Inductive construction of self-associated sets of points.

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§1. Introduction to the problem and historical background.

In 1889, Castelnuovo [Ca1] showed that if $P_1, \ldots, P_{2r+2}$ are $2r+2$ self-associated points in linearly general position in $\mathbb{P}^r$, not lying on a rational normal curve of degree r, then the $(r-2)$-plane of $\mathbb{P}^r$, spanned by any $r-1$ of them and denoted by Λ, is an $(r-1)$-secant plane to the unique rational normal curve of degree r, say C^r , through the remaining $r+3$ points. Moreover, the intersections of the $(r-2)$-plane and C^r together with the other points on Λ form a set of $2r-2 = 2(r-2)+2$ self-associated points in $\Lambda \cong \mathbb{P}^{r-2}$. Therefore, if we denote by Γ the set of self-associated points, we can divide it in two subsets Γ_1 and Γ_2 such that $\mid \Gamma_1 \mid= r+3$ and $\mid \Gamma_2 \mid= r-1$, respectively, where Γ_1 is the point set on the rational normal curve and Γ_2 spans Λ in such a way that $(\Lambda \cap C^r) \cup \Gamma_2$ is self-associated in Λ.

Castelnuovo defines two sets, each of $2r+2$ points, as **associated** if there exist two $(r+1)-gons$ (which are the configurations determined by linearly general $r+1$ points in $\mathbb{P}^r$) such that the points of one set projectively correspond to the $2r+2$ vertices of the two $(r+1)$- gons and the points of the other set to the $2r+2$ faces of the two $(r+1)$-gons. This means that the two sets of points are the set of vertices and face-baricenters of the two $(r+1)$-gons, in a suitable order. The particular case occurs when these two sets coincide, i.e. each point is homologous to itself, so that the points are called **self-associated**.

In a modern language, this is a particular case of the **Gale-Coble transform** (see, for example, [EP]).

Definition (Naive)

Let k be a field and $\Gamma \subset \mathbb{P}^r_k = \mathbb{P}(V)$ a set of γ labelled points, where $\gamma = r+s+2$, such that each $\gamma-1$ points span $\mathbb{P}^r_k$. If we consider the homogeneous coordinates of these γ points, we get a matrix $G \in M(\gamma \times (r+1); k)$ of rank $r+1$. If we dualize this matrix and take the kernel, we obtain a matrix

$$G' : k^{s+1} \longrightarrow (k^\gamma)^*.$$

An identification of $(k^\gamma)^*$ and k^γ allows us to see G' as a linear map

$$G' : k^{s+1} \longrightarrow k^\gamma.$$

The rows of this matrix determine a set, Γ', of γ points in $\mathbb{P}^s_k$, which is called the **Gale-Coble transform** of Γ. If $r = s$, then Γ has $2r+2$ points in $\mathbb{P}^r_k$ and Γ' has $2r+2$ points in the same projective space; up to the action of $PGL(r+1;k)$, we have $\Gamma' = \Gamma$.

The aim of this paper is to find conditions leading to a converse of Castelnuovo's result. From now on, we consider the projective space $\mathbb{P}^r$ over the complex field $\mathbb{C}$. For all the notation used and not explained the reader is referred to [H].

§2. Basic definitions and properties.

A fundamental property of a set of self-associated points, already explained by Castelnuovo [Ca1], is the following:

Each (hyper)quadric of $\mathbb{P}^r$, which passes through $2r+1$ points of a set of $2r+2$ self-associated points, passes also through the remaining one.

This is a characterization of self-associated points (see also [DO]).

Definition.

A linearly general point set is called **self-associated** if and only if its points impose one condition less on the quadrics of the space.

