=1}^n a_i\psi_i - \delta} Td(\overline{\mathcal{M}}_{0,n}).$$

By (0.44) and Remark 0.40, the claim follows. □

6. Symmetric functions on effective divisors of $\overline{\mathcal{M}}_{0,n}$ defined via those existing on $\overline{\mathcal{M}}_{0,n}$

In this section we show how to define symmetric functions on effective Cartier divisors of $\overline{\mathcal{M}}_{0,n}$ starting from $\xi_n^0(a_1,\ldots,a_n)$. Analogous computations hold for $\tau_n^0(a_1,\ldots,a_n)$.

Case 1 Take $A_0(a_1,\ldots,a_n) = \sum_{i=1}^n a_i\psi_i$, where $(a_1,\ldots,a_n) \neq (0,\ldots,0)$ are non-negative integers. Assume there exist at least two indices $1 \leq k \neq h \leq n$ such that $a_k, a_h \geq 1$. Take such indices and consider the divisor ψ_k which is an effective Cartier divisor - see (0.5). Now, tensor the exact sequence defining ψ_k by $\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1,\ldots,a_n)) = \mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\sum_{i=1}^n (a_i+1)\psi_i - 2\sum_{|S|\geq 2}\delta_{0,S})$. If we apply the adjunction formula on ψ_k, we get

$$
\begin{array}{rl}
0 \to \mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\sum_{i\neq k}(a_i+1)\psi_i + a_k\psi_k - 2\sum_{|S|\geq 2}\delta_{0,S}) & \longrightarrow \\
\to \mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1,\ldots,a_n)) & \longrightarrow \\
\to \mathcal{O}_{\psi_k}(K_{\psi_k} + \sum_{i\neq k}^n (a_i\psi_i + (a_k-1)\psi_k) & \to 0,
\end{array} \tag{0.47}
$$

since, by hypothesis, $a_k \geq 1$. Observe that

$$\sum_{i\neq k}(a_i+1)\psi_i + a_k\psi_k - 2\sum_{|S|\geq 2}\delta_{0,S} = K_{\overline{\mathcal{M}}_{0,n}} + (\sum_{i\neq k} a_i\psi_i) + (a_k-1)\psi_k.$$

Indeed, by assumption, $a_h \geq 1$, and the n-tuple $(a_1,\ldots,a_k-1,\ldots,a_h,\ldots,a_n) \neq (0,\ldots,0)$ is composed of non-negative integers. Therefore, from the Kodaira Vanishing Theorem, it follows that

$$H^j(\sum_{i\neq k}(a_i+1)\psi_i + a_k\psi_k - 2\sum_{|S|\geq 2}\delta_{0,S}) = H^j(K_{\overline{\mathcal{M}}_{0,n}} + D_0(a_1,\ldots,a_n)) = (0),\ \forall\, j>0.$$

Thus, we have

$$H^j(\mathcal{O}_{\psi_k}(K_{\psi_k} + \sum_{i\neq k} a_i\psi_i + (a_k-1)\psi_k)) = (0),\ \forall\, j>0.$$

Therefore, by the Hirzebruch-Riemann-Roch formula, we get the following claim.

Claim 0.48. *Let $(a_1,\ldots,a_n) \neq (0,\ldots,0)$ be non-negative integers, such that $a_k, a_h \geq 1$, for some $1 \leq k \neq h \leq n$. Let ψ_k be the effective Cartier divisor on $\overline{\mathcal{M}}_{0,n}$ (see Formula (0.5)). Then*

$$
\begin{array}{ll}
\zeta_{\psi_k}(a_1,\ldots,a_k-1,\ldots,a_h,\ldots,a_n) & := \\
h^0(\mathcal{O}_{\psi_k}(K_{\psi_k} + \sum_{i\neq k} a_i\psi_i + (a_k-1)\psi_k)) & = \\
\chi(\mathcal{O}_{\psi_k}(K_{\psi_k} + \sum_{i\neq k} a_i\psi_i + (a_k-1)\psi_k)), &
\end{array} \tag{0.49}
$$

is a symmetric function on ψ_k, which is defined for any n-tuple $(a_1,\ldots,a_k-1,\ldots,a_n) \neq (0,\ldots,0)$ of non-negative integers such that $a_h \geq 1$, for some index $h \neq k$. Furthermore, $\zeta_{\psi_k}(a_1,\ldots,a_k-1,\ldots,a_h,\ldots,a_n)$ is obtained from the difference of two symmetric functions on the ambient space $\overline{\mathcal{M}}_{0,n}$.

More precisely, we have that

$$\zeta_{\psi_k}(a_1,\ldots,a_k-1,\ldots,a_h,\ldots,a_n) = \xi_n^0(a_1,\ldots,a_n) - \xi_n^0(a_1,\ldots,a_k-1,\ldots,a_n),$$

where $a_k \geq 1$ and $a_h \geq 1$ for some index $h \neq k$.

Example. Consider the moduli space $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. Since each $\mathcal{L}_i \cong \mathcal{O}_{\mathbb{P}^1}(1)$, take, for example, $k=1$ and $h=2$; this means that $a_1, a_2 \geq 1$ and $a_3, a_4 \geq 0$. Therefore from our computations we have

$$
\begin{array}{rcl}
\xi_4^0(a_1,a_2,a_3,a_4) & = & h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(a_1+a_2+a_3+a_4)) \\
& = & a_1+a_2+a_3+a_4-1 \\
& = & \sigma_1(a_1,a_2,a_3,a_4) - 1,
\end{array}
$$

whereas

$$\begin{aligned}\xi^0_4(a_1-1,a_2,a_3,a_4) &= h^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-2)\otimes\mathcal{O}_{\mathbb{P}^1}(a_1-1+a_2+a_3+a_4))\\ &= a_1+a_2+a_3+a_4-2\\ &= \sigma_1(a_1,a_2,a_3,a_4)-2,\end{aligned}$$

where $a_1, a_2 \geq 1$. Furthermore,

$$\zeta_{\psi_1}(a_1,a_2,a_3,a_4)=1$$

is constant.

