EXTENSIONS OF LINE BUNDLES AND BRILL–NOETHER LOCI OF RANK-TWO VECTOR BUNDLES ON A GENERAL CURVE

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Abstract. In this paper we study Brill-Noether loci for rank-two, (semi)stable vector bundles on a general curve C. Our aim is to describe the general member $\mathcal{F}$ of some of its components just in terms of extensions of line bundles with suitable *minimality properties*, providing information both on families of irreducible, unisecant curves of the ruled surface $\mathbb{P}(\mathcal{F})$ and on the birational geometry of the component of the Brill-Noether locus to which $\mathcal{F}$ belongs.

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Introduction

Let C be a smooth, irreducible projective curve of genus g and $U_C(d)$ be the moduli space of (semi)stable, degree d, rank-two vector bundles on C. In this paper we will be mainly concerned with C of general moduli.

Our aim is to study the Brill-Noether loci $B^k_C(d) \subset U_C(d)$ parametrizing (classes of) vector bundles $[\mathcal{F}] \in U_C(d)$ having $h^0(C, \mathcal{F}) \geqslant k$, with k a non-negative integer.

The classical Brill-Noether theory for line bundles on a general curve is very important and well established (cf., e.g., [1]). Brill-Noether theory for higher-rank vector bundles is a very active research area (see References, for some results in the subject), but several basic questions concerning Brill–Noether loci, like non-emptiness, dimension, irreducibility, local structure, etc., are still open in general. Contrary to the rank-one case, the Brill-Noether loci for C general do not always behave as expected (cf. e.g. [7] and § 7.1).

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Apart from its intrinsic interest, Brill-Noether theory is important in view of applications to other areas, like birational geometry to mention just one (cf. e.g. [4, 6, 7, 22, 33]).

The most general existence result in the rank-two case is the following:

Theorem 0.1. *(see* [53]*) Let C be a curve with general moduli of genus $g \geqslant 1$. Let $k \geqslant 2$ and $i := k+2g-2-d \geqslant 2$ be integers. Let $\rho_d^k := 4g-3-ik$ and assume*

$$\rho_d^k \geqslant 1 \text{ when } d \text{ odd}, \quad \rho_d^k \geqslant 5 \text{ when } d \text{ even}.$$

Then $B_C^k(d)$ is not empty and it contains a component $\mathcal{B}$ of the expected dimension ρ_d^k.

This previous result is proved with a quite delicate degeneration argument (cf. also [13]); in some particular cases, one has improvements of it (cf., e.g., [45, 27, 50, 51, 47, 15, 52, 26]).

The degeneration technique used in [53], though powerful, does not provide a *geometric description* of the (isomorphism classes of) bundles $\mathcal{F}$ in $B_C^k(d)$, in particular of the general one in a component. By "geometric description" we mean a description of families of curves on the ruled surface $\mathbb{P}(\mathcal{F})$, in particular of unisecant curves to its fibres. This translates in turn to exhibiting $\mathcal{F}$ as an extension of line bundles

$$(*) \quad 0 \to N \to \mathcal{F} \to L \to 0$$

(cf. (2.2)), which we call a *presentation* of $\mathcal{F}$. Of particular interest is a presentation $(*)$ with suitable *minimality properties* on the quotient line bundle L, which translate into minimality properties for families of irreducible unisecant curves on the surface $\mathbb{P}(\mathcal{F})$ (cf. § 2 below).

This approach provides basic information about the vector bundle $\mathcal{F}$, which can be useful in a field in which so little is known and which has not been given so far: indeed, the description of such a minimal presentation is not known in general and, in particular, has not been provided in Theorem 0.1.

One of the main objective of this paper is to shed some light on this subject. As a consequence of our analysis, we provide explicit parametric representations (and so information about the birational geometry) of some components of $B_C^k(d)$ (cf. §'s 6, 7).

