CORRIGENDUM TO THE PAPER "ON THE K^2 OF DEGENERATIONS OF SURFACES AND THE MULTIPLE POINT FORMULA"

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Abstract. We correct an error in the Multiple Point Formula (7.3) in [1]. This correction propagates to Formulas (7.5), (7.6), (7.23), (8.18) and it affects minor results in Section 8, where few statements require an extra assumption, but it does not affect the main results of Section 8.

The Multiple Point Formula (7.3) in [1] is not correct as stated. The correct formula is

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) - r_3(\gamma) - \sum_{n\geqslant 4}(\rho_n(\gamma) + f_n(\gamma)) + \epsilon(\gamma) \geqslant d_\gamma \geqslant 0 \tag{1}$$

where $\epsilon(\gamma)$ is the number of E_4 points of the central fibre along γ, which are double points for the total space.

The absence of the correction term $\epsilon(\gamma)$ in (7.3) of [1] is a trivial error and the proof of (1) runs exactly as in [1], as we will now briefly explain freely referring to [1, pp. 383–387] for the setting and notation.

As noted on p. 384 of [1], since all computations are of a local nature, one may assume that the central fibre X of the degeneration has a single Zappatic singularity p along the double curve γ, which is the transverse intersection of two components X_1 and X_2 of X.

If p is not an E_4–point of X double for $\mathcal{X}$, the proof runs as in [1]. So we focus on the opposite case. As in [1], we blow–up p getting a new total space $\mathcal{X}'$. The new central fibre X' contains the strict transforms X_1' and X_2' of X_1 and X_2 respectively, and they intersect along the curve γ' isomorphic to γ. In addition, X' contains the exceptional divisor E' of the blow–up, with multiplicity 2. We denote by p_1 the intersection point of γ' with E'.

Assume first p is an ordinary double point of $\mathcal{X}$, so E' is a smooth quadric. Then p_1 is a smooth point for $\mathcal{X}'$, and we can apply Formula (7.16) from [1], which reads

$$\deg(\mathcal{N}_{\gamma'|X_1'}) + \deg(\mathcal{N}_{\gamma'|X_2'}) + f_3(\gamma') = d_{\gamma'}.$$

Since E' appears in X' with multiplicity 2, we have

$$f_3(\gamma') = f_3(\gamma) + 2.$$

On the other hand

$$\deg(\mathcal{N}_{\gamma'|X_i'}) = \deg(\mathcal{N}_{\gamma|X_i}) - 1, \quad \text{for} \quad 1 \leqslant i \leqslant 2, \quad \text{and} \quad d_{\gamma'} = d_\gamma$$

therefore we have

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = d_\gamma$$

which proves (1) in this case.

Assume next p is not an ordinary double point of $\mathcal{X}$, so E' is a singular quadric. Since p is an E_4–point, E' cannot be a rank 3 quadric, so it has to consist of 2 distinct planes. If p_1 is smooth for E', the proof goes exactly as before. So we only have to consider the case in which both components of E' pass through p_1, in which case p_1 is a double point for $\mathcal{X}'$ and a point of multiplicity 6 for X'.

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We blow–up p_1 getting a new total space $\mathcal{X}''$. The new central fibre X'' contains the strict transforms X_1'' and X_2'' of X_1' and X_2' respectively, which intersect along the curve γ'' isomorphic to γ. In addition, X'' contains the exceptional divisor E'' of the blow–up. We denote by p_2 the intersection point of γ'' with E''.

Suppose p_1 is an ordinary double point for $\mathcal{X}'$, so E'' is a smooth quadric. Then p_2 is a smooth point for $\mathcal{X}''$, and we can apply Formula (7.16) from [1]. First apply it to the intersection curve η of E'' with X_1'', which is a (-1)–curve on X_1'' whereas it has self–intersection 0 on E''. There are 2 triple points on η, one of them is p_2, which counts with multiplicity 1, the other one is the intersection of η with the strict transform of E', which counts with multiplicity 2. This implies that E'' appears with multiplicity 3 in X'' (which agrees with p_1 being a point of multiplicity 6 for X'). Apply now Formula (7.16) from [1] to γ''. We have

$$\deg(\mathcal{N}_{\gamma''|X_1''}) + \deg(\mathcal{N}_{\gamma''|X_2''}) + f_3(\gamma'') = d_{\gamma''}.$$

