(1a) $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$, $\det(A) = 2 - (-1) = 3$

$B = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$, $\det(B) = 3 - 10 = -7$

(1b) $A^T = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $\det(A^T) = 2 - (-1) = 3$

$B^T = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$, $\det(B^T) = 3 - 10 = -7$

} nota che $\det(A) = \det(A^T)$ sempre.

$AB = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 1 & -2 \end{bmatrix}$, $\det(AB) = -8 - 13 = -21$

$A+B = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}$, $\det(A+B) = 12 - 6 = 6$ } nota che $\det(A+B) \neq \det(A) + \det(B)$

(2a) $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & -1 & 2 & 0 \\ 1 & 2 & 2 & 1 & 2 \\ 3 & 2 & 4 & 3 & 2 \end{bmatrix}$ Uso la regola di Sarrus.

$$\det(A) = 16 + 0 - 2 - (-6 + 8 + 0) = 14 - 2 = 12$$

$B = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 1 & 5 \\ 4 & 2 & 3 \end{bmatrix}$, $\begin{vmatrix} 2 & 1 & 2 & 2 & 1 \\ -1 & 1 & 5 & -1 & 1 \\ 4 & 2 & 3 & 4 & 2 \end{vmatrix}$, $\det(B) = 6 + 20 - 4 - (8 + 20 - 3) =$

$= 22 - 25 = -3$

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(3) $A = \begin{bmatrix} 1 & -2 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ A ha una colonna nulla $\Longrightarrow \det(A) = 0$.

$B = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

Sviluppo di Laplace rispetto alla terza riga, in quanto è quella con più elementi uguale a zero.

$$\det(B) = \cancel{(-1)^{1+3} \cdot 0 \cdot \begin{vmatrix} -2 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}} + \cancel{(-1)^{2+3} \cdot 0 \cdot \begin{vmatrix} 1 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 3 \end{vmatrix}}$$

$$+ (-1)^{3+3} \cdot 1 \cdot \begin{vmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{vmatrix} + (-1)^{4+3} \cdot (-1) \cdot \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \underbrace{1 \cdot \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}}_{(*)} + \underbrace{(-1) \cdot 1 \cdot \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & +3 \\ 0 & 2 \end{vmatrix}}_{(**)}$$

(*) SVILUPPO RISP. 1° RIGA

(**) SVILUPPO RISP. 2° RIGA

$$= 1\cdot(6-2)+2(3)-1(-4-3)+2(2-3)=$$
$$= 4+6+7+4=21$$

$$C=\begin{bmatrix}1&2&0&2\\1&-2&3&0\\0&0&1&-1\\0&1&2&3\end{bmatrix},$$

notiamo che C si può ottenere da B scambiando le prime due righe, essendo il determinante alternante per scambio di righe o colonna, abbiamo:

$$\det(C) = -\det(B) = -21$$

$$D=\begin{bmatrix}1&-2&3&0\\1&2&0&4\\0&0&1&-2\\0&1&2&6\end{bmatrix},$$

notiamo che D si ottiene da B moltiplicando la quarta colonna per 2. Essendo il det multilineare sulle righe e sulle colonne, abbiamo:

$$\det(D) = 2\cdot\det(B) = 42$$

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(4) $$A=\begin{bmatrix}1&-5&3\\0&k+1&-1\\0&0&k-2\end{bmatrix}$$

- A ha un elemento non nullo (ad es: $a_{11}=1$) $\Rightarrow rg(A)\geq 1$
- $\begin{vmatrix}1&-5\\0&k+1\end{vmatrix} = k+1 \neq 0 \iff k\neq -1 \Rightarrow rg(A)\geq 2$ se $k\neq -1$
- se $k=-1$

$$A=\begin{bmatrix}1&-5&3\\0&0&-1\\0&0&-3\end{bmatrix}$$ e $\det(A)=0 \Rightarrow rg(A)=2$

- $\det(A) = 1\cdot(k+1)\cdot(k-2)\neq 0 \iff k\neq -1,2$
- se $k\neq -1,2$ $\det(A)\neq 0 \Rightarrow rg(A)=3$
- se $k=2$

$$A=\begin{bmatrix}1&-5&3\\0&3&-1\\0&0&0\end{bmatrix}$$ e $\det(A)=0 \Rightarrow rg(A)=2$

se A è triangolare il determinante è il prodotto degli elementi diagonali.

in conclusione

$$rg(A)=\begin{cases}3 & k\neq -1, 2\\ 2 & \text{altrimenti.}\end{cases}$$

$$B = \begin{bmatrix} k & -k & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & k & 1 \end{bmatrix}$$

- $b_{21} = 1 \ \forall k \in \mathbb{R} \Rightarrow \rho(B) \geq 1 \ \forall k \in \mathbb{R}$

- $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \Rightarrow \rho(B) \geq 2 \ \forall k \in \mathbb{R}$

- $\begin{vmatrix} -k & 0 & 1 \\ -2 & 1 & 0 \\ 1 & k & 1 \end{vmatrix} = -k \begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 1 \\ 1 & k \end{vmatrix} = -k - 2k - 1 = -1 - 3k \neq 0$
  $\Leftrightarrow k \neq -1/3$