Viewing rank two vector bundles as extensions of line bundles is very classical: by suitably interpreting the classical language, this goes back to C. Segre [42]. In recent times, it has been exploited, e.g., in [7, § 2,3], [33, § 8], where the case of canonical determinant and $g \leqslant 12$ has been treated. As noted in [7], this approach "*works well enough in low genera.... but seems difficult to implement in general* ". However, we tried to follow this route, with no upper-bounds on the genus but, as we will see, by bounding the speciality $i := h^1(C, \mathcal{F})$.

Our approach is as follows. We construct (semi)stable vector bundles $\mathcal{F}$ in Brill-Noether loci, as extensions of line bundles L and N: the Brill–Noether loci we hit in this way depend on the cohomology of L and N and on the behaviour of the *coboundary map* $H^0(C, L) \xrightarrow{\partial} H^1(C, N)$ associated to $(*)$, cf. §'s 4, 5; we exhibit explicit constructions of such vector bundles in Theorems 5.1, 5.4, 5.10, 5.13. These theorems provide existence results for $B_C^k(d)$ which are comparable to, though slightly worse but easier to prove than, Theorem 0.1 (cf. Remark. 5.5–(3)). At the same time, they imply non–emptiness for fibres of the determinant map $B_C^k(d) \to \mathrm{Pic}^d(C)$, i.e. for Brill–Noether loci with fixed determinant $\det(\mathcal{F}) := L \otimes N$, for any possible L and N as in the assumptions therein; this is in the same spirit of [25], where however only the case with fixed determinant of odd degree has been considered.

In any event, as we said, the main purpose of this paper is not the one of constructing new components of Brill-Noether loci, but of providing a minimal presentation for the *general element* of components of $B_C^k(d)$, for C with general moduli. To do this, we take line bundles L and N with assumptions as in Theorems 5.1–5.13 and let them vary in their own Brill-Noether loci of their Picard schemes. Accordingly, we let the constructed bundles $\mathcal{F}$ vary in suitable degeneracy loci $\Lambda \subseteq \mathrm{Ext}^1(L, N)$, defined in such a way that $\mathcal{F} \in \Lambda$ general has the desired speciality i (cf. § 6). In this way we obtain irreducible varieties parametrizing triples $(L, N, \mathcal{F})$, i.e. any such variety is endowed with a morphism π to $U_C(d)$, whose image is contained in a component $\mathcal{B}$ of a Brill-Noether locus. To find a minimal presentation $(*)$ for a general member $\mathcal{F}$ of $\mathcal{B}$, we are reduced to find conditions on L, N and on the coboundary map ∂ ensuring the morphism π to be dominant onto $\mathcal{B}$. We achieve this goal by using results in § 2, which deal with the study of some families of irreducible unisecants of given speciality on the ruled surface $\mathbb{P}(\mathcal{F})$ (cf. Lemmas 6.2, 6.8, Corollaries 6.5, 6.7, 6.9, 6.12 and Remarks 6.16, 6.17).

As is clear from the foregoing description, to find a presentation of $\mathcal{F}$ general in a component of a Brill-Noether locus is a difficult problem in general. We are able to solve it here for $i \leqslant 3$.

Our main results for Brill–Noether loci $B_C^k(d)$ are Theorems 7.1, 7.5, 7.11, which respectively deal with cases $i = 1, 2, 3$ and $k = d - 2g + 2 + i$. A first fact we prove therein is that (a component of) $B_C^k(d)$ is filled-up by vector bundles $\mathcal{F}$ having a minimal *special* presentation as

$$0 \to N \to \mathcal{F} \to \omega_C(-D_{i-1}) \to 0,$$

where $D_{i-1} \in \mathrm{Sym}^{i-1}(C)$, $N \in \mathrm{Pic}^{d-2g+1-i}$ and $\mathcal{F} \in \Lambda_{i-1}$ are general, where Λ_{i-1} is a *good component* of the degeneracy locus

$$\left\{\mathcal{F} \in \mathrm{Ext}^1(\omega_C(-D_{i-1}), N) \mid \dim \mathrm{Coker}\left(H^0(C, L) \xrightarrow{\partial} H^1(C, N)\right) \geqslant i-1\right\} \subseteq \mathrm{Ext}^1(\omega_C(-D_{i-1}), N)$$