We can translate this situation by using the cohomological language. Let Γ be a 0-dimensional closed subscheme of $\mathbb{P}^r$, $r > 1$, with I_Γ its ideal sheaf. The exact sequence of sheaves

$$0 \to I_\Gamma \to O_{\mathbb{P}^r} \to O_\Gamma \to 0$$

defines, after twisting by $O_{\mathbb{P}^r}(2)$, a cohomological exact sequence

$$0 \to H^0(\mathbb{P}^r, I_\Gamma(2)) \to H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) \to H^0(\Gamma, O_\Gamma(2)) \to H^1(\mathbb{P}^r, I_\Gamma(2)) \to 0.$$

Denote by φ_2 the map $H^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) \to H^0(\Gamma, O_\Gamma(2))$, then its kernel consists of all the quadrics which vanish at Γ. Consider

$$Coker(\varphi_2) \cong H^0(\Gamma, O_\Gamma(2))/Im(\varphi_2) \cong H^1(\mathbb{P}^r, I_\Gamma(2))$$

and put $\delta(\Gamma, 2) = dim(Coker(\varphi_2)) = h^1(\mathbb{P}^r, I_\Gamma(2))$.

By the exact sequence, we get

$$h^0(\mathbb{P}^r, I_\Gamma(2)) = h^0(\mathbb{P}^r, O_{\mathbb{P}^r}(2)) + \delta(\Gamma, 2) - h^0(\Gamma, O_\Gamma(2)).$$

If the subscheme Γ is a reduced scheme, then $h^0(\Gamma, O_\Gamma(2)) =\mid supp(\Gamma) \mid$. We expect that each point from Γ imposes one condition on the hypersurfaces of degree 2 in order they pass through it. Therefore, $\delta(\Gamma, 2)$ is the number of the extra linearly independent quadrics passing through Γ.

We can restate the definition of self-associated points by saying:

A point set Γ, such that $\mid supp(\Gamma) \mid= 2r+2$ and $P_i \neq P_j$, for $i \neq j$, is self-associated in $\mathbb{P}^r$ if and only if $\delta(\Gamma, 2) = 1$.

A very familiar example of this situation is given by the case when the $2r+2$ points lie on a rational normal curve of $\mathbb{P}^r$ which is contained in $\binom{r}{2}$ quadrics of the space; indeed,

$$\binom{r}{2} = \binom{r+2}{2} + \delta(\Gamma, 2) - (2r+2) \Rightarrow \delta(\Gamma, 2) = 1.$$

Another example is the set determined by the $2r+2$ intersections of a general quadric in $\mathbb{P}^r$ and an elliptic curve of degree $r+1$ in the same projective space. To see a simple case of this fact, consider $C \subset \mathbb{P}^3$, an elliptic quartic curve, which is a smooth normal quartic, set theoretical complete intersection of two quadrics. The intersection with a general quadric, linearly independent from those determining the elliptic quartic, gives us a set of 8 points which belongs to a net of 3 quadrics in $\mathbb{P}^3$ and which is a self-associated point set, since

$$3 = 10 + \delta(\Gamma, 2) - 8 \ \Rightarrow \delta(\Gamma, 2) = 1.$$

We can completely generalize this remark, by using a fundamental result of Castelnuovo [Ca2] and Mumford [Mum], which states that if $deg(D) \geq 2g+1$ then the natural maps

$$\rho_m : Sym^m(H^0(O_C(D)) \rightarrow H^0(O_C(mD))$$

are surjective for all $m \geq 0$. In other words, if $L = O_C(D)$, Φ_L embeds C as a **projectively normal curve**. This reduces the computation of the number of hypersurfaces of degree m passing through C, that is the dimension of $ker(\rho_m)$, to Riemann-Roch.

