SPECIAL SCROLLS WHOSE BASE CURVE HAS GENERAL MODULI

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Abstract. In this paper we study the Hilbert scheme of smooth, linearly normal, special scrolls under suitable assumptions on degree, genus and speciality.

1. Introduction

The classification of scroll surfaces in a projective space is a classical subject. Leaving aside the prehistory, the classical reference is C. Segre [29]. In this paper, Segre sets the foundations of the theory and proves a number of basic results concerning the family of unisecant curves of sufficiently general scrolls. Though important, Segre's results are far from being exhaustive and even satisfactory. This was already clear to F. Severi who, in [31], makes some criticism to some points of Segre's treatment, poses a good number of interesting questions concerning the classification of families of scrolls in a projective space, and proves some partial results on this subject. In particular, in the last two or three sections of his paper, Severi focuses on the study of what today we call the *Hilbert scheme* of scrolls and the related map to the moduli space of curves. Severi's paper does not seem to have received too much attention in modern times. By contrast Segre's approach has been reconsidered and improved in recent times by various authors (for a non comprehensive list, see the references).

The present paper has to be seen as a continuation of [7], [8] and [9]. In [7], inspired by some papers of G. Zappa (cf. [33], [34]) in turn motivated by Severi, we studied the Hilbert scheme of linearly normal, non-special scrolls and some of their degenerations. Recall that a scroll S is said to be *special* if $h^1(\mathcal{O}_S(1)) > 0$ and *non-special* otherwise (cf. Definition 3.1); the number $h^1(\mathcal{O}_S(1))$ is called the *speciality* of the scroll. The general linearly normal, non-special scroll corresponds to a stable, rank-two vector bundle on a curve with general moduli (see [8], cf. also [3]). This and the degeneration techniques in [7] enabled us to reconstruct and improve some enumerative results of Segre, reconsidered also by Ghione (cf. [20]). The same techniques turn out to be very effective in studying the Brill-Noether theory of sub-line bundles of a stable, rank-two vector bundle on a curve with general moduli, which translates into the study of families of unisecants of the corresponding scroll (cf. [9]).

In the present paper, we focus on the study of the Hilbert scheme of linearly normal, special scrolls. In contrast with what happens for non-special scrolls, the special ones, in the ranges considered by us for the degree and genus, always correspond to unstable — even sometimes decomposable — vector bundles (cf. Proposition 3.18 and Remark 5.25, cf. also [17]).

Our main result is Theorem 6.1, in which we give, under suitable assumptions on degree d, genus g and speciality h^1, the full description of all components of the Hilbert scheme of smooth, linearly normal, special scrolls whose base curve has general moduli. With a slight abuse of terminology, we will call such scrolls, *scrolls with general moduli*. Similarly, we will talk about *scrolls with special moduli*.

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All components of the Hilbert scheme of special scrolls with general moduli turn out to be generically smooth and we compute their dimensions. If the speciality is 1, there is a unique such component. Otherwise, there are several components, each one determined by the minimum degree m of a section contained by the general surface of the component. If the speciality is 2, all the components have the same dimension; if the speciality is larger than 2, the components have different dimensions depending on m: the larger m is the smaller is the dimension of the component. Sections 4, 5 and 6 are devoted to the proof of this theorem under suitable assumptions on d.

These results rely on the existence of a unique special section, which is the section of minimal degree and of the same speciality of the scroll in the range of interest for us. If $d \geq 4g - 2$, this is one of the main results by Segre in [29], which relies on a smart projective geometric argument, and could be easily exposed in modern terminology. However, we will use the more recent results in [18] and [19], which ensure the existence and uniqueness of the special section in a wider range than Segre's.

Section 7 contains further remarks on the Hilbert schemes. First we prove that in most cases projections of linearly normal, special scrolls to lower dimensional projective spaces fill up, rather unexpectedly, irreducible components of the Hilbert schemes (see Proposition 7.1; see also [5]). Moreover, we construct irreducible components of the Hilbert scheme parametrizing scrolls with special moduli (cf. Example 7.8). Our construction provides all components of the Hilbert scheme of scrolls of speciality 2, regardless to their number of moduli (cf. Proposition 7.13). Finally, in Propositions 7.15 and 7.17, we provide examples of smooth, linearly normal, special scrolls which correspond to singular points of their Hilbert schemes (cf. also Remark 5.23).

There are, in our opinion, three main subjects of interest in this field which have not been treated in this paper: (i) degenerations, (ii) Brill-Noether theory, (iii) special scrolls of *very low* degree, in particular special scrolls corresponding to stable vector bundles. We believe that the first two subjects can be attacked with techniques similar to the ones in [7], [8] and [9]. We hope to come back to this in the future. The last question is very intriguing and certainly requires an approach different from the one in this paper. Related results can be found in [32] (cf. also the expository paper [22]).

2. Notation and preliminaries

Let C be a smooth, irreducible, projective curve of genus g and let $F \xrightarrow{\rho} C$ be a *geometrically ruled surface* on C, i.e. $F = \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a rank-two vector bundle, equivalently a locally free sheaf, on C. We will set $d := \deg(\mathcal{F}) = \deg(\det(\mathcal{F}))$.

As in [8], we shall make the following:

Assumptions 2.1. *With notation as above,*

(1) $h^0(C, \mathcal{F}) = R + 1$, *with* $R \geq 3$;
(2) *the complete linear system* $|\mathcal{O}_F(1)|$ *is base-point-free and the morphism* $\Phi : F \to \mathbb{P}^R$ *induced by* $|\mathcal{O}_F(1)|$ *is birational to its image.*

Definition 2.2. The surface
$$\Phi(F) := S \subset \mathbb{P}^R$$
is said to be a *scroll of degree* d *and of (sectional) genus* g on C and F is its minimal desingularization; S is called the *scroll determined by the pair* $(\mathcal{F}, C)$. Note that S is smooth if and only if $\mathcal{F}$ is very ample.

For any $x \in C$, let $f_x := \rho^{-1}(x) \cong \mathbb{P}^1$; the line $l_x := \Phi(f_x)$ is called a *ruling* of S. Abusing terminology, the family $\{l_x\}_{x \in C}$ is also called the *ruling* of S.

For further details on ruled surfaces and on their projective geometry, we refer the reader to [21], [23, § V], and the references in [8].

Let $F \xrightarrow{\rho} C$ be as above. Then, there is a section $i : C \hookrightarrow F$, whose image we denote by H, such that $\mathcal{O}_F(H) = \mathcal{O}_F(1)$. Then

$$\mathrm{Pic}(F) \cong \mathbb{Z}[\mathcal{O}_F(H)] \oplus \rho^*(\mathrm{Pic}(C)).$$

Moreover,

$$\mathrm{Num}(F) \cong \mathbb{Z} \oplus \mathbb{Z},$$

generated by the classes of H and f, satisfying $Hf = 1$, $f^2 = 0$ (cf. [23, § 5, Prop. 2.3]). If $\underline{d} \in \mathrm{Div}(C)$, we will denote by $\underline{d}f$ the divisor $\rho^*(\underline{d})$ on F. A similar notation will be used when $\underline{d} \in \mathrm{Pic}(C)$. Any element of $\mathrm{Pic}(F)$ corresponds to a divisor on F of the form

$$nH + \underline{d}f, \; n \in \mathbb{Z}, \; \underline{d} \in \mathrm{Pic}(C).$$

As an element of $\mathrm{Num}(F)$ this is

$$nH + df, \; n \in \mathbb{Z}, \; d = \deg(\underline{d}) \in \mathbb{Z}.$$

We will denote by $\sim$ the linear equivalence and by $\equiv$ the numerical equivalence.

Definition 2.3. For any $\underline{d} \in \mathrm{Div}(C)$ such that $|H + \underline{d}f| \neq \emptyset$, any $B \in |H + \underline{d}f|$ is called a *unisecant curve* to the fibration $F \xrightarrow{\rho} C$ (or simply of F). Any irreducible unisecant curve B of F is smooth and is called a *section* of F.

There is a one-to-one correspondence between sections B of F and surjections $\mathcal{F} \twoheadrightarrow L$, with $L = L_B$ a line bundle on C (cf. [23, § V, Prop. 2.6 and 2.9]). Then, one has an exact sequence

$$0 \to N \to \mathcal{F} \to L \to 0, \tag{2.4}$$

where N is a line bundle on C. If $L = \mathcal{O}_C(\underline{m})$, with $\underline{m} \in \mathrm{Div}^m(C)$, then $m = HB$ and $B \sim H + (\underline{m} - \det(\mathcal{F}))f$.

For example, if $B \in |H|$, the associated exact sequence is

$$0 \to \mathcal{O}_C \to \mathcal{F} \to \det(\mathcal{F}) \to 0,$$

where the map $\mathcal{O}_C \hookrightarrow \mathcal{F}$ gives a global section of $\mathcal{F}$ corresponding to the global section of $\mathcal{O}_F(1)$ vanishing on B.

With B and F as in Definition 2.3, from (2.4), one has

$$\mathcal{O}_B(B) \cong N^\vee \otimes L \tag{2.5}$$

(cf. [23, § 5]). In particular,

$$B^2 = \deg(L) - \deg(N) = d - 2\deg(N) = 2m - d.$$

One has a similar situation if B_1 is a (reducible) unisecant curve of F such that $HB_1 = m$. Indeed, there is a section $B \subset F$ and an effective divisor $\underline{a} \in \mathrm{Div}(C)$, $a := \deg(\underline{a})$, such that

$$B_1 = B + \underline{a}f,$$

where $BH = m - a$. In particular there is a line bundle $L = L_B$ on C, with $\deg(L) = m - a$, fitting in the exact sequence (2.4). Thus, B_1 corresponds to the exact sequence

$$0 \to N \otimes \mathcal{O}_C(-\underline{a}) \to \mathcal{F} \to L \oplus \mathcal{O}_{\underline{a}} \to 0 \tag{2.6}$$

(for details, cf. [8]).

Definition 2.7. Let S be a scroll of degree d and genus g corresponding to the pair $(\mathcal{F}, C)$. Let $B \subset F$ be a section and L as in (2.4). Let $\Gamma := \Phi(B) \subset S$. If $\Phi|_B$ is birational to its image, then Γ is called a *section* of the scroll S. If $\Phi|_B$ is finite of degree n to its image, then Γ is called a *n-directrix* of S, and the general point of Γ has multiplicity at least n for S (cf. e.g. [17, Def. 1.9]). If $\Phi|_B$ is not finite, then S is a *cone*.

