EXTENSIONS OF LINE BUNDLES AND BRILL–NOETHER LOCI OF RANK-TWO VECTOR BUNDLES ON A GENERAL CURVE

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Abstract. In this paper we study Brill-Noether loci for rank-two vector bundles and describe the general member of some components as suitable extensions of line bundles.

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Introduction

Let \( C \) be a smooth, irreducible projective curve of genus \( g \) and \( U_C(d) \) be the moduli space of (semi)stable, degree \( d \), rank-two vector bundles on \( C \). In this paper we will be mainly concerned with the case \( C \) of general moduli.

Our objective is to study the Brill-Noether loci \( B^k_C(d) \subseteq U_C(d) \) parametrizing (classes of) vector bundles \([\mathcal{F}] \in U_C(d)\) having \( h^0(C, \mathcal{F}) \geq k \), with \( k \) a given non-negative integer.

By [25] and invariance of stability under standard operations (as dualizing or tensoring by a line bundle) and by Serre duality, one can restrict the analysis to \( 2g - 2 \leq d \leq 4g - 4 \).

The classical Brill-Noether theory for line bundles on a general curve is very important and well established (cf. e.g. [1]). Brill-Noether theory for higher-rank vector bundles is a very active research area (see References, just for some results in the subject), but several basic questions like non-emptiness, dimension, irreducibility, local structure, etc., are still open in general. With respect to the rank-one case, there are some surprises: e.g. the Brill-Noether loci \( B^k_C(d) \) for \( C \) general do not always behave as expected (cf. e.g. [7] and §7.1).
Apart from its intrinsic interest, Brill-Noether theory is important in view of applications to other areas, to say one to birational geometry (cf. e.g. [4, 6, 7, 21, 31]).

The most general result in the rank-two case is the following:

**Theorem 0.1.** (see [51]) Let $C$ be a curve with general moduli of genus $g \geq 1$. Let $k \geq 2$ and $i := k+2g-2-d \geq 2$ be integers. Let $\rho_d^k := 4g - 3 - ik$ and assume

$$\rho_d^k \geq 1 \text{ when } d \text{ odd, } \rho_d^k \geq 5 \text{ when } d \text{ even.}$$

Then $B_C^k(d)$ is not empty and it contains a component $\mathcal{B}$ of the expected dimension $\rho_d^k$.

This result is proved with a quite delicate degeneration argument.

Our approach to the study of Brill-Noether loci in this paper is, in a sense, more basic and elementary. It consists in describing the general vector bundle $\mathcal{F}$ of a component of $B_C^k(d)$ as an extension of two line bundles $L$ and $N$ as in (2.2) (we call this a presentation of $\mathcal{F}$), with suitable minimality properties on $L$, which translate in minimality properties for sections of the scroll $\mathbb{P}(\mathcal{F})$ (thus entering in the realm of projective geometry).

Such a presentation is not known in general, even in the cases described in Theorem 0.1, and it is clearly a basic information concerning the vector bundle $\mathcal{F}$.

This viewpoint is of course not new. For instance, it has been taken in some special situations in [7, §2,3], [31, §8], where the case of canonical determinant and $g \leq 12$ is treated. Indeed, as noted in [7], this viewpoint "works well enough in low genera... but seems difficult to implement in general". However, we tried to follow this route, with no upper-bounds on the genus but bounding the speciality $h^1(\mathcal{F})$.

In this paper, we do not exhibit new components of Brill-Noether loci; however we give a minimal presentation for the general element of some of the components whose existence is asserted in Theorem 0.1. Precisely, our main results are Theorems 7.1, 7.5, 7.11, which deal with cases of speciality $i = 1, 2, 3$ respectively, and Propositions 7.3, 7.10, 7.19, which for the same specialities deal with the canonical determinant case.

In principle, there is no obstruction in dealing with same ideas used in this paper, with the higher speciality cases. However, the treatment of $i \leq 3$ cases is increasingly complicated and therefore we limited ourselves to expose at the end of the paper (see §7.5) a few suggestions on how to proceed in general and a conjecture.

The paper is organized as follows. Sections 2, 3 and 4 are devoted to preliminaries and the useful technical Lemma 4.5. Section 5 contains the construction of vector bundles in Brill-Noether loci as extensions in (2.2), with $L$ and $N$ fixed line bundles. This construction depends on the cohomology of $L$ and $N$ and on the behaviour of the coboundary map of (2.2).

In §6 we let $L$ and $N$ vary in suitable Brill-Noether loci, thus constructing suitable parameter spaces for the above construction and maps from these parameter spaces to the moduli space $U_C(d)$. This technical machinery is finally used in §7 in order to prove the main results mentioned above.

1. **Notation and Terminology**

In this paper we work over $\mathbb{C}$. All schemes will be endowed with the Zariski topology. We will indifferently use the terms rank-$r$ vector bundle on a scheme $X$ and rank-$r$ locally free sheaf.

We will denote by $\sim$ the linear equivalence of divisors, by $\sim_{alg}$ their algebraic equivalence and by $\equiv$ their numerical equivalence. We may abuse notation and identify divisor classes with the corresponding line bundles, indifferently using additive and multiplicative notation.

If $\mathcal{P}$ is the parameter space of a flat family of subschemes of $X$ and if $Y$ is an element of the family, we will denote by $Y \in \mathcal{P}$ the point corresponding to $Y$. If $\mathcal{M}$ is a moduli space, parametrizing geometric objects modulo a given equivalence relation, we will denote by $[Z] \in \mathcal{M}$ the moduli point corresponding to the equivalence class of $Z$.

Let

• $C$ be a smooth, irreducible, projective curve of genus $g$, and
• $\mathcal{F}$ be a rank-two vector bundle on $C$.

Then, $F := \mathbb{P}(\mathcal{F}) \overset{p}{\to} C$ will denote the (geometrically) ruled surface (or the scroll) associated to $(\mathcal{F}, C)$; $f$ will denote the general $\rho$-fibre and $\mathcal{O}_F(1)$ the tautological line bundle. A divisor in $|\mathcal{O}_F(1)|$ will be usually denoted by $H$. If $\Gamma$ is a divisor on $F$, we will set $\deg(\Gamma) := \Gamma H$.

We will use the notation

$$d := \deg(\mathcal{F}) = \deg(\det(\mathcal{F})) = H^2 = \deg(H);$$

$i(\mathcal{F}) := h^1(\mathcal{F})$ is called the speciality of $\mathcal{F}$ and will be denoted by $i$, if there is no danger of confusion. $\mathcal{F}$ (and $F$) is non-special if $i = 0$, special otherwise.
As customary, $W^r_0(C)$ will denote the Brill-Noether locus, parametrizing line bundles $A \in \text{Pic}^g(C)$ such that $h^0(A) \geq r + 1$,

$$\rho(g, r, a) := g - (r + 1)(r + g - a)$$

the Brill-Noether number and

$$\mu_0(A) : H^0(C, A) \otimes H^0(\omega_C \otimes A^\vee) \to H^0(C, \omega_C)$$

(1.1)

the Petri map.

As for the rest, we will use standard terminology and notation as in e.g. [1], [20], etc.

2. Scrolls unisecants

We remind some basic facts on unisecant curves of the scroll $F$ (cf. [16, 18] and [20, V-2]). Recall that $\text{Pic}(F) \cong \mathbb{Z}[\mathcal{O}_F(1)] \oplus \rho^*(\text{Pic}(C))$ (cf. [20, §5, Prop. 2.3]). Let $\text{Div}_F$ be the scheme (not of finite type) of effective divisors on $F$, which is a sub-monoid of $\text{Div}(F)$. For any $k \in \mathbb{N}$, let $\text{Div}_F^k$ be the subscheme (not of finite type) of $\text{Div}_F$ formed by all divisors $\Gamma$ such that $\mathcal{O}_F(\Gamma) \cong \mathcal{O}_F(k) \otimes \rho^*(\mathcal{N}^\vee)$, for $N \in \text{Pic}(C)$ (this $N$ is uniquely determined); then one has a natural morphism

$$\Psi_k : \text{Div}_F^k \to \text{Pic}(C), \quad \Gamma \xrightarrow{\Psi_k} N.$$ 

If $D \in \text{Div}(C)$, then $\rho^*(D)$ will be denoted by $f_D$. Then $\Gamma \in \text{Div}_F^k$ if and only if $\Gamma \sim kH - f_D$, for some $D \in \text{Div}(C)$, and $\deg(\Gamma) = k\deg(\mathcal{F}) - \deg(D)$.

The curves in $\text{Div}_F^k$ are called unisectants of $F$. The irreducible unisectants are isomorphic to $C$ and are called sections of $F$.

For any positive integer $\delta$, we consider (cf. [18, §5])

$$\text{Div}_F^{1,\delta} := \{ \Gamma \in \text{Div}_F^k \mid \deg(\Gamma) = \delta \},$$

which is the Hilbert scheme of unisectants of degree $\delta$ of $F$ (w.r.t. $H$).

**Remark 2.1.** Let $\Gamma = \Gamma + f_A$ be a unisectant, with $\Gamma$ a section and $A$ effective. Equivalently, we have an exact sequence

$$0 \to N(-A) \to \mathcal{F} \to L \oplus \mathcal{O}_A \to 0$$

(2.1)

(cf. [10, 12]); in particular if $A = 0$, i.e. $\Gamma = \Gamma$ is a section, $\mathcal{F}$ fits in the exact sequence

$$0 \to N \xrightarrow{\mathcal{F}} L \to 0$$

and

$$N_{\Gamma/F} \cong L \oplus N^\vee, \quad \text{so } \Gamma^2 = \deg(L) - \deg(N) = 2\delta - d,$$

(2.3)

(cf. [20, § V, Prop. 2.6, 2.9]).

Accordingly $\Psi_{1,\delta} : \text{Div}_F^{1,\delta} \to \text{Pic}^{d-\delta}(C)$, the restriction of $\Psi_1$, endows $\text{Div}_F^{1,\delta}$ with a structure of Quot scheme: with notation as in [41, §4.4], one has

$$\Phi_{1,\delta} : \text{Div}_F^{1,\delta} \xrightarrow{\cong} \text{Quot}_{C,\delta+d-g+1} \xrightarrow{\mathcal{F}} \{ \mathcal{F} \to L \oplus \mathcal{O}_A \}.$$ 

(2.4)

From standard results (cf. e.g. [41, §4.4]), (2.4) gives identifications between tangent and obstruction spaces:

$$H^0(N_{\Gamma/F}) \cong T_{\mathcal{F}}(\text{Div}_F^{1,\delta}) \cong \text{Hom}(N(-A), L \oplus \mathcal{O}_A) \quad \text{and} \quad H^1(N_{\Gamma/F}) \cong \text{Ext}^1(N(-A), L \oplus \mathcal{O}_A)$$

(2.5)

Finally, if $\Gamma \sim H - f_D$, then one easily checks that

$$[\mathcal{O}_F(\Gamma)] \cong \mathbb{P}(H^0(\mathcal{F}(-D))).$$

(2.6)

**Definition 2.2.** $\tilde{\Gamma} \in \text{Div}_F^{1,\delta}$ is said to be:

(a) linearly isolated (li) if $\dim([\mathcal{O}_F(\tilde{\Gamma})]) = 0$,

(b) algebraically isolated (ai) if $\dim(\text{Div}_F^{1,\delta}) = 0$.

**Remark 2.3.** (1) If $\Gamma$ is ai, then it is also li but the converse is false (c.f. e.g. Example 2.9, Corollary 6.12).

(2) When $\text{Div}_F^{1,\delta}$ is of pure dimension, a sufficient condition for $\Gamma$ to be ai is $h^0(N_{\Gamma/F}) = 0$ (c.f. e.g. Theorem 5.4, Corollary 6.7 and §7.1 below).
2.1. The Segre-invariant.

**Definition 2.4.** The *Segre invariant* of \( \mathcal{F} \) is defined as:

\[
s(\mathcal{F}) := \deg(\mathcal{F}) - 2(\max \{\deg(N)\}),
\]

where the maximum is taken among all sub-line bundles \( N \) of \( \mathcal{F} \) (cf. e.g. [22]). The bundle \( \mathcal{F} \) is *stable* [resp. *semi-stable*] if \( s(\mathcal{F}) > 0 \) [resp. \( s(\mathcal{F}) \geq 0 \)].

Equivalently \( \mathcal{F} \) is stable [resp. semistable] if for every sub-line bundle \( N \subset \mathcal{F} \) one has \( \mu(N) < \mu(\mathcal{F}) \) [resp. \( \mu(N) \leq \mu(\mathcal{F}) \)], where \( \mu(\mathcal{E}) = \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E}) \) is the *slope* of a vector bundle \( \mathcal{E} \).

Note that, for any \( A \in \operatorname{Pic}(C) \), one has

\[
s(\mathcal{F}) = s(\mathcal{F} \otimes A). \tag{2.7}
\]

**Remark 2.5.** From (2.3), \( s(\mathcal{F}) \) coincides with the minimum self-intersection of sections of \( F \). In particular, if \( \Gamma \in \operatorname{Div}_F^{1,\delta} \) is a section s.t. \( \Gamma^2 = s(\mathcal{F}) \), then \( s(\mathcal{F}) = 2\delta - d \) and \( \Gamma \) is a section of *minimal degree* of \( F \), i.e. for any section \( \Gamma' \subset F \) one has \( \deg(\Gamma') \geq \deg(\Gamma) \).

We recall the following fundamental result.

**Proposition 2.6.** Let \( C \) be of genus \( g \geq 1 \) and let \( \mathcal{F} \) be indecomposable. Then, \( 2 - 2g \leq s(\mathcal{F}) \leq g \).

**Proof.** The lower-bound follows from \( \mathcal{F} \) being indecomposable (see e.g. [20, V, Thm. 2.12(b)]). The upper-bound is Nagata’s Theorem (see [32]). \( \square \)

2.2. Special scrolls unisecants.

In this paper we will be mainly concerned about the speciality of unisecants of a (necessarily special) scroll \( F \).

**Definition 2.7.** For \( \Gamma \in \operatorname{Div}_F \), we set \( \mathcal{O}_{\Gamma}(1) := \mathcal{O}_F(1) \otimes \mathcal{O}_\Gamma \). The *speciality* of \( \Gamma \) is \( i(\Gamma) := h^1(\mathcal{O}_{\Gamma}(1)) \). \( \Gamma \) is *special* if \( i(\Gamma) > 0 \).

If \( \Gamma \) is given by (2.1), then by (2.6) one has \( \Gamma \in [\mathcal{O}_F(1) \otimes \rho^*(N^\vee(A))]. \) Applying \( \rho_* \) to the exact sequence

\[
0 \rightarrow \mathcal{O}_F(1) \otimes \mathcal{O}_F(-\Gamma) \rightarrow \mathcal{O}_F(1) \rightarrow \mathcal{O}_F(1) \rightarrow 0
\]

and using \( \rho_*(\mathcal{O}_F(1) \otimes \mathcal{O}_F(-\Gamma)) \cong N(-A) \), \( \operatorname{R}^1\rho_*(\mathcal{O}_F(\rho^*(N(-A)))) = 0 \), we get

\[
i(\Gamma) = h^1(L \oplus \mathcal{O}_A) = h^1(L) = i(\Gamma), \tag{2.8}
\]

where \( \Gamma \subset \Gamma \) the unique section.

The following examples show that, in general, speciality is not constant even in linear systems or in algebraic families.

**Example 2.8.** Take \( g = 3 \), \( i = 1 \) and \( d = 9 = 4g - 3 \). There are smooth, linearly normal, special scrolls \( S \subset \mathbb{P}^3 \) of degree 9, speciality 1, sectional genus 3 with general moduli containing a unique special section \( \Gamma \) which is a genus 3 canonical curve (cf. [11, Thm. 6.1]). Moreover, \( \Gamma \) is the unique section of minimal degree 4 (cf. also [40]). There are lines \( f_1, \ldots, f_5 \) of the ruling, such that \( \Gamma := \Gamma + f_1 + \cdots + f_5 \in |H| \), where \( H \) the hyperplane section of \( S \). These curves \( \Gamma \) vary in a sub-linear system of dimension 2 contained in \( |H| \), whose movable part is the complete linear system \( |f_1 + \cdots + f_5| \). The curves as \( \Gamma \) are the only special unisecants in \( |H| \).

**Example 2.9.** Let \( C \) be a non-hyperelliptic curve of genus \( g \geq 3 \), \( d = 3g - 4 \) and \( N \in \operatorname{Pic}^{g-2}(C) \) general. \( N \) is non-effective with \( h^1(N) = 1 \). Consider \( \operatorname{Ext}^1(\omega_C, N) \). It has dimension \( 2g - 1 \) and its general point gives rise to a rank-two vector bundle \( \mathcal{F} \) of degree \( d \), fitting in an exact sequence like (2.2), with \( L = \omega_C \). By generality, the coboundary map \( \partial : H^0(\omega_C) \rightarrow H^1(N) \cong C \) is surjective (cf. Corollary 5.9 below); therefore \( i(\mathcal{F}_u) = 1 \). Since \( \mathcal{F} \) is of rank-two with \( \det(\mathcal{F}) = \omega_C \otimes N \), by Riemann-Roch one has \( h^0(\mathcal{F} \otimes N^\vee) = 1 \). From (2.6), the canonical section \( \Gamma \subset F \), corresponding to \( \mathcal{F} \rightarrow \omega_C \), is li. From (2.3), \( N_{\mathcal{F}/F} \cong \omega_C \otimes N^\vee \) hence \( h^i(N_{\mathcal{F}/F}) = 1 - i \), for \( i = 0, 1 \). Let \( \mathcal{D} \) be the irreducible, one-dimensional component of the Hilbert scheme containing the point corresponding to \( \Gamma \) (which is smooth for the Hilbert scheme). Therefore \( \mathcal{D} \) is an algebraic (non-linear) family whose general member is a li section. As a consequence of Proposition 2.12 below, \( \Gamma \) is the only special section in \( \mathcal{D} \). In particular, if all curves in \( \mathcal{D} \) are irreducible, then \( \Gamma \) is the only special curve in \( \mathcal{D} \) (see Lemma 2.11).

Note that \( \mathcal{F} \) is indecomposable. Indeed, assume \( \mathcal{F} = A \oplus B \), with \( A, B \) line bundles. Since \( h^0(\mathcal{F} \otimes N^\vee) = 1 \), we may assume \( h^0(A - N) = 1 \) and \( h^0(B - N) = 0 \). By the genericity of \( N \), \( A - N \) and \( B - N \) are both general of their degrees. Therefore \( \deg(A - N) = g \), hence \( \deg(A) = 2g - 2 \) and \( \deg(B) = g - 2 \). The image of \( A \) in
the surjection $\mathcal{F} \twoheadrightarrow \omega_C$ is zero, otherwise $A = \omega_C$ hence $B = N$ which is impossible, because $h^0(B - N) = 0$. Then we would have an injection $A \rightarrow N$ which is impossible by degree reasons.

Since $\text{Div}_F^{1,\delta}$ is a Quot-scheme, there is the universal quotient $\Omega_{1,\delta} \rightarrow \text{Div}_F^{1,\delta}$. Taking $\text{Proj}(\Omega_{1,\delta}) \rightarrow \text{Div}_F^{1,\delta}$, we can consider

$$S_F^{1,\delta} := \left\{ \tilde{\Gamma} \in \text{Div}_F^{1,\delta} \mid R^1p_*(\mathcal{O}_{\Omega_{1,\delta}}(1))_{\tilde{\Gamma}} \neq 0 \right\} \quad \text{and} \quad a_F(\delta) := \dim(S_F^{1,\delta}),$$

i.e. $S_F^{1,\delta}$ is the support of $R^1p_*(\mathcal{O}_{\Omega_{1,\delta}}(1))$. It parametrizes degree $\delta$, special unisecants of $F$.

**Definition 2.10.** Let $\tilde{\Gamma}$ be a special unisecant of $F$. Assume $\tilde{\Gamma} \in \mathfrak{F}$ where $\mathfrak{F} \subseteq \text{Div}_F^{1,\delta}$ is a subscheme.

- We will say that $\tilde{\Gamma}$ is:
  1. **specially unique (su)** in $\mathfrak{F}$, if $\tilde{\Gamma}$ is the only special unisecant in $\mathfrak{F}$, or
  2. **specially isolated (si)** in $\mathfrak{F}$, if $\dim_i \left( S_F^{1,\delta} \cap \mathfrak{F} \right) = 0$.

- In particular:
  1. when $\mathfrak{F} = |\mathcal{O}_F(\tilde{\Gamma})|$, $\tilde{\Gamma}$ is said to be **linearly specially unique (lsu)** in case (i) and **linearly specially isolated (lsi)** in case (ii);
  2. when $\mathfrak{F} = \text{Div}_F^{1,\delta}$, $\tilde{\Gamma}$ is said to be **algebraically specially unique (asu)** in case (i) and **algebraically specially isolated (asi)** in case (ii).

- When a section $\Gamma \subseteq F$ is asi, we will say that $\mathcal{F}$ is **rigidly specially presented (rsp)** as $\mathcal{F} \rightarrow L$ or by the sequence (2.2) corresponding to $\Gamma$. When $\Gamma$ is ai (cf. Def. 2.2), we will say that $\mathcal{F}$ is **rigidly presented (rp)** via $\mathcal{F} \rightarrow L$ or (2.2).

For examples, c.f. e.g. §7 below.

**Lemma 2.11.** Let $\Gamma \subseteq F$ be a section corresponding to a sequence as in (2.2). A section $\Gamma'$, corresponding to $\mathcal{F} \rightarrow L'$, is s.t. $\Gamma \sim \Gamma'$ if and only if $L \cong L'$. In particular

- (a) $i(\Gamma) = i(\Gamma')$;
- (b) $\Gamma$ is lsu if and only if it is lsi if and only if it is li.

**Proof.** The first assertion follows from (2.6). Then, (a) and (b) are both clear. \(\square\)

**Proposition 2.12.** Let $\mathcal{F}$ be indecomposable and let $\Gamma \in \mathfrak{F} \subseteq S_F^{1,\delta}$ be a section, where $\mathfrak{F}$ is an irreducible, projective scheme of dimension $k$. Assume:

- (a) $k \geq 1$, if $\mathfrak{F}$ is a linear system;
- (b) either $k \geq 2$, or $k = 1$ and $\mathfrak{F}$ with base points, if $\mathfrak{F}$ is not linear.

Then, $\mathfrak{F}$ contains reducible unisecants $\tilde{\Gamma}$ with

$$i(\tilde{\Gamma}) \geq i(\Gamma).$$

**Proof.** If $k \geq 2$, let $t$ be the unique integer such that $0 \leq k' := k - 2t \leq 1$. Let $f_1, \ldots, f_t$ be $t$ general $\rho$-fibres of $F$. Since $k' \geq 0$, by imposing to the curves in $\mathfrak{F}$ to contain fixed general pairs of points on $f_1, \ldots, f_t$, we see that

$$\mathfrak{F}' := \mathfrak{F} \left( - \sum_{i=1}^t f_i \right) \subseteq \mathfrak{F}$$

is non-empty, all components of it have dimension $k'$, and they all parametrize unisecants $\Gamma' \sim_{\text{alg}} \Gamma - \sum_{i=1}^t f_i$. Then $\mathfrak{F}$ contains reducible elements $\tilde{\Gamma}$, and they verify (2.10) by upper-semicontinuity. This proves the assertion when $k \geq 2$.