(cf. Def. 2.10, for precise definitions of special presentation and minimality, Rem. 5.7 and Defs. 5.12, 6.13, for goodness, and Thms. 5.8, 5.17, for existence of good components). The case $i = 1$ was already treated in [3] with different methods (cf. Remark 7.2); in cases $i = 2, 3$ our results are new.

Statements of our main results, to which the reader is referred, contain even more. Indeed they also describe families of special, irreducible unisecants of $\mathbb{P}(\mathcal{F})$ which are of minimal degree with respect to its tautological line bundle. Apart from its intrinsic interest, this description plays a fundamental role when one tries to construct components of the Hilbert scheme parametrizing linearly normal, genus g and degree d special scrolls in projective spaces and whose general point parametrizes a *stable scroll* (cf. e.g. [14]).

Finally, the proofs of Theorems 7.1, 7.5, 7.11 show in particular that the map π from the parameter space of triples $(L, N, \mathcal{F})$ to (the dominated component of) $B^k_C(d)$ is generically finite, sometimes even birational (cf. Rmk. 7.2–(3)), giving therefore information about the birational geometry of the Brill–Noether locus.

Other main results of the paper are given by Theorems 7.3, 7.10, 7.19 which deal with the canonical determinant case.

In principle, there is no obstruction in pushing further the ideas in this paper, to treat higher speciality cases. However, this is increasingly complicated and therefore we limited ourselves to expose at the end of the paper a few suggestions on how to proceed in general and propose a conjecture (see § 7.5).

The paper is organized as follows. Section 2 is devoted to preliminaries about families of special, irreducible unisecants on ruled surfaces $\mathbb{P}(\mathcal{F})$ and the corresponding special presentation of the bundle $\mathcal{F}$. Sections 3 and 4 are devoted to recalling basic facts on (semi)stable, rank–two vector bundles of degree d, extensions of line bundles, and useful results of Lange–Narashiman and Maruyama (cf. Proposition 4.4 and Lemma 4.5). Section 5 is the technical one, which contains our constructions of vector bundles in Brill-Noether loci as extensions of line bundles L and N. Section 6 is where we deal with parameter spaces of triples and maps from them to $U_C(d)$, landing in Brill-Noether loci. The general machinery developed in the previous sections is then used in § 7, in order to prove our main results mentioned above.

1. Notation and terminology

In this paper we work over $\mathbb{C}$. All schemes will be endowed with the Zariski topology. We will interchangeably use the terms rank-r vector bundle on a scheme X and rank-r locally free sheaf.

We denote by $\sim$ the linear equivalence of divisors, by $\sim_{alg}$ their algebraic equivalence and by $\equiv$ their numerical equivalence. We may abuse notation and identify divisor classes with the corresponding line bundles, interchangeably using additive and multiplicative notation.

If $\mathcal{P}$ is the parameter space of a flat family of subschemes of X and if Y is an element of the family, we denote by $Y \in \mathcal{P}$ the point corresponding to Y. If $\mathcal{M}$ is a moduli space, parametrizing geometric objects modulo a given equivalence relation, we denote by $[Z] \in \mathcal{M}$ the moduli point corresponding to the equivalence class of Z.

Let
- C be a smooth, irreducible, projective curve of genus g, and
- $\mathcal{F}$ be a rank-two vector bundle on C.

Then, $F := \mathbb{P}(\mathcal{F}) \xrightarrow{\rho} C$ will denote the *(geometrically) ruled surface* (or *the scroll*) associated to $(\mathcal{F}, C)$; f will denote the general ρ-fibre and $\mathcal{O}_F(1)$ the *tautological line bundle*. A divisor in $|\mathcal{O}_F(1)|$ will be usually denoted by H. If $\widetilde{\Gamma}$ is a divisor on F, we will set $\deg(\widetilde{\Gamma}) := \widetilde{\Gamma}H$.