We will say that the pair (S, Γ) is *associated with* (2.4) and that Γ *corresponds to the line bundle* L on C. If $m = \deg(L)$, then $m = nh$ and Γ has degree h; indeed, the map $\Phi|_B$ is determined by the linear series $\Lambda \subseteq |L|$, which is the image of the map

$$H^0(\mathcal{F}) \to H^0(L).$$

Similar terminology can be introduced for reducible unisecant curves.

When $g = 0$ we have rational scrolls and these are well-known (see e.g. [21]). Thus, from now on, we shall focus on the case $g \geq 1$.

Since ruled surfaces and scrolls are the projective counterpart of the theory of rank-two vector bundles, we finish this section by recalling some basic terminology on vector bundles. For details, we refer the reader to e.g. [28] and [30].

Let $\mathcal{E}$ be a vector bundle of rank $r \geq 1$ on C. The *slope* of $\mathcal{E}$, denoted by $\mu(\mathcal{E})$, is defined as

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{r}. \tag{2.8}$$

A rank-two vector bundle $\mathcal{F}$ on C is said to be *indecomposable*, if it cannot be expressed as a direct sum $L_1 \oplus L_2$, for some $L_i \in \mathrm{Pic}(C)$, $1 \leq i \leq 2$, and *decomposable* otherwise. Furthermore, $\mathcal{F}$ is said to be:

- *semistable*, if for any sub-line bundle $N \subset \mathcal{F}$, $\deg(N) \leq \mu(\mathcal{F})$;
- *stable*, if for any sub-line bundle $N \subset \mathcal{F}$, $\deg(N) < \mu(\mathcal{F})$;
- *strictly semistable*, if it is semistable and there is a sub-line bundle $N \subset \mathcal{F}$ such that $\deg(N) = \mu(\mathcal{F})$;
- *unstable*, if there is a sub-line bundle $N \subset \mathcal{F}$ such that $\deg(N) > \mu(\mathcal{F})$. In this case, N is called a *destabilizing* sub-line bundle of $\mathcal{F}$.

3. Preliminary results on scrolls

In this section, we recall some preliminary results concerning scrolls of degree d and genus $g \geq 1$ (cf. [29], [16], [17] and [8, § 3]). We will keep the notation introduced in § 2.

If K_F denotes a canonical divisor of F, one has

$$K_F \sim -2H + (\omega_C \otimes \det(\mathcal{F}))f,$$

hence

$$K_F \equiv -2H + (d + 2g - 2)f.$$

From Serre duality and the Riemann-Roch theorem, we have:

$$R + 1 := h^0(\mathcal{O}_F(1)) = d - 2g + 2 + h^1(\mathcal{O}_F(1)).$$

Definition 3.1. The non-negative integer $h^1(\mathcal{O}_F(1))$ is called the *speciality* of the scroll and will be denoted by $h^1(F)$, or simply by h^1, if there is no danger of confusion. Thus

$$R = d - 2g + 1 + h^1, \tag{3.2}$$

and the pair $(\mathcal{F}, C)$ determines $S \subset \mathbb{P}^R$ as a *linearly normal scroll of degree d, genus g and speciality h^1*. Such a scroll S is said to be *special* if $h^1 > 0$, *non-special* otherwise.

This definition coincides with the classical one given by Segre in [29, § 3, p. 128]: notice that Segre denotes by n the degree of the scroll, by p the sectional genus and by $i := g - h^1$. Obviously, a scroll S determined by a pair $(\mathcal{F}, C)$ is special if $\iota := h^1(C, \det(\mathcal{F})) > 0$. We will call such scrolls *strongly special scrolls*. In this paper, we will restrict our attention to special scrolls of degree strictly larger than $2g - 2$, which are therefore not strongly special.

Since $R \geq 3$, then $d \geq 2g + 2 - h^1$. One has an upper-bound for the speciality (cf. [29, § 14], [20]).

Lemma 3.3. *In the above setting, if* $\det(\mathcal{F})$ *is non-special, then* $h^1 \leq g$ *and, if* $d \geq 2g + 2$*, the equality holds if and only if* $\mathcal{F} = \mathcal{O}_C \oplus L$*, in which case* $\Phi = \Phi_{|\mathcal{O}_F(1)|}$ *maps* F *to a cone* S *over a projectively normal curve of degree* d *and genus* g *in* $\mathbb{P}^{d-g}$.

For a proof, the reader is referred to [8, Lemma 3.5].

Remark 3.4. It is not difficult to give similar bounds for the speciality of strongly special scrolls. Since for such a scroll $3 \leq R = d - 2g + 1 + h^1$ and $d \leq 2g - 2$, one has $h^1 \geq 4$.

If $\rho : H^0(C, \mathcal{F}) \to H^0(C, \det(\mathcal{F}))$ is the natural restriction map, then $h^1 = \iota + g - \dim(\mathrm{Coker}\,(\rho))$.

Using cones, one sees that every admissible value of h^1 can be obtained, in case ρ is surjective.

From (3.2) and from Lemma 3.3, we have

$$d - 2g + 1 \leq R \leq d - g + 1, \tag{3.5}$$

where the upper-bound is realized by cones if $d \geq 2g+2$, whereas the lower-bound is attained by non-special scrolls (cf. [7, Theorem 5.4]). Any intermediate value of h^1 can be realized, e.g. by using decomposable vector bundles (see, [29, pp. 144-145] and [8, Example 3.7]).

Definition 3.6. Let $\Gamma \subset S$ be a section associated to (2.4). Then:

(i) Γ is said to be *special* if $h^1(C, L) > 0$;
(ii) Γ is said to be *linearly normally embedded* if $H^0(\mathcal{F}) \twoheadrightarrow H^0(L)$.

Proposition 3.7. *(cf.* [16, Thm. 3.6 and Cor. 3.7]*) Let* $S \subset \mathbb{P}^R$ *be a linearly normal scroll of degree* d*, genus* $g \geq 1$ *and speciality* $h^1 > 0$*. Then* S *contains a special section of speciality* $i \leq h^1$*.*

We will need a more precise result, mainly contained in [19]. First we prove a lemma.

Lemma 3.8. *Let* C *be a smooth, projective curve of genus* $g \geq 2$*, with general moduli. Let* $0 < h^1 < g$ *be an integer.*

If $|M|$ *is a complete linear system on* C *having maximal degree and maximal dimension with respect to the condition* $h^1(C, M) = h^1$*, then*

$$\dim |M| = \overline{h} \qquad \text{and} \qquad \deg(M) = \overline{m}$$

where

$$\overline{h} := \lfloor \frac{g}{h^1} - 1 \rfloor \quad \text{and} \quad \overline{m} := \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1. \tag{3.9}$$

Proof. If we fix the index of speciality i and the genus g, the Brill-Noether number $\rho(g, r, d) = \rho(g, r, r + g - i) := g - (r + 1)i$ is a decreasing function of r. Since C has general moduli, there exists a complete, b.p.f. and special $\mathfrak{g}^h_m$, of given speciality $h^1 = i$, if and only if

$$\rho(g, h, m) = g - (h + 1)h^1 \geq 0. \tag{3.10}$$

Hence, the maximal dimension for such a linear series is given by the integer $\overline{h}$ as in (3.9). Accordingly, we get the expression for the maximal degree. $\square$

Theorem 3.11. *Let C be a smooth, projective curve of genus $g \geq 3$. Let $0 < h^1 < g$ and $d \geq 4g - 2h^1 - \mathrm{Cliff}(C) + 1$ be integers. Let $S \subset \mathbb{P}^R$ be a smooth, linearly normal, special scroll of degree d and speciality h^1 on C. Then:*
(i) S contains a unique, special section Γ. Furthermore, Γ is linearly normally embedded, its speciality equals the speciality of S, i.e. $h^1(\Gamma, \mathcal{O}_\Gamma(1)) = h^1$ and, if $deg(\Gamma) := m$, then

$$\Gamma \subset \mathbb{P}^h, \text{ with } 2h \leq m = h + g - h^1. \tag{3.12}$$

Moreover Γ is the curve on S, different from a ruling, with minimal degree.

If in addition $d \geq 4g - 3$, then Γ is the unique section with non-positive self-intersection.
(ii) If C has general moduli and $d \geq \frac{7g-\epsilon}{2} - 2h^1 + 2$, where $0 \leq \epsilon \leq 1$ and $\epsilon \equiv g \mod 2$, then

$$\textit{either} \;\; g \geq 4h^1, \; h \geq 3, \;\; \textit{or} \;\; g = 3, \; h^1 = 1, \; h = 2. \tag{3.13}$$

Moreover

$$2h \leq (h + g - h^1)\frac{h}{h+1} \leq m = h + g - h^1. \tag{3.14}$$

Furthermore Γ is the unique section with non-positive self-intersection.

Corollary 3.15. *Let $g \geq 4$, $0 < h^1 < g$ and $d \geq \frac{7g-\epsilon}{2} - 2h^1 + 2$ be integers, with ϵ as above. Assume that $g < 4h^1$. Then, if $\mathcal{H}$ denotes any irreducible component of the Hilbert scheme of linearly normal, smooth, special scrolls in $\mathbb{P}^R$ of degree d, genus g and speciality h^1, the general point of $\mathcal{H}$ parametrizes a scroll with special moduli.*

There are examples of such components of the Hilbert scheme (cf. Remark 7.14).

Remark 3.16. If $m = 2h$ then, by Clifford's theorem, either Γ is a canonical curve or it is hyperelliptic. However, the second possibility is not compatible with smoothness. Therefore, only the first alternative holds if S is smooth (cf. [29, p. 144] and [8, Example 3.7]).

Proof of Theorem 3.11. Part (i), except the assertion about Γ^2, follows from [18] and [19]. Note that (3.12) follows by Clifford's theorem.

The numerical hypotheses on d in Part (ii) come from the expression of the Clifford index for a curve with general moduli. Since S is smooth, then either $h \geq 3$ or $h = 2$, $g = 3$ and $h^1 = 1$, which gives (3.13). Furthermore, (3.10) reads $m(h+1) \geq h(h+g+1)$. Then, (3.14) holds since $\frac{h}{h+1}(h+g+1) = h(1 + \frac{g}{h+1})$ and $\frac{g}{h+1} \geq 1$.

From (2.5), one has

$$\Gamma^2 = 2m - d$$

and this is negative if either $d \geq 4g - 3$, for any curve C, or if C has general moduli as in part (ii), because of the assumptions on d and the bounds on m in Lemma 3.8.