So we are left with the case $k = 1$. Assume first that $\mathfrak{F}$ is a linear pencil. Since $\mathfrak{F} \subseteq |\mathcal{O}_F(\Gamma)|$, from the exact sequence

$$0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(\Gamma) \rightarrow \mathcal{O}_F(\Gamma) \rightarrow 0,$$

$\mathcal{O}_F(\Gamma)$ is effective so $\Gamma^2 \geq 0$. Let $\text{Bs}(\mathfrak{F})$ be the base locus of $\mathfrak{F}$. If $\Gamma^2 > 0$, take $p \in \text{Bs}(\mathfrak{F})$. We can clearly split off the fibre through $p$ with one condition, thus proving the result.

If $\Gamma^2 = 0$, $\mathfrak{F}$ is a base-point-free pencil. So $F$ contains two disjoint sections and this implies that $\mathcal{F}$ is decomposable, a contradiction.

Finally, if $\mathfrak{F}$ is non-linear, then $\text{Bs}(\mathfrak{F}) \neq \emptyset$ and we can argue as in the linear case with $\Gamma^2 > 0$. \(\square\)
3. Brill-Noether loci

As usual, $U_C(d)$ is the moduli space of (semi)stable, degree $d$, rank-two vector bundles on $C$. The subset $U_C^{ss}(d) \subseteq U_C(d)$ parametrizing (isomorphism classes of) stable bundles, is an open subset. The points in $U_C^{ss}(d) := U_C(d) \setminus U_C^{st}(d)$ correspond to (S-equivalence classes of) strictly semistable bundles (cf. e.g. [39, 42]).

**Proposition 3.1.** Let $C$ be a smooth curve of genus $g \geq 1$ and let $d$ be an integer.

(i) If $d \geq 4g - 3$, then for any $[\mathcal{F}] \in U_C(d)$, one has $i(\mathcal{F}) = 0$.

(ii) If $g \geq 2$ and $d \geq 2g - 2$, for $[\mathcal{F}] \in U_C(d)$ general, one has $i(\mathcal{F}) = 0$.

**Proof.** For (i), see [34, Lemma 5.2]; for (ii) see [25, p. 100] or [2, Rem. 3]. □

When $g \geq 2$ and
$$2g - 2 \leq d \leq 4g - 4,$$
it makes sense to consider the proper sub-loci of $U_C(d)$ parametrizing classes $[\mathcal{F}]$ such that $i(\mathcal{F}) > 0$.

**Definition 3.2.** Given non-negative integers $d$, $g$ and $i$, we set
$$k_i = d - 2g + 2 + i.$$ (3.2)

Given a curve $C$ of genus $g$, we define
$$B_{ki}^C(d) := \{ [\mathcal{F}] \in U_C(d) \mid h^0(\mathcal{F}) \geq k_i \} = \{ [\mathcal{F}] \in U_C(d) \mid h^1(\mathcal{F}) \geq i \}$$
which we call the $k_i$th-Brill-Noether locus.

**Remark 3.3.** The Brill-Noether loci $B_{ki}^C(d)$ have a natural structure of closed subschemes of $U_C(d)$:

(a) When $d$ is odd, $U_C(d) = U_C^{st}(d)$, then $U_C(d)$ is a fine moduli space and the existence of a universal bundle on $C \times U_C(d)$ allows one to construct $B_{ki}^C(d)$ as the degeneracy locus of a morphism between suitable vector bundles on $U_C(d)$ (see, e.g. [19, 28]). Accordingly, the expected dimension of $B_{ki}^C(d)$ is
$$\rho_{d,i}^{ki} := 4g - 3 - ik_i$$ (3.3)
is the Brill-Noether number. If $\emptyset \neq B_{ki}^C(d) \neq U_C(d)$, then $B_{ki}^{d+1}(d) \subseteq \text{Sing}(B_{ki}^C(d))$. Since any $[\mathcal{F}] \in B_{ki}^C(d)$ is stable, it is a smooth point of $U_C(d)$ and $T_{[\mathcal{F}]}(U_C(d))$ can be identified with $H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*)^\vee$. If $[\mathcal{F}] \in B_{ki}^C(d) \setminus B_{ki}^{d+1}(d)$, the tangent space to $B_{ki}^C(d)$ at $[\mathcal{F}]$ is the annihilator of the image of the cup–product, Petri map of $\mathcal{F}$ (see, e.g. [49])
$$P_\mathcal{F} : H^0(C, \mathcal{F}) \otimes H^0(C, \omega_C \otimes \mathcal{F}^*) \longrightarrow H^0(C, \omega_C \otimes \mathcal{F} \otimes \mathcal{F}^*).$$ (3.4)

If $[\mathcal{F}] \in B_{ki}^C(d) \setminus B_{ki}^{d+1}(d)$, then
$$\rho_{d,i}^{ki} = h^1(C, \mathcal{F} \otimes \mathcal{F}^*) - h^0(C, \mathcal{F})h^1(C, \mathcal{F})$$
and $B_{ki}^C(d)$ is non–singular, of the expected dimension at $[\mathcal{F}]$ if and only if $P_\mathcal{F}$ is injective.

(b) When $d$ is even, $U_C(d)$ is not a fine moduli space (because $U_C^{ss}(d) = \emptyset$). There is a suitable open, non-empty subscheme $Q^{ss} \subset Q$ of a certain Quot scheme $Q$ defining $U_C(d)$ via the GIT-quotient sequence
$$0 \rightarrow PGL(q) \rightarrow Q^{ss} \xrightarrow{\pi} U_C(d) \rightarrow 0$$
(cf. e.g. [44] for details); one can define $B_{ki}^C(d)$ as the image via $\pi$ of the degeneracy locus of a morphism between suitable vector bundles on $Q^{ss}$. The fibres of $\pi$ over strictly semistable bundle classes are not isomorphic to $PGL(q)$. It may happen for a component $\mathcal{B}$ of a Brill–Noether locus to be totally contained in $U_C^{ss}(d)$; in this case the lower bound $\rho_{d,i}^{ki}$ for the expected dimension of $\mathcal{B}$ is no longer valid (cf. Corollary 6.9 below and [8, Remark 7.4]). The lower bound $\rho_{d,i}^{ki}$ is still valid if $\mathcal{B} \cap U_C^{ss}(d) \neq \emptyset$.

**Definition 3.4.** Assume $B_{d}^{ki}(C) \neq \emptyset$. A component $\mathcal{B} \subseteq B_{d}^{ki}(C)$ such that $\mathcal{B} \cap U_C^{ss}(d) \neq \emptyset$ will be called regular, if $\dim(\mathcal{B}) = \rho_{d,i}^{ki}$, superabundant, if $\dim(\mathcal{B}) > \rho_{d,i}^{ki}$.

4. (Semi)stable vector bundles and extensions

In this section we discuss how to produce special, (semi)stable vector bundles $\mathcal{F}$ as extensions of line bundles $L$ and $N$ as in (2.2). This is the same as considering vector bundles $\mathcal{F}$, with a sub-line bundle $N$ s.t. $\mathcal{F} \otimes N^\vee$ has a nowhere vanishing section.

If $g = 2$, in the range (3.1) one has bundles $\mathcal{F}$ with slope $1 \leq \mu(\mathcal{F}) \leq 2$ on a hyperelliptic curve, which have been studied in [8, 9, 27, 29]. Thus, we will assume $C$ non-hyperelliptic of genus $g \geq 3$, with $d$ as in (3.1).
4.1. Extensions and a result of Lange-Narashiman. Let $\delta \leq d$ be a positive integer. Consider $L \in \text{Pic}^\delta(C)$ and $N \in \text{Pic}^{d-\delta}(C)$; $\text{Ext}^1(L, N)$ parametrizes (strong) isomorphism classes of extensions (cf. [15, p. 31]). Any $u \in \text{Ext}^1(L, N)$ gives rise to a degree $d$, rank-two vector bundle $\mathcal{F} = \mathcal{F}_u$ as in (2.2).

In order to get $\mathcal{F}_u$ (semi)stable, a necessary condition is (cf. Remark 2.5)

$$2\delta - d \geq 0. \quad (4.1)$$

Therefore, by Riemann-Roch theorem, we have

$$m := \dim(\text{Ext}^1(L, N)) = \begin{cases} 2\delta - d + g - 1 & \text{if } L \not\cong N \\ g & \text{if } L \cong N. \end{cases} \quad (4.2)$$

**Lemma 4.1.** Let $\mathcal{F}$ be a (semi)stable, special, rank-two vector bundle on $C$ of degree $d \geq 2g-2$. Then $\mathcal{F} = \mathcal{F}_u$, for $L$ a special, effective line bundle on $C$ and $u \in \text{Ext}^1(L, N)$, with $N$ as in (2.2).

**Proof.** By Serre duality, $i(\mathcal{F}) > 0$ gives a non-zero morphism $\mathcal{F} \rightarrow \omega_C$. The line bundle $L := \text{Im}(\sigma^\vee) \subseteq \omega_C$ is special. Set $\delta = \deg(L)$. Since $\mathcal{F}$ is (semi)stable, then (4.1) holds hence $\delta \geq \frac{d}{2} \geq g - 1$, therefore $\chi(L) \geq 0$, so $L$ is effective. $\square$

**Remark 4.2.** In the setting of Lemma 4.1, consider the scroll $F = \mathbb{P}(\mathcal{F})$ and let $\Gamma \subset F$ be the section corresponding to $L$, with $L \in \text{Pic}^\delta(C)$ a special, effective quotient of minimal degree of $\mathcal{F}$. Suppose $\mathcal{F}$ indecomposable. From Proposition 2.12, one has

$$\Gamma \subset F \text{ is li with } a_F(\delta) \leq 1, \quad (4.3)$$

where $a_F(\delta)$ as in (2.9). Then $\mathcal{F}$ is rsp via $L$ if $a_F(\delta) = 0$, and even rp if $\Gamma$ is ai.

Fix $L$ a special, effective line bundle on $C$ of degree $\delta$ and $N$ a line bundle of degree $d - \delta$, where $d$ satisfies (3.1) and (4.1) (so $\deg(L) \geq \deg(N) \geq 0$). We fix the following notation

$$j := h^1(L) > 0, \quad \ell := h^0(L) = \delta - g + 1 + j > 0,$$

$$r := h^1(N) \geq 0, \quad n := h^0(N) = d - \delta - g + 1 + r \geq 0. \quad (4.4)$$

Any $u \in \text{Ext}^1(L, N)$ gives rise to the following diagram

$$(u) : \begin{array}{cccccc} 0 & \to & N & \to & \mathcal{F}_u & \to & L & \to & 0 \\ \deg & & d - \delta & & \delta & & \\ h^0 \ & & n & & \ell & & \\ h^1 \ & & r & & j & & \end{array} \quad (4.5)$$

Let

$$\partial_u : H^0(L) \to H^1(N)$$

be the coboundary map (simply denoted by $\partial$ if there is no danger of confusion) and let $\text{cork}(\partial_u) := \dim(\text{Coker}(\partial_u))$. Then

$$i(\mathcal{F}_u) = j + \text{cork}(\partial_u).$$

As for (semi)stability of $\mathcal{F}_u$, information can be obtained by using [22, Prop. 1.1] (see Proposition 4.4 below) and the projection technique from [14] (see Theorem 5.4 below).

For the reader’s convenience, we recall [22, Prop. 1.1] (cf. also [6, §3.4, 7, §3]). Take $u \in \text{Ext}^1(L, N)$. Tensor by $N^\vee$ and consider $\mathcal{E}_e := \mathcal{F}_u \otimes N^\vee$, which is an extension

$$(e) : \begin{array}{cccc} 0 & \to & \mathcal{O}_C & \to & \mathcal{E}_e & \to & L \otimes N^\vee & \to & 0, \end{array}$$

where $e \in \text{Ext}^1(L \otimes N^\vee, \mathcal{O}_C)$. Then deg$(\mathcal{E}_e) = 2\delta - d$. From (2.7), one has $s(\mathcal{F}_u) = s(\mathcal{E}_e)$ and, by Serre duality, $u$ and $e$ define the same point in

$$P := \mathbb{P}(H^0(K_C + L - N)^\vee). \quad (4.6)$$

**Remark 4.3.** If $\deg(L - N) = 2\delta - d \geq 2$, then $\dim(P) \geq g \geq 3$ and the map $\varphi := \varphi|_{K_C + L - N} : C \to \mathbb{P}$ is a morphism. Set $X := \varphi(C) \subset \mathbb{P}$. For any positive integer $h$ denote by $\text{Sec}_h(X)$ the $h^\text{th}$-secant variety of $X$, defined as the closure of the union of all linear subspaces $\langle \varphi(D) \rangle \subset \mathbb{P}$, for all effective general divisors of degree $h$. One has

$$\dim(\text{Sec}_h(X)) = \min\{\dim(P), 2h - 1\}. \quad (4.7)$$
Proposition 4.4. (see [22, Prop. 1.1]) Let $2\delta - d \geq 2$. For any integer
\[ \sigma = 2\delta - d \mod 2 \] and \[ 4 + d - 2\delta \leq \sigma \leq 2\delta - d, \]
one has
\[ s(\mathcal{E}_e) \geq \sigma \Leftrightarrow e \in \text{Sec}_{2(2\delta - d + \sigma - 2)}(X). \]

4.2. A technical lemma. Later on we will need the following technical result.

Lemma 4.5. Let $L$ and $N$ be as in (4.5) and such that $h^0(N - L) = 0$. Take $u, u' \in \text{Ext}^1(L, N)$ such that:
(i) $\mathcal{F}_u$ and $\mathcal{F}_{u'}$ are stable bundles, and
(ii) there exists an isomorphism $\varphi$
\[ 0 \to N \xrightarrow{i_1} \mathcal{F}_{u'} \to L \to 0 \]
\[ 0 \to N \xrightarrow{i_2} \mathcal{F}_u \to L \to 0 \]
such that $\varphi \circ i_1 = \lambda i_2$, for some $\lambda \in \mathbb{C}^*$.

Then $\mathcal{F}_u \cong \mathcal{F}_{u'}$, i.e. $u, u'$ are proportional vectors in $\text{Ext}^1(L, N)$.

Proof. If $\{U_i\}_{1 \leq i \leq N}$ is a sufficiently fine open covering of $C$, on any $U_i$ we can choose local coordinates
\[ \left( \begin{array}{c} u_i \\ v_i \end{array} \right) \text{ for } \mathcal{F}_u|U_i \text{ and } \left( \begin{array}{c} u'_{i} \\ v'_{i} \end{array} \right) \text{ for } \mathcal{F}_{u'}|U_i, \]
such that:
\begin{itemize}
  \item the transition matrices on $U_i \cap U_j$ are
  \[ \left( \begin{array}{cc} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{array} \right) \text{ for } \mathcal{F}_u|U_i \text{ and } \left( \begin{array}{cc} a'_{ij} & c'_{ij} \\ 0 & b'_{ij} \end{array} \right) \text{ for } \mathcal{F}_{u'}|U_i, \]
  where $a_{ij}, a'_{ij}, b_{ij}, b'_{ij} \in \mathcal{O}_C^*(U_i \cap U_j)$, $c_{ij}, c'_{ij} \in \mathcal{O}_C(U_i \cap U_j)$, for $1 \leq i, j \leq n$,
  \item $N$ is defined by
  \[ v_i = 0 \text{ for } \mathcal{F}_u|U_i, \text{ and by } v'_i = 0 \text{ for } \mathcal{F}_{u'}|U_i, \]
  and its transition functions on $U_i \cap U_j$ are given by
  \[ a_{ij} \text{ when } N \subset \mathcal{F}_u \text{ and by } a'_{ij} \text{ when } N \subset \mathcal{F}_{u'}, \text{ for } 1 \leq i, j \leq n. \]
\end{itemize}
In the above setting, transition functions on $U_i \cap U_j$ for $L$ are given by
\[ b_{ij} \text{ when } \mathcal{F}_u \to L \text{ and by } b'_{ij} \text{ for } \mathcal{F}_{u'} \to L, \text{ for } 1 \leq i, j \leq n. \]
The map $\varphi$ induces isomorphisms
\[ \varphi_i := \varphi|U_i : \mathcal{F}_{u'}|U_i \xrightarrow{\cong} \mathcal{F}_u|U_i, \text{ for any } 1 \leq i \leq n. \]
By (ii), one has
\[ \varphi_i = \left( \begin{array}{c} \lambda \\ 0 \end{array} \right), 1 \leq i \leq n, \]
where $\lambda \in \mathbb{C}^*$, $\beta_i \in \mathcal{O}_C^*(U_i)$, $\gamma_i \in \mathcal{O}_C(U_i)$. Moreover, any $\varphi_i$ has to commute with the transition matrices, i.e.
\[ \left( \begin{array}{cc} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{array} \right) \left( \begin{array}{c} \lambda \\ 0 \end{array} \right) = \left( \begin{array}{c} \lambda \\ 0 \end{array} \right) \left( \begin{array}{cc} a'_{ij} & c'_{ij} \\ 0 & b'_{ij} \end{array} \right), \text{ for } 1 \leq i, j \leq n, \]
which read
\[ a_{ij} = a'_{ij}, b_{ij} = \frac{\beta_i}{\beta_j} b_{ij}, \text{ and } \gamma_i a_{ij} - \gamma_i \frac{\beta_i}{\beta_j} b_{ij} = \lambda c'_{ij} - \beta_j c_{ij}, \text{ for } 1 \leq i, j \leq n. \]
(4.9)
Since $b_{ij}, b'_{ij} \in \mathcal{O}_C^*(U_i \cap U_j)$ are both transition functions for $L$, the second equality in (4.9) implies that, on $U_i \cap U_j$, $\beta_i$ and $\beta_j$ differ by a coboundary, i.e. there exist $h_i \in \mathcal{O}_C^*(U_i)$, for $1 \leq i \leq n$, such that $\frac{\beta_j}{\beta_i} = \frac{h_j}{h_i}$ on $U_i \cap U_j$, for $1 \leq i, j \leq n$. Therefore,
\[ \frac{\beta_i}{h_i} \in \mathcal{O}_C^*(U_i) \text{ and } \frac{\beta_j}{h_i} \in \mathcal{O}_C^*(U_i \cap U_j), \text{ for } 1 \leq i, j \leq n, \]
i.e.
\[ \frac{\beta_i}{h_i} = \mu \in \mathbb{C}^*, \text{ for any } 1 \leq i \leq n. \]
(4.10)
Make the following change of local coordinates on $\mathcal{F}_{u'|U_i}$,
\[
u_i' = x_i', \quad \nu_i' = \frac{1}{h_i} y_i', \quad \text{for any } 1 \leq i \leq n.
\]
In these coordinates, one has that:
- from (4.8) and (4.10), the representation of $\varphi_i$ becomes
\[
\varphi_i = \begin{pmatrix} \lambda & \tilde{\gamma}_i \\ 0 & \mu \end{pmatrix}, \quad \text{for } 1 \leq i \leq n
\]
where $\tilde{\gamma}_i := \frac{\varphi_i}{h_i} \in \mathcal{O}_C(U_i)$;
- the transition matrices for $\mathcal{F}_{u'}$ on $U_i \cap U_j$ become
\[
\begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}, \quad \text{for } 1 \leq i, j \leq n
\]
where $c_{ij} := \frac{\varphi_i}{h_i} \in \mathcal{O}_C(U_i \cap U_j)$, $b_{ij} := \frac{\varphi_i b_i}{h_i}$, \(\tilde{\gamma}_i = \frac{\varphi_i}{h_i} \in \mathcal{O}_C(U_i \cap U_j)$; and
- the compatibility conditions as in (4.9) become
\[
a_{ij} = a_{ij}', \quad b_{ij} = b_{ij}' \quad \text{and} \quad \tilde{\gamma}_j a_{ij} - \tilde{\gamma}_i b_{ij} = \lambda \tilde{\gamma}_i' - \mu c_{ij}, \quad 1 \leq i, j \leq n. \tag{4.12}
\]
For the third equality in (4.12), two cases have to be discussed.

(a) Assume first $\lambda \tilde{\gamma}_i' - \mu c_{ij} = 0$. Thus,
\[
\tilde{\gamma}_i = \frac{a_{ij}}{b_{ij}} \tilde{\gamma}_j, \quad \text{for } 1 \leq i, j \leq n,
\]
i.e. the collection $\{\tilde{\gamma}_i\}_{1 \leq i \leq n}$ defines a global section of $N - L$. Since $h^0(N - L) = 0$, from (4.11) one has
\[
\varphi = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbb{C}^*.
\]
If $\lambda = \mu$, then $\varphi = \lambda \text{Id}$ so $\mathcal{F}_u = \mathcal{F}_{u'}$ as vector bundles, $\varphi \in \text{End}(\mathcal{F}_u)$ and $u, u' \in \text{Ext}^1(L, N)$ are proportional.

Next we exclude that $\lambda \neq \mu$. Indeed, in this case we get the diagram
\[
\begin{array}{cccccc}
0 & \to & N \otimes L^\vee & \to & \mathcal{F}_{u'} \otimes L^\vee & \to & \mathcal{O}_C \to 0 \\
\downarrow & & \downarrow \lambda \mu^{-1} & & \downarrow \tilde{\varphi} & & \downarrow \text{id} \\
0 & \to & N \otimes L^\vee & \to & \mathcal{F}_u \otimes L^\vee & \to & \mathcal{O}_C \to 0,
\end{array}
\]
where $\tilde{\varphi} = \mu^{-1} \varphi$. Taking the coboundary maps
\[
\partial_{u'} : H^0(\mathcal{O}_C) \to H^1(N - L) \quad \text{and} \quad \partial_u : H^0(\mathcal{O}_C) \to H^1(N - L),
\]
we get
\[
\partial_{u'}(1) = (\lambda \mu^{-1}) \partial_u(1),
\]
which implies again that $u, u' \in \text{Ext}^1(L, N)$ are proportional vectors, i.e. $\mathcal{F}_u = \mathcal{F}_{u'}$ as vector bundles. In this case, $\varphi \in \text{End}(\mathcal{F}_u) \setminus \mathbb{C}^*$, contradicting assumption (i) (\(\mathcal{F}_u\) stable implies that $\mathcal{F}_u$ is simple, cf. e.g. [42, Prop. 6-c, p.17]). So the case $\lambda \neq \mu$ cannot occur, and we are done.

(b) Assume now $\lambda \tilde{\gamma}_i' - \mu c_{ij} \neq 0$. In this case, we argue as in [26, Lemma 1]. Indeed, for $1 \leq i \leq n$, we consider the following further change of coordinate
\[
\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} := \begin{pmatrix} 1 & -\frac{\varphi_i}{h_i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_i' \\ \eta_i' \end{pmatrix} \quad \text{for } \mathcal{F}_{u'|U_i} \quad \text{and} \quad \begin{pmatrix} u_i' \\ v_i' \end{pmatrix} := \begin{pmatrix} 1 & -\frac{\varphi_i}{h_i} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \quad \text{for } \mathcal{F}_{u|U_i}.
\]
Taking into account (4.12), in these coordinates the transition matrices become
\[
\begin{pmatrix} \xi_i' \\ \eta_i' \end{pmatrix} = \begin{pmatrix} a_{ij} & \frac{1}{\lambda} (2 \mu c_{ij} - \lambda \tilde{\gamma}_i') \\ 0 & b_{ij} \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}, \quad \text{for } \mathcal{F}_{u'|U_i}
\]
and
\[
\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} a_{ij} & \frac{1}{\mu} (2 \mu c_{ij} - \lambda \tilde{\gamma}_i') \\ 0 & b_{ij} \end{pmatrix} \begin{pmatrix} \xi_i' \\ \eta_i' \end{pmatrix}, \quad \text{for } \mathcal{F}_{u|U_i},
\]
whereas $\varphi_i$ reads as
\[
\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.
\]
One can then conclude as in case (a). \(\square\)
5. Stable bundles as extensions of line bundles

In this section we start with line bundles $L$ and $N$ on a curve $C$, and consider rank-2 vector bundles $\mathcal{F}$ on $C$ as extensions as in (2.2). We give conditions under which $\mathcal{F}$ is stable, with a given speciality, and $L$ is a quotient with suitable minimality properties.