Since E'' appears in X_0'' with multiplicity 3, we have

$$f_3(\gamma'') = f_3(\gamma) + 3.$$

On the other hand

$$\deg(\mathcal{N}_{\gamma''|X_i''}) = \deg(\mathcal{N}_{\gamma|X_i}) - 2, \;\; 1 \leqslant i \leqslant 2, \quad \text{and} \quad d_{\gamma''} = d_\gamma$$

therefore we have

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = d_\gamma + 1 > d_\gamma \tag{2}$$

which proves (1) in this case.

If p_1 is not an ordinary double point, we repeat the argument. As at the end of p. 386 of [1], this blow-up procedure stops after finitely many, say h, steps, i.e., we find infinitely near double point $p_1, \ldots, p_h$ to p, whereas p_{h+1} is smooth. Then one sees that Formula (2) has to be replaced by

$$\deg(\mathcal{N}_{\gamma|X_1}) + \deg(\mathcal{N}_{\gamma|X_2}) + f_3(\gamma) = d_\gamma + h > d_\gamma \tag{3}$$

concluding the proof of (1).

Coming to the other corrections, Formula (7.5) in [1] has to be changed accordingly by adding $\epsilon(\gamma)$ to the leftmost side of the inequality. Formula (7.6) has to be changed too by adding $4\,\epsilon_{\mathcal{X}}$ to the leftmost side of the inequality, where $\epsilon_{\mathcal{X}}$ is the number of E_4 points of the central fibre which are double points for the total space $\mathcal{X}$. Also the formula in Remark 7.23 of [1] has to be changed accordingly.

The corrected Formula (7.5) implies the corrected (7.6). This, in turn, is used in the proof of Theorem 8.4, in the proof of Proposition 8.16 and in Remark 8.18 of [1].

In the former case, Formula (7.6) is used to prove inequality $(*)$ in the last line of the first formula in the proof of Theorem 8.4. The proof of $(*)$ runs by applying the correct version of (7.6) as well: on the right side of $(*)$ now appears

$$\frac{1}{2}f_3 + 2f_4 - 2\epsilon_{\mathcal{X}} + \frac{1}{2}f_5 \geqslant 0$$

since $\epsilon_{\mathcal{X}} \leqslant f_4$. This in particular proves Formula (8.5) and Zappa's original statement in Theorem 8.1.

Moreover if equality in (8.5) holds, then the same conclusion of Theorem 8.4 holds if one assumes that each E_4-point is not double for the total space $\mathcal{X}$ (in particular if $f_4 = 0$ as in Zappa's original statement).

Finally, if $\mathcal{X}_t$ is assumed to be of general type, then (8.5) holds. If moreover each E_4 point is not a double point for $\mathcal{X}$ (in particular if $f_4 = 0$), then (8.6) holds.

As a consequence:

- Corollary 8.10 holds verbatim as stated in [1],

• Corollaries 8.11, 8.13 still hold as stated in [1], under the assumption that each E_4 point is not double for $\mathcal{X}$ (in particular if $f_4 = 0$);
• Corollary 8.12 still holds as in [1], if each E_4 point is not double for $\mathcal{X}$ (in particular if $f_4 = 0$); otherwise one has $g \leqslant 6\chi + 7$.

A similar argument used above for the proof of (8.5) works for the proof of Proposition 8.16. As for Remark 8.18, the only change to be made is in the lower bound for δ on line -4 of p. 392, which now reads

$$\delta \geqslant 3f_3 + r_3 + \sum_{n\geqslant 4}(12-n)f_n + \sum_{n\geqslant 4}(n-1)\rho_n - 4\epsilon_{\mathcal{X}} - k.$$

This does not affect the rest of the Remark.

References

[1] Calabri, A., Ciliberto C., Flamini, F., Miranda, R., On the K^2 of degenerations of surfaces and the multiple point formula, Annals of Math., **165** (2007), 335–395.

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