Finally, let $\Gamma' \equiv \Gamma + \alpha l$ be a section different from Γ. Then $0 \leq \Gamma' \cdot \Gamma = \Gamma^2 + \alpha$, hence $\alpha \geq -\Gamma^2 > 0$. If $0 \geq (\Gamma')^2 = \Gamma^2 + 2\alpha = \Gamma' \cdot \Gamma + \alpha > \Gamma' \cdot \Gamma$ we get a contradiction. □

Remark 3.17. It is not true that the minimal section Γ in Theorem 3.11 is the only special curve on S (cf. e.g. [29, p. 129]). In [16, Proposition 2.2] the authors prove that all the irreducible bisecant sections are special with speciality h^1. It would be interesting to make an analysis of the speciality of multi-secant curves.

Under the assumption of Theorem 3.11, the existence of a special section provides information on the rank-two vector bundle on C determining the scroll S.

Proposition 3.18. *Same hypotheses as in Theorem 3.11, Part (i), with $d \geq 4g - 3$, or as in Part (ii), if C has general moduli. Suppose S determined by a pair $(\mathcal{F}, C)$. Then $\mathcal{F}$ is unstable. If, moreover, $d \geq 6g - 5$ then $\mathcal{F} = L \oplus N$, where $\mathcal{F} \twoheadrightarrow L$ corresponds to the special section as in Theorem 3.11.*

Proof. Let Γ be the section as in Theorem 3.11, corresponding to the exact sequence (2.4). One has $\mu(\mathcal{F}) = \frac{d}{2}$ and $\deg(N) = d - m$. From $\Gamma^2 = 2m - d < 0$, one has $\deg(N) > \mu(\mathcal{F})$, hence $\mathcal{F}$ is unstable.

If $d \geq 6g - 5$, from (2.4), we have $[\mathcal{F}] \in \mathrm{Ext}^1(L, N) \cong H^1(C, N \otimes L^\vee)$. Since L is special, $\deg(N \otimes L^\vee) = d - 2m \geq 2g - 1$, thus $N \otimes L^\vee$ is non-special, so (2.4) splits. □

4. Projections and degenerations

Let S be a smooth, special scroll of genus g and degree d as in Theorem 3.11, determined by a pair $(\mathcal{F}, C)$. Let Γ be the special section of S as in Theorem 3.11 and let (2.4) be the associated exact sequence.

Let $l = l_q$ be the ruling of S corresponding to the point $q \in C$ and let p be a point on l. We want to consider the ruled surface $S' \subset \mathbb{P}^{R-1}$ which is the projection of S from p. We will assume that S' is smooth.

Proposition 4.1. *In the above setting, one has:*

(i) *if $p \notin \Gamma$, then S' corresponds to a pair $(\mathcal{F}', C)$, where $\mathcal{F}'$ fits into an exact sequence*
$$0 \to N(-q) \to \mathcal{F}' \to L \to 0; \tag{4.2}$$
(ii) *when $\mathcal{F}$ varies in the set of all extensions as in* (2.4) *and p varies on l, then $\mathcal{F}'$ varies in the set of all extensions as in* (4.2)*;*

(iii) *if $p \in \Gamma$, then S' corresponds to a pair $(\mathcal{F}', C)$, where $\mathcal{F}'$ fits into an exact sequence*
$$0 \to N \to \mathcal{F}' \to L(-q) \to 0; \tag{4.3}$$
(iv) *when $\mathcal{F}$ varies in the set of all extensions as in* (2.4) *then $\mathcal{F}'$ varies in the set of all extensions as in* (4.3).

If $d \geq 4g - 2h^1 - \mathrm{Cliff}(C) + 2$, then the exact sequence (4.2) *(resp.* (4.3)*) corresponds to the unique special section of S'.*

Proof. Assertions (i) and (iii) are clear since S' contains a section Γ' such that, in the former case $\mathcal{O}_{\Gamma'}(1) \cong L$, in the latter $\mathcal{O}_{\Gamma'}(1) \cong L \otimes \mathcal{O}_C(-q)$. More specifically, there are surjective maps
$$\mathrm{Ext}^1(L, N(-q)) \cong H^1(N(-q) \otimes L^\vee) \longrightarrow H^1(N \otimes L^\vee) \cong \mathrm{Ext}^1(L, N)$$
$$\mathrm{Ext}^1(L, N) \cong H^1(N \otimes L^\vee) \longrightarrow H^1(N(q) \otimes L^\vee) \cong \mathrm{Ext}^1(L(-q), N).$$
The former map has the following geometric interpretation: any surface S' corresponding to a pair $(\mathcal{F}', C)$ as in (4.2) comes as a projection of a surface S corresponding to a pair $(\mathcal{F}, C)$ as in (2.4). The surface S is obtained by blowing-up the intersection point of the ruling l' corresponding to q with the section Γ' and then contracting the proper transform of l'. The latter map corresponds to the projection of the surface S to S' from the intersection point of l with Γ.

The surjectivity of the above maps imply assertions (ii) and (iv). The final assertion is clear. □

The following corollary will be useful later.

Corollary 4.4. *In the above setting, let q be a point in C such that $L(-q)$ is very ample on C. Then there is a flat degeneration of S to a smooth scroll S' corresponding to a pair $(\mathcal{G}, C)$ with $\mathcal{G}$ fitting in an exact sequence*
$$0 \to N(q) \to \mathcal{G} \to L(-q) \to 0.$$

Proof. Proposition 4.1 tells us that S comes as a projection of a scroll T in $\mathbb{P}^{R+1}$ corresponding to a pair $(\mathcal{F}', C)$ with $\mathcal{F}'$ fitting in an exact sequence

$$0 \to N(q) \to \mathcal{F}' \to L \to 0.$$

By moving the centre of projection along the ruling of T corresponding to q and letting it go to the intersection with the unisecant corresponding to L and applying again Proposition 4.1, we see that S degenerates to a surface S' as needed. $\square$

Remark 4.5. The hypotheses in Proposition 4.1 and Corollary 4.4 are too strong. In fact, one only needs the existence of the exact sequence (2.4). However, (4.2) (resp. (4.3)) will not correspond in general to the minimal section. Moreover, the assumption that $L(-q)$ is very ample in Corollary 4.4 is not strictly necessary: in case this does not hold the limit scroll S' is simply not smooth. However, we will never need this later.

5. Special scrolls with general moduli

From now on we will focus on smooth, linearly normal scrolls of degree d, genus g and speciality h^1, with general moduli.

In what follows we shall use the following result.

Lemma 5.1. *Let C be a smooth, projective curve of genus $g \geq 2$, with general moduli. Let L be a special line bundle on C such that $|L|$ is a base-point-free $\mathfrak{g}^r_m$, with $i := h^1(C, L) \geq 2$. Assume also that $[L] \in W^r_m(C)$ is general when the Brill-Noether number*

$$\rho(g, L) := g - h^0(C, L)h^1(C, L) \tag{5.2}$$

is positive. Then the complete linear system $|\omega_C \otimes L^\vee|$ is base-point-free.

Proof. Assume by contradiction that the base divisor of $|\omega_C \otimes L^\vee|$ is $p_1, \ldots, p_h$, $h \geq 1$. Then

$$|L + p_1 + \cdots + p_h|$$

is a $\mathfrak{g}^{r+h}_{m+h}$ on C; moreover the series $|B| := |K - L - p_1 - \cdots - p_h|$ is base-point-free. We have a rational map

$$\phi : W^r_m(C) \dashrightarrow W^{r+h}_{m+h}(C)$$

which maps the general $[L] \in W^r_m(C)$ to $[L + p_1 + \cdots + p_h] \in W^{r+h}_{m+h}(C)$. Let Φ_L be the component of the fibre of $[L]$ via ϕ containing $[L]$ and let $[L']$ be a general point of Φ_L. Hence $L + p_1 + \cdots + p_h = L' + q_1 + \cdots + q_h$, for some $q_i \in C$, $1 \leq i \leq h$, which are the base points of $|K - L'|$. There is a rational map

$$\psi_L : \Phi_L \dashrightarrow \mathrm{Sym}^h(C)$$

which maps $[L']$ to $q_1 + \cdots + q_h$. The map ψ_L is clearly birational to its image. Therefore

$$\dim(W^{r+h}_{m+h}(C)) \geq \dim(\mathrm{Im}(\phi)) \geq \dim(W^r_m(C)) - h =$$

$$\dim(W^{r+h}_{m+h}(C)) + hi - h$$

which is a contradiction to $i \geq 2$. $\square$

The main result of this section is the following proposition.

Proposition 5.3. *Let h^1 and g be positive integers as in* (3.13). *Let d be as in Theorem 3.11 - (ii), if $h^1 \neq 2$, whereas $d \geq 4g - 3$, if $h^1 = 2$.*

Let $S \subset \mathbb{P}^R$ be a smooth, linearly normal, special scroll of degree d, genus g, speciality h^1, with general moduli, and let Γ be the unique, special section of S as in Theorem 3.11.

Let $m = \deg(\Gamma)$ *and let* L *be the line bundle on* C *associated to* Γ *as in* (2.4). *Assume further that*

- $L = \omega_C$, *when* $h^1 = 1$,
- $[L] \in W^h_m(C)$ *is such that* $|\omega_C \otimes L^\vee|$ *is base-point-free (in particular* $[L] \in W^h_m(C)$ *is general, when* $h^1 \geq 2$ *and* $\rho(g, L)$ *as in* (5.2) *is positive).*

If $\mathcal{N}_{S/\mathbb{P}^R}$ *denotes the normal bundle of* S *in* $\mathbb{P}^R$*, then:*

(i) $h^0(S, \mathcal{N}_{S/\mathbb{P}^R}) = 7(g-1) + (R+1)(R+1-h^1) + (d-m-g+1)h^1 - (d-2m+g-1)$;
(ii) $h^1(S, \mathcal{N}_{S/\mathbb{P}^R}) = h^1(d-m-g+1) - (d-2m+g-1)$;
(iii) $h^2(S, \mathcal{N}_{S/\mathbb{P}^R}) = 0$.

Proof of Proposition 5.3. First notice that $4g-3 \geq \frac{7g-\epsilon}{2} - 2h^1 + 2$, so that Theorem 3.11, Part (ii), can be applied also to the case $h^1 = 2$.

