SOLUZIONI TUTORATO 10

1) a) - $f(0,0) = (0,0,0)$ ok

- $\forall (x,y),(x',y') \in \mathbb{R}^2 \Rightarrow f(x+x', y+y') = (x+y+x'+y', 2x+2x', x-y+x'-y')$
$= (x+y, 2x, x-y) + (x'+y', 2x', x'-y')$
$= f(x,y) + f(x',y')$ ok

- $\forall (x,y) \in \mathbb{R}^2, \forall \lambda \in \mathbb{R} \Rightarrow f(\lambda x, \lambda y) = (\lambda(x+y), \lambda(2x), \lambda(x-y)) =$
$= \lambda(x+y, 2x, x-y)$
$= \lambda f(x,y)$ ok

b) Siano $(0,-1)$ e $(0,1)$ in $\mathbb{R}^2$

$g(0,-1) + g(0,1) = (0,1) + (0,+1) = (0,2)$
$g((0,-1)+(0,1)) = g(0,0) = (0,0)$ $\times$ non e' lineare

c) $\ker(f)$: $\begin{cases} x+y=0 \\ 2x=0 \\ x-y=0 \end{cases} \longrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \rightarrow \ker(f) = \{\vec{0}\}$

CONOSCO SUBITO LE SUE EQ. CARTESIANE

$\mathrm{Im}(f) = \mathrm{span}\left\{ f\begin{pmatrix}1\\0\end{pmatrix}, f\begin{pmatrix}0\\1\end{pmatrix} \right\} = \mathrm{span}\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}$, $\dim \mathrm{Im}(f) = 2$

d) f e' iniettiva, f non e' suriettiva

e) $M_{EE}(f) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix}$

(f) $f(1,2) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

2) a) $M_{EE}(f) = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$f(x,y,z) = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y+2z \\ 2x+2z \\ x-y \end{bmatrix}$

②

b) $\ker(\varphi) = \begin{cases} 2y+2z=0 \\ 2x+2z=0 \\ x-y=0 \end{cases} \longrightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \rightarrow \ker(\varphi)=\{\underline{0}\}$

$\dim \operatorname{Im}(\varphi) = 3 - \dim\ker(\varphi) = 3 \implies \operatorname{Im}(\varphi) = \mathbb{R}^3 \implies \left\{ \begin{pmatrix}0\\2\\1\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix} \right\}$ è una base di $\operatorname{Im}(\varphi)$

c) φ è iniettiva e suriettiva $\implies$ φ è biettiva

d) ok

e)

$(4,4,0) \in \operatorname{Im}(\varphi) \iff \exists\,(x,y,z): \begin{cases} 2y+2z=4 \\ 2x+2z=4 \\ x-y=0 \end{cases} \rightarrow \begin{cases} z=2-y \\ y=1 \\ x=y \end{cases} \rightarrow \begin{cases} x=1 \\ y=1 \\ z=1 \end{cases}$

3) $\varphi(x,y,z,w) = (-x+z,\ 2y,\ x-2z,\ w)$

a) $M_{\mathcal{E}\mathcal{E}}(\varphi) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$|M_{\mathcal{E}\mathcal{E}}(\varphi)| = (+1)\cdot\begin{vmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = (1)\cdot(2)\cdot\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 2\cdot(1) = 2 \neq 0$

$\rho(M_{\mathcal{E}\mathcal{E}}(\varphi)) = 4 \implies \dim\operatorname{Im}(\varphi) = 4 \implies \dim\ker(\varphi) = 0$

b) φ è invertibile, φ^{-1} ha come matrice associata $M_{\mathcal{E}\mathcal{E}}(\varphi)^{-1}$

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&-1&0&1&0 \\ 0&1&0&0&0&2&0&0 \\ 0&0&1&0&1&0&-2&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right] \xrightarrow{R_3 \to R_3+R_1} \left[\begin{array}{cccc|cccc} 1&0&0&0&-1&0&1&0 \\ 0&1&0&0&0&2&0&0 \\ 1&0&1&0&0&0&-1&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1+R_3} \left[\begin{array}{cccc|cccc} 2&0&1&0&-1&0&0&0 \\ 0&1&0&0&0&2&0&0 \\ 1&0&1&0&0&0&-1&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right] \xrightarrow{\substack{R_1 \to -R_1 \\ R_2 \to \frac{1}{2}R_2 \\ R_3 \to -R_3}} \left[\begin{array}{cccc|cccc} -2&0&-1&0&1&0&0&0 \\ 0&\frac{1}{2}&0&0&0&1&0&0 \\ -1&0&-1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{array}\right]$$

$\implies \varphi^{-1}(x,y,z,w) = (-2x-z,\ \tfrac{1}{2}y,\ -x-z,\ w)$

4) $B = \{(2,1,0),\ (1,1,0),\ (0,1,1)\}$ base di $\mathbb{R}^3$,

$\varphi(x,y,z) = (3x-2y,\ x+y+z,\ 2x-3y-z)$

a) $$M_{\mathcal{E}\mathcal{E}}(\varphi) = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix}$$

b)
$$\begin{tikzcd} \varphi : \mathbb{R}^3_B \arrow[r, "M_{\mathcal{E}B}(\varphi)"] \arrow[d, "M_{\mathcal{E}B}(id)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R}^3_{\mathcal{E}} \\ \mathbb{R}^3_{\mathcal{E}} \arrow[ur, "M_{\mathcal{E}\mathcal{E}}(\varphi)"'] & \end{tikzcd}$$

TROVARE $M_{\mathcal{E}B}(\varphi)$

: $M_{\mathcal{E}B}(\varphi) = M_{\mathcal{E}\mathcal{E}}(\varphi) \cdot M_{\mathcal{E}B}(id)$

$$M_{\mathcal{E}B}(id) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ MATRICE CAMBIO BASE

$$M_{\mathcal{E}B}(\varphi) = \begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -2 \\ 3 & 2 & 2 \\ 1 & -1 & -4 \end{bmatrix}$$

c) TROVARE $M_{B\mathcal{E}}(\varphi)$

$$\begin{tikzcd} \mathbb{R}^3_{\mathcal{E}} \arrow[r, "M_{B\mathcal{E}}(\varphi)"] \arrow[dr, "M_{\mathcal{E}\mathcal{E}}(\varphi)"'] \arrow[dr, phantom, shift left=3, "\circlearrowright"] & \mathbb{R}^3_B \\ & \mathbb{R}^3_{\mathcal{E}} \arrow[u, "M_{B\mathcal{E}}(id)"'] \end{tikzcd}$$

: $M_{B\mathcal{E}}(\varphi) = M_{B\mathcal{E}}(id) \cdot M_{\mathcal{E}\mathcal{E}}(\varphi)$

$$M_{B\mathcal{E}}(id) = \left[M_{\mathcal{E}B}(id)\right]^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{B\mathcal{E}}(\varphi) = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 1 & 1 & 1 \\ 2 & -3 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -6 & -2 \\ -5 & 10 & 4 \\ 2 & -3 & -1 \end{bmatrix}$$