5.1. The case $N$ non-special. In this section we focus on the case $N$ non-special. Notation as in (4.4), (4.5), with $r = 0$.

**Theorem 5.1.** Let $j \geq 1$ and $g \geq 3$ be integers. Let $C$ be of genus $g$ with general moduli. Let $\delta$ and $d$ be integers such that

\[ \rho(g, \ell - 1, \delta) \geq 0, \text{ i.e. } \delta \leq g - 1 + \frac{g}{j} - j, \]

(5.1)

\[ \delta + g - 1 \leq d \leq 2\delta - 2. \]

(5.2)

Let $N \in \text{Pic}^{d-\delta}(C)$ be general and $L \in W_{\delta - g + j}^{g+\delta}(C)$ be a smooth point. Then, for $u \in \text{Ext}^1(L, N)$ general, $\mathcal{F}_u$ is indecomposable with

(i) $s(\mathcal{F}_u) = 2\delta - d$. In particular $2 \leq s(\mathcal{F}_u) \leq \frac{g}{j} - j$, hence $\mathcal{F}_u$ is stable;

(ii) $i(\mathcal{F}_u) = j$;

(iii) $L$ is a quotient of minimal degree of $\mathcal{F}_u$.

**Remark 5.2.** (1) In (5.2), the upper-bound on $d$ reads $2\delta - d \geq 2$, which is required to apply Proposition 4.4. The lower-bound reads $\text{deg}(N) = d - \delta \geq g - 1$, hence the general $N \in \text{Pic}^{d-\delta}(C)$ is non-special. Moreover, (5.2) implies $\delta \geq g + 1$.

(2) From (5.1) and $\delta \geq g + 1$, one has $g \geq j^2 + 2j$ i.e. $1 \leq j \leq \sqrt{g+1} - 1$.

**Proof.** The line bundle $N$ is non-special, so (ii) holds. By Remark 5.2-(1), we can use Proposition 4.4 with $\sigma := 2\delta - d$. One has $\text{dim}(\mathcal{P}) = 2\delta - d + g - 2$. From (4.7), we have

\[ \text{dim} \left( \text{Sec}_{\frac{1}{2}(2\delta - d - 2)}(X) \right) = \min \{ \text{dim}(\mathcal{P}), 2(2\delta - d) - 3 \} = 2(2\delta - d) - 3 \]

since (5.1) and (5.2) yield $2\delta - d < \frac{g}{j} - j$ which implies $2(2\delta - d) - 3 < \text{dim}(\mathcal{P})$.

From Proposition 4.4, $u \in \text{Ext}^1(L, N)$ general is such that $s(\mathcal{F}_u) \geq 2\delta - d$. If $\Gamma$ is the section corresponding to $\mathcal{F}_u \rightarrow L$, one has $\text{Dim}^2 = 2\delta - d$ as in (2.3) so $s(\mathcal{F}_u) = 2\delta - d$, hence (i) and (iii) are also proved.

**Corollary 5.3.** Assumptions as in Theorem 5.1. Let $\Gamma$ be the section corresponding to $\mathcal{F}_u \rightarrow L$. Then $\Gamma$ is of minimal degree. In particular, $\Gamma$ is li and $0 \leq \text{dim}(\text{Div}^1_{\mathcal{F}_u, \delta}) \leq 1$.

**Proof.** The fact that $\Gamma$ is of minimal degree follows from (i) of Theorem 5.1. The rest is a consequence of minimality and Proposition 2.12.

**Theorem 5.4.** Let $j \geq 1$ and $g \geq 3$ be integers. Let $C$ be of genus $g$ with general moduli. Let $\delta$ and $d$ be integers such that (5.1) holds and moreover

\[ \delta + g + 3 \leq d \leq 2\delta. \]

(5.3)

Let $N \in \text{Pic}^{d-\delta}(C)$ and $L \in W_{\delta - g + j}^{g+\delta}(C)$ be general points. Then, for any $u \in \text{Ext}^1(L, N)$, $\mathcal{F}_u$ is very-ample with $i(\mathcal{F}_u) = j$. Moreover, for $u \in \text{Ext}^1(L, N)$ general

(i) $L$ is the quotient of minimal degree of $\mathcal{F}_u$, thus

\[ 0 \leq s(\mathcal{F}_u) = 2\delta - d \leq \frac{g - 4j - j^2}{j}, \]

so $\mathcal{F}_u$ is stable when $2\delta - d > 0$, strictly semistable when $d = 2\delta$;

(ii) if $\Gamma$ is the section corresponding to $\mathcal{F}_u \rightarrow L$, then $\text{Div}^1_{\mathcal{F}_u, \delta} = \{ \Gamma \}$ and $\mathcal{F}_u$ is rp via $L$.

**Proof.** The proof is as in [14, Theorem 2.1], and it works also in the case $d = 2\delta$, not considered there.

**Remark 5.5.** (1) The lower-bound in (5.3) reads $\text{deg}(N) = d - \delta \geq g + 3$, hence $N \in \text{Pic}^{d-\delta}(C)$ general is non-special and $\delta \geq g + 3$.

(2) From (5.1) and $\delta \geq g + 3$, one has $g \geq j^2 + 4j$ i.e. $1 \leq j \leq \sqrt{g+4} - 2$.

(3) The bounds on $d$ in (5.2) and (5.3) are in general slightly worse than those in Theorem 0.1 (cf. [14, Remark 1.7]). For $j$ close to the upper-bound (see Remark 5.2-(2), respectively (2) above), the difference is of the order of $\sqrt{g}$. However our approach gives the additional information of the description of vector bundles in irreducible components of $B^2_{\mathcal{C}^L}(d)$ (see also §7 as line bundle extensions).
5.2. The case $N$ special. In this section $N \in \text{Pic}^{d-\delta}(C)$ is assumed to be special; hence, in (4.4), we have $\ell, j, r > 0$ whereas $n \geq 0$.

For any integer $t > 0$, consider
\[ \mathcal{W}_t := \{ u \in \text{Ext}^1(L, N) \mid \text{cork}(\partial_u) \geq t \} \subseteq \text{Ext}^1(L, N), \]
which has a natural structure of determinantal scheme; as such, $\mathcal{W}_t$ has \textit{expected codimension}
\[ c(\ell, r, t) := t(\ell - r + t) \] (5.5)
From (4.2), if $m > 0$ and $\mathcal{W}_t \neq \emptyset$, then any irreducible component $\Lambda_t \subseteq \mathcal{W}_t$ is such that
\[ \dim(\Lambda_t) \geq \min\{m, m - c(\ell, r, t)\}, \] (5.6)
where the right-hand-side is the \textit{expected dimension}. These loci have been considered also in [7, §2,3], [31, §6,8] for low genus and canonical determinant.

\textbf{Remark 5.6.} One has $\dim(\ker(\partial_u)) = 1 + \dim(\Gamma)$, where $\Gamma$ is the section corresponding to the quotient $\mathcal{F} \rightarrow L$. Note that $\dim(\Gamma) = -1$ if and only if $\dim(\ker(\partial_u)) = 0$, i.e. $H^0(\mathcal{F}) = H^0(N)$. This happens if and only if $\Gamma$ is a fixed component of $[\mathcal{O}_C(1)]$, i.e. if and only if the image of the map $\Phi_{|\mathcal{O}_C(1)}$ has dimension smaller than 2. If $d \geq 2g - 2$ and this happens, then $n \geq i \geq j \geq 1$, where $i = i(\mathcal{F})$.

\textbf{Remark 5.7.} The map $\partial_u$ can be interpreted in terms of multiplication maps among global sections of suitable line bundles on $C$. Indeed, consider $r \geq t$ and $\ell \geq \max\{1, r - t\}$. Denote by
\[ U : H^0(L) \otimes H^1(N - L) \rightarrow H^1(N), \]
the cup-product: for any $u \in H^1(N - L) \cong \text{Ext}^1(L, N)$, one has $\partial_u(-) = - \cup u$. By Serre duality, the consideration of $\cup$ is equivalent to the one of the multiplication map
\[ \mu := \mu_{L,K_C-N} : H^0(L) \otimes H^0(K_C - N) \rightarrow H^0(K_C + L - N) \] (5.7)
when $N = L$, $\mu$ coincides with $\mu_0(L)$ as in (1.4)). For any subspace $W \subseteq H^0(K_C - N)$, we set
\[ \mu_W := \mu|_W : H^0(L) \otimes W \rightarrow H^0(K_C + L - N). \] (5.8)
Imposing $\text{cork}(\partial_u) \geq t$ is equivalent to ask that
\[ V_t := \text{Im}(\partial_u)^\perp \subseteq H^0(K_C - N) \]
has at least dimension $t$. Therefore
\[ \mathcal{W}_t = \{ u \in H^0(K_C + L - N)^\vee \mid \exists V_t \subseteq H^0(K_C - N), \text{ s.t.} \dim(V_t) \geq t \text{ and } \text{Im}(\mu_V_t) \subseteq \{u = 0\}\}, \]
where $\{u = 0\} \subset H^0(K + L - N)$ is the hyperplane defined by $u \in H^0(K_C + L - N)^\vee$.

\textbf{Theorem 5.8.} Let $C$ be a smooth curve of genus $g \geq 3$. Let
\[ r \geq 1, \quad \ell \geq \max\{1, r - 1\}, \quad m \geq \ell + 1 \]
be integers. Then
(i) $c(\ell, r, 1) = \ell - r + 1 \geq 0$;
(ii) $\mathcal{W}_1$ is irreducible of the expected dimension $\dim(\mathcal{W}_1) = m - c(\ell, r, 1) \geq r$. In particular $\mathcal{W}_1 = \text{Ext}^1(L, N)$ if and only if $\ell = r - 1$.

\textbf{Proof.} Part (i) and $m - c(\ell, r, 1) \geq r$ are obvious. Let us prove (ii). Since $\ell, r \geq 1$, both $L$ and $K_C - N$ are effective. One has an inclusion
\[ 0 \rightarrow L \rightarrow K_C + L - N \]
obtained by tensoring by $L$ the injection $O_C \hookrightarrow K_C - N$ given by a given non-zero section of $K_C - N$. Thus, for any $V_1 \subseteq \mathbb{P}(H^0(K_C - N))$, $\mu_V_1$ as in (5.8) is injective. Since $m \geq \ell + 1$, one has $\dim(\text{Im}(\mu_V_1)) = \ell \leq m - 1$, i.e. $\text{Im}(\mu_V_1)$ is contained in some hyperplane of $H^0(K_C + L - N)$. Let
\[ \Sigma := \{ \sigma := H^0(L) \otimes V_1^\vee \subseteq H^0(L) \otimes H^0(K_C - N) \mid V_1^\vee \subseteq \mathbb{P}(H^0(K_C - N)) \}. \]
Thus $\Sigma \cong \mathbb{P}(H^0(K_C - N))$, so it is irreducible of dimension $r - 1$. Since $\mathbb{P}(H^0(K_C + L - N)^\vee) = \mathbb{P}$ as in (4.6), we can define the incidence variety
\[ \mathcal{J} := \{ (\sigma, \pi) \in \Sigma \times \mathbb{P} \mid \mu_{V_1^\vee}(\sigma) \subseteq \pi \} \subset \Sigma \times \mathbb{P}. \]
Let
\[ \Sigma \xrightarrow{\text{pr}_1} \mathcal{J} \xrightarrow{\text{pr}_2} \mathbb{P} \]
be the two projections. As we saw, $pr_1$ is surjective. In particular $\mathcal{J} \neq \emptyset$ and, for any $\sigma \in \Sigma$, 
$$pr_{1}^{-1}(\sigma) = \{\pi \in \mathbb{P} \mid \mu_{\psi}(\sigma) \subseteq \pi\} \cong [J_{\mathbb{P}}(\mathcal{V}(1))],$$
where $\hat{\sigma} := \mathbb{P}(\mu_{\psi}(\sigma))$. Since $\dim(\hat{\sigma}) = \ell - 1$, then $\dim(pr_{1}^{-1}(\sigma)) = m - 1 - \ell \geq 0$.

This shows that $\mathcal{J}$ is irreducible and $\dim(\mathcal{J}) = m - 1 - c(\ell, r, t) \leq m - 1$. Then, $W_1 := \mathbb{P}(W_1) = pr_2(\mathcal{J})$. Recalling (5.6), $W_1 \neq \emptyset$ is irreducible, of the expected dimension $m - c(\ell, r, t)$.

**Corollary 5.9.** Assumptions as in Theorem 5.8. If $\ell \geq r$, then

(i) for $u \in \text{Ext}^1(L, N)$ general, $\partial_u$ is surjective, in which case $i(\mathcal{F}_u) = h^1(L) = j$;

(ii) the inclusion $W_1 \subset \text{Ext}^1(L, N)$ is strict and, for $v \in W_1$ general, $\text{cork}(\partial_v) = 1$, hence $i(\mathcal{F}_v) = j + 1$.

5.2.1. Surjective coboundary. Take $0 \neq u \in \text{Ext}^1(L, N)$ and assume $\partial_u$ is surjective (from Corollary 5.9, this happens e.g. when $\ell \geq r$, $m \geq \ell + 1$ and $u$ general).

**Theorem 5.10.** Let $j \geq 1$ and $g \geq 3$ be integers. Let $C$ be of genus $g$ with general moduli. Let $\delta$ and $d$ be integers such that (5.1) holds and moreover

$$2g - 2 \leq d \leq 2\delta - g. \quad (5.9)$$

Let $L \in W_{g-3}^{g-j}(C)$ be a smooth point and $N \in \text{Pic}^{d-\delta}(C)$ be any point. Then, for $u \in \text{Ext}^1(L, N)$ general, $\mathcal{F}_u$ is indecomposable with

(i) $i(\mathcal{F}_u) = j$.

(ii) $s(\mathcal{F}_u) = g - \epsilon$, $\epsilon \in \{0, 1\}$ such that $\epsilon \equiv d + g \pmod{2}$. In particular, $\mathcal{F}_u$ is stable.

(iii) The minimal degree of a quotient of $\mathcal{F}_u$ is $\frac{d + g - \epsilon}{2}$ and $1 - \epsilon \leq \dim \left(\text{Div}_{\mathcal{F}_u} \frac{d^2 + g - 2}{2}\right) \leq 1$;

(iv) $L$ is a minimal degree quotient of $\mathcal{F}_u$ if and only if $\epsilon = 0$ and $d = 2\delta - g$.

**Remark 5.11.** (1) From (5.9) we get $\delta \geq \frac{3}{4}g - 1$ hence from (5.1), $j \leq \frac{\sqrt{g^2 + 16g} - g}{4}$.

(2) Since $L$ is special, then $\delta \leq 2g - 2$. Therefore, the upper-bound in (5.9) implies $d - \delta \leq -\delta \leq g - 2$, i.e., any $N \in \text{Pic}^{d-\delta}(C)$ is special too.

(3) The inequalities (5.1), (5.9) imply $\ell \geq r$, $m \geq \ell + 1$ as in the assumptions of Corollary 5.9. Indeed:

- $\ell \geq r$ reads

$$\delta \geq g - 1 + r - j. \quad (5.10)$$

Since $r = \delta - d + g - 3 + n$, then (5.10) is equivalent to $d \geq 2g - 2 - j + n$. Thus (5.10) holds by (5.9), if $n \leq 1$. If $n \geq 2$, $C$ with general moduli implies $r \leq \frac{g}{2} \leq \frac{g}{4} \leq g - 1 - j \leq \frac{3}{2}g - 1 - j$ and (5.10) holds because $\delta \geq \frac{3}{2}g - 1$.

- We have $d - \delta \leq \delta - g < \delta$ by (5.9). So from (4.2) we have $m = 2\delta - d + g - 1$. Thus $m \geq \ell + 1$ reads $d \leq \delta + 2g - 3 - j$. By (5.9), to prove this it suffices to prove $2\delta - g \leq \delta + 2g - 3 - j$. This in turn is a consequence of (5.1).

(4) Notice that, under hypotheses of Theorem 5.10, when $\epsilon = 1$ $L$ is not of minimal degree: from (iii), one would have $d = 2\delta - g - 1$ which is out of range in (5.9). Indeed, if $d = 2\delta - g + 1$ and e.g. $\delta = 2g - 2$, then $d = 3g - 3$, $\deg(N) = d - \delta = 2g - 1$, thus if $N$ is general, it is non-special, which is a case already considered in Theorem 5.1. From (1) above, to allow minimality for $L$ also for $\epsilon = 1$, one should replace (5.1), (5.9) in the statement of Theorem 5.10 with the more annoying conditions $\delta \leq \min\{g - 1 + \frac{\ell}{2} - j, 2g - 2\}$ and $d \leq 2\delta - g + \epsilon$, respectively.

**Proof of Theorem 5.10.** By Remark 5.11-(2), $N$ is special. Moreover, by Remark 5.11-(3) and Corollary 5.9, for $u \in \text{Ext}^1(L, N)$ general, $\partial_u$ is surjective. Hence (i) holds.

From the upper-bound in (5.9) and $g \geq 3$, we can apply Proposition 4.4 with the choice $\sigma := g - \epsilon$, i.e., the maximum for which $\sigma \equiv 2\delta - d \pmod{2}$, $\sigma \leq 2\delta - d$ and one has a strict inclusion

$$\text{Sec}_{\frac{1}{2}(2\delta - d + g - \epsilon - 2)}(X) \subset \mathbb{P}.$$ 

If $\epsilon = 0$, (ii) follows from Propositions 2.6, 4.4. Let $\Gamma \subset F_0$ be a section of minimal degree, which we denote by $m_0$. Then, $\Gamma^2 = 2m_0 - d = g$ (cf. (2.3) and Remark 2.5). In particular, $m_0 = \frac{d^2 + g}{2}$ and

$$1 = \Gamma^2 - g + 1 \leq \chi(N_{\Gamma/F_0}) \leq \dim \left(\text{Div}_{F_0} \frac{d^2 + g}{2}\right) \leq 1,$$
where the upper-bound holds by the minimality condition (cf. proof of Proposition 2.12). This proves (iii) in this case.

When $\epsilon = 1$, by Propositions 2.6, 4.4, one has $g - 1 \leq s(F_v) \leq g$ and, by parity, the leftmost equality holds. As above, part (iii) holds also for $\epsilon = 1$.

Finally, $L$ is a minimal degree quotient if and only if $2\delta = d + g - \epsilon$ which by (5.9) is only possible for $\epsilon = 0$, proving (iv) (cf. Remark 5.11-(4)).

5.2.2. Non-surjective coboundary. From Corollary 5.9, when $\ell \geq r$ and $m \geq \ell + 1$, for $v \in W_t \subseteq \text{Ext}^1(L, N)$ general, one has $\text{cork}(F_v) = t$.

**Definition 5.12.** Take $\ell \geq r \geq t \geq 1$ integers. Assume
(1) there exists an irreducible component $\Lambda_t \subseteq W_t$ with the expected dimension $\dim(\Lambda_t) = m - c(\ell, r, t)$;
(2) for $v \in \Lambda_t$ general, $\text{cork}(F_v) = t$.

Any such $\Lambda_t$ is called a good component of $W_t$.

By Theorem 5.8, $\Lambda_1 = W_1$ is good. In §5.3 we shall give sufficient conditions for goodness, when $t \geq 2$.

With notation as in (4.6), for any $t \geq 1$ and any good component $\Lambda_t$, we set
$$\hat{\Lambda}_t := \mathbb{P}(\Lambda_t) \subset \mathbb{P} \quad (5.11)$$
(cf. notation as in the proof of Theorem 5.8 for $W_1$).

**Theorem 5.13.** Let $g \geq 3, d \geq 2g - 2$, $j \geq 1$, $\ell \geq r \geq t \geq 1$ be integers. Take $\epsilon \in \{0, 1\}$ such that
$$d + g - c(\ell, r, t) \equiv \epsilon \quad (\text{mod } 2).$$

Take $\delta$ be such that $\ell = \delta - g + 1 + j$ and assume
$$g \geq c(\ell, r, t) + \epsilon, \quad (5.12)$$
$$g + r - j - 1 \leq \delta \leq g - 1 + \frac{g}{j} - j, \quad (5.13)$$
$$2\delta - d \geq \max\{2, g - c(\ell, r, t) - \epsilon\}. \quad (5.14)$$

Let $C$ be a curve of genus $g$ with general moduli. Let $L \in W_{d-1}^\ell(1)$ be a smooth point, $N \in \text{Pic}^{d-\delta}(C)$ of speciality $r$.

Then, for any good component $\Lambda_t$ and for $v \in \Lambda_t$ general, $F_v$ is such that:
(i) $i(F_v) = j + t$;
(ii) $s(F_v) \geq g - c(\ell, r, t) - \epsilon \geq 0$; in particular, when $g - c(\ell, r, t) > 0$, $F_v$ is stable, hence indecomposable.
(iii) If $2\delta - d \geq 2 \geq 2\delta - d \equiv 2\delta - d \equiv \epsilon$ (mod 2), then $L$ is a quotient of minimal degree.

**Remark 5.14.** (1) The upper-bound on $\delta$ in (5.13) is $\rho(g, \ell - 1, \delta) \geq 0$; the lower bound is equivalent to $\ell \geq r$.
(2) Condition $2\delta - d \geq 2$ in (5.14) is as in Proposition 4.4; the other condition in (5.14) will be explained in the proof of Theorem 5.13.
(3) As in Remark 5.11-(3), one shows that $m \geq \ell + 1$.

**Proof of Theorem 5.13.** We apply Proposition 4.4 with $\sigma := g - c(\ell, r, t) - \epsilon$, which is non-negative by (5.12) and is the integer such that $\sigma \leq 2\delta - d$, $\sigma \equiv 2\delta - d \equiv \epsilon$ (mod 2) and $\dim(\hat{\Lambda}_t) > \dim(\text{Sec}^{2\delta - d - g + 1 + \epsilon} \setminus X))$. Then, if $v \in \hat{\Lambda}_t$ general, the assertions hold.