First, we prove (iii). Since S is linearly normal, from the Euler sequence we get:

$$\cdots \to H^0(\mathcal{O}_S(H))^\vee \otimes H^2(\mathcal{O}_S(H)) \to H^2(\mathcal{T}_{\mathbb{P}^R}|_S) \to 0;$$

since S is a scroll, then $h^2(\mathcal{O}_S(H)) = 0$, which implies $h^2(\mathcal{T}_{\mathbb{P}^R}|_S) = 0$. Thus (iii) follows from using the tangent sequence

$$0 \to \mathcal{T}_S \to \mathcal{T}_{\mathbb{P}^R}|_S \to \mathcal{N}_{S/\mathbb{P}^R} \to 0. \tag{5.4}$$

Since S is a scroll,

$$\chi(\mathcal{T}_S) = 6 - 6g. \tag{5.5}$$

From the Euler sequence, since S is linearly normal, we get

$$\chi(\mathcal{T}_{\mathbb{P}^R}|_S) = (R+1)(R+1-h^1) + g - 1. \tag{5.6}$$

Thus, from (iii) and from (5.5), (5.6) we get

$$\chi(\mathcal{N}_{S/\mathbb{P}^R}) = h^0(\mathcal{N}_{S/\mathbb{P}^R}) - h^1(\mathcal{N}_{S/\mathbb{P}^R}) = 7(g-1) + (R+1)(R+1-h^1). \tag{5.7}$$

The rest of the proof will be concentrated on the computation of $h^1(\mathcal{N}_{S/\mathbb{P}^R})$.

Claim 5.8. *One has* $h^1(\mathcal{N}_{S/\mathbb{P}^R}(-\Gamma)) = h^2(\mathcal{N}_{S/\mathbb{P}^R}(-\Gamma)) = 0$*. In other words,*

$$h^1(\mathcal{N}_{S/\mathbb{P}^R}) = h^1(\mathcal{N}_{S/\mathbb{P}^R}|_\Gamma). \tag{5.9}$$

Proof of Claim 5.8. Look at the exact sequence

$$0 \to \mathcal{N}_{S/\mathbb{P}^R}(-\Gamma) \to \mathcal{N}_{S/\mathbb{P}^R} \to \mathcal{N}_{S/\mathbb{P}^R}|_\Gamma \to 0. \tag{5.10}$$

From (5.4) tensored by $\mathcal{O}_S(-\Gamma)$ we see that $h^2(\mathcal{N}_{S/\mathbb{P}^R}(-\Gamma)) = 0$ follows from $h^2(\mathcal{T}_{\mathbb{P}^R}|_S(-\Gamma)) = 0$ which, by the Euler sequence, follows from $h^2(\mathcal{O}_S(H-\Gamma)) = h^0(\mathcal{O}_S(K_S - H + \Gamma)) = 0$, since $K_S - H + \Gamma$ intersects the ruling of S negatively.

As for $h^1(\mathcal{N}_{S/\mathbb{P}^R}(-\Gamma)) = 0$, this follows from $h^1(\mathcal{T}_{\mathbb{P}^R}|_S(-\Gamma)) = h^2(\mathcal{T}_S(-\Gamma)) = 0$. By the Euler sequence, the first vanishing follows from $h^2(\mathcal{O}_S(-\Gamma)) = h^1(\mathcal{O}_S(H-\Gamma)) = 0$. Since $K_S + \Gamma$ meets the ruling negatively, one has $h^0(\mathcal{O}_S(K_S+\Gamma)) = h^2(\mathcal{O}_S(-\Gamma)) = 0$. Moreover, Theorem 3.11 implies $h^1(\mathcal{O}_S(H-\Gamma)) = h^1(C, N) = 0$.

In order to prove $h^2(\mathcal{T}_S(-\Gamma)) = 0$, consider the exact sequence

$$0 \to \mathcal{T}_{rel} \to \mathcal{T}_S \to \rho^*(\mathcal{T}_C) \to 0 \tag{5.11}$$

arising from the structure morphism $S \cong F = \mathbb{P}(\mathcal{F}) \xrightarrow{\rho} C$. The vanishing we need follows from $h^2(\mathcal{T}_{rel} \otimes \mathcal{O}_S(-\Gamma)) = h^2(\mathcal{O}_S(-\Gamma) \otimes \rho^*(\mathcal{T}_C)) = 0$. The first vanishing holds since $\mathcal{T}_{rel} \cong \mathcal{O}_S(2H - df)$ and therefore, $\mathcal{O}_S(K_S + \Gamma) \otimes \mathcal{T}_{rel}^\vee$ restricts negatively to the ruling. Similar considerations also yield the second vanishing.

□

Next, we want to compute $h^1(\Gamma, \mathcal{N}_{S/\mathbb{P}^R}|_\Gamma)$. To this aim, consider the exact sequence

$$0 \to \mathcal{N}_{\Gamma/S} \to \mathcal{N}_{\Gamma/\mathbb{P}^R} \to \mathcal{N}_{S/\mathbb{P}^R}|_\Gamma \to 0. \tag{5.12}$$

First we compute $h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R})$ and $h^1(\mathcal{N}_{\Gamma/S})$.

Claim 5.13. *One has*

$$h^1(\mathcal{N}_{\Gamma/S}) = d - 2m + g - 1 \tag{5.14}$$

and

$$h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) = (d - g - m + 1)h^1. \tag{5.15}$$

Proof of Claim 5.13. From (2.5),

$$\deg(\mathcal{N}_{\Gamma/S}) = \Gamma^2 = 2m - d < 0$$

by Theorem 3.11. Thus,

$$h^0(\mathcal{O}_\Gamma(\Gamma)) = 0, \quad h^1(\mathcal{O}_\Gamma(\Gamma)) = d - 2m + g - 1$$

which gives (5.14).

To compute $h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R})$ we use the fact that S is a scroll with general moduli. First, consider $\Gamma \subset \mathbb{P}^h$ and the Euler sequence of $\mathbb{P}^h$ restricted to Γ. By taking cohomology and by dualizing, we get

$$0 \to H^1(\mathcal{T}_{\mathbb{P}^h}|_\Gamma)^\vee \to H^0(\mathcal{O}_\Gamma(H)) \otimes H^0(\omega_\Gamma(-H)) \xrightarrow{\mu_0} H^0(\omega_\Gamma),$$

where μ_0 is the usual Brill-Noether map of $\mathcal{O}_\Gamma(H)$. Since $\Gamma \cong C$ and since C has general moduli, then μ_0 is injective by Gieseker-Petri (cf. [2]) so

$$h^1(\mathcal{T}_{\mathbb{P}^h}|_\Gamma) = 0. \tag{5.16}$$

From the exact sequence

$$0 \to \mathcal{T}_\Gamma \to \mathcal{T}_{\mathbb{P}^h}|_\Gamma \to \mathcal{N}_{\Gamma/\mathbb{P}^h} \to 0$$

we get

$$h^1(\mathcal{N}_{\Gamma/\mathbb{P}^h}) = 0. \tag{5.17}$$

From the inclusions

$$\Gamma \subset \mathbb{P}^h \subset \mathbb{P}^R$$

we have the sequence

$$0 \to \mathcal{N}_{\Gamma/\mathbb{P}^h} \to \mathcal{N}_{\Gamma/\mathbb{P}^R} \to \mathcal{N}_{\mathbb{P}^h/\mathbb{P}^R}|_\Gamma \cong \oplus_{i=1}^{R-h}\mathcal{O}_\Gamma(H) \to 0. \tag{5.18}$$

By (5.17), (5.18), we have

$$\begin{aligned} h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) &= (R-h)h^1(\mathcal{O}_\Gamma(H)) = (d - 2g + 1 + h^1 - h)h^1 \\ &= (d - g - m + 1)h^1, \end{aligned}$$

proving (5.15).

□

Next we show that

$$h^1(\mathcal{N}_{S/\mathbb{P}^R}|_\Gamma) = h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) - h^1(\mathcal{N}_{\Gamma/S}).$$

Claim 5.19. *The map*

$$H^0(\Gamma, \mathcal{N}_{\Gamma/\mathbb{P}^R}) \to H^0(\Gamma, \mathcal{N}_{S/\mathbb{P}^R}|_\Gamma),$$

coming from the exact sequence (5.12) *is surjective.*

Proof of Claim 5.19. To show this surjectivity is equivalent to showing the injectivity of the map

$$H^1(\mathcal{N}_{\Gamma/S}) \to H^1(\mathcal{N}_{\Gamma/\mathbb{P}^R})$$

or equivalently, from (2.5), the surjectivity of the dual map

$$H^0(\omega_\Gamma \otimes \mathcal{N}^\vee_{\Gamma/\mathbb{P}^R}) \to H^0(\omega_\Gamma \otimes \mathcal{N}^\vee_{\Gamma/S}) \cong H^0(\omega_C \otimes N \otimes L^\vee).$$

From (2.4) and the non-speciality of N, we get

$$0 \to H^0(L)^\vee \to H^0(\mathcal{F})^\vee \to H^0(N)^\vee \to 0.$$

Since $H^0(\mathcal{O}_S(H)) \cong H^0(C, \mathcal{F})$ and $\mathcal{O}_\Gamma(H) \cong L$, the Euler sequences restricted to Γ give the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to & \mathcal{O}_\Gamma & \to & H^0(L)^\vee \otimes \mathcal{O}_\Gamma(H) & \to & \mathcal{T}_{\mathbb{P}^h}|_\Gamma & \to 0 \\
 & || & & \downarrow & & \downarrow & \\
0 \to & \mathcal{O}_\Gamma & \to & H^0(\mathcal{F})^\vee \otimes \mathcal{O}_\Gamma(H) & \to & \mathcal{T}_{\mathbb{P}^R}|_\Gamma & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & H^0(N)^\vee \otimes \mathcal{O}_\Gamma(H) & \stackrel{\cong}{\to} & \mathcal{N}_{\mathbb{P}^h/\mathbb{P}^R}|_\Gamma & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

which shows that $\mathcal{N}_{\mathbb{P}^h/\mathbb{P}^R}|_\Gamma \cong H^0(N)^\vee \otimes \mathcal{O}_\Gamma(H)$ (cf. also (5.18)).

