SOLUZIONI TUTORATO 8

1a) $\begin{vmatrix} x & y & z \\ 1 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = 0 \iff x\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix} - y\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + z\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 0$

$\iff -2x + y - z = 0$

1b) $\vec{v}_1 \wedge \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 1 & 1 & -1 \end{vmatrix} = -2\vec{i} + \vec{j} - \vec{k} = \begin{pmatrix} -2 \\ +1 \\ -1 \end{pmatrix}$

1c) $U = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$, dato che $\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$ e' una base di U posso scrivere

$U^\perp: \begin{cases} x + 2y = 0 \\ x + y - z = 0 \end{cases} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{array}{c} \begin{array}{ccc} & & \lambda \end{array} \\ \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \end{pmatrix} \end{array}$

$U^\perp: \begin{cases} x + 2y = 0 \\ -y - \lambda = 0 \\ z = \lambda \end{cases} \iff \begin{cases} x = 2\lambda \\ y = -\lambda \\ z = \lambda \end{cases}, \lambda \in \mathbb{R} \iff \left\{\lambda\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}\right\} \iff \left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}$ e' una base di $U^\perp$

2a) $r: \left\{\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R}\right\}$

2b) $s: \left\{\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}\right\}$

3a) $\vec{AB} = (-2, 1, -3)$, $\vec{AC} = (-1, 1, k)$

$\pi: \left\{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 1 \\ k \end{pmatrix}\right\}, \quad k \in \mathbb{R}$ eq. parametrica

$\lambda, \mu \in \mathbb{R}$

$: \begin{vmatrix} x & y & z \\ -2 & 1 & -3 \\ -1 & 1 & k \end{vmatrix} = 0 \iff (k+3)x - y(-2k-3) + z(-1) = 0$ eq. cartesiana

$\iff (k+3)x + (2k+3)y - z = 0$

3b) AREA TRIANGOLO ABC $= \frac{1}{2} \| \vec{AB} \wedge \vec{AC} \|$

$$= \frac{1}{2} \| (k+3, 2k+3, -1) \| =$$

$$\vec{AB} \wedge \vec{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & -3 \\ -1 & 1 & k \end{vmatrix} = (k+3,\ 2k+3,\ -1)$$

$$= \frac{1}{2} \sqrt{k^2+6k+9+4k^2+12k+9+1} =$$

$$= \frac{1}{2} \sqrt{5k^2+18k+19} = 1$$

$$\Longleftrightarrow \quad 5k^2+18k+19-4=0$$

$$\Longleftrightarrow \quad k_{12} = \frac{-9 \pm \sqrt{81-75}}{5} = \frac{-9 \pm \sqrt{6}}{5}$$

4a)

$$\begin{cases} 2x-y=1 \\ x+2y+z=0 \\ x-2z=0 \end{cases} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & 4 & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 + \frac{1}{2} R_2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 4+\frac{3}{2} & 1 \end{pmatrix}$$

$$\pi_1 \cap \pi_2 \cap \pi_3 : \begin{cases} x-2z=0 \\ 2y+3z=0 \\ \frac{11}{2} z = 1 \end{cases} \longrightarrow \begin{cases} x = 4/11 \\ y = -\frac{3}{2} z = -\frac{3}{\cancel{2}} \cdot \frac{\cancel{2}}{11} = -\frac{3}{11} \\ z = \frac{2}{11} \end{cases} \longrightarrow \pi_1 \cap \pi_2 \cap \pi_3 = \begin{pmatrix} 4/11 \\ -3/11 \\ 2/11 \end{pmatrix} = P$$

4b) $\pi_1 \cap \pi_2 : \begin{cases} 2x-y=1 \\ x+2y+z=0 \end{cases}$

Siccome siamo in $\mathbb{R}^3$ possiamo calcolarne la direzione con il prodotto vettoriale

$$\vec{d} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = (-1, -2, 5)$$

siccome π_4 e' perpendicolare a d possiamo scriverne le eq. cartesiane

$$\pi_4 : -x-2y+5(z-1)=0$$

5a) $\begin{cases} 3x + y = 1 \\ 4x + y - z = 1 \\ 2x + z = -1 \\ (k+1)x - y + 2z = 1 \end{cases} \longrightarrow \left(\begin{array}{ccc|c} 3 & 1 & 0 & 1 \\ 4 & 1 & -1 & 1 \\ 2 & 0 & 1 & -1 \\ k+1 & -1 & 2 & 1 \end{array}\right) = (A|b)$, ne calcolo il rango

$M = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix}$, $\quad A = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \\ k+1 & -1 & 2 \end{pmatrix}$

$$\det(M) = \begin{vmatrix} 3 & 1 & 0 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = -2-1 = -3 \neq 0$$

$\Longrightarrow$ $rg(A) = 3$ in quanto ha una sottomatrice quadrata 3×3 con $\det \neq 0$

$$\det(A|b) = \begin{vmatrix} 3 & 1 & 0 & 1 \\ 4 & 1 & -1 & 1 \\ 2 & 0 & 1 & -1 \\ k+1 & -1 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 1 & 1 \\ 4 & 1 & 1 \\ k+1 & -1 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 3 & 1 & 0 \\ 4 & 1 & -1 \\ k+1 & -1 & 2 \end{vmatrix} =$$

$$= 2\left[\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}\right] + \left[(k+1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}\right] +$$

$$+ \left[(k+1)\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}\right] =$$

$$= 2\left[-3+1\right] + \left[0-1-1\right] + \left[-(k+1)-3-2\right] =$$

$$= -4-2-k-1-3-2$$

$$= -k-12$$

In conclusione se $k = -12$ $\det(A|b) \neq 0 \Longrightarrow rg(A|b) = 4$ ma $rg(A) = 3$
$\Longrightarrow r_1$ e r_2 sono sghembe

se $k \neq -12$ $\det(A|b) = 0 \Longrightarrow r_1$ e r_2 sono incidenti

5b) Siccome siamo in $\mathbb{R}^3$, posso trovare le direzioni di r_1 e r_2 col prodotto vettoriale

$$\vec{d}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 0 \\ 4 & 1 & -1 \end{vmatrix} = (-1, 3, -1)$$

$$\vec{d}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ k+1 & -1 & 2 \end{vmatrix} = (1, -(4-k-1), -2) = (1, k-3, -2)$$

↳ r_1 e r_2 sono perpendicolari se $\langle \vec{d}_1, \vec{d}_2 \rangle = 0$

$$\Leftrightarrow \left\langle \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ k-3 \\ -2 \end{pmatrix} \right\rangle = -1 + 3(k-3) + 2 = 3k - 8 \overset{!}{=} 0 \Leftrightarrow k = \frac{8}{3}$$