In our case, we are considering an elliptic smooth curve in $\mathbb{P}^r$ such that $deg(D) = r+1$, then $h^0(O_C(D)) = r+1$ and $h^1(O_C(D)) = 0$. It is an elliptic curve of degree $r+1$ in $\mathbb{P}^r$, called the **normal elliptic curve**. By using the above mentioned result, since $deg(D) = r+1 \geq 3 (\Leftrightarrow r \geq 2)$, the map ρ_2 is surjective. It follows that

$$h^0(I_C(2)) = \binom{r+2}{2} - 2(r+1) = \frac{(r+1)(r-2)}{2},$$

i.e. the elliptic (r+1)-curve of $\mathbb{P}^r$ lies on exactly $\frac{(r+1)(r-2)}{2}$ quadrics. If we consider the intersection of this curve with a general quadric of $\mathbb{P}^r$, the $2r+2$ points lie on $\frac{(r+1)(r-2)}{2}+$

$1 = \binom{r}{2}$ quadrics of the space, then $\delta(\Gamma, 2) = 1$; therefore, Γ is a set of self-associated points in $\mathbb{P}^r$.

In low dimensions we have a complete characterization of sets of self-associated points (see [Ba]).

In $\mathbb{P}^1$ any 4 points are self-associated, since two sets of 4 points are associated, in the sense of Castelnuovo, if and only if they are projectively equivalent, i.e. they have the same cross-ratio.

In $\mathbb{P}^2$ six points are self-associated if and only if they lie on a conic, whereas in $\mathbb{P}^3$ the general case of 8 self-associated points is determined by the base locus of a net of 3 quadrics.

In $\mathbb{P}^4$, apart from any 10 points on a rational normal quartic, Bath [Bat] has shown that the general case of 10 self-associated points occurs when they lie on a normal elliptic quintic, being the intesections of this curve with a general quadric. Always in this general case, the rational normal quartic through any 7 of the 10 points meets the 2-plane, spanned by the remaining 3, in further 3 points such that the 6 points on the plane lie on a conic; this is Castelnuovo's statement.

§3. First examples for a converse.

The very first example is the trivial case of a point p, viewed as a projective space of dimension 0, i.e. $\mathbb{P}^0$. Consider it as a point of the projective plane $\mathbb{P}^2$ and choose 4 general points of the plane, say $\{p_1, \ldots, p_4\}$. We denote by C^2 the unique conic of the plane through the 5 points $\{p,\ p_1, \ldots,\ p_4\}$. A general point on C^2, together with the other 5 points, determines a set of 6 self-associated points of the plane. Therefore, also in this trivial case, by starting from a point, the set of 6 self-associated points is not uniquely determined.

We now start with a set of 4 distinct points in $\mathbb{P}^1$, which are always self-associated. Consider this $\mathbb{P}^1$ embedded as a line L in the projective space and denote the 4 points by $\{s_1,\ s_2,\ p_7,\ p_8\}$. Consider 4 general points in $\mathbb{P}^3$, none belonging to the line L, say $\{p_1,\ p_2,\ p_3,\ p_4\}$. The unique twisted cubic C^3, passing through $\{p_1,\ p_2,\ p_3,\ p_4,\ s_1,\ s_2\}$ has the line L as its chord through s_1 and s_2. This configuration is the intersection of 2 quadrics of the space, say Q_1 and Q_2, containing the twisted cubic, since this intersection is the union of a divisor of type (1,2) and one of type (1,0). Take Q_3 the general quadric through the 6 points $\{p_1,\ p_2,\ p_3,\ p_4,\ p_7,\ p_8\}$, which does not contain the line L or the twisted cubic. Therefore,

$$Q_1 \cap Q_2 \cap Q_3 = (C^3 \cup L) \cap Q_3 = 8 \text{ points.}$$

Among these 8 points we have:

a. 4 points on C^3, $\{p_1,\ p_2,\ p_3,\ p_4\}$;

b. 2 points on L, $\{p_7,\ p_8\}$;

the other 2 points must lie on the twisted cubic. Denote these points by $\{p_5,\ p_6\}$. The set $\Gamma_1 = \{p_1, \ldots, p_6\}$ lies on C^3, whereas $\Gamma_2 = \{p_7,\ p_8\}$ lies on L; $\Gamma_2 \cup \{s_1,\ s_2\}$ are self-associated in L and such that $L \cap C^3 = \{s_1,\ s_2\}$. Finally, $\Gamma = \Gamma_1 \cup \Gamma_2$ lies on a net of 3 quadrics or, more precisely, it is the base locus of the net

$$\{\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3 \mid \ \lambda_i \in k\};$$

by definition, this means that Γ is a set of 8 self-associated points in $I\!P^3$.