From (5.17), (5.18) and the above identification, we have

$$H^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) \cong H^0(C, N)^\vee \otimes H^1(C, L)$$

i.e.

$$H^0(C, N) \otimes H^0(C, \omega_C \otimes L^\vee) \cong H^0(\omega_\Gamma \otimes \mathcal{N}^\vee_{\Gamma/\mathbb{P}^R}).$$

By taking into account (5.12) and (5.18), we get a commutative diagram

$$\begin{array}{cccc}
H^0(N) \otimes H^0(\omega_C \otimes L^\vee) & \stackrel{\cong}{\longrightarrow} & H^0(\omega_\Gamma \otimes \mathcal{N}^\vee_{\Gamma/\mathbb{P}^R}) & \\
 & \searrow & \downarrow & \\
 & & H^0(\omega_C \otimes N \otimes L^\vee) & \cong H^0(\omega_\Gamma \otimes \mathcal{N}^\vee_{\Gamma/S}).
\end{array} \tag{5.20}$$

Therefore, if the diagonal map is surjective then also the vertical one will be surjective.

There are two cases.

- If $h^1 = 1$, then $L = \omega_C$ and the diagonal map is the identity; so we are done in this case.
- If $h^1(L) \geq 3$, the surjectivity of the diagonal map follows by applying [6, Lemma 2.9]. In the notation there, $L_1 = \omega_C \otimes L^\vee$, $L_2 = N$, $p = 2g + 2 + 2m - d$ and the conditions (ii) and (iii) in [6, Lemma 2.9] ensuring the surjectivity follow from the hypothesis on d.
- If $h^1(L) = 2$ the required surjectivity follows as above by using Castelnuovo's Lemma (cf. [27, Theorem 2] rather than [6]. Indeed, $|\omega_C \otimes L^\vee|$ is base-point-free and $h^1(N \otimes L \otimes \omega_C^\vee) = 0$ since $\deg(N \otimes L \otimes \omega_C^\vee) = d - 2g + 2 \geq 2g - 1$. In particular, for $[L] \in W^r_m(C)$ general, one can conclude by using Lemma 5.1.

□

From Claims 5.8 and 5.19, we get

$$h^1(\mathcal{N}_{S/\mathbb{P}^R}) = h^1(\mathcal{N}_{S/\mathbb{P}^R}|_\Gamma) = h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) - h^1(\mathcal{N}_{\Gamma/S}).$$

Thus, from Claim 5.13, we get

$$h^1(\mathcal{N}_{S/\mathbb{P}^R}) = (d-m-g+1)h^1 - (d-2m+g-1)$$

which is (ii) in the statement. By using (5.7) and (ii), we get (i). This completes the proof of Proposition 5.3. $\square$

Remark 5.21. From (5.7), we have:

$$\chi(\mathcal{N}_{S/\mathbb{P}^R}) = 7(g-1) + (R+1)(R+1-h^1) = 7(g-1) + (R+1)^2 - h^1(L)h^0(\mathcal{F}).$$

Thus, formula (i) reads as

$$h^0(S, \mathcal{N}_{S/\mathbb{P}^R}) = \chi(\mathcal{N}_{S/\mathbb{P}^R}) + h^1(L)h^0(N) - \chi(N \otimes L^\vee) - 2(g-1),$$

or equivalently

$$h^0(S, \mathcal{N}_{S/\mathbb{P}^R}) = 5(g-1) + (R+1)^2 - h^1(L)h^0(L) - \chi(N \otimes L^\vee). \tag{5.22}$$

Remark 5.23. The proof of Claim 5.19 shows that if $|\omega_C \otimes L^\vee|$ has t base-points then

$$h^1(\mathcal{N}_{S/\mathbb{P}^R}) = h^1(\mathcal{N}_{\Gamma/\mathbb{P}^R}) - h^1(\mathcal{N}_{\Gamma/S}) + t$$

and therefore

$$h^0(\mathcal{N}_{S/\mathbb{P}^R}) = 7(g-1) + (R+1)(R+1-h^1) + (d-m-g+1)h^1 - (d-2m+g-1) + t.$$

Remark 5.24. If one applies different surjectivity criteria for multiplication maps, one may also have different ranges in which the previous proposition holds. For instance, by applying [6, Theorem 1], one may prove that the conclusion in Proposition 5.3 holds if $d > 4g - 4h^1$ and $g > 6h^1 - 2$.

Remark 5.25. A very interesting question is to look for components of the Hilbert scheme whose general point corresponds to a smooth, linearly normal, special scroll with general moduli corresponding to a stable vector bundle. Such components are related to irreducible components of Brill-Noether loci in $U_C(d)$, the moduli space of degree d, semistable, rank-two vector bundles on C, a curve of genus g with general moduli. There are several open questions on this subject: as a reference, see e.g. the overview in [22].

6. Hilbert schemes of linearly normal, special scrolls

From now on, we will denote by $\mathrm{Hilb}(d, g, h^1)$ the open subset of the Hilbert scheme parametrizing smooth scrolls in $\mathbb{P}^R$ of genus $g \geq 3$, degree $d \geq 2g+2$ and speciality h^1, with $0 < h^1 < g$ and $R = d - 2g + 1 + h^1$ as in (3.2).

Theorem 3.11 can be used to describe the irreducible components of $\mathrm{Hilb}(d, g, h^1)$ whose general point represents a smooth, scroll with general moduli. We will denote by $\mathcal{H}_{d,g,h^1}$ the union of these components.

Theorem 6.1. *Let $g \geq 3$, and $h^1 > 0$ be integers as in* (3.13). *Let $d \geq \frac{7g-\epsilon}{2} - 2h^1 + 2$, where $0 \leq \epsilon \leq 1$ and $\epsilon \equiv g \mod 2$, if $h^1 \neq 2$, whereas $d \geq 4g-3$, if $h^1 = 2$. Let m be any integer such that either*

$$m = 4, \text{ if } g = 3, h^1 = 1, \quad \text{or} \quad g + 3 - h^1 \leq m \leq \overline{m} := \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1, \text{ otherwise.} \tag{6.2}$$

(i) If $h^1 = 1$ then $\mathrm{Hilb}(d, g, 1)$ consists of a unique, irreducible component $\mathcal{H}^{2g-2}_{d,g,1}$ whose general point parametrizes a smooth, linearly normal and special scroll $S \subset \mathbb{P}^R$, $R = d-2g+2$, whose special section is a canonical curve. Furthermore,

(1) *$dim(\mathcal{H}^{2g-2}_{d,g,1}) = 7(g-1) + (d-2g+3)^2 - (d-2g+3)$,*

(2) $\mathcal{H}^{2g-2}_{d,g,1}$ *is generically smooth and dominates* $\mathcal{M}_g$.

Moreover, scrolls with speciality 1*, whose special section is not canonical of degree* $m < 2g-2$ *fill up an irreducible subscheme of* $\mathcal{H}^{2g-2}_{d,g,1}$ *which also dominates* $\mathcal{M}_g$ *and whose codimension in* $\mathcal{H}^{2g-2}_{d,g,1}$ *is* $2g-2-m$.

(ii) If $h^1 \geq 2$ *then, for any* $g \geq 4h^1$ *and for any* m *as in* (6.2), $\mathrm{Hilb}(d, g, h^1)$ *contains a unique, irreducible component* $\mathcal{H}^m_{d,g,h^1}$ *whose general point parametrizes a smooth, linearly normal and special scroll* $S \subset \mathbb{P}^R$, $R = d - 2g + 1 + h^1$, *having general moduli whose special section* Γ *has degree* m *and speciality* h^1. *Furthermore,*

(1) $dim(\mathcal{H}^m_{d,g,h^1}) = 7(g-1) + (R+1)(R+1-h^1) + (d-m-g+1)h^1 - (d-2m+g-1)$,

(2) $\mathcal{H}^m_{d,g,h^1}$ *is generically smooth.*

Remark 6.3. Let us comment on the bounds on m in (6.2). From Theorem 3.11, S contains a unique, special section Γ of speciality h^1, which is the image of C via a complete linear series $|L|$, which is a $\mathfrak{g}^h_m$. In order to have C with general moduli, by Lemma 3.8, $m \leq \overline{m} = \lfloor \frac{g}{h^1} - 1 \rfloor + g - h^1$. This condition is empty if $h^1 = 1$.

On the other hand, since S is smooth, we have $h = m - g + h^1 \geq 2$. If $h^1 \geq 2$ and the scroll has general moduli, then $h \geq 3$ by (3.13). If $h^1 = 1$ and $h = 2$, then $m = 4$ and $g = 3$.

Remark 6.4. As a consequence of Theorem 6.1, $\mathrm{Hilb}(d, g, h^1)$ is reducible as soon as $h^1 \geq 2$, and when $h^1 \geq 3$ it is also not equidimensional. Indeed, the component of maximal dimension is $\mathcal{H}^{g+3-h^1}_{d,g,h^1}$ whereas the component of minimal dimension is $\mathcal{H}^{\overline{m}}_{d,g,h^1}$, with $\overline{m}$ as in (6.2).

By Proposition 3.18, the scrolls in $\mathcal{H}^m_{d,g,h^1}$ correspond to unstable bundles.

Proof of Theorem 6.1. The proof is in several steps.

Step 1. Construction of $\mathcal{H}^m_{d,g,h^1}$. As usual, for any smooth, projective curve C of genus g and for any integer m, denote by $W^h_m(C)$ the subscheme of $\mathrm{Pic}^m(C)$ consisting of special, complete linear series $|A|$ on C such that $\deg(A) = m$ and $h^0(C, A) \geq h + 1$ and by $G^h_m(C)$ the scheme parametrizing special $\mathfrak{g}^h_m$'s on C (see, e.g., [2]). Consider the morphism $G^h_m(C) \to W^h_m(C)$.