**Remark 5.15.** When $N$ is non-effective, of speciality $r$ and degree $d - \delta$, then $r = \delta - d + g - 1$. By (3.2),
$$c(\ell, r, t) = t(d - 2g + 2 + j + t) = tk_j + t$$
and the conditions in Theorem 5.13 can be replaced by
$$\delta \leq g - 1 + \frac{g}{j} - j, \quad (5.12)$$
$$d \leq g - 1 - t + \min\left\{\delta, g - 1 - j + \frac{g - \epsilon}{t}\right\},$$
where $\epsilon \in \{0, 1\}$ such that $d + g - tk_{j+1} \equiv \epsilon$ (mod 2), and
$$2\delta - d \geq \max\{2, g - tk_{j+1} - \epsilon\}.$$
Remark 5.16. When \( N \) is effective, then \( d \geq \delta + g - r \). Moreover, \( \rho(g, n-1, d-\delta) \geq 0 \) gives \( d \leq \delta + g - 1 + \frac{g}{r} - r \). Thus, the conditions in Theorem 5.13 can be replaced by

\[
g - 1 - j + r \leq \delta \leq g - 1 - j + \min \left\{ \frac{g}{j}, \frac{g - \epsilon}{t} + r - t - 1 \right\},
\]

\[
g + \delta - r \leq d \leq \delta + g - 1 + \frac{g}{r} - r,
\]

\[
2\delta - d \geq \max \{2, g - c(\ell, r, t) - \epsilon\},
\]

with \( \epsilon \) and \( c(\ell, r, t) \) as in Theorem 5.13.

5.3. Existence of good components.

Theorem 5.17. Let \( C \) be a smooth curve of genus \( g \geq 3 \). Assume \( \ell \geq r \geq t \geq 2 \). Take any integer \( \eta \) such that

\[
0 \leq \eta \leq \min\{t(r-t), \ell(t-1)\} \quad \text{and} \quad m \geq \ell t + 1 - \eta.
\]

Suppose, in addition, that the subvariety \( \Sigma_\eta \subseteq \mathbb{G}(t, H^0(K_C - N)) \) := \( \mathbb{G} \), parametrizing \( V_t \in \mathbb{G} \) s.t.

\[
\dim(\ker(\mu_{V_t})) \geq \eta.
\]

has pure codimension \( \eta \) in \( \mathbb{G} \) and that, for the general point \( V_t \) in any irreducible component of \( \Sigma_\eta \), equality holds in (5.16). Then:

(i) \( c(\ell, r, t) > 0 \);

(ii) \( \emptyset \neq W_t \subset W_1 \subset \text{Ext}^1(L, N) \), where all the inclusions are strict;

(iii) there exists a good component \( \Lambda_t \) of \( W_t \).

Proof. By (5.15) one has \( m \geq \ell + 1 \); moreover \( \ell \geq r \) by assumption. Thus, from Corollary 5.9, \( \emptyset \neq W_t \subset \text{Ext}^1(L, N) \) and the inclusion is strict. By definition \( W_t \subset W_1 \), where the inclusion is strict by Corollary 5.9. Then the same argument as in the proof of Theorem 5.8 applies.

Corollary 5.18. Let \( C \) be of genus \( g \geq 3 \) with general moduli. Let \( j \geq 1 \), \( \ell \geq r \geq t \geq 2 \) and \( m \geq \ell t + 1 \) be integers. Let \( L \in \text{Pic}^{d-g+j}(C) \) be a smooth point.

If \( j \geq t \), \( N \in \text{Pic}^{d-\delta}(C) \) is general and \( \ell \leq 2\delta - d \), then for \( V_t \in \mathbb{G}(t, H^0(K_C - N)) \) general, \( \mu_{V_t} \) is injective. In particular, there exists a good component \( \emptyset \neq \Lambda_t \subset W_t \).

Proof. Set \( h := 2\delta - d \) and let \( N_0 := L - D \in \text{Pic}^{d-\delta}(C) \), with \( D = \sum_{i=1}^{h} p_i \in C^{(h)} \) general. Since \( 0 < \ell \leq h \), we have \( h^0(N_0) = 0 \). Thus, \( N \in \text{Pic}^{d-\delta}(C) \) general is also non-effective, so \( h^1(N) = h^1(N_0) \).

Let \( \mu \) be as in (5.7). To prove injectivity of \( \mu_{V_t} \) as in (5.8) for \( N \) and \( V_t \) general, it suffices to prove a similar condition for

\[
\mu^0 : H^0(L) \otimes H^0(K_C - L + D) \rightarrow H^0(K_C + D).
\]

Consider

\[
W := H^0(K_C - L) \subset H^0(K_C - L + D).
\]

One has \( \dim(W) = j \). We have the diagram

\[
H^0(L) \otimes W \cong H^0(L) \otimes H^0(K_C - L) \xrightarrow{\mu_{W}^0 \otimes \mu_{L}} H^0(K_C + D)
\]

\[
\xrightarrow{\mu_{V_t}} H^0(K_C).
\]

where \( \mu_{W}^0 = \mu_{W}^0|_{H^0(L) \otimes W} \), \( \mu_{L}(L) \) is as in (1.1) and \( \iota \) is the obvious inclusion.

By Gieseker–Petri theorem \( \mu_{L}(L) \) is injective. By composition with \( \iota \), \( \mu_{W}^0 \) is also injective. Since by assumptions \( t \leq j \), then for any \( V_t \in \mathbb{G}(t, W) \), \( \mu_{V_t}^0 \) is also injective. By semicontinuity, for \( N \in \text{Pic}^{d-\delta}(C) \) and \( V_t \in \mathbb{G}(t, H^0(K_C - N)) \) general, \( \mu_{V_t} \) is injective. Then, one can conclude by using Theorem 5.17. \( \square \)
6. Parameter Spaces

Let \( C \) be a projective curve of genus \( g \) with general moduli. Given a sequence as in (4.5), we set
\[
\rho(L) := \rho(g, \ell - 1, \delta) \quad \text{and} \quad \rho(N) := \rho(g, n - 1, d - \delta),
\]
\[
W_L := \begin{cases} \mathbb{P}^{L-1}(C) & \text{if } \rho(L) > 0 \end{cases} \quad \text{and} \quad W_N := \begin{cases} \mathbb{P}^{N-1}(C) & \text{if } \rho(N) > 0 \end{cases}.
\]
Both \( W_L \) and \( W_N \) are irreducible, generically smooth, of dimensions \( \rho(L) \) and \( \rho(N) \) (cf. [1, p. 214]). Let
\[
N \to C \times \text{Pic}^{d-\delta}(C) \quad \text{and} \quad \mathcal{L} \to C \times \text{Pic}^{\delta}(C)
\]
be the Poincaré line-bundles. With an abuse of notation, we will denote by \( \mathcal{L} \) (resp., by \( N \)) the restriction of the Poincaré line-bundle to the Brill-Noether locus.

Set
\[
Y := \text{Pic}^{d-\delta}(C) \times W_L \quad \text{and} \quad Z := W_N \times W_L \subset Y.
\]
They are both irreducible, of dimensions
\[
\dim(Y) = g + \rho(L) \quad \text{and} \quad \dim(Z) = \rho(N) + \rho(L).
\]
Consider the projections
\[
C \times \text{Pic}^{d-\delta}(C) \xrightarrow{pr_{1,2}} C \times Y \xrightarrow{pr_{2,3}} Y \xrightarrow{\gamma} \mathbb{P}(\mathcal{E}) \xrightarrow{\gamma} U,
\]
As in [1, p. 164-179]), we define
\[
\mathcal{E}_\delta := R^1(pr_{2,3})_*(pr_{1,2}(N) \otimes pr_{1,3}^*(\mathcal{L}^\vee))
\]
(Shortly denoted by \( \mathcal{E} \) if no confusion arises). By (4.2), when \( 2\delta - d \geq 1 \), \( \mathcal{E} \) is a vector bundle of rank \( m = 2\delta - d + g - 1 \) on \( Y \) whereas, when \( d = 2\delta \), \( \mathcal{E} \) is a vector bundle of rank \( g - 1 \) on \( Y \setminus \Delta_y \), where \( \Delta_y = \{ (M, M) \mid M \in W_L \} \cong W_L \). We set
\[
\mathcal{U} := \left\{ \begin{array}{ll} y & \text{if } 2\delta - d \geq 1 \\ y \setminus \Delta_y & \text{if } d = 2\delta \end{array} \right. \quad \text{and} \quad \mathbb{P}(\mathcal{E}) \xrightarrow{\gamma} \mathcal{U},
\]
where \( \gamma \) is the projective bundle morphism: for \( y = (N, L) \in \mathcal{U} \), its \( \gamma \)-fibre is \( \mathbb{P}(\text{Ext}^1(L, N)) = \mathbb{P} \) as in (4.6).

From (4.2) and (6.1), one has
\[
\dim(\mathbb{P}(\mathcal{E})) = g + \rho(L) + m - 1 \quad \text{and} \quad \dim(\mathbb{P}(\mathcal{E})|_Z) = \rho(N) + \rho(L) + m - 1.
\]
Since (semi)stability is an open condition (cf. e.g. [42, Prop. 6-c, p. 17]), for \( g, d, \delta \) as in §’s 5.1, 5.2, there is an open, dense subset \( \mathbb{P}(\mathcal{E})^0 \subset \mathbb{P}(\mathcal{E}) \) and a morphism
\[
\pi_{d, \delta} : \mathbb{P}(\mathcal{E})^0 \to U_C(d).
\]
We set
\[
\mathcal{V}_d^{\delta, j} := \text{Im}(\pi_{d, \delta}) \quad \text{and} \quad \nu_d^{\delta, j} = \dim(\mathcal{V}_d^{\delta, j}).
\]

6.1. Non-special \( N \). We will put ourselves in the hypotheses either of Theorem 5.1 or of Theorem 5.4. In either case, \( \deg(N) \geq g - 1 \) so \( N \) can be taken general in \( \text{Pic}^{d-\delta}(C) \) and \( \mathcal{V}_d^{\delta, j} \subset B_C^k(d) \).

6.1.1. Case \( 2\delta - d \geq 1 \). One has \( \mathcal{V}_d^{\delta, j} \subset B_C^k(d) \cap U_C(d) \). Therefore any irreducible component of \( B_C^k(d) \) intersected by \( \mathcal{V}_d^{\delta, j} \) has at least dimension \( k^{\delta, j}_d \) (cf. Remark 3.3 and Definition 3.4).

**Proposition 6.1.** Assumptions as in Theorem 5.4, with \( 2\delta - d \geq 1 \). Then, for any \( j, \delta, d \) therein, there exists an irreducible component \( \mathcal{B} \subset B_C^k(d) \) such that:

(i) \( \mathcal{V}_d^{\delta, j} \subset \mathcal{B} \);

(ii) \( \mathcal{B} \) is regular and generically smooth;

(iii) for \( [\mathcal{E}] \in \mathcal{B} \) general, \( \mathcal{E} \) is stable, with \( s(\mathcal{E}) \geq 2\delta - d \) and \( i(\mathcal{E}) = j \).

**Proof.** Parts (i) and (iii) follow from Theorem 5.4 (note that the Segre invariant is lower-semicontinuous; cf. also [22, § 3]). Assertion (ii) follows from the fact that, for \( [\mathcal{F}] \in \mathcal{V}_d^{\delta, j} \) general, the Petri map \( P_{\mathcal{F}} \) is injective (cf. Remark 3.3 and [14, Lemma2.1]).

**Lemma 6.2.** In the hypotheses of Theorem 5.4, with \( 2\delta - d \geq 1 \), the morphism \( \pi_{d, \delta} \) is generically injective.
Proof. Let $[\mathcal{F}] \in \mathcal{V}^{\delta,j}_d$ be general; then $\mathcal{F} = \mathcal{F}_u$, for $u \in \text{Ext}^1(L,N)$ and $y = (N,L) \in \mathcal{U}$ general. Then
\[ \pi_{d,\delta}^{-1}(\mathcal{F}_u) = \{ (N', L', u') \in \mathcal{P}(\mathcal{E})^g \mid \mathcal{F}_{u'} \cong \mathcal{F}_u \}. \]
Assume by contradiction there exists $(N', L', u') \neq (N, L, u)$ in $\pi_{d,\delta}^{-1}(\mathcal{F}_u)$. Then $N \otimes L \cong N' \otimes L'$.

(1) If $L \cong L' \in \mathcal{W}_L$, then $N \cong N' \in \text{Pic}^{d-\delta}(C)$. Thus, $u, u' \in \mathcal{P}$. Let $\varphi : \mathcal{F}_{u'} \rightarrow \mathcal{F}_u$ be the isomorphism between the two bundles. Since $\mathcal{F}_u$ is stable, then $u \neq u' \in \mathcal{P}$ (notation as in (4.6)) and we have the diagram
\[
\begin{array}{c}
0 \rightarrow N \rightarrow \mathcal{F}_{u'} \rightarrow L \rightarrow 0 \\
0 \rightarrow N \rightarrow \mathcal{F}_u \rightarrow L \rightarrow 0.
\end{array}
\]
The maps $\varphi \circ \iota_1$ and $\iota_2$ determine two non-zero sections $s_1 \neq s_2 \in H^0(\mathcal{F}_u \otimes N')$. They are linearly dependent, otherwise the section $\Gamma \subset \mathcal{F}_u$, corresponding to $\mathcal{F}_u \rightarrow L$ would not be li (cf. (2.6) and Theorem 5.4-(ii)). So $s_1 = \lambda s_2$. But then Lemma 4.5 implies $u = u'$, a contradiction.

(2) If $L \cong L' \in \mathcal{W}_L$, (in particular, $\rho(L) > 0$), sections $\Gamma \neq \Gamma'$, corresponding respectively to $\mathcal{F}_u \rightarrow L$ and $\mathcal{F}_u \rightarrow L'$, would be such that $\Gamma \sim_{alg} \Gamma'$ on $\mathcal{F}_u$, contradicting Theorem 5.4-(ii).

\[\square\]

Example 6.3. One can have loci $\mathcal{V}^{\delta,j}_d$ of the same dimension for different values of $\delta$. For instance, let $j = 2$, $g \geq 18$ and $d = 2g + 9$ in Theorem 5.4. Then $g + 5 \leq d \leq g + 6$ are admissible values in (5.1) and for both one has $2d - d > 0$. Now $\rho(g,7,g + 5) = g - 16 > g - 18 = \rho(g,8,g + 6)$; by (6.5) and Lemma 6.2 one has $\nu_{g+5}^{d+2} = \nu_{g+6}^{d+2} = 3g - 17$.

Remark 6.4. From (3.3), (6.5) and Lemma 6.2, for $\mathcal{V}^{\delta,j}_d$ as in Proposition 6.1 one has
\[ \nu_{d,j}^{\delta,j} - \rho_{dj}^{k_l} = d(j - 1) - \delta(j - 2) - (g - 1)(j + 1). \]
(1) For $j = 1$, $\nu_{d}\delta^{\delta - 2} - \rho_{dj}^{k_l} = \delta - 2g + 2 \leq 0$, since $L$ special. Equality holds if and only if $\delta$ reaches the upper-bound in (5.1).

(2) For $j \geq 2$, using the upper-bound in (5.1) and $d < 2\delta$, one gets
\[ \nu_{d,j}^{\delta,j} - \rho_{dj}^{k_l} < \delta j - gj - g + j \leq 1 - j^2 < 0. \]
Thus, $\mathcal{V}^{\delta,j}_d$ can never fill up a regular component of $B^{\delta,j}_{C}(d)$ unless $j = 1$ and $\delta = 2g - 2$.

Corollary 6.5. Let $C$ be of genus $g > 5$ with general moduli. For $3g + 1 \leq d \leq 4g - 5$, $\mathcal{V}^{2g-2,1}_d$ is dense in a regular, generically smooth component $\mathcal{B} \subset B^{2g-2,1}_C(d)$. Moreover:

(i) $[\mathcal{F}_u] \in \mathcal{V}^{2g-2,1}_d$ general is stable and comes from $u \in \text{Ext}^1(\omega_C, N)$ general, with $N \in \text{Pic}^{2g-2+2}(C)$ general. In particular, $i([\mathcal{F}_u]) = 1$.

(ii) The minimal degree quotient of $\mathcal{F}_u$ is $\omega_C$, so $s([\mathcal{F}_u]) = 4g - 4 - d > 0$.

(iii) $\text{Div}^{1,2g-2}_{\mathcal{F}_u} = \{1\}$, where $\Gamma$ is the section corresponding to the quotient $\mathcal{F}_u \twoheadrightarrow \omega_C$ (i.e., $\mathcal{F}_u$ is rp via $\omega_C$).

Proof. It follows from Theorem 5.4, with $2\delta - d \geq 1$ and $j = 1$, from Proposition 6.1 and from Remark 6.4. \[\square\]

Remark 6.6. Using Theorem 5.1 and Corollary 5.3, one can prove results similar to Proposition 6.1 and Corollary 6.5 with slightly different numerical bounds. As in Remark 6.4, $\mathcal{V}^{\delta,j}_d$ can never fill up a regular component of $B^{2g-2,1}_C(d)$, unless $\delta = 2g - 2$ and $j = 1$. The numerical bounds in this case are $3g - 3 \leq d \leq 4g - 6$ with $g \geq 3$, hence the cases not already covered are $3g - 3 \leq d \leq 4g - 6 \leq \min\{3g, 4g - 6\}$.

Corollary 6.7. Let $C$ be of genus $g \geq 6$ with general moduli. For $3g - 3 \leq d \leq 3g$, $\mathcal{V}^{2g-2,1}_d$ is dense in a regular, generically smooth component $\mathcal{B} \subset B^{2g-2}_C(d)$. Moreover:

(i) $[\mathcal{F}_u] \in \mathcal{V}^{2g-2,1}_d$ general is stable and comes from $u \in \text{Ext}^1(\omega_C, N)$ general, with $N \in \text{Pic}^{d-2g+2}(C)$ general (so non-special). In particular, $i([\mathcal{F}_u]) = 1$.

(ii) The minimal degree quotient of $\mathcal{F}_u$ is $\omega_C$, thus $s([\mathcal{F}_u]) = 4g - 4 - d \geq g - 4$.

(iii) $\text{Div}^{1,2g-2}_{\mathcal{F}_u} = \{1\}$, where $\Gamma$ the section corresponding to $\mathcal{F}_u \twoheadrightarrow \omega_C$ (i.e., $\mathcal{F}_u$ is rp via $\omega_C$).

Proof. We need to prove that $\pi_{d,2g-2}$ is generically injective. The proof of Lemma 6.2 shows that, for $[\mathcal{F}_u] \in \mathcal{V}^{2g-2,1}_d$ general, one has $\dim(\pi_{d,2g-2}([\mathcal{F}_u])) \leq \dim(\text{Div}^{1,2g-2}_{\mathcal{F}_u})$. By construction of $\mathcal{V}^{2g-2,1}_d$ and by (2.3), $N_{T/\mathcal{F}_u} \cong K_C - N$. Since $N$ is general of degree $d - 2g + 2$, one has $h^1(N) = 0$. From Remark 2.3 we conclude. To prove the injectivity of $P_{\mathcal{F}_u}$ one can argue as in [14, Lemma 2.1] (we leave the easy details to the reader). \[\square\]
6.1.2. Case $d = 2\delta$. From Theorem 5.4, for any $j \geq 1$, $g$ and $\delta$ as in (5.1) and in Remark 5.5, we have

$$V_{\delta j}^{\delta j} \subseteq B_{\delta j}^2(2\delta) \cap U_{\delta j}^2(2\delta).$$

Lemma 6.8. The morphism $\pi_{d, \delta}$ contracts the $\gamma$-fibres, with $\gamma$ as in (6.2). Thus, $\nu_{d, \delta}^{\delta j} \leq g + \rho(L)$.

Proof. For any $y = (N, L) \in \mathcal{U}$, $\gamma^{-1}(y) \cong \mathbb{P}$ as in (4.6). For any $u \in \mathbb{P}$, one has $\text{gr}(\mathcal{F}_u) = L + N$, where $\text{gr}(\mathcal{F}_u)$ is the graded object associated to $\mathcal{F}_u$ (cf. [42, Thm. 4]). Therefore, all elements in a $\gamma$-fibre determine $S$-equivalent bundles (cf. e.g. [28, 44]). This implies that $\pi_{d, \delta}$ contracts any $\gamma$-fibre. □

Corollary 6.9. Let $C$ be of genus $g \geq 5$ with general moduli. One has

$$B_{d j}^{k_1}(4g - 4) = V_{d g - 4}^{2g - 2}.$$

Thus:

(i) $B_{d j}^{k_1}(4g - 4)$ is irreducible, of dimension $g < \rho_{4g-4}^{k_1} = 2g - 2$. In particular, it is birational to $\text{Pic}^{2g-2}(C)$, with $B_{d j}^{k_2}(4g - 4) = \{[\omega_C \oplus \omega_C] \}$;

(ii) $[\mathcal{F}] \in B_{d j}^{k_1}(4g - 4)$ general comes from $u \in \text{Ext}^1(\omega_C, N)$ general, with $N \in \text{Pic}^{2g-2}(C)$ general. Hence, $i(\mathcal{F}_u) = 1$.

(iii) The minimal degree quotient of $\mathcal{F}_u$ is $\omega_C$, thus $s(\mathcal{F}_u) = 0$ and $\mathcal{F}_u$ is strictly semistable.

(iv) $\text{Div} V_{d j}^{2g - 2} = \{\Gamma\}$, where $\Gamma$ the section corresponding to $\mathcal{F}_u \mapsto \omega_C$ (i.e. $\mathcal{F}_u$ is $\text{rp}$ via $\omega_C$).

Proof. From Theorem 5.4, the only case for $d = 4g - 4$ is $j = 1$ and $\delta = 2g - 2$. Since $d = 2\delta$, from (6.2) we have $\mathcal{U} \cong \text{Pic}^{2g-2}(C) \setminus \{\omega_C\}$ and $\mathcal{E}$ is a vector bundle of rank $g - 1$ on $\mathcal{U}$. From Lemma 6.8, the moduli map $\pi_{d, 2g-2}$ factors through a map from $\mathcal{U}$ to $B_{d j}^{k_1}(4g - 4)$, which is injective by Chern class reasons.

Next we prove that $B_{d j}^{k_1}(4g - 4)$ is irreducible. Consider $[\mathcal{F}]$ general in a component of $B_{d j}^{k_1}(4g - 4)$; it can be presented via an exact sequence as in (2.2), with $L$ special and effective (cf. Lemma 4.1). Since $s(\mathcal{F}) = 0$, then $\text{deg}(L) = 2g - 2$, i.e. $L \cong \omega_C$. Thus, we are in the image of $\mathcal{U}$ to $B_{d j}^{k_1}(4g - 4)$.

The remaining assertions are easy to check and can be left to the reader. □

Corollary 6.9 has been proved already in [8, Theorems 7.2, 7.3 and Remark 7.4]. Our proof is completely independent.

6.2. Special $N$. Under the assumptions of Theorem 5.10, any $N \in \text{Pic}^{d-\delta}(C)$ is special (cf. Remark 5.11-(2)) and, for $u \in \text{Ext}^1(L, N)$ general, $\partial_u$ is surjective (cf. Remark 5.11-(3)). Hence $i(\mathcal{F}_u) = h^1(L) = j$. We have:

Proposition 6.10. Assumptions as in Theorem 5.10. For any $j$, $\delta$ and $d$ therein, there exists an irreducible component $\mathcal{B} \subseteq B_{d j}^{k_1}(d)$ such that:

(i) $V_{d j}^{\delta j} \subseteq \mathcal{B}$;

(ii) $\mathcal{B} \cap U_{d j}^*(d) \neq \emptyset$;

(iii) For $[\mathcal{E}] \in \mathcal{B}$ general, $\mathcal{E}$ is stable, with $s(\mathcal{E}) \geq g - \epsilon$ and $\epsilon$ as in Theorem 5.10. The minimal degree quotients of $\mathcal{E}$ as well as the the minimal degree sections of $\mathcal{P}(\mathcal{E})$ are as in (iii) and (iv) of Theorem 5.10. In particular, $L$ is of minimal degree if and only if $d = 2\delta - g$.