- se $k = -1/3$

$$B = \begin{bmatrix} -1/3 & 1/3 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1/3 & 1 \end{bmatrix}$$

$$\begin{vmatrix} -1/3 & 1/3 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1/3 \end{vmatrix} \begin{matrix} -1/3 & 1/3 \\ 1 & -2 \\ 0 & 1 \end{matrix} = \left(-\frac{2}{9} + 0 + 0\right) - \left(0 - 1/3 - 1/9\right) = -\frac{2}{9} + \frac{4}{9} = \frac{2}{9}$$

- Conclusioni

$$\rho(B) = \begin{cases} 3 & \forall k \in \mathbb{R} \end{cases}$$

$$C = \begin{bmatrix} 1 & 2 & k \\ 0 & 1 & 0 \\ 3 & 2 & k \\ k & k+1 & 0 \end{bmatrix}$$

- $\rho(C) \geq 1$ perché $c_{11} = 1$

- $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow \rho(C) \geq 2 \ \forall k \in \mathbb{R}$

- $\begin{vmatrix} 1 & 2 & k \\ 0 & 1 & 0 \\ 3 & 2 & k \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & k \\ 3 & k \end{vmatrix} = k - 3k = -2k \neq 0 \Leftrightarrow k = 0$

- se $k = 0$

$C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 3 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ e $\begin{vmatrix} 0 & 1 & 2 \\ 3 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 6 \neq 0$

$\Rightarrow \rho(C) = 3 \ \forall k \in \mathbb{R}$

5) $\begin{bmatrix} 1 & -4 & 2 & 0 \\ 1 & t+1 & -1 & 1 \\ 0 & -2 & t-3 & 1 \end{bmatrix} = (A|b)$

il sistema ha un'unica soluzione sse $rg(A) = 3 \Leftrightarrow \det(A) \neq 0$

$$\begin{vmatrix} 1 & -4 & 2 \\ 1 & t+1 & -1 \\ 0 & -2 & t-3 \end{vmatrix} = (-1)\cdot(-2)\cdot\begin{vmatrix} 1 & 2 \\ 0 & t-3 \end{vmatrix} + (t-3)\begin{vmatrix} 1 & -4 \\ 1 & t+1 \end{vmatrix} =$$

$$= 2(t-3) + (t-3)[(t+1)+4] =$$

$$= 2(t-3) + (t-3)(t+5) =$$

$$= (t-3)(t+5+2) = (t-3)(t+7)$$

$\Rightarrow \det(A) \neq 0 \Leftrightarrow t \neq 3, -7$. Inoltre

$$sol(Ax=b) = \left(\begin{vmatrix} 0 & -4 & 2 \\ 1 & t+1 & -1 \\ 1 & -2 & t-3 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & t-3 \end{vmatrix}, \begin{vmatrix} 1 & -4 & 0 \\ 1 & t+1 & 1 \\ 0 & -2 & 1 \end{vmatrix} \right) \cdot \frac{1}{(t-3)(t+7)}$$

$$= (2t-14, t, t+7) \cdot \frac{1}{(t-3)(t+7)}$$

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6) $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow A^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$, Calcolo B^{-1} usando GAUSS.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 2 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 5 \end{array}\right] \xrightarrow{R_4 \to \frac{1}{5} R_4} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 2 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 2 \\ 0 & 1/2 & 1/10 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{array}\right] \xleftarrow{R_2 \to \frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 4 & 2 \\ 0 & 1 & 1/5 & 0 & 2 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_2$
$R_1 \to R_1 - 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2/5 - 1/5 & 1 & 0 & 0 \\ 0 & 1/2 & 1/10 & 0 & 1 & 0 \\ 0 & 0 & 1/5 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & -3/5 \\ 0 & 1/2 & 1/10 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$C = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & 2 \\ 2 & 0 & 7 \end{bmatrix}$, calcolo C^{-1} usando la matrice dei cofattori.

$\det(C_{1,1}) = \begin{vmatrix} 1 & 2 \\ 0 & 7 \end{vmatrix} = 7 \qquad \det(C_{2,1}) = \begin{vmatrix} 1 & 3 \\ 0 & 7 \end{vmatrix} = 7$

$\det(C_{1,2}) = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7-4=3 \qquad \det(C_{2,2}) = \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} = 3$

$\det(C_{1,3}) = \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2 \qquad \det(C_{2,3}) = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 2$

$\det(C_{3,1}) = \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} = -5 \qquad \det(C_{3,3}) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

$\det(C_{3,2}) = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1 \qquad \det(C) = 4$

$$C^{-1} = \frac{1}{\det(C)} \cdot \begin{bmatrix} (-1)^{1+1}\det(C_{1,1}) & (-1)^{2+1}\cdot\det(C_{2,1}) & (-1)^{3+1}\det(C_{3,1}) \\ (-1)^{1+2}\det(C_{1,2}) & (-1)^{2+2}\det(C_{2,2}) & (-1)^{2+3}\det(C_{2,3}) \\ (-1)^{1+3}\det(C_{1,3}) & (-1)^{3+2}\det(C_{3,2}) & (-1)^{3+3}\det(C_{3,3}) \end{bmatrix}$$

$$= \frac{1}{4} \cdot \begin{bmatrix} 7 & 7 & -5 \\ -3 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$