If C has general moduli, this map is birational and $G^h_m(C)$ is smooth (see, e.g [2]). However, for any curve C, if $h = m - g + 1$ and if the general element of a component Z of $G^h_m(C)$ has index of speciality 1, again the map is birational and Z is birational to $\mathrm{Sym}^{2g-2-m}(C)$, hence it is generically smooth. In any case, for any m as in (6.2),

$$\dim(W^h_m(C)) = \dim(G^h_m(C)) = \rho(g, h, m) = g - (h+1)(h - m + g) \geq 0.$$

Let $\mathcal{M}^0_g$ be the Zariski open subset of the moduli space $\mathcal{M}_g$, whose points correspond to equivalence classes of curves of genus g without non-trivial automorphisms. By definition, $\mathcal{M}^0_g$ is a fine moduli space, i.e. we have a universal family $p : \mathcal{C} \to \mathcal{M}^0_g$, where $\mathcal{C}$ and $\mathcal{M}^0_g$ are smooth schemes and p is a smooth morphism. $\mathcal{C}$ can be identified with the Zariski open subset $\mathcal{M}^0_{g,1}$ of the moduli space $\mathcal{M}_{g,1}$ of smooth, pointed, genus g curves, whose points correspond to equivalence classes of pairs (C, x), with $x \in C$ and C a smooth curve of genus g without non-trivial automorphisms. On $\mathcal{M}^0_{g,1}$ there is again a universal family $p_1 : \mathcal{C}_1 \to \mathcal{M}^0_{g,1}$, where $\mathcal{C}_1 = \mathcal{C} \times_{\mathcal{M}^0_g} \mathcal{C}$. The family p_1 has a natural regular global section δ whose image is the diagonal. By means of δ, for any integer k, we have the universal family of Picard varieties of order k, i.e.

$$p_1^{(k)} : \mathcal{P}ic^{(k)} \to \mathcal{M}^0_{g,1}$$

(cf. [11, § 2]), with a Poincarè line-bundle on $\mathcal{C}_1 \times_{\mathcal{M}^0_{g,1}} \mathcal{P}ic^{(k)}$ (cf. a relative version of [2, p. 166-167]). For any closed point $[(C,x)] \in \mathcal{M}^0_{g,1}$, its fibre via $p_1^{(k)}$ is isomorphic to $\mathrm{Pic}^{(k)}(C)$.

Then, one can consider the map

$$\mathcal{W}^h_m \xrightarrow{p_1^{(m)}} \mathcal{M}^0_{g,1},$$

with $\mathcal{W}^h_m$ a subscheme of $\mathcal{P}ic^{(m)}$: for any closed point $[(C,x)] \in \mathcal{M}^0_{g,1}$, its fibre via $p_1^{(m)}$ is isomorphic to $W^h_m(C)$. One has:
(a) if $\rho(g,h,m) > 0$, by a result of Fulton-Lazarsfeld in [15], $W^h_m(C)$ is irreducible and generically smooth of dimension $\rho(g,h,m)$ for $[C] \in \mathcal{M}_g$ general. Moreover, for $[L] \in W^h_m(C)$ a smooth point, $h^0(C,L) = h+1$. Therefore, there is a unique, reduced component $\mathcal{W}$ of $\mathcal{W}^h_m$ which dominates $\mathcal{M}^0_{g,1}$ via the map $p_1^{(m)}$.
(b) If $\rho(g,h,m) = 0$, from [13, Theorem 1], again there is a unique, reduced component $\mathcal{W}$ of $\mathcal{W}^h_m$ dominating $\mathcal{M}^0_{g,1}$. For $[C,x] \in \mathcal{M}^0_{g,1}$ general, the fibre of $\mathcal{W} \longrightarrow \mathcal{M}^0_{g,1}$ is $W^h_m(C)$ which is a finite number of points $[L]$ with $h^0(C,L) = h+1$.

Let

$$\mathcal{Y}_{h,m} := \mathcal{W} \times_{\mathcal{M}^0_{g,1}} \mathcal{P}ic^{(d-m)},$$

which is irreducible. For $y \in \mathcal{Y}_{h,m}$ general, then $y = (L,N) \in W^h_m(C) \times Pic^{d-m}(C)$ for $[(C,x)] \in \mathcal{M}^0_{g,1}$ general.

Note also the existence of Poincarè line bundles $\mathcal{L}$ and $\mathcal{N}$ on $\mathcal{Y}_{h,m} \times_{\mathcal{M}^0_{g,1}} \mathcal{C}_1 \xrightarrow{\pi_{h,m}} \mathcal{Y}_{h,m}$: for $y \in \mathcal{Y}_{h,m}$ general, corresponding to $(L,N) \in W^h_m(C) \times Pic^{d-m}(C)$ with $[(C,x)] \in \mathcal{M}^0_{g,1}$ general, the restriction of $\mathcal{L}$ (respectively, $\mathcal{N}$) to $\pi^{-1}_{h,m}(y) \cong C$ is L (respectively, N). Hence, we can consider $\mathcal{R}_{h,m} := R^1(\pi_{h,m})_*(\mathcal{N} \otimes \mathcal{L}^\vee)$ on $\mathcal{Y}_{h,m}$.

There is a dense, open subset $\mathcal{U}_{h,m} \subset \mathcal{Y}_{h,m}$ on which the sheaf $\mathcal{R}_{h,m}$ is locally-free and therefore it gives rise to a vector bundle $\mathcal{E}_{h,m}$ whose rank we denote by t: for a general point $y \in \mathcal{U}_{h,m}$, corresponding to $(L,N) \in W^h_m(C) \times Pic^{d-m}(C)$, with $[(C,x)] \in \mathcal{M}^0_{g,1}$ general, the fibre of $\mathcal{E}_{h,m}$ on y is $H^1(C, N \otimes L^\vee) \cong \mathrm{Ext}^1(L,N)$.

Note that, if $t = 0$ (this is certainly the case if $d \geq 6g-5$, cf. Proposition 3.18), then $\mathcal{E}_{h,m} \cong \mathcal{U}_{h,m}$. On the other hand, if $t > 0$, we take into account the *weak isomorphism classes* of extensions (cf. [14, p. 31]). Therefore, we consider

$$\mathcal{S}_{h,m} := \begin{cases} \mathcal{U}_{h,m} & \text{if } t = 0 \\ \mathbb{P}(\mathcal{E}_{h,m}) & \text{otherwise.} \end{cases}$$

On $\mathcal{U}_{h,m}$ there is a universal family $\mathcal{C}_{h,m}$ of curves and on $\mathcal{S}_{h,m} \times_{\mathcal{U}_{h,m}} \mathcal{C}_{h,m}$ there is a universal vector-bundle $\mathcal{F}_{h,m}$. A general point $z \in \mathcal{S}_{h,m}$ corresponds to a pair $(L,N) \in W^h_m(C) \times Pic^{d-m}(C)$, with $[(C,x)] \in \mathcal{M}^0_{g,1}$ general, together with an element $\xi \in \mathbb{P}(\mathrm{Ext}^1(L,N))$ if $t > 0$; the fibre of $\mathcal{F}_{h,m}$ on z is the extension $\mathcal{F}_z$ of L with N on C corresponding to ξ. Given the projection π_1 of $\mathcal{S}_{h,m} \times_{\mathcal{U}_{h,m}} \mathcal{C}_{h,m}$ to the first factor, the sheaf $(\pi_1)_*(\mathcal{F}_{h,m})$ is free of rank $R+1 = d-2g+2+h^1$ on a suitable dense, open subset of $\mathcal{S}_{h,m}$. We will abuse notation and denote this open subset with $\mathcal{S}_{h,m}$. Therefore, on $\mathcal{S}_{h,m}$, we have functions $s_0, \ldots, s_R$ such that, for each point $z \in \mathcal{S}_{h,m}$, $s_0, \ldots, s_R$ computed at z span the space of sections of the correspoding vector bundle $\mathcal{F}_z$.

There is a natural map $\psi_{h,m} : \mathcal{S}_{h,m} \times \mathrm{PGL}(R+1, \mathbb{C}) \to \mathrm{Hilb}(d,g,h^1)$: given a pair (z, ω), embed $\mathbb{P}(\mathcal{F}_z)$ to $\mathbb{P}^R$ via the sections $s_0, \ldots, s_R$ computed at z, compose with ω and take the image.

We define $\mathcal{H}^m_{d,g,h^1}$ to be the closure of the image of the above map to the Hilbert scheme. By construction, $\mathcal{H}^m_{d,g,h^1}$ dominates $\mathcal{M}_g$ and its general point represents a smooth, linearly normal scroll S in $\mathbb{P}^R$ of degree d, genus g, speciality h^1 and containing a unique section of degree m, speciality h^1, which is linearly normally embedded in S. The general point of $\mathcal{H}^m_{d,g,h^1}$ corresponds to an indecomposable scroll if $t > 0$; however, in this case, decomposable scrolls fill up a proper subscheme of $\mathcal{H}^m_{d,g,h^1}$.

In the next steps, we will show that:

- $\mathcal{H}^{2g-2}_{d,g,1}$ strictly contains any $\mathcal{H}^m_{d,g,1}$, for $m < 2g-2$, and it is an irreducible component of $\mathrm{Hilb}(d, g, 1)$.
- for $h^1 \geq 2$, $\mathcal{H}^m_{d,g,h^1}$ is an irreducible component of $\mathrm{Hilb}(d, g, h^1)$, for any $g + 3 - h^1 \leq m \leq \overline{m}$,

Step 2. A Lemma concerning automorphisms. Here we prove the following:

Lemma 6.5. *Assume* $\mathrm{Aut}(C) = \{Id\}$ *(in particular, this happens if C has general moduli). Let* (2.4) *be the exact sequence corresponding to the pair (S, Γ), where S is general and Γ is the unique special section of S.*

If $G_S \subset \mathrm{PGL}(R + 1, \mathbb{C})$ denotes the sub-group of projectivities of $\mathbb{P}^R$ fixing S, then $G_S \cong$ $\mathrm{Aut}(S)$ *and*

$$\dim(G_S) = \begin{cases} h^0(N \otimes L^\vee) & \text{if } \mathcal{F} \text{ is indecomposable} \\ h^0(N \otimes L^\vee) + 1 & \text{if } \mathcal{F} \text{ is decomposable} \end{cases} \tag{6.6}$$

Proof of Lemma 6.5. There is an obvious inclusion $G_S \hookrightarrow \mathrm{Aut}(S)$. We want to show that this is an isomorphism. Let σ be an automorphism of S. By Theorem 3.11, $\sigma(\Gamma) = \Gamma$ and since $\mathrm{Aut}(C) = \{Id\}$, σ fixes Γ pointwise. Now $H \sim \Gamma + \rho^*(N)$ and by the above, $\sigma^*(H) = \sigma^*(\Gamma) + \sigma^*(\rho^*(N)) = \Gamma + \rho^*(N) \sim H$. Therefore, σ is induced by a projective trasformation.