We will use the notation

$$d := \deg(\mathcal{F}) = \deg(\det(\mathcal{F})) = H^2 = \deg(H);$$

$i(\mathcal{F}) := h^1(\mathcal{F})$ is called the *speciality* of $\mathcal{F}$ and will be denoted by i, if there is no danger of confusion. $\mathcal{F}$ (and F) is *non-special* if $i = 0$, *special* otherwise.

As customary, $W^r_a(C)$ will denote the *Brill-Noether locus*, parametrizing line bundles $A \in Pic^a(C)$ such that $h^0(A) \geqslant r+1$,

$$\rho(g, r, a) := g - (r+1)(r+g-a)$$

the *Brill-Noether number* and

$$\mu_0(A) : H^0(C, A) \otimes H^0(\omega_C \otimes A^\vee) \to H^0(C, \omega_C) \tag{1.1}$$

the *Petri map*. As for the rest, we will use standard terminology and notation as in e.g. [1], [21], etc.

2. SCROLLS UNISECANTS

We recall some basic facts on unisecant curves of the scroll F (cf. [17, 19] and [21, V-2]).

One has $\mathrm{Pic}(F) \cong \mathbb{Z}[\mathcal{O}_F(1)] \oplus \rho^*(\mathrm{Pic}(C))$ (cf. [21, § 5, Prop. 2.3]). Let Div_F be the scheme (not of finite type) of effective divisors on F, which is a sub-monoid of $\mathrm{Div}(F)$. For any $k \in \mathbb{N}$, let Div^k_F be the subscheme (not of finite type) of Div_F formed by all divisors $\widetilde{\Gamma}$ such that $\mathcal{O}_F(\widetilde{\Gamma}) \cong \mathcal{O}_F(k) \otimes \rho^*(N^\vee)$, for some $N \in \mathrm{Pic}(C)$ (this N is uniquely determined); then one has a natural morphism

$$\Psi_k : \mathrm{Div}^k_F \to \mathrm{Pic}(C), \qquad \widetilde{\Gamma} \xrightarrow{\Psi_k} N.$$

If $D \in \mathrm{Div}(C)$, then $\rho^*(D)$ will be denoted by f_D. Then $\widetilde{\Gamma} \in \mathrm{Div}^k_F$ if and only if $\widetilde{\Gamma} \sim kH - f_D$, for some $D \in \mathrm{Div}(C)$, and $\deg(\widetilde{\Gamma}) = k \deg(\mathcal{F}) - \deg(D)$.

The curves in Div^1_F are called *unisecants* of F. Irreducible unisecants are isomorphic to C and called *sections* of F. For any positive integer δ, we consider (cf. [19, § 5])

$$\mathrm{Div}^{1,\delta}_F := \{\widetilde{\Gamma} \in \mathrm{Div}^1_F \mid \deg(\widetilde{\Gamma}) = \delta\},$$

which is the *Hilbert scheme of unisecants of degree* δ *of* F (w.r.t. H).

Remark 2.1. Let $\widetilde{\Gamma} = \Gamma + f_A$ be a unisecant, with Γ a section and A effective. Equivalently, we have an exact sequence

$$0 \to N(-A) \to \mathcal{F} \to L \oplus \mathcal{O}_A \to 0 \tag{2.1}$$

(cf. [10, 12]); in particular if $A = 0$, i.e. $\widetilde{\Gamma} = \Gamma$ is a section, $\mathcal{F}$ fits in the exact sequence

$$0 \to N \to \mathcal{F} \to L \to 0 \tag{2.2}$$

and

$$\mathcal{N}_{\Gamma/F} \cong L \otimes N^\vee, \ \text{ so } \ \Gamma^2 = \deg(L) - \deg(N) = 2\delta - d, \tag{2.3}$$