§4. Case in $I\!P^4$.

Here the case is the one with 10 self-associated points in linearly general position in $I\!P^4$, which was considered by Bath [Bat]. From the discussion in section 2, we know that simple examples of this configuration are given by the intersection of a normal elliptic quintic with a general quadric not containing the curve or by any set of 10 points on a rational normal quartic of $I\!P^4$ (which lies on $\binom{4}{2} = 6$ quadrics).

Bath proved that the first of these examples is the general case of $I\!P^4$, and, always in this general case, the unique rational normal quartic through any 7 points out of that set meets the 2-plane spanned by the remaining 3 in 3 further points (so the 2-plane is a 3-secant plane to the rational quartic). These 6 points on the 2-plane lie on a conic so that this set is self-associated in $I\!P^2$.

This was the statement of Castelnuovo, but the arguments he used to prove it are very different from the ones of Bath. He also pointed out that the method of the proof of this second result only can be used to generalize the situation to 2r+2 self-associated points in $I\!P^r$. In fact, the first result of Bath does not extend to the general case of $I\!P^r$. This is because, for example, Babbage [Ba] proved that the most general configuration of 12 points, which are self-associated in $I\!P^5$, is determined by the fact that the normal elliptic sextic, through any 9 of the 12 points, meets the 2-plane, spanned by the other 3 points, in 3 further points and these 6 lie on a conic. This means that in the general case of $I\!P^5$, 12 self-associated points do not lie on a normal elliptic sextic. So, in dimension $r \geq 5$, we do not have a characterization of the most general configuration of such points.

We want to use Bath result for a construction of a Castelnuovo converse in $I\!P^4$; to do this, we start by analyzing a rational normal scroll of degree 3 in $I\!P^4$.

Let $I\!P^2(x_0)$ be the blowing-up of the projective plane at a point and consider the linear system given by the divisor

$$H = 2l - E_0,\ \textit{such that}\ l^2 = 1,\ E_0^2 = -1,\ E_0 \cdot l = 0,$$

which is the linear system of the conics through x_0. This is very ample of projective dimension 4, so it defines an embedding Φ_H in $\mathbb{P}^4$ whose image is a smooth surface S of degree $H \cdot H = 3$ not contained in any hyperplane; therefore, it is a surface of minimal degree. The image of a line l not passing through x_0 is a conic on S, since $l \cdot H = 2$; on the other hand, if l' is a line through the point x_0, its image is a line contained in S, whereas x_0 becomes the exceptional divisor E_0, which is skew with respect to the 2-plane that contains the conic coming from the line l. Therefore, by Del Pezzo theorem (see [GH]), this surface is a rational normal scroll of type $S_{1,2}$. Let $F \in | 3l - 2E_0 |$ be a divisor belonging to the linear series defined by the plane cubics with a double points in x_0; we get

$$dim \mid 3l - 2E_0 \mid = 6$$

and $deg(\Phi_H(F)) = (3l - 2E_0) \cdot (2l - E_0) = 4$, i.e. the image of a divisor in the given linear series is a quartic contained in the scroll. This quartic is not contained in any hyperplane, since $| H - F | = \emptyset$; therefore this is a rational normal quartic.

In the same way, we consider a divisor D in the linear system $| 3l - E_0 |$; then $dim | 3l - E_0 | = 8$, and the image of D is an elliptic quintic not contained in a hyperplane of $\mathbb{P}^4$. On the other hand, a divisor $G \in | l |$ (the linear series of the lines not through x_0), maps to a conic on the scroll and $dim | l | = 2$.