The rest of the claim directly follows from cases (2) and (3) of [26, Theorem 2] and from [26, Lemma 6]. Indeed, since $\mathrm{Aut}(C) = \{Id\}$ therefore, in the notation of [26, Lemma 6] one has $\mathrm{Aut}(S) \cong \mathrm{Aut}_C(S)$. Furthermore, from Theorem 3.11, Γ is the unique section of minimal degree on S. Thus, one can conclude by using the description of $\mathrm{Aut}_C(S)$ in [26, Theorem 2]. $\square$

Step 3. The dimension of $\mathcal{H}^m_{d,g,h^1}$. Given a general point of $\mathcal{H}^m_{d,g,h^1}$ corresponding to a scroll S, the base of the scroll C and the line bundles L and N on C are uniquely determined. From the previous steps, $\dim {\psi_{h,m}}^{-1}([S]) = \dim(G_S)$. An easy computation shows that, in any case, one has

$$\dim(\mathcal{H}^m_{d,g,h^1}) = 5g - 5 + (R + 1)^2 - h^0(L)h^1(L) - \chi(N \otimes L^\vee). \tag{6.7}$$

Step 4. The case $h^1 = 1$. Let S correspond to a general point in $\mathcal{H}^m_{d,g,1}$, i.e. S is determined by a pair $(\mathcal{F}, C)$, with $\mathcal{F}$ fitting in an exact sequence like (2.4), with $|L|$ a linear series $\mathfrak{g}^h_m$ of speciality 1. Suppose that $L \neq \omega_C$. In particular, $g \geq 4$. The residual series is $|\omega_C \otimes L^\vee|$ is a $\mathfrak{g}^0_{2g-2-m}$, i.e. $\omega_C \otimes L^\vee \cong \mathcal{O}_C(p_1 + \cdots + p_{2g-2-m})$, where p_j are general points on C, $1 \leq j \leq 2g - 2 - m$, hence $L \cong \omega_C(-p_1 - \cdots - p_{2g-2-m})$. From Corollary 4.4 we have that $\mathcal{H}^m_{d,g,1}$, with $m < 2g-2$, sits in the closure of $\mathcal{H}^{2g-2}_{d,g,h^1}$. The dimension count for $\mathcal{H}^m_{d,g,1}$'s follows by (6.7). The generic smoothness of $\mathcal{H}^{2g-2}_{d,g,1}$ follows by comparing the formula for $h^0(\mathcal{N}_{S/\mathbb{P}^R})$ from Proposition 5.3 (equivalently (5.22)) with (6.7).

The dimension count for $\mathcal{H}^m_{d,g,1}$ follows by the construction.

Step 5. The case $h^1 \geq 2$. In Step 1 we constructed the irreducible subschemes $\mathcal{H}^m_{d,g,h^1}$ of the Hilbert scheme $\mathrm{Hilb}(d,g,h^1)$ for $g+3-h^1 \leq m \leq \overline{m}$. From the dimension count (6.7) and from the formula for $h^0(\mathcal{N}_{S/\mathbb{P}^R})$ in Proposition 5.3 (equivalently (5.22)) one has that each $\mathcal{H}^m_{d,g,h^1}$ is a generically smooth, irreducible component of the Hilbert scheme $\mathrm{Hilb}(d,g,h^1)$.

We have finished the proof of Theorem 6.1. □

Remark 6.8. Observe that (6.6) can be also computed via cohomological arguments. Indeed, $\dim(\mathrm{Aut}(S)) = h^0(S, \mathcal{T}_S)$. From (5.11), one has $H^0(\mathcal{T}_S) \cong H^0(\mathcal{T}_{rel})$. Since $\mathcal{T}_{rel} \cong \omega_S^\vee \otimes \rho^*(\omega_C)$, we get $\mathcal{T}_{rel} \cong \mathcal{O}_S(2H - \rho^*(\det(\mathcal{F})))$. As $\Gamma \cdot (2H - \rho^*(\det(\mathcal{F}))) = 2m - d = \Gamma^2 < 0$, Γ is a fixed component of $|2H - \rho^*(\det(\mathcal{F}))|$. Since $H \sim \Gamma + \rho^*(N)$, then $h^0(\mathcal{O}_S(2H - \rho^*(\det(\mathcal{F})))) = h^0(\mathcal{O}_S(H - \rho^*(L)))$. Moreover, $\Gamma \cdot (H - \rho^*(L)) = 0$. By the projection formula, we get $H^0(S, H - \rho^*(L)) \cong H^0(C, \mathcal{F} \otimes L^\vee)$. Therefore, $h^0(S, \mathcal{T}_S) = h^0(C, \mathcal{F} \otimes L^\vee)$.

The map $H^0(\mathcal{F} \otimes L^\vee) \to H^0(\mathcal{O}_C)$ arising from the exact sequence (2.4) identifies with the restriction map $H^0(\mathcal{O}_S(H - \rho^*(L))) \to H^0(\mathcal{O}_\Gamma)$.

• If $\mathcal{F}$ is decomposable, this map is surjective. Therefore, $h^0(\mathcal{F} \otimes L^\vee) = 1 + h^0(N \otimes L^\vee)$. In particular, this happens when $\mathrm{Ext}^1(L, N) \cong H^1(N \otimes L^\vee) = 0$, e.g. if $d \geq 6g-5$ (cf. Proposition 3.18).

• If $\mathcal{F}$ is indecomposable, then the coboundary map $H^0(\mathcal{O}_C) \to H^1(N \otimes L^\vee)$, arising from (2.4), is injective since it corresponds to the choice of $\mathcal{F}$ as an element of $\mathrm{Ext}^1(L, N)$. In particular, $h^0(\mathcal{F} \otimes L^\vee) = h^0(N \otimes L^\vee)$.

The above discussion proves (6.6).

Remark 6.9. As an alternative to the construction of $\mathcal{H}^m_{d,g,h^1}$ in the above proof of Theorem 6.1, one could start from the main result in [18]. One has that the general scroll in $\mathcal{H}^{2g-2}_{d,g,1}$ can be obtained as a general internal projection of a scroll S corresponding to a decomposable vector-bundle on a genus g curve C of the type $\omega_C \oplus M$, with M a non-special, line-bundle of degree greater or equal than $2g-2$. Such a scroll has a unique, special section which is canonical.

One may prove that the general scroll in $\mathcal{H}^m_{d,g,h^1}$ arises as the projection of S from suitable $2g-2-m$ points on the canonical section and from other general points on S.

7. Further considerations on Hilbert schemes

7.1. **Components of the Hilbert scheme of non-linearly normal scrolls.** In this section we consider the following problem. Let $r = d - 2g + 1 + k$, with $0 \leq k < l$ and consider the family $\mathcal{Y}^m_{k,l}$ whose general element is a general projection to $\mathbb{P}^r$ of the general scroll in $\mathcal{H}^m_{d,g,l}$. Is $\mathcal{Y}^m_{k,l}$ contained in $\mathcal{H}^n_{d,g,k}$ for some n, if $k > 0$, or in $\mathcal{H}_{d,g}$, if $k = 0$? Recall that $\mathcal{H}_{d,g}$ is the component of the Hilbert scheme of linearly normal, non-special scrolls of degree d and genus g in $\mathbb{P}^{d-2g+1}$ (cf. [7]).

Proposition 7.1. *In the above setting:*

(i) *if $k > 0$, $\mathcal{Y}^m_{k,l}$ sits in an irreducible component of the Hilbert scheme different from $\mathcal{H}^n_{d,g,k}$, for any n;*

(ii) *if $l > 1$, $\mathcal{Y}^m_{0,l}$ sits in an irreducible component of the Hilbert scheme different from $\mathcal{H}_{d,g}$, for any m;*

(iii) *$\mathcal{Y}^{2g-2}_{0,1}$ is a divisor inside $\mathcal{H}_{d,g}$ whose general point is a smooth point for $\mathcal{H}_{d,g}$.*

Proof. (i) It suffices to prove that

$$\dim(\mathcal{Y}^m_{k,l}) \geq \dim(\mathcal{H}^m_{d,g,k}). \tag{7.2}$$

Indeed, if $\mathcal{Y}^m_{k,l}$ is contained in a component $\mathcal{H}^n_{d,g,k}$, then $n \geq m$ and the conclusion follows from Remark 6.4.

In order to prove (7.2), we count the number of parameters on which $\mathcal{Y}^m_{k,l}$ depends. Let $[S] \in \mathcal{H}^m_{d,g,l}$ be general. Then, $S \subset \mathbb{P}^R$, where $R = d - 2g + 1 + l$ and $S \cong \mathbb{P}(\mathcal{F})$ with $\mathcal{F}$ a vector bundle on a curve C sitting in an exact sequence like (2.4). Let $S' \subset \mathbb{P}^r$ be the general projection of S, with r as above. Let $G_{S'} \subset \mathrm{PGL}(r+1, \mathbb{C})$ be the subgroup of projectivities which fix S'.

The parameters on which $\mathcal{Y}^m_{k,l}$ depends are the following:

- $3g - 3$, for the parameters on which C depends, plus
- g, for the parameters on which N depends, plus
- $\rho(g, L) = g - l(m - g + l + 1)$, for the parameters on which L depends, plus
- $\epsilon = 0$ (respectively, $h^1(N \otimes L^\vee) - 1$) if the general bundle is decomposable (respectively, indecomposable), plus
- $(r+1)(l-k) = dim(\mathbb{G}(r, R))$, which are the parameters for the projections, plus
- $(r+1)^2 - 1 = dim(\mathrm{PGL}(r+1, \mathbb{C}))$, minus
- $\dim(G_{S'})$.

We remark that, since $G_S \cong \mathrm{Aut}(S)$ (cf. Lemma 6.5), one has $\dim(G_{S'}) \leq \dim(G_S)$.

Therefore, by recalling (6.6), in any case we get

$$\dim(\mathcal{Y}^m_{k,l}) \geq 5g - 5 + (r+1)^2 - \chi(N \otimes L^\vee) - l(m - g + l + 1) + (l - k)(r + 1). \quad (7.3)$$

By Theorem 6.1 and (5.22),

$$\dim(\mathcal{H}^m_{d,g,k}) = 5g - 5 + (r+1)^2 - k(m - g + k + 1) - \chi(N \otimes L^\vee).$$

Then

$$\dim(\mathcal{Y}_k) - \dim(\mathcal{H}^m_{d,g,k}) \geq (l - k)(d - m - g + 1 - k - l).$$

From the assumptions on d and the facts that L is special and $k < l \leq \frac{g}{4}$, then $\dim(\mathcal{Y}^m_{k,l}) > \dim(\mathcal{H}^m_{d,g,k})$, i.e. we proved (7.2).

(ii) In this case we have to prove

$$\dim(\mathcal{Y}^m_{0,l}) \geq \dim(\mathcal{H}_{d,g}) = (r+1)^2 + 7(g-1).$$

Arguing as above, we see that this is a consequence of

$$l(d - g - l + 1 - m) \geq g - 1 + d - 2m,$$

which holds since $l \geq 2$ and by the assumptions on d.