(cf. [21, § V, Prop. 2.6, 2.9]). Accordingly $\Psi_{1,\delta} : \mathrm{Div}^{1,\delta}_F \to \mathrm{Pic}^{d-\delta}(C)$, the restriction of Ψ_1, endows $\mathrm{Div}^{1,\delta}_F$ with a structure of Quot scheme: with notation as in [43, § 4.4], one has

$$\begin{array}{rccc} \Phi_{1,\delta} : & \mathrm{Div}^{1,\delta}_F & \xrightarrow{\cong} & \mathrm{Quot}^C_{\mathcal{F},\delta+t-g+1} \\ & \widetilde{\Gamma} & \longrightarrow & \{\mathcal{F} \twoheadrightarrow L \oplus \mathcal{O}_A\}\,. \end{array} \tag{2.4}$$

From standard results (cf. e.g. [43, § 4.4]), (2.4) gives identifications between tangent and obstruction spaces:

$$H^0(\mathcal{N}_{\widetilde{\Gamma}/F}) \cong T_{[\widetilde{\Gamma}]}(\mathrm{Div}^{1,\delta}_F) \cong \mathrm{Hom}(N(-A), L \oplus \mathcal{O}_A) \quad \text{and} \quad H^1(\mathcal{N}_{\widetilde{\Gamma}/F}) \cong \mathrm{Ext}^1(N(-A), L \oplus \mathcal{O}_A) \tag{2.5}$$

Finally, if $\widetilde{\Gamma} \sim H - f_D$, then one easily checks that

$$|\mathcal{O}_F(\widetilde{\Gamma})| \cong \mathbb{P}(H^0(\mathcal{F}(-D))). \tag{2.6}$$

Definition 2.2. $\widetilde{\Gamma} \in \mathrm{Div}^{1,\delta}_F$ is said to be:
(a) *linearly isolated (li)* if $\dim(|\mathcal{O}_F(\widetilde{\Gamma})|) = 0$,
(b) *algebraically isolated (ai)* if $\dim(\mathrm{Div}^{1,\delta}_F) = 0$.

Remark 2.3. (1) If $\widetilde{\Gamma}$ is ai, then it is also li but the converse is false (c.f. e.g. Example 2.9, Corollary 6.12).
(2) When $\mathrm{Div}^{1,\delta}_F$ is of pure dimension, a sufficient condition for $\widetilde{\Gamma}$ to be ai is $h^0(\mathcal{N}_{\widetilde{\Gamma}/F}) = 0$ (cf. e.g. Theorem 5.4, Corollary 6.7 and § 7.1 below).

2.1. **The Segre-invariant.**

Definition 2.4. The *Segre invariant* of $\mathcal{F}$ is defined as

$$s(\mathcal{F}) := \deg(\mathcal{F}) - 2(\max\{\deg(N)\}),$$

where the maximum is taken among all sub-line bundles N of $\mathcal{F}$ (cf. e.g. [24]). The bundle $\mathcal{F}$ is *stable* [resp. *semi–stable*], if $s(\mathcal{F}) > 0$ [resp. if $s(\mathcal{F}) \geqslant 0$].

Equivalently $\mathcal{F}$ is stable [resp. semistable] if for every sub-line bundle $N \subset \mathcal{F}$ one has $\mu(N) < \mu(\mathcal{F})$ [resp. $\mu(N) \leq \mu(\mathcal{F})$], where $\mu(\mathcal{E}) = \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ is the *slope* of a vector bundle $\mathcal{E}$.

Note that, for any $A \in \mathrm{Pic}(C)$, one has

$$s(\mathcal{F}) = s(\mathcal{F} \otimes A). \tag{2.7}$$

Remark 2.5. From (2.3), $s(\mathcal{F})$ coincides with the minimum self-intersection of sections of F. In particular, if $\Gamma \in \mathrm{Div}_F^{1,\delta}$ is a section s.t. $\Gamma^2 = s(\mathcal{F})$, then $s(\mathcal{F}) = 2\delta - d$ and Γ is a section of *minimal degree* of F, i.e. for any section $\Gamma' \subset F$ one has $\deg(\Gamma') \geqslant \deg(\Gamma)$.