(iv) If moreover $d \geq \delta + g - 3$ (so $\delta \geq 2g - 3$), then $\mathcal{B}$ is also regular and generically smooth.

Proof.Assertions (i), (ii) and (iii) follow from Theorem 5.10, the map (6.4) and the fact that the Segre invariant is lower-semicontinuous (cf. e.g. [22, § 3]).

To prove (iv), we argue as in [14, Lemma 2.1.]. Take $\mathcal{F}_0 = L \oplus N$, with $N \in \text{Pic}^{d-\delta}(C)$ general. Then, $N$ is non-effective and $1 \leq h^1(N) \leq 2$. The Petri map $P_{\mathcal{F}_0}$ decomposes as $\mu_L(L) \oplus \mu_N$, where $\mu_L(L)$ is the Petri map of $L$ as in (1.1) and $\mu_N$ is as in (5.7). Since $C$ has general moduli, $\mu_L(L)$ is injective (cf. [1, (1.7), p. 215]). The injectivity of $\mu$ is immediate when $h^1(N) = 1$ (cf. the proof of Theorem 5.8). When $h^1(N) = 2$, the generality of $N$ implies that $[K_C - N]$ is a base-point-free pencil so the injectivity of $\mu$ follows from the base-point-free pencil trick, since $h^0(N - (K_C - L)) = 0$ (because $K_C - L$ is effective and $N$ non-effective). By semicontinuity on the elements of $\text{Ext}^1(L, N)$ and the fact that $V_{d j}^{\delta j} \subseteq \mathcal{B}$, the Petri map $P_{\mathcal{E}}$ is injective. One concludes by Remark 3.3. □

Remark 6.11. Computing $\text{dim}(\mathcal{P}(\mathcal{E})) - \rho_{d j}^{k_1}$ one finds the right-hand-side of (6.6). Since $d < 2\delta$ (see (5.9)), as in Remark 6.4-(2) one sees that $\text{dim}(\mathcal{P}(\mathcal{E})) - \rho_{d j}^{k_1} < 0$, unless $j = 1$ and $\delta = 2g - 2$, in which case $\text{dim}(\mathcal{P}(\mathcal{E})) - \rho_{d j}^{k_1} = 0$. As in Lemma 6.2, we see that $\pi_{d, 2g-2}$ is generically injective. Thus, with notation as in (6.5), one has $\nu_{d j}^{\delta j} \geq \rho_{d j}^{k_1}$ only if $j = 1$, $\delta = 2g - 2$ and $N \in \text{Pic}^{d-\delta}(C)$ is general, in which case $\nu_{d j}^{2g-2,1} = \rho_{d j}^{k_1}$. 

Corollary 6.12. Let $C$ be of genus $g \geq 3$ with general moduli. For $2g - 2 \leq d \leq 3g - 4$, one has $\nu_{d}^{2g-2,1} = \rho_{d}^{k_{1}} = 6g - 6 - d$. Moreover:

(i) $[\mathcal{F}] \in V_{d}^{g-2,1}$ general is stable and comes from $u \in \text{Ext}^{1}(\omega_{C}, N)$ general, with $N \in \text{Pic}^{g-2g+2}(C)$ general (hence special, non-effective). In particular, $i(\mathcal{F}) = 1$.

(ii) If $3g - 5 \leq d \leq 3g - 4$, then $V_{d}^{g-2,1}$ is dense in a regular, generically smooth component of $B_{C}^{d}(d)$.

(iii) $s(\mathcal{F}) = g - \epsilon$, with $\epsilon$ as in Theorem 5.10. Quotients of minimal degree of $\mathcal{F}$ (equivalently sections of minimal degree on $\mathcal{F}$) are as in (iii) and (iv) of Theorem 5.10. In particular, these are li sections.

(iv) The canonical section $\Gamma \subset F_{d}$ is the only special section; it is lsu and asu but not ai. Moreover, it is of minimal degree only when $d = 3g - 4$.

(v) $\mathcal{F}$ is rsp but not rp via $\omega_{C}$.

Proof. (i), (ii) and (iii) follow from Theorem 5.10, Proposition 6.10 and Remarks 3.3, 6.11. Sections of minimal degree are li (see the proof of Proposition 2.12).

As for (iv) and (v), from Serre duality and the fact that $\mathcal{F}$ is of rank-two with $\det(\mathcal{F}) = \omega_{C} \otimes N$, one has

$$h^{0}(\mathcal{F} \otimes N^{\vee}) = h^{1}(\mathcal{F}^{\vee} \otimes \omega_{C} \otimes N) = h^{1}(\mathcal{F}).$$

(6.7)

Since $i(\mathcal{F}) = 1$, from (2.6) $\Gamma$ is li. Since $N$ is special and non-effective, from (2.3), $\text{div}_{F}^{1,2g-2}$ is smooth, of dimension $3g - 3 - d \geq 1$ at $\Gamma$. Thus, $\Gamma$ is not ai but, since $W_{g-2g+2}(C) = \{\omega_{C}\}$, it is asu (see the proof of Proposition 2.12 and Remark 6.11). For the same reason, from Theorem 5.10-(iv), the only possibility for $\omega_{C}$ to be a minimal quotient is $d = 3g - 4$. Finally, the fact that $\Gamma \subset F_{d}$ is the only special section follows from Remark 6.11.

When $N \in \text{Pic}^{g-2}(C)$ is special and $L \in W_{d}^{g-2g+j}(C)$ is a smooth point, assumptions as in Theorem 5.8 imply that $\partial_{u}$ is surjective for $u \in \text{Ext}^{1}(L, N)$ general (cf. Corollary 5.9), thus $i(\mathcal{F}) = j$. Therefore, to have $i(\mathcal{F}) > j$, we are forced to use degeneracy loci (5.4). To do this, let $y = (N, L)$ be general in $\mathcal{U}$ (resp., in $\mathcal{Z}$), when $N$ is non-effective (resp., effective). Set $P(y) := \gamma^{-1}(y) \cong \mathbb{P}$. Take the numerical assumptions as in Remark 5.15 (resp., Remark 5.16), when $N$ is non-effective (resp., effective). With notation as in (5.11), for any good component $\hat{\Lambda}_{t}(y) \subseteq \hat{\mathcal{W}}_{t}(y) \subseteq P(y)$ we have

$$\emptyset \neq \hat{\mathcal{W}}_{t}^{\text{Tot}} \subseteq \mathcal{W}_{t} \subseteq P(\mathcal{E}),$$

where a point in $\hat{\mathcal{W}}_{t}^{\text{Tot}}$ corresponds to the datum of a pair $(y, u)$, with $y = (N, L)$ and $u \in \mathcal{W}_{t}(y)$. Any irreducible component of $\hat{\mathcal{W}}_{t}^{\text{Tot}}$ has dimension at least $\dim(P(\mathcal{E})) - c(\ell, r, t)(c(\ell, r, t)$ as in (5.5) and $\dim(P(\mathcal{E}))$ as in (6.3)). From the generality of $y$, for any good component $\hat{\Lambda}_{t}(y)$, we have an irreducible component

$$\hat{\Lambda}_{t}^{\text{Tot}} \subseteq \hat{\mathcal{W}}_{t}^{\text{Tot}} \subseteq P(\mathcal{E})$$

such that

(i) $\hat{\Lambda}_{t}^{\text{Tot}}$ dominates $\mathcal{U}$ (resp., $\mathcal{Z}$);

(ii) $\dim(\hat{\Lambda}_{t}^{\text{Tot}}) = \dim(P(\mathcal{E})) - c(\ell, r, t)$;

(iii) for $(y, u) \in \hat{\Lambda}_{t}^{\text{Tot}}$ general, cork($\partial_{u}$) = $t$;

(iv) if $\lambda := \gamma|_{\hat{\Lambda}_{t}^{\text{Tot}}}$, for $y$ general one has $\lambda^{1}(y) = \hat{\Lambda}_{t}(y)$.

Definition 6.13. Any component $\hat{\Lambda}_{t}^{\text{Tot}}$ satisfying (i)-(iv) above will be called a (total) good component of $\hat{\mathcal{W}}_{t}^{\text{Tot}}$.

We set

$$\nu_{d}^{k, j, t} := \text{Im} \left( \pi_{d, \delta}|_{\hat{\Lambda}_{t}^{\text{Tot}}(y)} \right) \subseteq B_{C}^{k, j, t}(d)$$

and $\nu_{d}^{k, j, t} := \dim(\nu_{d}^{k, j, t})$.

(6.8)

As usual, we will have to discuss two cases.

6.2.1. $N$ non-effective. With assumptions as in Remark 5.15, $N$ can be taken general in Pic$^{g-2}(C)$; the general bundle in $V_{d}^{k, j, t}$ is stable (by Theorem 5.13 and by the open nature of stability). Set

$$\varphi_{0}(\delta, j, t) := \dim(\hat{\Lambda}_{t}^{\text{Tot}}) - \rho_{d}^{k+1} = d(j - 1) - \delta(j - 2) - (g - 1)(j + 1) + jt.$$  

(6.9)

One has $\varphi_{0}(\delta, j, t) \geq \nu_{d}^{k, j, t} - \rho_{d}^{k+1}$ with equality if and only if $\pi_{d, \delta}|_{\hat{\Lambda}_{t}^{\text{Tot}}}$ is generically finite. Thus, from Remark 3.3, $\nu_{d}^{k, j, t}$ cannot fill up a component of $B_{C}^{k, j, t}(d)$ if $\varphi_{0}(\delta, j, t) < 0$.

- For $j = 1$, one has

$$\varphi_{0}(\delta, 1, t) = \delta - 2g + 2 + t.$$  

(6.10)
When \( j \geq 2 \), from Remark 5.15 and arguing as in Remark 6.4, one gets
\[
\varphi_0(\delta, j, t) \leq j(t - j).
\]
Thus, \( \nu^{\delta,j,t}_d \) never fills up a component of \( B^{k,j,t+1}_d \) as soon as \( j > t \geq 1 \).

**6.2.2. N effective.** With assumptions as in Remark 5.16, \( N \) is general in \( W_{\rho(N)} \). From the second equality in (6.3), for any \( n \geq 1 \), one has
\[
\varphi_n(\delta, j, t) := \dim(\Lambda^T_{\delta, j, t}) - \rho^j_{d, t+1} = \varphi_0(\delta, j, t) - n(r - t),
\]
where \( \varphi_0(\delta, j, t) \) as in (6.9).

**Remark 6.14.** For a total good component \( \Lambda^T_{\delta, j, t} \) and for \( (L, N, u) \in \Lambda^T_{\delta, j, t} \) general, one has \( n(r - t) = h^0(N) \) \( \text{rk}(\partial_u) \). Hence, \( n(r - t) \) is non-negative and it is zero if and only if \( r = t \), i.e. \( \partial_u \) is the zero map. Therefore, \( \varphi_n(\delta, j, t) \leq \varphi_0(\delta, j, t) \) and equality holds if and only if \( r = t \). The possibility for a \( \nu^{\delta,j,t}_d \) to fill up a component of \( B^{k,j,t+1}_d \) can be discussed as in \S 6.2.1.

From (6.8), (6.9) and (6.12), we have \( \nu^{\delta,j,t}_d - \rho^j_{d, t+1} \leq \varphi_n(\delta, j, t) \) for any \( n \geq 0 \). Next proposition easily follows.

**Proposition 6.15.** Assumptions as in Theorem 5.13 (more precisely, either as in Remark 5.15, when \( N \) is non-effective, or as in Remark 5.16, when \( N \) is effective). Then for any \( j, \delta \) and \( d \), there exists an irreducible component \( B \subseteq B^{k,j,t+1}_d \) such that:

(i) \( \nu^{\delta,j,t}_d \subseteq B \);

(ii) For \( \mathcal{E} \in B \) general, \( s(\mathcal{E}) \geq g - c(\ell,r,t) - \epsilon \geq 0 \), where \( c(\ell,r,t) \) as in (5.5) and \( \epsilon \in \{0,1\} \) such that \( d + g - c(\ell,r,t) \equiv \epsilon \pmod{2} \);

(iii) \( B \cap U^\mathcal{E}_2(d) \neq \emptyset \), if \( g - c(\ell,r,t) - \epsilon > 0 \).

If \( \nu^{\delta,j,t}_d \geq \rho^j_{d, t+1} \), then \( \varphi_n(\delta, j, t) \geq 0 \).

**Remark 6.16.** In order to estimate \( \nu^{\delta,j,t}_d \), one has to estimate \( \dim(\pi_{d, \delta, \varphi}^{-1}([\mathcal{F}])) \) for \( [\mathcal{F}] \in \nu^{\delta,j,t}_d \) general. A rough estimate is
\[
\dim\left(\pi_{d, \delta, \varphi}^{-1}\left([\mathcal{F}] \right)\right) \leq a_F(\delta),
\]
where \( F = \mathbb{P}(\mathcal{F}) \) and \( a_F(\delta) \) as in (2.9). In particular, if for \( L \in W_{\rho(L)} \) general (4.3) holds, then \( \nu^{\delta,j,t}_d \geq \rho^j_{d, t+1} + \varphi_n(\delta, j, t) - 1 \).

**Remark 6.17.** Assume \( j = 1 \) in Proposition 6.15.

(1) When \( N \in \text{Pic}^{d-\delta}(C) \) is general, assumptions as in Remark 5.15 give \( \delta \leq 2g - 2 \) and \( N \) non-effective, for any \( t \geq 1 \). The only case to consider is \( \varphi_0(\delta, 1, t) \). From Proposition 6.15, a necessary condition for \( \nu^{\delta,j,t}_d \) to have dimension at least \( \rho^j_{d, t+1} \) is \( \varphi_0(\delta, 1, t) \geq 0 \), i.e. \( \delta \geq 2g - 2 - t \) (cf. (6.10)). Thus:

- when \( \delta = 2g - 2 - t \), then \( L = \omega_C(-D_t) \), with \( D_t \in C^{(t)} \), \( t < g \), imposing independent conditions to \( |\omega_C| \).

Since \( \varphi_0(\delta, 1, t) = 0 \), for \( D_t \in C^{(t)} \) general, (6.13) and a parameter count suggest that for \( [\mathcal{F}] \in \nu^{\delta,j,t}_d \) general one has \( a_F(2g - 2 - t) = 0 \), i.e. \( \mathcal{F} \) is rsp via \( \omega_C(-D_t) \), and \( B = \nu^{\delta,j,t}_d \) is regular.

- to the opposite, when \( \delta = 2g - 2 \), then \( L = \omega_C \). Let \( [\mathcal{F}] \in \nu^{2g-2,1,t}_d \) be general, and let \( \Gamma \subseteq F = \mathbb{P}(\mathcal{F}) \) be the canonical section corresponding to \( \mathcal{F} \rightarrow \omega_C \). By definition of \( \nu^{2g-2,1,t}_d \), \( \mathcal{F} \in \mathcal{S}_v \) for \( v \in \Lambda_t \subseteq \text{Ext}^1(N, \omega_C) \) general in a good component. By (2.6) and (6.7), one has \( \dim(\mathcal{O}_F(\Gamma)) = t \). Thus, \( [\mathcal{F}] \in \nu^{2g-2,1,t}_d \) general is not rsp via \( \omega_C \), since the general fibre of \( \pi_{d,2g-2}^{-1}\left([\mathcal{F}] \right) \) has dimension at least \( t \).

It is therefore natural to expect that the component \( B \) in Proposition 6.15 is such that
\[
B = \nu^{2g-2,1,t}_d = \nu^{2g-3,1,t}_d = \cdots = \nu^{2g-2-t,1,t}_d,
\]
where \( [\mathcal{F}] \in B \) general is rsp only when \( [\mathcal{F}] \) is considered as element in \( \nu^{2g-2-t,1,t}_d \).

(2) One may expect something similar when \( j = 1 \) and \( N \) effective general in \( W_{\rho(N)} \). In this case, \( \varphi_n(\delta, 1, t) \geq 0 \) gives \( \delta \geq 2g - 2 + rn - t(n + 1) \) whereas, from the first line of bounds on \( \delta \) in Remark 5.16, we get \( \delta \leq \min\{2g - 2, g - 2 + r - t + \frac{d^2}{t^2}\} \). A necessary condition for \( \nu^{\delta,j,t}_d \geq \rho^j_{d, t+1} \) is therefore
\[
\rho^j_{d, t+1} - \rho^j_{d, t} \geq 0, \quad (6.14)
\]
otherwise either \( L \) would be non special, contradicting Lemma 4.1, or \( L \equiv \omega_C \), so \( \mathcal{S}_v \) would be not rsp as in (1) above. In the next section, we will discuss these questions.
7. Low Speciality, Canonical Determinant

In this section we apply the results of §’s 5.1, 5.2 and 6 to describe Brill-Noether loci of vector bundles with canonical determinant and Brill-Noether loci of vector bundles of fixed degree $d$ and low speciality on a curve $C$ with general moduli. For irreducible components arising from constructions in § 6, we determine a rigidly specially presentation of their general point.

For any $i \geq 1$, $2g - 2 \leq d \leq 4g - 4$, we set

$$\tilde{B}^k_C(d) := \begin{cases} B^k_C(d) & \text{if either $d$ odd or $d = 4g - 4$} \\ B^k_C(d) \cap U^*_C(d) & \text{otherwise} \end{cases}$$

(7.1)

7.1. Vector bundles with canonical determinant. Given an integer $d$ and any $\xi \in \text{Pic}^d(C)$, there exists the moduli space of (semi)stable, rank-two vector bundles with determinant $\xi$. Following [30, 31], we denote it by $M_C(2, \xi)$ (sometimes a different notation is used, see e.g. [42, 6, 7, 52, 37, 53, 5, 24]).

The scheme $M_C(2, \xi)$ is defined as the fibre over $\xi \in \text{Pic}^d(C)$ of the determinantal map

$$U_C(d) := \text{Pic}^d(C).$$

(7.2)

For any $\xi \in \text{Pic}^d(C)$, $M_C(2, \xi)$ is smooth, irreducible, of dimension $3g - 3$ (cf. [33, 42]).

Brill-Noether loci can be considered in $M_C(2, \xi)$. Recent results for arbitrary $\xi$ are given in [37, 38, 23].

A case which has been particularly studied (for its connections with Fano varieties) is $M_C(2, \omega_C)$. Seminal papers on the subject are [7, 30]; other important results are contained in [52, 53, 24]. If $[\mathcal{F}] \in M_C(2, \omega_C)$, Serre duality gives

$$i(\mathcal{F}) = k_i(\mathcal{F}) := k.$$ 

(7.3)

For $[\mathcal{F}] \in M_C(2, \omega_C) \subset U_C(2g - 2)$, the Petri map $P_{\mathcal{F}}$ (3.4) splits as $P_{\mathcal{F}} = \lambda_{\mathcal{F}} \oplus \mu_{\mathcal{F}}$, where

$$\lambda_{\mathcal{F}} : 2 H^0(\mathcal{F}) \to H^0(\omega_C) \quad \text{and} \quad \mu_{\mathcal{F}} : \text{Sym}^2(H^0(\mathcal{F})) \to H^0(\text{Sym}^2(\mathcal{F}));$$

the latter is called the symmetric Petri map.

For $[\mathcal{F}] \in M_C(2, \omega_C)$ general, one has $k = 0$ (cf. [31, §4], after formula (4.3)). For any $k \geq 1$, one sets

$$M^k_C(2, \omega_C) := \{[\mathcal{F}] \in M_C(2, \omega_C) \mid h^0(\mathcal{F}) = h^1(\mathcal{F}) \geq k\}$$

which is called the $k^{th}$-Brill-Noether locus in $M_C(2, \omega_C)$. In analogy with (7.1), we set

$$\tilde{M}^k_C(2, \omega_C) := \tilde{M}^k_C(2, \omega_C) \cap U^*_C(2g - 2).$$

(7.4)

By (7.3) and, one has

$$\text{expcodim}_{M_C(2, \omega_C)}(\tilde{M}^k_C(2, \omega_C)) = \frac{k(k + 1)}{2} \leq k^2 = i(\mathcal{F})k_i(\mathcal{F}).$$

Similarly to $\tilde{B}^k_C(d)$, if $[\mathcal{F}] \in \tilde{M}^k_C(2, \omega_C)$, then $\tilde{M}^k_C(2, \omega_C)$ is smooth and regular (i.e. of the expected dimension) at $[\mathcal{F}]$ if and only if $\mu_{\mathcal{F}}$ is injective (see [7, 30, 31]).

Several basic questions on $M^k_C(2, \omega_C)$, like non-emptiness, irreducibility, etc., are still open. A complete description is available only for some $k$ on $C$ general of genus $g \leq 12$ (cf. [31, §4], [7]); see also [50, 24]. On the other hand, if one assumes $[\mathcal{F}] \in M^k_C(2, \omega_C)$, injectivity of $\mu_{\mathcal{F}}$ on $C$ general of genus $g \geq 1$ has been proved in [52] (cf. [5] for $k < 6$ with a different approach).

7.2. Case $i = 1$. In this case $\rho^{k_1}_d = 6g - 6 - d$. Using notation and results as in § 6, we get:

**Theorem 7.1.** Let $C$ be of genus $g \geq 5$, with general moduli. For $2g - 2 \leq d \leq 4g - 4$, one has

$$\bar{B}^{k_1}_C(d) = \mathcal{F}^{g^2 - 2 - 1},$$

as in Corollaries 6.5, 6.7, 6.9 and 6.12. In particular,

(i) $\bar{B}^{k_1}_C(d)$ is non-empty, irreducible. For $2g - 2 \leq d \leq 4g - 5$ it is regular, whereas $\dim(\bar{B}^{k_1}_C(4g - 4)) = g < \rho^{k_1}_{4g - 4} = 2g - 2$.

(ii) For $3g - 5 \leq d \leq 4g - 4$, $\bar{B}^{k_1}_C(d)$ is generically smooth.

(iii) $[\mathcal{F}] \in \bar{B}^{k_1}_C(d)$ general is stable for $2g - 2 \leq d \leq 4g - 5$, and strictly semistable for $d = 4g - 4$, fitting in a (unique) sequence

$$0 \to N \to \mathcal{F} \to \omega_C \to 0,$$
where \( N \in \text{Pic}^{d-2g+2}(C) \) is general, the coboundary map is surjective and \( i(\mathcal{F}) = 1 \).

(iv) For \( 3g - 4 \leq d \leq 4g - 4 \) and \([\mathcal{F}] \in \widetilde{B}^k_C(d)\) general, one has \( s(\mathcal{F}) = 4g - 4 - d \), the quotient of minimal degree being \( \omega_C \). The section \( \Gamma \subset F \) corresponding to \( \mathcal{F} \rightarrow \omega_C \) is the only special section of \( F \). Moreover:

- for \( d \geq 3g - 3 \), \( \Gamma \) is ai,
- for \( d = 3g - 4 \), \( \Gamma \) is lsu and asu but not ai.