(iii) The same computation as above shows that

$$\dim(\mathcal{Y}^{2g-2}_{0,1}) \geq \dim(\mathcal{H}_{d,g}) - 1.$$

We want to prove that equality holds and that $\mathcal{Y}^{2g-2}_{0,1} \subset \mathcal{H}_{d,g}$. Consider the *Rohn exact sequence*

$$0 \to \mathcal{O}_S(H) \to \mathcal{N}_{S/\mathbb{P}^{r+1}} \to \mathcal{N}_{S'/\mathbb{P}^r} \to 0$$

(see, e.g. [12], p. 358, formula (2.2)). From Proposition 5.3 (ii), we have $h^1(\mathcal{N}_{S/\mathbb{P}^{r+1}}) = 0$, therefore also $h^1(\mathcal{N}_{S'/\mathbb{P}^r}) = 0$. Hence $\mathcal{Y}^{2g-2}_{0,1}$ is contained in a component $\mathcal{H}$ of the Hilbert scheme of dimension $\chi(\mathcal{N}_{S'/\mathbb{P}^r}) = 7(g-1) + (r+1)^2$ and the general point of $\mathcal{Y}^{2g-2}_{0,1}$ is a smooth point of $\mathcal{H}$. The map $H^0(\mathcal{N}_{S/\mathbb{P}^{r+1}}) \to H^0(\mathcal{N}_{S'/\mathbb{P}^r})$ is not surjective: its cokernel is $H^1(\mathcal{O}_S(H))$, which has dimension one (cf. Proposition 5.3 (ii)). This shows that the general point of $\mathcal{H}$ correspond to a smooth scroll which is linearly normal in $\mathbb{P}^r$. By the results in [7], $\mathcal{H} = \mathcal{H}_{d,g}$. $\square$

Remark 7.4. The previous result extends and makes more precise the contents of [7, Example 5.12].

7.2. **Components of the Hilbert scheme with special moduli.** There are irreducible components of $\mathrm{Hilb}(d,g,h^1)$ which do not dominate $\mathcal{M}_g$, i.e. components with *special moduli.* We prove the existence of some of these components in the next example. To do this we first recall some preliminary results we will use.

Given any integer $g \geq 3$, let

$$\gamma := \begin{cases} \frac{g+2}{2} & \text{if } g \text{ even}, \\ \frac{g+3}{2} & \text{if } g \text{ odd}. \end{cases} \tag{7.5}$$

Set

$$\mathcal{M}^1_{g,t} := \{[C] \in \mathcal{M}_g | \ C \text{ possesses a } \mathfrak{g}^1_t\},$$

which is called the *t-gonal locus* in $\mathcal{M}_g$.

It is well-known that $\mathcal{M}^1_{g,t}$ is irreducible, of dimension $2g+2t-5$, when $t<\gamma$, whereas $\mathcal{M}^1_{g,t} = \mathcal{M}_g$, for $t \geq \gamma$ (see e.g. [1]). Moreover, the general curve in $\mathcal{M}^1_{g,t}$ has no non-trivial automorphism (cf. the computations as in [21, p. 276]). Finally, for $t<\gamma$, the general curve in $\mathcal{M}^1_{g,t}$ has a unique base-point-free $\mathfrak{g}^1_t$ ([1, Theorem 2.6]).

Proposition 7.6. *(cf.* [4, Prop. 1]*) Fix positive integers g, t, r and a, with $a \geq 3$, $(a-2)(t-1) < g \leq (a-1)(t-1)$. Let $|D|$ be the b.p.f. linear series $\mathfrak{g}^1_t$ on a general t-gonal curve C of genus g. Then*

$$\dim(|rD|) = r, \ \text{ if } r \leq a-2.$$

With assumptions as in Proposition 7.6, if we consider

$$L_r := \mathcal{O}_C(K_C - rD) \tag{7.7}$$

then L_r is a special line bundle, of speciality $r+1$; in particular, $h^0(L_r) = g - r(t-1)$.

Example 7.8. Let a, g and t be positive integers as in Proposition 7.6. Let $[C] \in \mathcal{M}^1_{g,t}$ be general, with $2<t<\gamma$. Denote by D the divisor on C such that $|D|$ is the $\mathfrak{g}^1_t$ on C. Let $2 \leq l \leq a-1$ be any positive integer. As in (7.7), let

$$L := L_{l-1} = \omega_C \otimes \mathcal{O}_C(-(l-1)D).$$

Then

$$m := \deg(L) = 2g-2-(l-1)\,t \ \text{ and } \ h^1(L) = l.$$

If we further assume that

$$l \leq \frac{2g}{t(t-1)} - \frac{1}{t} - 1, \tag{7.9}$$

then L is also very ample (cf. [24, Theorem B]).

Let $d \geq 6g-5$ be an integer. Let $N \in Pic^{d-m}(C)$ be general. Then $\deg(N) = d+(l-1)t-2g+2 \geq 4g-3+(l-1)t$. Therefore, N is very ample and non-special.

Let $\mathcal{F} = N \oplus L$: the choice of $\mathcal{F}$ decomposable is not restrictive; indeed, since $d \geq 6g-5$ and L is special, from Proposition 3.18 any scroll in $\mathrm{Hilb}(d,g,l)$ is associated with a splitting vector bundle. From Theorem 3.11, the pair $(\mathcal{F}, C)$ determines a smooth, linearly normal scroll $S \subset \mathbb{P}^R$, $R = d-2g+1+l$, of degree d, genus g, speciality l, with special moduli and containing a unique special section Γ, which corresponds to L (cf. Theorem 3.11 (i)).

Scrolls arising from this constructions fill-up closed subschemes $\mathcal{Z}_{t,l}$ of $\mathrm{Hilb}(d,g,l)$, which depend on the following parameters:

- $2g+2t-5$, since C varies in $\mathcal{M}^1_{g,t}$, plus

- g, which are the parameters on which N depends, plus
- $(R+1)^2 - 1 = dim(\mathrm{PGL}(R+1,\mathbb{C}))$, minus
- $\dim(G_S)$, which is the dimension of the projectivities of $\mathbb{P}^R$ fixing a general S arising from this construction.

Since $\mathcal{F}$ is decomposable, from (6.6) it follows that

$$\dim(\mathcal{Z}_{t,l}) = (R+1)^2 + 8(g-1) - 4 - d - 2t(l-2). \tag{7.10}$$

On the other hand, if we assume $g \geq 4l$, it makes sense to consider also the irreducible components $\mathcal{H}^m_{d,g,l}$ of $\mathrm{Hilb}(d,g,l)$ with general moduli, which have been constructed in Theorem 6.1.

We claim that $\mathcal{Z}_{t,l}$ is not contained in any component of the type $\mathcal{H}^m_{d,g,l}$, for any m. From Remark 6.4 it suffices to show that $\dim(\mathcal{Z}_{t,l}) \geq \dim(\mathcal{H}^m_{d,g,l})$, with $m = 2g-2-(l-1)t$. In this case, by Theorem 6.1 - (ii), we get

$$\dim(\mathcal{H}^m_{d,g,l}) = (10-l)(g-1) - d - l^2 + t(l-1)(l-2) + (R+1)^2. \tag{7.11}$$

Thus, by using (7.10) and (7.11), we get

$$\dim(\mathcal{Z}_{t,l}) - \dim(\mathcal{H}^m_{d,g,l}) = (l-2)(g+1+l-t(l+1)). \tag{7.12}$$

From (7.9), one has

$$g+1+l-t(l+1) \geq 2+l+g\,\frac{t-3}{t-1} > 0$$

since $t \geq 3$. This implies that the difference is non-negative and therefore the assertion.

In the case $h^1 = 2$, we can be even more precise.

Proposition 7.13. *The irreducible components of* $\mathrm{Hilb}(d,g,2)$ *are either* $\mathcal{H}^m_{d,g,2}$ *or* $\mathcal{Z}_{t,2}$.

Proof. By Theorem 3.11, any scroll in $\mathrm{Hilb}(d,g,2)$ either belongs to a component $\mathcal{H}^m_{d,g,2}$ or to $\mathcal{Z}_{t,2}$. Since, from Remark (6.4) and from (7.12), all of them have the same dimension (in particular, independent from t) this proves the assertion. □

Remark 7.14. There are examples of components of the Hilbert scheme of linearly normal, smooth, special scrolls in $\mathbb{P}^R$, of degree d, genus g and speciality h^1, with $g < 4h^1$. According to Corollary 3.15, they have special moduli. Examples of such components are e.g. the $\mathcal{Z}_{3,l}$'s, with $l > 4$ and $g = 3l+4 < 4l$. Note that (7.9) is verified in this case with equality. From Proposition 7.6, $a = \lceil \frac{g}{2} \rceil + 1$ hence $l \leq a-1$. Therefore, the construction in Example 7.8 works and produces the examples in questions.

7.3. **Singular points of the Hilbert scheme.** Finally we will prove the existence of points of components of the type $\mathcal{H}^m_{d,g,h^1}$ corresponding to smooth scrolls with general moduli which are singular points of the Hilbert scheme.

Proposition 7.15. *Assumptions as in Theorem 6.1. Let* $[S] \in \mathcal{H}^m_{d,g,h^1}$ *be a smooth scroll with the special section* Γ *of degree* m *corresponding to* $[L] \in W^r_m(C)$ *such that* $\omega_C \otimes L^\vee$ *has base points. Then* $[S]$ *is a singular point of the Hilbert scheme.*

Proof. It is an immediate consequence of Theorem 6.1 and of Remark 5.23. □

Remark 7.16. One has examples of such singular points of the Hilbert scheme as soon as $\rho(g,h+1,m+1) = \rho(g,h^1-1,2g-3-m) \geq 0$. This is equivalent to the inequality $g \geq h^1 \frac{m+h^1+2}{h^1+1}$.

Similarly, the presence of base points for the linear series $|L|$ also produces singular points of the Hilbert scheme.

Proposition 7.17. *Let $[S] \in \mathcal{H}^m_{d,g,h^1}$ correspond to a smooth scroll with general moduli and with the special section Γ of degree $n < m$. Then $[S]$ is a singular point of the Hilbert scheme.*

Proof. Such a point $[S]$ also belongs to the component $\mathcal{H}^n_{d,g,h^1}$. □

Remark 7.18. The existence of such singular points is ensured by Corollary 4.4. This implies that $\mathcal{H}_{d,g,h^1}$ is connected.

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