We recall the following fundamental result.

Proposition 2.6. *Let C be of genus $g \geqslant 1$ and let $\mathcal{F}$ be indecomposable. Then, $2 - 2g \leqslant s(\mathcal{F}) \leqslant g$.*

Proof. The lower-bound follows from $\mathcal{F}$ being indecomposable (see e.g. [21, V, Thm. 2.12(b)]). The upper-bound is Nagata's Theorem (see [34]). ☐

2.2. **Special scrolls unisecants.** In the paper we will be mainly concerned about the speciality of unisecants of a (necessarily special) scroll F.

Definition 2.7. For $\widetilde{\Gamma} \in \mathrm{Div}_F$, we set $\mathcal{O}_{\widetilde{\Gamma}}(1) := \mathcal{O}_F(1) \otimes \mathcal{O}_{\widetilde{\Gamma}}$. The *speciality* of $\widetilde{\Gamma}$ is $i(\widetilde{\Gamma}) := h^1(\mathcal{O}_{\widetilde{\Gamma}}(1))$. $\widetilde{\Gamma}$ is *special* if $i(\widetilde{\Gamma}) > 0$.

If $\widetilde{\Gamma}$ is given by (2.1), then by (2.6) one has $\widetilde{\Gamma} \in |\mathcal{O}_F(1) \otimes \rho^*(N^\vee(A))|$. Applying ρ_* to the exact sequence

$$0 \to \mathcal{O}_F(1) \otimes \mathcal{O}_F(-\widetilde{\Gamma}) \to \mathcal{O}_F(1) \to \mathcal{O}_{\widetilde{\Gamma}}(1) \to 0$$

and using $\rho_*(\mathcal{O}_F(1) \otimes \mathcal{O}_F(-\widetilde{\Gamma})) \cong N(-A)$, $R^1\rho_*(\mathcal{O}_F(\rho^*(N(-A))) = 0$, we get

$$i(\widetilde{\Gamma}) = h^1(L \oplus \mathcal{O}_A) = h^1(L) = i(\Gamma), \tag{2.8}$$

where Γ the unique section in $\widetilde{\Gamma}$.

The following examples show that, in general, speciality is not constant either in linear systems or in algebraic families.

Example 2.8. Take $g = 3$, $i = 1$ and $d = 9 = 4g - 3$. There are smooth, linearly normal, special scrolls $S \subset \mathbb{P}^5$ of degree 9, speciality 1, sectional genus 3 with general moduli containing a unique special section Γ which is a genus 3 canonical curve (cf. [11, Thm. 6.1]). Moreover, Γ is the unique section of minimal degree 4 (cf. also [42]). There are lines $f_1, \ldots, f_5$ of the ruling, such that $\widetilde{\Gamma} := \Gamma + f_1 + \ldots + f_5 \in |H|$, where H the hyperplane section of S. These curves $\widetilde{\Gamma}$ vary in a sub-linear system of dimension 2 contained in $|H|$, whose movable part is the complete linear system $|f_1 + \cdots + f_5|$. The curves as $\widetilde{\Gamma}$ are the only special unisecants in $|H|$.