(v) For \( 2g - 2 \leq d \leq 3g - 5 \) and \([\mathcal{F}] \in \widetilde{B}^k_C(d)\) general, one has \( s(\mathcal{F}) = g - \epsilon \), with \( \epsilon \in \{0, 1\} \) such that \( d + g \equiv \epsilon \pmod{2} \). The section \( \Gamma \subset F \) is the only special section; it is asu but not ai. Moreover, \( \Gamma \) is not of minimal degree; indeed:

- when \( d + g \) is even, minimal degree sections of \( F \) are li sections of degree \( \frac{d + g}{2} \); \( \dim(\text{Ext}^1_{\mathcal{F}}^{d+g}) = 1 \);
- when \( d + g \) is odd, minimal degree sections are li of degree \( \frac{d + g - 1}{2} \), and \( \dim(\text{Ext}^1_{\mathcal{F}}^{d+g-1}) \leq 1 \).

(vi) In particular, for \( 2g - 2 \leq d \leq 4g - 4 \), \([\mathcal{F}] \in \widetilde{B}^k_C(d)\) general is rp via \( \omega_C \).

Proof. All the assertions, except the irreducibility, follow from Corollaries 6.5, 6.7, 6.9 and 6.12. For \( d = 4g - 4 \) irreducibility has been proved in Corollary 6.9. Thus, we focus on \( 2g - 2 \leq d \leq 4g - 5 \).

Let us consider an irreducible component \( \mathcal{B} \subseteq \widetilde{B}^k_C(d) \). From Lemma 4.1, \([\mathcal{F}] \in \mathcal{B}\) general is as in (2.2), with \( h^1(L) = j \geq 1 \) and \( L \) of minimal degree among special, effective quotient line bundles. Moreover \( \dim(\mathcal{B}) \geq r^k_d \) (cf. Remark 3.3). Two cases have to be considered.

1. If \( i(\mathcal{F}) = 1 \), then \( j = 1 \) (notation as in (4.4), (4.5)) and \( \partial : H^0(L) \rightarrow H^1(N) \) is surjective. In particular \( \ell \geq r \). If \( r = 0 \) then we are in cases of Corollaries 6.5, 6.7, and \( \mathcal{B} = \mathcal{V}_{d}^{2g-2,1} \). If \( r > 0 \), as in Remark 6.11 one has

\[
0 \leq \dim(\mathcal{P}(\mathcal{E})) - \dim(\mathcal{B}) \leq \delta - 2g + 2
\]

(cf. (6.6)). Hence \( \delta = 2g - 2 \) and \( \mathcal{B} = \mathcal{V}_{d}^{2g-2,1} \) as in Corollary 6.12.

2. Assume \( i(\mathcal{F}) = i > 1 \). As in Remarks 6.4, 6.6, 6.11 one has \( L = \omega_C \). Thus \( i > 1 \) forces \( r \geq \text{cork}(\partial) = i - 1 > 0 \). Recalling (4.2) and (4.6), one has \( \dim(\mathcal{P}) = 5g - 6 - d \). Therefore, \( \mathcal{B} \) must be regular and \( \mathcal{F} \) corresponds to the general point of \( \text{Ext}^1(\omega_C, N) \), with \( N \in \text{Pic}^{d-2g+2}(C) \) general, so non-effective. In particular, one has \( 2g - 2 \leq d \leq 3g - 4 \) and \( r = 3g - 3 - d \). On the other hand, since

\[
\ell = g, \quad 1 \leq r \leq g - 1, \quad 2g - 1 \leq m = 5g - 5 - d \leq 3g - 3
\]

we are in the hypotheses of Corollary 5.9, hence \( \text{cork}(\partial) = 0 \), a contradiction. \( \square \)

Remark 7.2. (1) Theorem 7.1 gives alternative proofs of results in [43, 25, 8] for the rank-two case, additionally providing a description of \( \widetilde{B}_C^k(b) \), with \( 0 \leq b \leq 2g - 2 \). The same description is given in [3], with a different approach, i.e. using general negative elementary transformations as in [25]. For the case of speciality 1, cf. also [17, Theorem 3.9].

(2) The Segre invariant does not stay constant on a component of the Brill-Noether locus. For example, the general element of \( \widetilde{B}_C^k(4g - 7) \) has \( s = 3 \) and \( i = 1 \) (cf. Theorem 7.1). On the other hand, in Theorem 5.4, we constructed vector bundles in \( \mathcal{V}_{4g-7}^{2g-3,1} \subset \widetilde{B}_C^k(4g - 7) \) with \( s = i = 1 \). The minimal special quotient of the latter vector bundles is the canonical bundle minus a point, whereas for the general vector bundle in \( \widetilde{B}_C^k(4g - 7) \) is the canonical bundle.

Proposition 7.3. Let \( C \) be of genus \( g \geq 5 \), with general moduli. Then \( \widetilde{M}_C^k(2, \omega_C) \neq \emptyset \). Moreover, there exists an irreducible component which is

(i) generically smooth

(ii) regular (i.e. of dimension \( 3g - 4 \)), and

(iii) its general point \( [\mathcal{F}_u] \) comes from \( u \in \mathbb{P}(\text{Ext}^1(\omega_C, \mathcal{O}_C)) \) general. In particular, \( s(\mathcal{F}_u) = g - \epsilon, \) where \( \epsilon \in \{0, 1\} \) such that \( g \equiv \epsilon \pmod{2} \).

Proof. Take \( u \in \mathbb{P}(\text{Ext}^1(\omega_C, \mathcal{O}_C)) \) general. With notation as in (4.4), (4.5) one has

\[
\ell = r = g, \quad \text{and} \quad m = 3g - 3 \geq \ell + 1.
\]

Thus, from Corollary 5.9 and from (7.3), \( h^0(\mathcal{F}_u) = h^1(\mathcal{F}_u) = 1 \). From (4.2), \( \dim(\mathbb{P}(\text{Ext}^1(\omega_C, \mathcal{O}_C))) = 3g - 4 \). Thus, \( \mathcal{F}_u \) stable with \( s(\mathcal{F}_u) = g - \epsilon \) follows from Proposition 4.4. This shows that \( \widetilde{M}_C^k(2, \omega_C) \neq \emptyset \).
Since $\mu_{\mathcal{F}_n}$ is injective if and only if $P_{\mathcal{F}_n}$ is. On the other hand, one has $H^0(\mathcal{F}_n) \otimes H^0(\omega_C \otimes \mathcal{F}_n^*) \cong \mathbb{C}$. Therefore, one needs to show that $P_{\mathcal{F}_n}$ is not the zero-map. This follows by limit of $P_{\mathcal{F}_n}$ when $u$ tends to 0, so that $\mathcal{F}_n = O_C \oplus \omega_C$: then the limit of $P_{\mathcal{F}_n}$ is the map $H^0(O_C) \otimes H^0(O_C) \rightarrow H^0(O_C)$.

To get (i)-(iii) at once, one observes that $\pi_{2g-2,2g-2}^D(\mathbb{P}(\text{Ext}^1(\omega_C, O_C)))$ is generically injective, since the exact sequence

$$0 \rightarrow O_C \rightarrow \mathcal{F}_n \rightarrow \omega_C \rightarrow 0$$

is unique: indeed, the surjection $\mathcal{F}_n \twoheadrightarrow \omega_C$ is unique and $h^0(\mathcal{F}_n) = 1$ (cf. (2.6) and computations as in (6.7)), moreover, by Lemma 4.5, two general vector bundles in $\mathbb{P}(\text{Ext}^1(\omega_C, O_C))$ cannot be isomorphic. \hfill \qed

Remark 7.4. (1) For a similar description, cf. [7]. Generic smoothness for components of $M_{1,1}^2(2, \omega_C)$ follows also from results in [52, 5].

(2) From Theorem 7.1, $[\mathcal{F}] \in B^\sim_{C^2}(2g - 2)$ general fits in a sequence $0 \rightarrow \eta \rightarrow \mathcal{F} \rightarrow \omega_C \rightarrow 0$, with $\eta \in \text{Pic}^0(C)$ general. Hence the map $\delta := \det|_{\mathcal{F}^t/(2g - 2)}$ is dominant. Since $\text{Pic}^{2g-2}(C)$ and $B^\sim_{C^2}(2g - 2)$ are irreducible and generically smooth, then $\delta^{-1}(\eta) = M_{1,1}^1(2, \omega_C \otimes \eta)$ is equidimensional and each component is generically smooth. Proposition 7.3 yields that in this situation each component of $M_{1,1}^1(2, \omega_C \otimes \eta)$ has dimension $3g - 4$ (equal to the expected dimension). This agrees with [37, Theorem 1.1].

7.3. Case $i = 2$. In this case, $\rho_{d^2} = 8g - 11 - 2d$.

Theorem 7.5. Let $C$ be of genus $g \geq 3$, with general moduli. For $2g - 2 \leq d \leq 3g - 6$, one has $B^\sim_{C^2}(d) \neq \emptyset$.

(i) $V_{d}^{2g-3,1,1}$ is the unique component of $B^\sim_{C^2}(d)$, whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 2$. Moreover, $V_{d}^{2g-3,1,1} = V_{d}^{2g-2,1,1}$ and it is regular.

(ii) For $[\mathcal{F}] \in V_{d}^{2g-3,1,1}$ general, one has $s(\mathcal{F}) \geq 3g - 4 - d - \epsilon > 0$ and $\mathcal{F}$ fits in a sequence

$$0 \rightarrow N \rightarrow \mathcal{F} \rightarrow \omega_C(-p) \rightarrow 0,$$

with

- $p \in C$ general,
- $N \in \text{Pic}^{d-2g+3}(C)$ general, and
- $\mathcal{F} = \mathcal{F}_v$ and $v$ is general in the good locus $W_1 \subset \text{Ext}^1(\omega_C(-p), N)$.

(iii) A section $\Gamma \subset F$, corresponding to a quotient $\mathcal{F} \twoheadrightarrow \omega_C(-p)$, is not of minimal degree. However, it is of minimal degree among special sections and it is asi but not ai (i.e. $\mathcal{F}$ is rsp but not rp via $\omega_C(-p)$).

(iv) For $g \geq 13$ and $2g + 6 \leq d \leq 3g - 7$, $V_{d}^{2g-3,1,1}$ is generically smooth.

Proof. Once part (i) has been proved, parts (ii)-(iii) follow from Theorem 5.13 and Proposition 6.15, with $\delta = 2g - 3$, $j = t = 1$, whereas part (iv) follows from Proposition 6.1-(ii), with $j = 2$.

The proof of part (i) consists of four steps.

Step 1. In this step, we show that if $\mathcal{B}$ is an irreducible component of $B^\sim_{C^2}(d)$ such that, for $[\mathcal{F}] \in \mathcal{B}$ general, $i(\mathcal{F}) = 2$, then $\mathcal{B}$ comes from a total good component $\mathcal{W}_{1}^{\text{tot}} \subseteq \mathbb{P}(\mathcal{E}_d)$, for some $\delta$ (cf. Theorem 5.8 and Definition 5.12).

Indeed, let (2.2) be a special presentation of $\mathcal{F}$ with $L$ of minimal degree. Then, from Remarks 6.4, 6.6, 6.11 one has $h^1(L) = j = 1$. Hence, with notation as in (4.4), (4.5) and (5.4), $t = 1$, $\ell \geq r - 1$. Moreover, $d' \leq 3g - 3$, $\delta \geq g - 1$ and $j = 1$ imply $m = 2\delta + d - g - 1 \geq 2 - 3 = \ell + 1$. Therefore, we can apply Theorem 5.8, finishing the proof of this step.

Step 2. In this step we determine which of the loci $V_{d}^{1,j,1}$, as in (6.5), or $V_{d}^{\delta,j,t}$, as in (6.8), has general point $[\mathcal{F}]$ such that $i(\mathcal{F}) = 2$ and dimension at least $\rho_{d^2} = 8g - 11 - 2d$ (hence, it can be conjecturally dense in a component of $B^\sim_{C^2}(d)$); we will prove that this only happens for $2g - 3 \leq \delta \leq 2g - 2$ and $j = t = 1$. The presentation is specially rigid only if $\delta = 2g - 3$.

Let $\mathcal{V}$ be such a locus, presented as in (2.2) with special quotient $L$. As in Step 1, $j = 1$ hence $N$ has to be special and $t = 1$, so $\mathcal{V}$ has to be of the form $V_{d}^{1,1,1}$. We have two cases: (a) $N$ non-effective, (b) $N$ effective.

Case (a). As in Remark 6.17-(1), a necessary condition for $\dim(V_{d}^{1,1,1}) = \varphi_{d,1} \geq \rho_{d^2} = \varphi_{0}(\delta, 1)$, $\geq 0$, i.e. $2g - 3 \leq \delta \leq 2g - 2$.

In case $\delta = 2g - 2$, $[\mathcal{F}] \in V_{d}^{2g-2,1,1}$ general is not rsp via $\omega_C$ (it follows from the fact that $i = 2$ and computation as in (6.7)).
In case $\delta = 2g - 3$, the hypotheses $2g - 2 \leq d \leq 3g - 6$ ensure stability for $\mathcal{F}$ (cf. Theorems 5.8, 5.13 and Proposition 6.15). By definition, $V_d^{2g-3,1,1} = \text{Im}(\pi_{d,2g-3}|_{\tilde{W}_1^{\text{Tot}}})$, where $\tilde{W}_1^{\text{Tot}} \subset \mathbb{P}(\mathcal{E}_{2g-3})$ is the good locus. To accomplish the proof, we need to show that the fibre of $\pi_{d,2g-3}|_{\tilde{W}_1^{\text{Tot}}}$ over $[\mathcal{F}] \in V_d^{2g-3,1,1}$ general is finite. As in (6.13), it suffices to prove the following:

**Claim 7.6.** $a_F(2g - 3) = 0$.

**Proof of the Claim.** Assume by contradiction this is not zero. Since $\mathcal{F}$ is stable, hence unsplit, from $\varphi_0(2g - 3, 1, 1) = 0$ and Remark 4.2, $a_F(2g - 3) = 0$ must be 1 (cf. Proposition 2.12).

Let $\mathfrak{F}$ be the corresponding one-dimensional family of sections of $F$, which has positive self-intersection, since $\mathcal{F}$ is stable. From Proposition 2.12 and Step 1, the system $\mathfrak{F}$ cannot be contained in a linear system, otherwise we would have sections of degree lower than $2g - 3$.

Thus, from the proof of Proposition 2.12, there is an open, dense subset $C^0 \subset C$ such that, for any $q \in C^0$, $\mathcal{F} = \mathcal{F}_v$ with $v = v_q \in \text{Ext}^1(\omega_C(-q), N_q)$, where $\{N_q\}_{q \in C^0}$ is a 1-dimensional family of non-isomorphic line bundles of degree $d - 2g + 3$, whose general member is in $\text{Pic}^{d-2g+3}(C)$ . Let $\Gamma_q \subset F_i$ be the section corresponding to $\mathcal{F}_v \mapsto \omega_C(-q)$; so the one-dimensional family is $\mathfrak{F} = \{\Gamma_q\}_{q \in C^0}$.

We set $\Gamma_q := \Gamma_q + f_q$, for $q \in C^0$. From (2.1), $\Gamma_q$ corresponds to $\mathcal{F}_v \mapsto \omega_C(-q) \oplus O_q$, whose kernel we denote by $N_q^\prime$. Then $\tilde{\mathfrak{F}} = \{\Gamma_q\}_{q \in C^0}$ is a one-dimensional family of unisecants of $F_v$ of degree $2g - 2$ and speciality 1 (cf. (2.8)). For $h, q \in C^0$, we have

$$c_1(N_q^\prime) = \text{det}(\mathcal{F}_v) \otimes \omega_C^\vee = c_1(N_q^\prime).$$

Therefore, from (2.6), $\tilde{\mathfrak{F}}$ is contained in a linear system $|\mathcal{O}_{F_v}(\Gamma)|$. By Bertini’s theorem, the general member of $|\mathcal{O}_{F_v}(\Gamma)|$ is a section of degree $2g - 2$. In particular, $\dim(|\mathcal{O}_{F_v}(\Gamma)|) \geq 2$.

If $L_\Gamma$ is the corresponding quotient line bundle, since $\Gamma \sim \Gamma_q$, then $c_1(L_\Gamma) = \omega_C$, i.e. $\Gamma$ is a canonical section. This is a contradiction: indeed, if $M_{\omega_C}$ is the kernel of the surjection $\mathcal{F}_v \mapsto \omega_C$, we have (cf. (6.7))

$$2 \leq \dim(|\mathcal{O}_{F_v}(\Gamma)|) = h^0(\mathcal{F}_v \otimes M_{\omega_C}^\vee) - 1 = i(\mathcal{F}_v) - 1 = 1.$$

Thus, case (b) cannot occur.

**Step 3.** In this step we prove that $V_d^{2g-3,1,1}$ is a component of $B_d^{2g-3,1,1}$.

Let $\mathcal{B} \subseteq B_d^{2g-3,1,1}$ be a component containing $V_d^{2g-3,1,1}$ and let $[\mathcal{F}] \in \mathcal{B}$ general. By semicontinuity, $\mathcal{F}$ has speciality $i = 2$. It has also a special presentation as in (2.2), with $2g - 3 \leq \text{deg}(L) = \delta \leq 2g - 2$. Since $C$ has general moduli, then $h^1(L) = j = 1$ so the corank of the coboundary map is $t = 1$. If $\delta = 2g - 3$, from Step 2 we are done.

Assume therefore $\delta = 2g - 2$, so $L = \omega_C$. Notice that: (i) $r = h^1(N) \leq g$; (ii) $m = \dim(\text{Ext}^1(N, \omega_C)) \geq g + 1$.

Indeed, (i) is trivial if $N$ is effective; if otherwise $h^0(N) = 0$, then $h^1(N) = 3g - 3 - d < g$ since $d \geq 2g - 2$. As for (ii), $m = 5g - 5 - d$ (cf. (4.2)), hence (ii) follows since $d \leq 3g - 6$. So we are in position to apply Theorem 5.8 and Corollary 5.9, which yield that $\tilde{W}_1^{\text{Tot}} \subseteq \mathbb{P}(\mathcal{E}_{2g-2})$ is irreducible and good. Hence $\dim(\tilde{W}_1^{\text{Tot}}) \leq 8g - 10 - 2d$ (equality holds when $N$ is general, i.e. non effective).

On the other hand, $\mathcal{B}$ is the image of $\tilde{W}_1^{\text{Tot}}$ via $\pi_{d,2g-2}$ (cf. Step 1) and the general fibre of this map has dimension at least 1 because $h^1(\mathcal{F}) = 2$ (cf. (2.6) and computation as in (6.7)). Thus

$$8g - 11 - 2d \geq \dim(\mathcal{B}) \geq \dim(V_d^{2g-3,1,1}) = 8g - 11 - 2d = \rho_d^{k_2}.$$

This proves that $\mathcal{B} = V_d^{2g-3,1,1}$ is a regular component.

The previous argument also shows that $V_d^{2g-3,1,1} = V_d^{2g-2,1,1}$ (cf. Remark 6.17) and that the dimension of the general fibre of $\pi_{d,2g-2}|_{\tilde{W}_1^{\text{Tot}}}$ onto $V_d^{2g-2,1,1}$ has exactly dimension 1 (actually, it is a $\mathbb{P}^1$, cf. Lemma 2.11).

**Step 4.** Assume we have a component $\mathcal{B} \subseteq B_d^{2g-3,1,1}$, whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 2$. From Step 1, $[\mathcal{F}] \in \mathcal{B}$ general can be specially presented as in (2.2), with $h^1(L) = j = 1$, so $N$ is special. The same discussion as in Steps 2 and 3 shows that $\mathcal{B} = V_d^{2g-3,1,1}$.
Remark 7.7. (i) For \([F] \in \mathbb{V}_d^{2g-3,1,1}\) general, one has \(\ell = g - 1\), \(r = 3g - 4 - d\) and \(m = 5g - 7 - d\) (cf. (4.2), (4.4)). So \(\ell \geq r + 1\) (because \(d \geq 2g - 2\)), moreover \(m \geq \ell + g\) (because \(d \leq 3g - 6\)). Note that the inequality \(m \geq \ell + 1\) is necessary to ensure \(\emptyset \neq W_1^{\alpha_1} \subset \mathbb{P}(E_{2g-2})\) (see the proof of Theorem 5.8).

(ii) Step 4 of Theorem 7.5 shows that, if \(B\) is a component of \(B_C^{2g}(d)\), different from \(B_C^{2g}(d)\), then \(B\) is a component of \(B_C^{2g}(d)\), for some \(i \geq 3\), and as such it is not regular. Otherwise, we would have

\[8g - 11 - 2d \leq \dim(B) = 4g - 3 - i(d - 2g + 2 + i) \leq 10 - 3d - 16,\]

i.e. \(d \leq 2g - 7\) which is out of our range for \(d\).

Remark 7.8. (1) Take \(\Gamma_p\) as in the proof of Claim 7.6. Then, \(N_{\Gamma_p/F_v}\) is non-special on the (reducible) unisecant \(\Gamma_p\). Indeed, \(\omega_{\Gamma_p} \otimes N_{\Gamma_p/F_v} \mid v \cong \mathcal{O}_F(K_F)\) whereas \(\omega_{\Gamma_p} \otimes N_{\Gamma_p/F_v} \mid F_v \cong \mathcal{O}_V(-2)\). Thus \(\Gamma_p \in \text{Div}_{F_v}^{1,2g-2}\) is a smooth point. Moreover, \(h^0(N_{\Gamma_p/F_v}) = 3g - 2 - d \geq 2\) for \(d \leq 3g - 4\). From the generality of \(v\) in the good locus \(W_1\), (2.6) and from computation as in (6.7), one has that \(\Gamma_p \subset F_v\) is a (reducible) unisecant, moving in a complete linear pencil of special unisecants whose general member is a canonical section, and \(\Gamma_p\) is algebraically equivalent on \(F_v\) to non-special sections of degree \(2g - 2\).

As soon as \(d \leq 3g - 6\), there are in \(\text{Div}_{F_v}^{1,2g-2}\) unisecants containing two general fibres (cf. Proposition 2.12) hence the ruled surface \(F_v\) has (non-special) sections of degree smaller than \(2g - 3\).

(2) Take \(N \in \text{Pic}^k(C)\) general with \(0 < k \leq g - 2\). Since \(N\) is special, non-effective, from Corollary 5.9 and Remark 5.15, \(v \in \Lambda_1 \subset \text{Ext}^1(\omega_C,N)\) general determines \(\mathcal{F} := \mathcal{F}_v\) stable, with \(i(\mathcal{F}) = 2\). If \(\Gamma\) denotes the canonical section corresponding to \(\mathcal{F} \to \omega_C\), from (2.6) one has \(\dim(\mathcal{O}_{F_v}(\Gamma)) = 1\) and all unisecants in this linear pencil are special (cf. Lemma 2.11). Since \(F\) is indecomposable, \(\mathcal{O}_{F_v}(\Gamma)\) has base-points (cf. the proof of Proposition 2.12, from which we keep the notation). Thus, \(\mathcal{F}\) is r.s.p. via \(\omega_C(-p)\), for \(p = \rho(q)\) and \(q \in F\) a base point of the pencil (recall Remark 6.17).