Case 2 Let us now introduce a different symmetric function. Take $D_0(a_1,\dots,a_n) = \sum_{i=1}^n a_i\psi_i$ and $D_0(a_1+1,\dots,a_n+1) = \sum_{i=1}^n (a_i+1)\psi_i$, where $(a_1,\dots,a_n)\neq(0,\dots,0)$ are non-negative integers. From the fact that each $\delta_{0,S}$ is a subvariety in $\overline{\mathcal{M}}_{0,n}$, $\sum_{|S|\geq 2}\delta_{0,S}$ is effective, since it is a sum of effective Weil divisors.

Take $D\sim 2\sum_{|S|\geq 2}\delta_{0,S}$ as an effective Cartier divisor on $\overline{\mathcal{M}}_{0,n}$. Therefore, if we tensor the exact sequence defining D by $\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\sum_{i=1}^n(a_i+1)\psi_i)$, we get

$$\begin{aligned} 0\to\ &\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\textstyle\sum_{i=1}^n(a_i+1)\psi_i - 2\sum_{|S|\geq 2}\delta_{0,S}) &&\to\\ \to\ &\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\textstyle\sum_{i=1}^n(a_i+1)\psi_i)\to\mathcal{O}_D(\sum_{i=1}^n(a_i+1)\psi_i) &&\to 0.\end{aligned}\tag{0.50}$$

By (0.33), we have

$$\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(\sum_{i=1}^n(a_i+1)\psi_i-2\sum_{|S|\geq 2}\delta_{0,S})\cong\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(K_{\overline{\mathcal{M}}_{0,n}}+D_0(a_1,\dots,a_n)).$$

By Proposition 1 in [13], we observe that

$$H^j(\overline{\mathcal{M}}_{0,n},\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(D_0(a_1+1,\dots,a_n+1)))=(0),\ \forall\, j>0.$$

Moreover, by the usual Kodaira Vanishing Theorem, we get

$$H^j(\overline{\mathcal{M}}_{0,n},\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(K_{\overline{\mathcal{M}}_{0,n}}+D_0(a_1,\dots,a_n)))=(0),\ \forall\, j>0.$$

This implies that

$$H^j(\overline{\mathcal{M}}_{0,n},\mathcal{O}_D(D_0(a_1+1,\dots,a_n+1)))=(0),\ \forall\, j>0.$$

Therefore, by the Hirzebruch-Riemann-Roch formula, we get the following claim.

Claim 0.51. *Let $(a_1,\dots,a_n)\neq(0,\dots,0)$ be non-negative integers and let $D\sim 2\sum_{|S|\geq 2}\delta_{0,S}$ be an effective, Cartier divisor on $\overline{\mathcal{M}}_{0,n}$. Then*

$$\begin{aligned}&\zeta_D(a_1+1,\dots,a_n+1) &&:=\\ &h^0(\mathcal{O}_D(D_0(a_1+1,\dots,a_n+1))) &&=\\ &\chi(\mathcal{O}_D(D_0(a_1+1,\dots,a_n+1))&&\end{aligned}\tag{0.52}$$

is a symmetric function on D, which is defined for any n-tuple $(a_1+1,\dots,a_n+1)\neq(1,\dots,1)$ of positive integers. Furthermore, $\zeta_D(a_1+1,\dots,a_n+1)$ is obtained from the difference of two symmetric functions on the ambient space $\overline{\mathcal{M}}_{0,n}$.

More precisely, if $\gamma_n(a_1,\dots,a_n)$ denotes the symmetric function in [13], then we have

$$\zeta_D(a_1+1,\dots,a_n+1)=\gamma_n(a_1+1,\dots,a_n+1)-\xi^0_n(a_1,\dots,a_n),$$

for any n-tuple $(a_1,\dots,a_n)\neq(0,\dots,0)$ of non-negative integers.

Finally, from the previous discussion, it easily follows that

$$\xi^0_n(a_1,\dots,a_n)\leq\gamma_n(a_1+1,\dots,a_n+1)$$

for any $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ n-tuple of non-negative integers.

Example. As before, we give a concrete example in the case of $\overline{\mathcal{M}}_{0,4}$. First, note that $\mathcal{L}_i \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and $\mathcal{O}_{\overline{\mathcal{M}}_{0,4}}(D) \cong \mathcal{O}_{\mathbb{P}^1}(6)$. Therefore we have

$$\begin{aligned} \xi_4^0(a_1, a_2, a_3, a_4) &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(a_1 + a_2 + a_3 + a_4)) \\ &= a_1 + a_2 + a_3 + a_4 - 1 \\ &= \sigma_1(a_1, a_2, a_3, a_4) - 1, \end{aligned}$$

whereas $\gamma_4(a_1{+}1, a_2{+}1, a_3{+}1, a_4{+}1) = 5{+}\sigma_1(a_1, a_2, a_3, a_4)$. Furthermore, $\zeta_D(a_1, a_2, a_3, a_4) = 6$ is constant.

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