An easy calculation shows

$$F \cdot D = (3l - 2E_0) \cdot (3l - E_0) = 9 - 2 = 7 \, points;$$

$$G \cdot F = (3l - 2E_0) \cdot (l) = 3 \, points;$$

$$G \cdot D = (l) \cdot (3l - E_0) = 3 \, points;$$

this suggests us a way to find a possible converse of the Castelnuovo statement in $\mathbb{P}^4$.

We start, as in the cases above, with a $\mathbb{P}^2$ and 6 self-associated points on this plane, say $\{s_1, \ s_2, \ s_3, \ p_8, \ p_9, p_{10}\}$, so they lie on a conic C^2. We can see this $\mathbb{P}^2$ as a 2-plane in $\mathbb{P}^4$; take a line ($\mathbb{P}^1$), which is skew to this 2-plane and then construct a scroll $S_{1,2}$ of degree 3 in $\mathbb{P}^4$. By the calculations above, we know that on this scroll a linear system of rational normal quartics "lives" of (projective) dimension 6; therefore, we can impose to the quartics of this linear system to pass through the points $\{s_1, \ s_2, \ s_3\}$. In the same way, we know that on the scroll there are ∞^8 normal elliptic quintics and we can impose the 3 independent conditions of passing through the other 3 points on the conic, $\{p_8, \ p_9, \ p_{10}\}$. The 0-dimensional scheme Γ, obtained by

$$D \cdot (F + G)$$

is a set of 10 points on an elliptic quintic. We only have to prove that this divisor on D is equivalent to a divisor cut by a quadric of the space. If we consider $F+G$ as a divisor on the scroll S, we get

$$F+G \sim 3l-2E_0+l=4l-2E_0=2H,$$

i.e. the divisor $F+G$ belongs to the linear series $\mid 2H \mid$ on the scroll. Since the rational normal scroll is arithmetically Cohen-Macaulay, i.e.

$$(i)\ H^0(O_{I\!P^4}(h)) \to H^0(O_S(h)) \to 0,\ \forall\, h \in \mathbf{Z};$$

$$(ii)\ H^1(O_S(h))=0,\ \forall\, h \in \mathbf{Z},$$

the linear series cut, on the scroll, by the quadrics of $I\!P^4$ are complete linear series; therefore, $2 \mid H \mid = \mid 2H \mid$ (in general, we denote by $d \mid H \mid$ the linear system E_d cut out on S by the hypersurfaces of degree d, see [ACGH]). This means that the subscheme Γ is the intersection of a normal elliptic quintic and a quadric of $I\!P^4$; moreover, since each normal elliptic quintic is contained in 5 quadrics of the space, the 10 points lie on 6 linearly independent quadrics, that is $\delta(\Gamma,2)=1$.

§5. Conjecture in higher dimensions.

As mentioned above, the situation in $I\!P^r$, for $r>4$, is much more complicated, because the case of the intersection of a normal elliptic curve of degree $r+1$ with a general quadric not containing it is not the general case in this dimension. The situation now is, as always, the projective space $I\!P^{r-2}$, viewed as an $(r-2)$-plane Λ in $I\!P^r$, with a set of $2r-2$ self-associated points. We can divide this set of points in two subsets, say Δ_1 and Δ_2, each of cardinality $r-1$ and then consider 4 general points in $I\!P^r$ and the unique rational normal curve of degree r passing through these 4 points and those of one of these sets, for example Δ_1.

This set will play the role of the $r-1$ point set of intersection of the $(r-2)$-plane spanned by the others in Δ_2, i.e. Δ_2 coincides with the set Γ_2 of §1. We would like to find further $r-1$ points on this rational normal curve in such a way they form, together with the 4 general chosen points, the set Γ_1.