Remark 7.9. In [48, 49] the locus \(B_C^{2g}(b)\) is studied, for \(g \geq 2\) and \(3 \leq b \leq 2g - 1\). It is proved there with different arguments that, when \(C\) has general moduli, then \(B_C^{2g}(b)\) is not empty, irreducible, regular (with \(\rho_C^2 = 2b - 3\)), generically smooth and \([\mathcal{E}] \in B_C^{2g}(b)\) general is stable, with \(h^0(\mathcal{E}) = 2\), fitting in a sequence

\[0 \to \mathcal{O}_C(p) \to \mathcal{F}_v \to \omega_C(-p) \to 0.\]  

(7.4)

Considering the natural isomorphism \(\widetilde{B}_C^{2g}(b) \cong B_C^{2g}(4g - 4 - b)\), when \(d = 4g - 4 - b\) is as in Theorem 7.5 we recover Teixidor’s results (without irreducibility) via a different approach, which provides in addition the rigidly special presentation of the general element of the component.

Teixidor’s results and our analysis imply that, for any \(2g - 2 \leq d \leq 4g - 7\), \(\widetilde{B}_C^{2g}(d) = \mathbb{V}_d^{2g-3,1,1}\).

Proposition 7.10. Let \(C\) be of genus \(g \geq 3\), with general moduli. Then, \(\widetilde{M}_C^{2g}(2, \omega_C) \neq \emptyset\) and irreducible. Moreover, it is regular (i.e. of dimension \(3g - 6\)), and its general point \([\mathcal{F}_v]\) fits in a sequence

\[0 \to \mathcal{O}_C(p) \to \mathcal{F}_v \to \omega_C(-p) \to 0,\]

where

- \(p \in C\) is general, and
- \(v \in \Lambda_1 = W_1 \subset \text{Ext}^1(\omega_C(-p), \mathcal{O}_C(p))\) is general.

Proof. Irreducibility follows from [38, Thm. 1.3]. With notation as in (4.4), (4.5), one has

\[\ell = r = g - 1, \quad m = h^1(2p - K_C) = 3g - 5 \geq \ell + 1;\]

from Corollary 5.9, \(W_1\) is good and \(v \in W_1\) general is such that \(\text{cork}(\partial_v) = 1\). In particular, \(\dim(W_1) = 3g - 6\).

Stability of \(\mathcal{F}_v\), with \(1 < s(\mathcal{F}_v) = \sigma < g\), follows from Proposition 4.4. Finally one uses the same approach as in Claim 7.6 to deduce that \(\pi_{2g-2,2g-3}^{\alpha_1}W_{7,1}\) is generically finite (cf. (6.4)), since \(\mathcal{F}_v\) is r.s.p. via \(\omega_C(-p)\). \(\square\)

Generic smoothness of the components of \(\widetilde{M}_C^{2g}(2, \omega_C)\) follows from results in [52, 5]. Proposition 7.10 can be interpreted in the setting of [7] as saying that, for a curve \(C\) of general moduli of genus \(g \geq 3\), \(\mathbb{P}(\text{Ext}^1(\omega_C, \mathcal{O}_C))\) is not contained in the divisor \(D_1\) considered in that paper.
7.4. Case $i = 3$. One has $s_d^{k_3} = 10g - 18 - 3d$. We have the following:

Theorem 7.11. Let $C$ be of genus $g \geq 8$, with general moduli. For any $2g - 2 \leq d \leq \frac{5}{2}g - 6$, one has $B_C^{k_3}(d) \neq \emptyset$. Moreover:

(i) $V_d^{2g-4,1,2}$ is the unique component of $B_C^{k_3}(d)$ of types either (6.5) or (6.8), whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 3$. Furthermore, it is regular and $V_d^{2g-4,1,2} = V_d^{2g-3,1,2} = V_d^{2g-2,1,2}$.

(ii) For $[\mathcal{F}] \in V_d^{2g-4,1,2}$ general, one has $s(\mathcal{F}) \geq 5g - 10 - 2d - \epsilon \geq 2 - \epsilon$ and $\mathcal{F}$ fits in a sequence 

$$0 \to N \to \mathcal{F} \to \omega_C(-D_2) \to 0,$$

where

- $D_2 \in C^{(2)}$ is general,
- $N \in \text{Pic}^{d-2g+4}(C)$ is general (special, non-effective),
- $\mathcal{F} = \mathcal{F}_v$ with $v$ general in a good component $\Lambda_2 \subset \text{Ext}^1(N, \omega_C(-D_2))$ (cf. Definition 5.12).

(iii) Any section $\Gamma \subset F$, corresponding to a quotient $\mathcal{F} \to \omega_C(-D_2)$, is not of minimal degree. However, it is minimal among special sections of $F$; moreover, $\Gamma$ is as in but not ai (i.e., $\mathcal{F}$ is rsp via $\omega_C(-D_2)$).

Proof. As in Theorem 7.5, once (i) has been proved, parts (ii)-(iii) follow from results proved in previous sections. Precisely, by definition of $V_d^{2g-4,1,2}$ one has $L = \omega_C(-D_2)$, with $D_2 \in C^{(2)}$, $t = 2$ and $N \in \text{Pic}^{d-2g+4}(C)$ of speciality $r \geq 2$. From regularity, Proposition 6.15 and (6.9), (6.10), (6.12) give

$$0 = \nu_d^{2g-4,1,2} - \rho_d^{k_3} \leq \dim(\Lambda_2^{\text{Tot}}) - \rho_d^{k_3} = \varphi_n(2g - 4, 1, 2) = \varphi_0(2g - 4, 1, 2) - n(r - 2) = -n(r - 2).$$

Thus, $n(r - 2) = 0$. This implies that the general fibre of $\pi_{d,2g-4(2)}$, is finite, i.e. $[\mathcal{F}_v] \in V_d^{2g-4,1,2}$ general is rsp via $\omega_C(-D_2)$ (correspondingly $\Gamma \subset F_v$ is as in (iii)).

Since $n(r - 2) = 0$, then either $n = 0$ or $r = 2$. The latter case cannot occur otherwise we would have $n = d - 3g + 7 < -\frac{2}{2} + 1 < 0$, by the assumptions on $d$. Thus $n = 0$ and $r = 3g - 5 - d$. Moreover, from (4.2), (5.5), one has $m = 5g - 10 - d$ and $c(\ell, r, 2) = 2d + 10 - 4g$, so a good component $\Lambda_2 \subset \mathbb{P}(\text{Ext}^1(\omega_C(-D_2), N))$ has dimension $9g - 20 - 3d$. If we add up to this quantity $g$, for the parameters of $N$, we get $10g - 20 - 3d$.

Therefore, regularity forces $D_2$ to be general in $C^{(2)}$.

Now, $N_{\Gamma/F_v} \cong K_C - D_2 - N$ (cf. (2.3)) so $h^i(N_{\Gamma/F_v}) = h^1(-i(N + D_2))$ for $0 \leq i \leq 1$. By the assumptions on $d$, $\deg(N + D_2) = d - 2g + 6 \leq \frac{5}{2}g$, thus generality of $N$ implies that $N + D_2$ is also general, so $h^0(N + D_2) = 0$ and $h^1(N + D_2) = 3g - 7 - d \geq \frac{5}{2}g - 1$. This implies that $\Gamma$ is not ai and not of minimal degree among quotient line bundles of $\mathcal{F}$.

Numerical conditions of Theorem 5.13 (see also Remark 5.15) are satisfied for $j = 1, t = 2$ and $\delta = 2g - 4$, under the assumptions $d \leq \frac{5}{2}g - 6$.

Finally, the fact that $\Gamma$ is of minimal degree among special, quotient line bundles of $\mathcal{F}$ follows from the proof of part (i) below.

The proof of part (i) consists of the following steps.

Step 1. In this step we determine which of the loci of the form $V_d^{k,j}$, as in (6.5), or $V_d^{k,j,t}$, as in (6.8):

(a) has the general point $[\mathcal{F}]$ with $i(\mathcal{F}) = 3$,

(b) is the image, via $\pi_{d,\delta}$, of a parameter space in $\mathbb{P}(\mathcal{E}_d)$ of dimension at least $\rho_d^{k_3} = 10g - 18 - 3d$.

Let $\mathcal{V}$ be such locus and use notation as in (4.4), (4.5). From Remarks 6.4, 6.6, 6.11, conditions (a) and (b) are both satisfied only if the presentation of $[\mathcal{F}] \in \mathcal{V}$ general as in (2.2) with $L$ special and effective, is such that $N \in \text{Pic}^{d-2g}(C)$ is special and the coboundary map $\partial : H^0(L) \to H^1(N)$ is not surjective. Possibilities are:

(i) $j = 1$ and $t = \text{cork}(\partial) = 2$;

(ii) $j = 2$ and $t = 1$.

In any event, one has $\ell \geq r$ (in particular, we will be in position to apply Theorems 5.8, 5.17; cf. e.g the proof of Claim 7.13 below). Indeed:

- in case (i), the only possibilities for $\ell < r$ are $r - 2 \leq \ell \leq r - 1$. Then, $\dim(\mathbb{P}(\text{Ext}^1(L, N)) = 2\delta - d + g - 2$, $\rho(L) = g - (\delta - g + 2)$, $\rho(N) = g - rn$, so the number of parameters is $\delta + 4(g - 1) - d - rn < \rho_d^{k_3}$ since $d \leq \frac{5}{2}g - 6$.

- in case (ii), the only possibility for $\ell < r$ is $\ell = r - 1$. The same argument as above applies, the only difference is that $\rho(L) = g - 2(\delta - g + 3)$.
Since \( \ell \geq r \), we see that case (ii) cannot occur by (6.11) and (6.12). Thus, we focus on \( \mathcal{V}_{d,1}^{5,1,2} \), investigating for which \( \delta \) it satisfies (b). We will prove that this only happens for \( 2g - 4 \leq \delta \leq 2g - 2 \).

We have two cases: (1) \( N \) effective, (2) \( N \) non-effective. We will show that only case (2) occurs.

**Case (1).** When \( N \) is effective, from Remark 6.17, a necessary condition for (b) to hold is (6.14), which reads 
\[ (r - 2)n - 2 < 0. \]
This gives \( 2 \leq r \leq 3 \), since \( r \geq t = 2 \). We can apply Theorem 5.13 (more precisely, Remark 5.16): the first line of bounds on \( \delta \) in Remark 5.16 gives \( \delta \leq \frac{3g - \epsilon}{2} + r - 4 \). In particular, one must have 
\[ \delta \leq \frac{3g - \epsilon}{2} - 1. \]
On the other hand, another necessary condition for (b) to hold is \( \varphi_n(\delta, 1, 2) \geq 0 \) (cf. Proposition 6.15).

From Remark 6.14, \( \varphi_n(\delta, 1, 2) \leq \varphi_0(\delta, 1, 2) = 2 - 2g + 4 \) (cf. (6.10)) and \( \varphi_0(\delta, 1, 2) \geq 0 \) gives \( \delta \geq 2g - 4 \) which contradicts \( \delta \leq \frac{3g - \epsilon}{2} - 1 \), since \( g \geq 8 \).

**Case (2).** When \( N \) is non-effective, we apply Theorem 5.13 (more precisely, Remark 5.15), with \( j = 1 \) and \( t = 2 \). By the same argument as in case (1), we see that \( 2g - 4 \leq \delta \leq 2g - 2 \).

**Step 2.** In this step we prove that the loci \( V_{d,1}^{5,1,2} \), with \( 2g - 4 \leq \delta \leq 2g - 2 \), are not empty. Precisely, we will exhibit components \( V_{d,1}^{5,1,2} \) which are the image, via \( \pi_{d,1}^{5,1,2} \), of a total good component \( \Lambda_2^{\text{Tot}} \subset \mathbb{P}(\mathcal{E}) \), of dimension \( \rho_{d,1}^{5,1,2} + 2g + 4 \) (cf. Definition 6.13 and (6.8)).

We will treat only the case \( \delta = 2g - 4 \), i.e. \( L = \omega_C(-D_2) \), with \( D_2 \in C(2) \), since the cases \( L = \omega_C, \omega_C(-p) \) can be dealt with similar arguments and can be left to the reader.

**Claim 7.12.** Let \( N \in \text{Pic}^{d-2g+4}(C) \) be general. For \( V_2 \in \mathbb{G}(2, H^0(K_C - N)) \) general, the map \( \mu_{V_2} \) as in (5.8) is injective.

**Proof of Claim 7.12.** The general \( V_2 \in \mathbb{G}(2, H^0(K_C - N)) \) determines a base point free linear pencil on \( C \). Indeed, \( H^0(K_C - N) = 3g - 5 - d \geq 5 \). Take \( \sigma_1, \sigma_2 \in H^0(K_C - N) \) general sections. If \( p \in C \) is such that \( \sigma_i(p) = 0 \), for \( i = 1, 2 \), by the generality of the sections we would have \( p \in \text{Bs}(|K_C - N|) \) so \( H^0(N + p) = 1 \). This is a contradiction because \( N \) is general and \( \deg(N) < g - 1 \). The injectivity of \( \mu_{V_2} \) follows from the base-point-free pencil trick: indeed, \( \ker(\mu_{V_2}) = H^0(N(-D_2)) \) which is zero since \( N \) is non-effective. \( \Box \)

**Claim 7.13.** Let \( N \in \text{Pic}^{d-2g+4}(C) \) and \( D_2 \in C^{(2)} \) be general. Then, there exists a unique good component \( \Lambda_2 \subset \text{Ext}^1(\omega_C(-D_2), N) \) whose general point \( v \) is such that \( \ker(\partial_v)^\vee \) is general in \( \mathbb{G}(2, H^0(K_C - N)) \) (cf. Remark 5.7).

**Proof of Claim 7.13.** With notation as in (4.2), (4.4), we have \( \ell = g - 2 \), \( m = 5g - 9 - d \) and \( r = 3g - 5 - d \). Then assumptions on \( d \) and \( g \) imply 
\[ m \geq 2\ell + 1 \text{ and } \ell \geq r \geq t = 2 \] (7.5)
(cf. Step 1 for \( \ell \geq r \)). From (7.5) and Claim 7.12, we are in position to apply Theorem 5.17, with \( \eta = 0 \) and 
\[ \Sigma_\eta = \mathbb{G}(2, H^0(K_C - N)). \]
This yields the existence of a good component \( \Lambda_2 \subset W_2 \subset \text{Ext}^1(\omega_C(-D_2), N) \). Actually, \( \Lambda_2 \) is the only good component whose general point \( v \) gives \( \ker(\partial_v)^\vee = V_2 \) general in \( \mathbb{G}(2, H^0(K_C - N)) \).

Indeed any component of \( W_2 \), whose general point \( v \) is such dim(\( \ker(\partial_v) \)) = 2, is obtained in the following way (cf. the proofs of Theorems 5.8, 5.17):

- take any \( \Sigma \subset \mathbb{G}(2, H^0(K_C - N)) \) irreducible, of codimension \( \eta \geq 0 \);
- for \( V_2 \) general in \( \Sigma \), consider \( H^0(\omega_C(-D_2)) \otimes V_2 \) and the map \( \mu_{V_2} \) as in (5.8);
- let \( \tau := \dim(\ker(\mu_{V_2})) \geq 0 \) and \( \mathbb{P} := \mathbb{P}(\text{Ext}^1(\omega_C(-D_2), N)) = \mathbb{P}(H^0(2K_C - D_2 - N)^\vee) \) (cf. (4.6));
- consider the incidence variety 
\[ \mathcal{J}_{\Sigma} := \{ (\sigma, \pi) \in \Sigma \times \mathbb{P} \mid \ker(\mu_{V_2}) \supset \pi \}. \]
Since \( m \geq 2\ell + 1 - \tau \), one has \( \mathcal{J}_{\Sigma} \neq \emptyset \) (cf. the proofs of Theorems 5.8, 5.17);
- consider the projections \( \Sigma \overset{pr_1}{\rightarrow} \mathcal{J}_{\Sigma} \overset{pr_2}{\rightarrow} \mathbb{P} \);
- the fibre of \( pr_1 \) over any point \( V_2 \) in the image is \( \{ \pi \in \mathbb{P} \mid \ker(\mu_{V_2}) \supset \pi \}, \) i.e. it is isomorphic to the linear system of hyperplanes of \( \mathbb{P} \) passing through the linear subspace \( \mathbb{P}(\ker(\mu_{V_2})) \). For \( V_2 \in \Sigma \) general, this fibre is irreducible of dimension \( m - 1 - 2\ell + \tau = 3g - 6 - d + \tau \). In particular, there exists a unique component \( \mathcal{J} \subset \mathcal{J}_{\Sigma} \) dominating \( \Sigma \) via \( pr_1 \);
- since \( r = 3g - 5 - d \), one has 
\[ \dim(\mathcal{J}) = 9g - 20 - 3d + \tau - \eta = \expdim(\hat{W}_2) + \tau - \eta. \]
where \( \hat{W}_2 = \mathbb{P}(W_2) \subseteq \mathbb{P} \) (notation as in the proof of Theorem 5.8). By construction, \( pr_2(\mathcal{J}) \subseteq \hat{W}_2 \). Moreover, if \( \epsilon \) denotes the dimension of the general fibre of \( pr_2(\mathcal{J}) \), then \( pr_2(\mathcal{J}) \) can fill up a component \( X \) of \( \hat{W}_2 \) only if \( \tau - \eta - \epsilon \geq 0 \); the component \( X \) is good when equality holds.

When \( \Sigma = G(2, H^0(K_C - N)) \), then \( \eta = 0 \) and, by Claim 7.12, also \( \tau = 0 \). Since \( \mathcal{J} \subseteq \mathcal{J}_{G(2, H^0(K_C - N))} \) is the unique component dominating \( G(2, H^0(K_C - N)) \), then \( pr_2(\mathcal{J}) \) fills up a component \( \Lambda_2 \) of \( \hat{W}_2 \), i.e. \( \epsilon = 0 \). Thus \( \hat{\Lambda}_2 \) is good.

By the generality of \( N \in \text{Pic}^{d-2g+4}(C) \) and of \( D_2 \in C^{(2)} \), Claim 7.13 ensures the existence of a total good component \( \hat{\Lambda}_2^{\text{tot}} \subseteq \mathbb{P}(E_{2g-4}) \).

For \( 2g - 4 \leq \delta \leq 2g - 2 \), we will denote by \( V_d^\delta \) the total good component we constructed in this step. To ease notation, we will denote by \( V_d^\delta \) its image in \( B_C^{2g}(d) \) via \( \pi_{d,\delta} \), which is a \( V_d^{\delta,1} \) as in Step 1.

**Step 3.** In this step, we prove that \( V_d^{2g-2} \) has dimension \( \rho_d^{k_3} \).

From Step 2, one has \( \dim(V_d^{2g-2}) = \rho_d^{k_3} + 2 = 10g - 16 - 3d \). We want to show that the general fibre of \( \pi_{d,2g-2}\mid V_d^{2g-2} \) has dimension two. To do this, we use similar arguments as in the proof of Lemma 6.2.

Let \( [\mathcal{F}] \in V_d^{2g-2} \) be general; by Step 2, \( \mathcal{F} = \mathcal{F}_u \), for \( u \in \hat{\Lambda}_2 \subseteq \mathbb{P}(H^0(2K_C - N)^\vee) \) general and \( N \in \text{Pic}^{d-2g+4}(C) \) general, where \( \hat{\Lambda}_2 = pr_2(\mathcal{J}) \) and \( \mathcal{J} \subseteq G(2, H^0(K_C - N)) \times \mathbb{P}(H^0(2K_C - N)^\vee) \) the unique component dominating \( G(2, H^0(K_C - N)) \) (cf. the proof of Claim 7.13). Then

\[
(\pi_{d,2g-2}\mid V_d^{2g-2})^{-1}([\mathcal{F}_u]) = \left\{ (\mathcal{N}, \omega_C, u') \in V_d^{2g-2} \mid \mathcal{F}_{u'} \cong \mathcal{F}_u \right\}.
\]

In particular, one has \( N \cong N' \) so \( u, u' \in \hat{\Lambda}_2 \subseteq \mathbb{P}(H^0(2K_C - N)^\vee) \).

Let \( \varphi : \mathcal{F}_{u'} \cong \mathcal{F}_u \) be the isomorphism between the two bundles and consider the diagram

\[
\begin{array}{ccc}
0 & \to & N & \xrightarrow{i} & \mathcal{F}_{u'} & \to & \omega_C & \to & 0 \\
\downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} \\
0 & \to & N & \xrightarrow{i_2} & \mathcal{F}_u & \to & \omega_C & \to & 0.
\end{array}
\]

If \( u = u' \), then \( \varphi = \lambda \in \mathbb{C}^* \) (since \( \mathcal{F}_u \) is simple) and the maps \( i_1 \) and \( i_2 \) determine two non-zero sections \( s_1 \neq s_2 \in H^0(\mathcal{F}_u \otimes N'^\vee) \). Similar computation as in (6.7) shows that \( h^0(\mathcal{F}_u \otimes N'^\vee) = i(\mathcal{F}_u) = 3 \), since \( u \in \hat{\Lambda}_2 \) general. Therefore, if \( \Gamma \subset F \) denotes the section corresponding to \( \mathcal{F}_u \to \omega_C, (\pi_{d,2g-2}\mid V_d^{2g-2})^{-1}([\mathcal{F}_u]) \) contains a \( \mathbb{P}^2 \) isomorphic to \( |O_{F_\Gamma}()| \) (cf. (2.6) and Lemma 2.11).

The case \( u \neq u' \) cannot occur. Indeed, for any inclusion \( i_1 \) as above, there exist an inclusion \( i_2 \) and a \( \lambda = \lambda(i_1, i_2) \in \mathbb{C}^* \) such that \( \varphi \circ i_1 = \lambda i_2 \), otherwise we would have \( \dim(|O_{F_\Gamma}()|) > 2 \), a contradiction. One concludes by Lemma 4.5.

In conclusion, the general fibre of \( \pi_{d,2g-2}\mid V_d^{2g-2} \) has dimension two (actually, this fibre is a \( \mathbb{P}^2 \)).

**Step 4.** In this step we prove that \( V_d^{2g-2} := V_d^{2g-2} \subseteq \mathbb{V} \) will be specially rigid only for \( \delta = 2g - 4 \).

From Step 2 one has \( \dim(V_d^\delta) = \rho_d^{k_3} + \delta - 2g + 4 \), for \( 2g - 4 \leq \delta \leq 2g - 2 \). Moreover, the general element of \( V_d^\delta \) can be identified with a pair \( (F, \Gamma) \), where \( F = \mathbb{P}(\mathcal{F}), \Gamma \subset F \) a section corresponding to \( \mathcal{F} \to \omega_C(-D) \), where \( D \in C^{(2g-2-\delta)} \) and, for \( \delta = 2g - 2 \), one has \( D = 0 \) and \( \dim(|O_{F_\Gamma}()|) = 2 \).

We will now prove that there exist dominant, rational maps:

(a) \( r_1 : V_d^{2g-2} \times C \to V_d^{2g-3} \), such that \( r_1((F, \Gamma, p)) = (F, \Gamma_p) \), where \( \Gamma_p \subset F \) is a section corresponding to \( \mathcal{F} \to \omega_C(-p) \),

(b) \( r_2 : V_d^{2g-2} \to V_d^{2g-4} \), where \( V_d^{2g-2} \) is a finite cover \( \varphi : V_d^{2g-2} \to V_d^{2g-2} \) endowed with a rational map \( \psi : V_d^{2g-2} \to C^{(2)} \); if \( \xi \in V_d^{2g-2} \) is general and \( \varphi(\xi) = (F, \Gamma) \), then \( r_2(\xi) = (F, \Gamma') \), with \( \Gamma' \) a section corresponding to \( \mathcal{F} \to \omega_C(-\psi(\xi)) \).