A useful observation is the fact that in $I\!P^r$ there are $r-1$ linearly independent quadrics which contain the rational normal curve and the $(r-2)$-plane. Suppose, in fact, coordinates are chosen in the projective space in such a way that the $(r-2)$-plane has equations

$$x_0=x_1=0,$$

then the quadrics of the space containing this $(r-2)$-plane are of the form

$$F_r := \{x_0 l_0 + x_1 l_1 \mid l_o,\ l_1 \in (\mathbb{C}[x_0, \dots, x_r])_1\},$$

whose linear dimension is $2r+1$. In order to contain the rational normal curve, we have to impose they pass through further $r-2$ points on it, since Γ_2 already lies on the $r-2$-plane. Therefore, we get $2r+1-(r-2) = r-1$ linearly independent such quadrics.

We think that, by using the fact that the $2r-2$ points in $\mathbb{P}^{r-2}$ lie on exactly $\binom{r-2}{2}$ quadrics, the fundamental step is to find a suitable rational normal scroll of degree $r-1$ in $\mathbb{P}^r$ such that it passes through the 4 points on C^r, the points on the $(r-2)$-plane of Γ_2 and cuts on C^r further $r-1$ points; moreover, since each scroll is the intersection of $\binom{r-1}{2}$ quadrics, we have to find this scroll in such a way that the quadrics determining it are linearly independent from those containing $C^r \cup \Lambda$. This would imply that these $2r+2$ points form a set Γ which lies on $(r-1) + \binom{r-1}{2} = \binom{r}{2}$ quadrics, then $\delta(\Gamma, 2) = 1$.

There is a result of Fano [Fa] about the rational normal scrolls in a projective space $\mathbb{P}^r$. He proved that there are ∞^{r-1} scrolls of degree $r-1$ which contain a fixed rational normal curve of degree r. This fact suggests to consider also the other rational normal curve in our configuration; more precisely, the one containing the set Γ_1 and the 4 general points. Denote it by D^r. Therefore, these rational normal curves share the 4 general points and are such that one passes through Γ_1 and the other through Γ_2. We know that in the ideal I_{C^r} we can find $r-1$ quadrics containing Λ and by Fano result there are ∞^{r-1} scrolls of degree $r-1$ containing D^r. We would like to find a way to impose a suitable number of conditions on these scrolls in such a way we can find the desired numbers of points and of independent quadrics, as explained before.

Further investigations in dimension $r > 4$ might lead to a general construction in $\mathbb{P}^r$.

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References

[ACGH] Arbarello E., Cornalba M., Griffiths P.A. and Harris J.,*Geometry of Algebraic Curves*, Springer, Berlin, 1985.

[Ba] Babbage D.W., Twelve associated points in [5], *J. London Math. Soc.*, **23** (1948), 58-64.

[Bat] Bath F., Ten associated points and quartic and quintic curves in [4], *J. London Math. Soc.*, **13** (1938), 198-201.

[Ca1] Castelnuovo G., Su certi gruppi associati di punti, *Rendiconti Circ. Matematico Palermo*, **13** (1889), 179-192.

[Ca2] Castelnuovo G., Sui multipli di una serie lineare di gruppi di punti appartenenti ad una curva algebrica, *Rendiconti Circ. Matematico Palermo*, **17** (1893), 99-119.

[DO] Dolgachev I. and Ortland D., Point sets in projective spaces and theta functions, *Astérisque* **165**, Société Mathématique de France, 1988.

[EP] Eisenbud D. and Popescu S., Gale Duality and Free Resolutions of Ideals of Points, (to appear) (1996).

[Fa] Fano G., A proposito di un lavoro del Sig. Ramamurti (Sulle rigate razionali normali), *Atti Scienze To*, **71** (1936), 105-109.

[GH]Griffiths P. and Harris J., *Principles of Algebraic Geometry.*, J. Wiley and Sons, New York, 1978.

[H] Hartshorne R., *Algebraic Geometry* , Springer, Berlin, 1977.

[Mum] Mumford D., Varieties defined by quadratic equations, Corso CIME 1969, *Questions on Algebraic Varieties*, Rome (1970), 30-100.