The existence of these maps clearly proves that \( \varphi = \varphi^{2g-2} \) and \( \varphi', \varphi^{2g-2} \subseteq \mathbb{C}^2 \).

(a) Take \( (F, \Gamma) \) general in \( V_d^{2g-2} \) and \( p \in C \) general. Then, the restriction map

\[
C \cong H^0(O(\Gamma)) \to H^0(O_{F_p}(\Gamma)) \cong \mathbb{C}^2
\]

is surjective, because the general member of \( |O_{F_\Gamma}()| \) is irreducible. Hence there is a unique \( \Gamma_p \in |O_{F_\Gamma}(-p)| \).

We claim that \( \Gamma_p \) is irreducible, i.e. it is a section. If not, \( \Gamma_p \) would be a section plus a number \( n \geq 1 \) of fibres. As we saw, \( n \leq 1 \) (cf. Step 1) so \( n = 1 \). This determines an automorphism of \( C \) and, since \( C \) has general moduli, this automorphism must be the identity. This is impossible because the map \( \Phi_F : F \to \mathbb{P}^2 \), given by \( |O_{F_\Gamma}()| \), is dominant hence it is ramified only in codimension one.
In conclusion, $\Gamma_p$ corresponds to $\mathcal{F} \rightarrow \omega_C(-p)$ and $(F, \Gamma_p)$ belongs to $V_d^{2g-3}$, and this defines $r_1$. The proof that $(F, \Gamma_p)$ belongs to $V_d^{2g-3}$ is postponed for a moment (cf. Claim 7.14).

(b) Given $(F, \Gamma)$ general in $V_d^{2g-2}$, we can consider the map $\Phi_F$ as in Case (a). Since $\Phi_F$ maps the rulings of $F$ to lines, it determines a morphism $\Psi: C \rightarrow C' \subset (\mathbb{P}^2)^\vee$. From Step 1, no (scheme-theoretical) fibre of $\Psi$ can have length bigger than two. Therefore, since $C$ has general moduli, $\Psi: C \rightarrow C'$ is birational and moreover, since $g \geq 8$, $C'$ has a certain number $n$ of double points, corresponding to curves of type $\Gamma_D + f_D$, with $D \in C''$ fibre of $\Psi$ over a double point of $C'$.

Then the general point $\xi$ of $V_d^{2g-2}$ corresponds to a triple $(F, \Gamma, D)$ (with $D \in C'')$ as above), the pair $(F, \Gamma_D)$ belongs to $V_d^{2g-4}$ (cf. Claim 7.14) and $r_2(F, \Gamma, D) = (F, \Gamma_D)$, $\psi(F, \Gamma, D) = D$.

Claim 7.14. With the above notation, $(F, \Gamma_p)$ belongs to $V_d^{2g-3}$ and $(F, \Gamma_D)$ belongs to $V_d^{2g-4}$.

Proof of Claim 7.14. We prove the claim for $(F, \Gamma_D)$, since the proof is similar in the other case. Take $(F, \Gamma)$ general in $V_d^{2g-2}$; this determines a sequence

$$0 \rightarrow N \rightarrow \mathcal{F} \rightarrow \omega_C \rightarrow 0,$$  

(7.6)

where $N$ is general of degree $d - 2g + 2$ and the corresponding extension is general in the unique (good) component $\mathcal{A}_2 \subset \mathcal{P}(\text{Ext}^1(\omega_C, N))$ dominating $\mathbb{G}(2, H^0(K_C - N))$ (cf. the proof of Step 2); thus, if $\partial$ is the coboundary map, then $\text{Coker}(\partial)$ is a general two-dimensional quotient of $H^1(N)$.

On the other hand, $(F, \Gamma_D)$ determines a sequence

$$0 \rightarrow N(D) \rightarrow \mathcal{F} \rightarrow \omega_C(-D) \rightarrow 0.$$  

Since $\text{deg}(N(D)) = d - 2g + 4 \leq \frac{d}{2} - 2 < g - 1$ and $N(D)$ is general of its degree, then $H^0(N(D)) = 0$. In view of

$$0 \rightarrow N \rightarrow N(D) \rightarrow \mathcal{O}_D \rightarrow 0,$$

one has the exact sequence

$$0 \rightarrow H^0(\mathcal{O}_D) \cong \mathbb{C}^2 \rightarrow H^1(N) \xrightarrow{\alpha} H^1(N(D)) \rightarrow 0.$$  

The existence of the unisecant $\Gamma_D + f_D$ on $F$ gives rise to the sequence

$$0 \rightarrow N \rightarrow \mathcal{F} \rightarrow \omega_C(-D) \oplus \mathcal{O}_D \rightarrow 0$$  

(7.7)

(cf. (2.1)). This sequence corresponds to an element $\xi \in \text{Ext}^1(\omega_C(-D) \oplus \mathcal{O}_D, N)$, which by Serre duality, is isomorphic to $H^0(\mathcal{O}_D)^\vee \oplus \text{Ext}^1(\omega_C(-D), N)$ (cf. [20, Prop. III.6.7, Thm. III.7.6]). So $\xi = (\sigma, \eta)$, with $\sigma \in H^0(\mathcal{O}_D)^\vee$ and $\eta \in \text{Ext}^1(\omega_C(-D), N) \cong H^1(N(D) \otimes \omega_C^\vee)$.

We have the following diagram

$$\begin{array}{ccc}
H^0(\mathcal{O}_D) \oplus H^0(\omega_C(-D)) & \xrightarrow{\partial_0} & H^1(N) \\
\uparrow & & \downarrow \alpha \\
H^0(\omega_C(-D)) & \xrightarrow{\partial'} & H^1(N(D)) \\
\uparrow & & \downarrow \\
0 & & 0
\end{array}$$

where $\partial_0, \partial'$ are the coboundary maps. The action of $\xi$ on $\{0\} \oplus H^0(\omega_C(-D))$ coincides with the action of $\eta$ on $H^0(\omega_C(-D))$ via cup-product. This yields an isomorphism $\text{Coker}(\partial') \cong \text{Coker}(\partial_0)$.

Notice that (7.7) can be seen as a limit of (7.6). Since $\text{Coker}(\partial)$ is a general two-dimensional quotient of $H^1(N)$, then also $\text{Coker}(\partial_0)$ is general. The above argument implies that $\text{Coker}(\partial')$ is also general, proving the assertion (cf. the proof of Claim 7.13).

Finally, to prove that $r_1, r_2$ are dominant, it suffices to prove the following:

Claim 7.15. The general fibre of $r_i$ has dimension two, for $1 \leq i \leq 2$.

Proof of Claim 7.15. It suffices to prove that there are fibres of dimension two. For $r_1$, take $(F, \Gamma, p)$ general in $V_d^{2g-2} \times C$. The fibre of $r_1$ containing this triple consists of all triples $(F, \Gamma', p)$, with $\Gamma' \in |\mathcal{O}_F(\Gamma)|$ so it has dimension two since $i(\mathcal{F}) = 3$ (cf. computation as in (6.7)). The same argument works for $r_2$.

Step 5. In this step we prove that $V$ is an irreducible component of $\overline{B_{C}^{\infty}(d)}$.
Claim 7.16. Let \((F, \Gamma_D) \in V_d^{2g-2-1}\) be general, with \(1 \leq i \leq 2\) and \(D \in C^{(i)}\). Then \(|\Gamma_D + f_D|\) has dimension two, its general member \(\Gamma\) is smooth and it corresponds to a sequence \(0 \to N(-D) \to \mathcal{F} \to \omega_C \to 0\). Consequently, the pair \((F, \Gamma_D)\) is in the image of \(r_i\).

Proof of Claim 7.16. Given the first part of the statement, the conclusion is clear. To prove the first part, note that the existence of \(\Gamma \subset F\) gives an exact sequence

\[0 \to N \to \mathcal{F} \to \omega_C(-D) \to 0,\]  

hence \(h^0(\mathcal{O}_F(\Gamma_D + f_D)) = h^0(\mathcal{F} \otimes N^\vee) = h^0(\mathcal{F} \otimes \omega_C \otimes \det(\mathcal{F})^\vee) = h^1(\mathcal{F}) = 3\) (cf. (2.6)). This implies the assertion. \(\square\)

Let now \(\mathcal{B} \subseteq \widehat{B}^{k_3}_C(\delta)\) be a component containing \(\widehat{\nabla}\). From Step 4, \([\mathcal{F}] \in \mathcal{B}\) general has speciality \(i = 3\) and a special presentation as in (2.2) with \(L\) of minimal degree \(\delta\). Thus \(2g - 4 \leq \delta \leq 2g - 2\), since the Segre invariant is lower semi-continuous (cf. Remark 2.5 and also [22, \S 3]).

By Claim 7.16, \(2g - 3 \leq \delta \leq 2g - 2\) does not occur under the minimality assumption on \(L\). Indeed, in both cases we have a two-dimensional linear system \([\Gamma]\), whose general member is a section, corresponding to a surjection \(\mathcal{F} \to \omega_C\) and we proved that there would be curves in this linear system containing two rulings.

If \(\delta = 2g - 4\), we have an exact sequence as in (7.8). By specializing to a general point of \(\overline{\nabla} = \overline{V}_d^{2g-4,1,2}\), because of Claim 7.16, we see that in (7.8) one has \(h^0(N) = 0\). Hence, \(h^1(N)\) is constant. Since for the general element of \(\overline{\nabla}\), \(\text{Ker}(\mu_{i_2}) = 0\) the same happens for the general element of \(\mathcal{B}\) i.e., with notation as in the proof of Claim 7.13, \(\tau\) is constant equal to zero. Therefore, also \(\eta = \epsilon = 0\) for the general point of \(\mathcal{B}\) (see (1.c.), which implies the assertion. \(\square\)

With our approach, we cannot conclude that \(\overline{V}_d^{2g-4,1,2}\) in Theorem 7.11 is the unique regular component, whose general point \([\mathcal{F}]\) is such that \(i([\mathcal{F}]) = 3\), because we do not know if \(\Lambda_3 \subset \mathcal{W}_2\) is the only good component when \(N \in \text{Pic}^{d - 2g + 4}(C)\) and \(D_2 \in C^{(2)}\) general. However, results in [45, 49] imply that \(\tilde{B}^{k_2}_C(d)\) is irreducible for \(d \leq \frac{10g - 7}{3}\), though they say nothing on rigid special presentation of the general element. Putting all together, we have:

Corollary 7.17. Under the assumptions of Theorem 7.11, one has \(\tilde{B}^{k_3}_C(d) = \overline{V}_d^{2g-4,1,2}\).

Remark 7.18. (1) Theorem 5.4, for \(j = 3\), shows the existence of elements of \(\tilde{B}^{k_3}_C(d)\) with injective Petri map in the range \(g \geq 21\), \(g + 3 \leq \delta \leq \frac{4}{3}g - 4\), \(2g + 6 \leq d \leq \frac{5}{3}g - 9\). This gives a proof, alternative to the one in [49], of generic smoothness of \(\tilde{B}^{k_3}_C(d)\) in the above range.

(2) If \(\frac{5}{3}g - 5 \leq d \leq \frac{10}{3}g - 7\), \(B^{k_2}_C(d)\) is, as we saw, irreducible but in general it is no longer true that it is determined by a (total) good component. To see this, we consider a specific example.

Take \(B^{k_3}_C(3g - 4)\), which is non-empty, irreducible, generically smooth, of dimension \(g - 6\) and \([\mathcal{F}] \in \tilde{B}^{k_3}_C(3g - 4)\) general is such that \(i(\mathcal{F}) = 3\) by [45, 49]. By Lemma 4.1, \(\mathcal{F}\) can be rigidly presented as in (2.2), where \(L \in W_\delta^{-g+\delta}(C)\) and \(1 \leq j \leq 3\).

The cases \(j = 2, 3\) cannot occur: the stability condition (4.1) imposes \(\delta \geq \frac{2}{3}g - 2\), but if \(j = 3\), \(\rho(L) \geq 0\) forces \(\delta \leq \frac{5}{3}g - 3\) whereas if \(j = 2\), \(\rho(L) \geq 0\) implies \(\delta \leq \frac{3}{2}g - 3\); in both cases we get a contradiction.

The only possible case is therefore \(j = 1\), so the corank of the coboundary map is \(t = 2\), which implies that \(N\) is of speciality \(r \geq 2\). Since \(\chi(N) = 2g - 3 - \delta\), the case \(N\) non-effective would give \(\delta > 2g - 3\), i.e. \(L \cong \omega_C\). But in this case, \(a_p(2g - 2) \geq 2\) (usual computations as in (6.7)) against the rigidity assumptions.

Therefore \(N\) must be effective, with \(n = h^0(N) = 2g - 3 - \delta + r\). We want to show that the hypotheses of Corollary 5.9 hold. Assume by contradiction \(\ell < r\); then

\[\delta < g - 2 + r.\]  

From stability \(3g - 4 < 2\delta < 2g - 4 + 2r\), i.e. \(g - 2r < 0\). Since \(C\) has general moduli, one has \(\rho(N) \geq 0\), hence \(h^0(N) = 1\). So \(\delta = g - r\) and (7.9) yields \(d = \delta + d - \delta < 2g - 2 - 2r\) a contradiction. Thus, \(\ell \geq r\).

Now, from (4.2), \(m = 2\delta - 2g + 3\) since \(N\) is not isomorphic to \(L\). Thus, \(m \geq \ell + 1\): this is equivalent to \(\delta \geq g\), which holds by stability.

In conclusion, by Corollary 5.9, \(W_1^{\text{Tot}}\) is irreducible, of the expected dimension. Assume that \(\hat{W}_1^{\text{Tot}}\) contains a total good component \(\hat{\Lambda}_2\), whose image via \(\pi_{3g - 4,\delta}\) is \(\tilde{B}^{k_2}_C(3g - 4)\). Thus, \(r \geq 2\). On the other hand \(r = 2\) cannot occur since

\[c(\ell, 2, 2) = 2(\delta - g + 2) > 2\delta - 2g + 2 = m - 1 = \text{dim}(\mathcal{F}(\text{Ext}^1(L, N))).\]
a contradiction. Therefore, one has $r \geq 3$.

From the second equality in (6.3)
$$\dim(\mathcal{P}(\mathfrak{C})|_{\mathcal{Z}}) = (r + 1)\delta - (2r - 1)g - r(r - 3)$$
and the codimension of $\hat{\Lambda}^{\text{Tot}}$ is
$$c(\ell, r, 2) = 2(\delta - g + 4 - r).$$
Set $a := a_{F}(\delta)$. From Remark 4.2 we can assume $a \leq 1$. Therefore,
$$\dim(\text{Im}(\pi_{d, \delta}|_{\hat{\Lambda}^{\text{Tot}}})) = g - 6$$
gives $(r - 1)\delta = 2rg - 2g + r^2 - 5r + 2 + a$, i.e.
$$\delta = 2g + r - 4 + \frac{a - 2}{r - 1}.$$ 
This yields a contradiction. Indeed, since $0 \leq a \leq 1$, $r \geq 3$ and $\delta$ is an integer, the only possibility is $r = 3$, $a = 0$, $\delta = 2g - 2$ which we already saw to contradict the rigidity assumption.

**Proposition 7.19.** Let $C$ be of genus $g \geq 4$, with general moduli. Then, $\tilde{M}^3_C(2, \omega_C) \neq \emptyset$. Moreover, there exists an irreducible component which is regular (i.e. of dimension $3g - 9$), whose general point $[\mathcal{F}]$ fits in a sequence
$$0 \to \mathcal{O}_C(p + q) \to \mathcal{F} \to \omega_C(-p - q) \to 0,$$
where
- $p + q \in C^{(2)}$ general, and
- $\mathcal{F} = \mathcal{F}_v$ with $v \in \Lambda \subset \mathcal{W}_2 \subset \text{Ext}^1(\omega_C(-p), \mathcal{O}_C(p))$ general in $\Lambda$, which is a component of $\mathcal{W}_2$ of dimension $3g - 10$ (hence, not good).

**Proof.** With notation as in (4.4), (4.5), for $\mathcal{F} = \mathcal{F}_v$ as in (7.10), we have
$$\ell = r = g - 2, \quad t = 2, \quad m = h^1(2p + 2q - K_C) = 3g - 7.$$ 
Consider the map (notation as in (5.7) and (5.8))
$$\mu : H^0(\omega_C(-p - q)) \otimes H^0(\omega_C(-p - q)) \to H^0(\omega_C^2(-2p - 2q)).$$
For $V_2 \in \mathbb{G}(2, H^0(\omega_C(-p - q)))$ general, $\mu_{V_2}$ has kernel of dimension 1 (cf. computations as in Claim 7.13). Arguing as in the proofs of Theorem 5.17 and Claim 7.13, there is a component $\Lambda \subset \mathcal{W}_2 \subset \text{Ext}^1(\omega_C(-p - q), \mathcal{O}_C(p + q))$ (dominating $\mathbb{G}(2, H^0(\omega_C(-p - q)))$, hence not good) of dimension $3g - 10$.

Stability of $\mathcal{F}$ follows from Proposition 4.4. This shows that $\tilde{M}^3_C(2, \omega_C) \neq \emptyset$. Regularity and generic smoothness follow from the injectivity of the symmetric Petri map as in [52, 5].

The fact that $[\mathcal{F}]$ general has a presentation as in (7.10) follows from an obvious parameter computation. \qed

### 7.5. A conjecture for $i \geq 4$

For any $i \geq 4$ and $d$ as in Theorems 5.1, 5.4, 5.10, 5.13, one has $B^k_{C}(d) \neq \emptyset$. In particular, when $d$ is as in Theorem 5.4, with $j = i$, one deduces that $B^k_{C}(d)$ contains a regular, generically smooth component. This gives results in the same flavour as Theorem 0.1.

One may wish to give a special, rigid presentation of the general point of all components of $\tilde{B}^k_{C}(d)$. The following less ambitious conjecture is inspired by the results in this paper.

**Conjecture 7.20.** Let $i \geq 4$ and $g > i^2 + i + 1$ be integers. Let $C$ be of genus $g$, with general moduli. Let $d$ be an integer such that
$$2g - 2 \leq d < 2g - 2 - i + \frac{g - \epsilon}{i - 1},$$
with $\epsilon \in \{0, 1\}$ such that $d + g - (i - 1)k_i \equiv \epsilon \pmod{2}$. Then, there exists an irreducible, regular component $\mathcal{B} \subseteq B^k_{C}(d)$, s.t.
$$\mathcal{B} = \mathbb{V}_d^{2g - 1 - i, 1, i - 1}.$$ 
In particular, $[\mathcal{F}] \in \mathcal{B}$ general is stable, with $i(\mathcal{F}) = i$, $s(\mathcal{F}) \geq g - (i - 1)k_i - \epsilon > 0$ and it is rigidly specially presented as
$$0 \to N \to \mathcal{F} \to \omega_C(-D_{i-1}) \to 0,$$
where
- $D \in C^{(i - 1)}$ is general,
- $N \in \text{Pic}^{d - 2g + 1 + 1}(C)$ is general (i.e. special, non-effective),
The bounds on $g$ and $d$ in Conjecture 7.20 ensure the following:

1. $\rho_d^k \geq 0$ which is equivalent to $d \leq 2g - 2 - i + \frac{4g-3}{2} \ (\text{cf. (3.3)})$.
2. $N \in \text{Pic}^{d-2g+1+i}(C)$ general is special, non-effective; indeed $r = 3g - 2 - i - d > 0$.
3. $r \geq i - 1 = \text{cork}(\partial_e)$, which is equivalent to $d \leq 3g - 1 - 2i$.
4. There are no obstructions for a good component $\Lambda_{i-1} \subseteq \text{Ext}^1(\omega_C(-D), N)$ to exist; indeed, one has
   \[
   \dim(P) = m - 1 = 5g - 4 - 2i - d
   \]
   from (4.2), and from (5.5), Remark 5.15, we have
   \[
   c(\ell, r, t) = (i - 1)k_i = (i - 1)(d - 2g + 2 + i),
   \]
   since $t = i - 1$ and $N$ non-effective. Therefore
   \[
   \dim(\Lambda_{i-1}) = m - 1 - c(\ell, r, t) = 3g - 2 - i(d - 2g + 2 + i)
   \]
   is non-negative as soon as $d \leq 2g - 3 - i + \frac{3g-2}{2}$.
5. From Remark 5.15, $v \in \Lambda_{i-1}$ general is such that $s(F_v) \geq g - (i - 1)k_i - \epsilon$, which is positive because of the upper-bound on $d$. Thus $F_v$ is stable.
6. The interval $2g - 2 \leq d < 2g - 2 - i + \frac{g-\epsilon}{i}$ is not empty.

From Remark 4.2 and (6.9), to prove Conjecture 7.20 it would suffice to prove the following two facts:

(a) For $i \geq 4$, $D \in C^{(i-1)}$ general and $L = \omega_C(-D)$, there is a good component $\Lambda_{i-1} \subseteq \mathcal{W}_{i-1} \subset \text{Ext}^1(\omega_C(-D), N)$.

(b) For $v \in \Lambda_{i-1}$ general, $F_v$ is r.s.p via $\omega_C(-D)$.

Concerning (a), notice that for no $V_{i-1} \in \mathcal{G}(i-1, H^0(K_C - N))$ the map $\mu_{V_{i-1}}$ is injective. Indeed, $d \geq 2g - 2$ and $i \geq 4$ imply
   \[
   \dim(H^0(K_C - D) \otimes V_{i-1}) = (i - 1)g - (i - 1)^2 > 5g - 3 - d - 2i = h^0(2K_C - N - D).
   \]

Hence, according to Theorem 5.17, one should find an irreducible subvariety $\Sigma_i \subset \mathcal{G}(i-1, H^0(K_C - D))$ of codimension $\eta := d + (i - 6)g + 7 - (i - 2)^2$ such that $\dim(\ker(\mu_{V_{i-1}})) = \eta$ for $V_{i-1} \in \Sigma_i$ general.

Concerning (b), the minimality assumption implies $a_{F_v}(2g - 1 - i) \leq 1$, for $v \in \Lambda_{i-1}$ general. To prove rigidity, one has to show that $a_{F_v}(2g - 1 - i) = 0$. This is equivalent to prove a regularity statement for a Severi variety of nodal curves on $F$. Indeed, for any section $\Gamma$ corresponding to a quotient $F_v \rightarrow \omega_C(-D)$ as above, the linear system $[\Gamma_D + f_D]$ has dimension $i - 1$, is independent on $D$ and its general member $\Gamma$ is a section corresponding to a quotient $F_v \rightarrow \omega_C$. The curve $\Gamma_D + f_D$ belongs to the Severi variety of $(i-1)$-nodal curves in $[\Gamma]$. So rigidity is equivalent to show that this Severi variety has the expected dimension zero. Proving this is equivalent to prove that $D$, considered as a divisor on $\Gamma_D$, imposes independent condition to $[\Gamma]$. Unfortunately, the known results on regularity of Severi varieties (see [36, 46, 47]) do not apply in this situation.

References