Example 2.9. Let C be a non-hyperelliptic curve of genus $g \geqslant 3$, $d = 3g - 4$ and $N \in \mathrm{Pic}^{g-2}(C)$ general. N is non-effective with $h^1(N) = 1$. Consider $\mathrm{Ext}^1(\omega_C, N)$. It has dimension $2g - 1$ and its general point gives rise to a rank-two vector bundle $\mathcal{F}$ of degree d, fitting in an exact sequence like (2.2), with $L = \omega_C$. By generality, the coboundary map $\partial : H^0(\omega_C) \to H^1(N) \cong \mathbb{C}$ is surjective (cf. Corollary 5.9 below); therefore $i(\mathcal{F}_u) = 1$. Since $\mathcal{F}$ is of rank-two with $\det(\mathcal{F}) = \omega_C \otimes N$, by Riemann-Roch one has $h^0(\mathcal{F} \otimes N^\vee) = 1$. From (2.6), the canonical section $\Gamma \subset F$, corresponding to $\mathcal{F} \twoheadrightarrow \omega_C$, is li. From (2.3), $\mathcal{N}_{\Gamma/F} \cong \omega_C \otimes N^\vee$ hence $h^i(\mathcal{N}_{\Gamma/F}) = 1 - i$, for $i = 0, 1$. Let $\mathcal{D}$ be the irreducible, one-dimensional component of the Hilbert scheme containing the point corresponding to Γ (which is smooth for the Hilbert scheme). Therefore $\mathcal{D}$ is an algebraic (non-linear) family whose general member is a li section. As a consequence of Proposition 2.12 below, Γ is the only special section in $\mathcal{D}$. In particular, if all curves in $\mathcal{D}$ are irreducible, then Γ is the only special curve in $\mathcal{D}$ (see Lemma 2.11).

Note that $\mathcal{F}$ is indecomposable. Indeed, assume $\mathcal{F} = A \oplus B$, with A, B line bundles. Since $h^0(\mathcal{F} \otimes N^\vee) = 1$, we may assume $h^0(A - N) = 1$ and $h^0(B - N) = 0$. By the genericity of N, $A - N$ and $B - N$ are both general of their degrees. Therefore $\deg(A - N) = g$, hence $\deg(A) = 2g - 2$ and $\deg(B) = g - 2$. The image of A in the surjection $\mathcal{F} \twoheadrightarrow \omega_C$ is zero, otherwise $A = \omega_C$ hence $B = N$ which is impossible, because $h^0(B - N) = 0$. Then we would have an injection $A \hookrightarrow N$ which is impossible by degree reasons.

Since $\mathrm{Div}_F^{1,\delta}$ is a Quot-scheme, there is the universal quotient $\mathcal{Q}_{1,\delta} \to \mathrm{Div}_F^{1,\delta}$. Taking $\mathrm{Proj}(\mathcal{Q}_{1,\delta}) \xrightarrow{p} \mathrm{Div}_F^{1,\delta}$, we can consider

$$\mathcal{S}_F^{1,\delta} := \{\widetilde{\Gamma} \in \mathrm{Div}_F^{1,\delta} \ \mid \ R^1p_*(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,\delta})}(1))_{\widetilde{\Gamma}} \neq 0\} \quad \text{and} \quad a_F(\delta) := \dim(\mathcal{S}_F^{1,\delta}), \tag{2.9}$$

i.e. $\mathcal{S}_F^{1,\delta}$ is the support of $R^1p_*(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_{1,\delta})}(1))$. It parametrizes degree δ, special unisecants of F.

Definition 2.10. Let $\widetilde{\Gamma}$ be a special unisecant of F. Assume $\widetilde{\Gamma} \in \mathfrak{F}$, where $\mathfrak{F} \subseteq \mathrm{Div}_F^{1,\delta}$ is a subscheme.

- We will say that $\widetilde{\Gamma}$ is:

(i) *specially unique (su)* in $\mathfrak{F}$, if $\widetilde{\Gamma}$ is the only special unisecant in $\mathfrak{F}$, or

(ii) *specially isolated (si)* in $\mathfrak{F}$, if $\dim_{\widetilde{\Gamma}}\left(\mathcal{S}^{1,\delta}_F \cap \mathfrak{F}\right) = 0$.

- In particular:

(a) when $\mathfrak{F} = |\mathcal{O}_F(\widetilde{\Gamma})|$, $\widetilde{\Gamma}$ is said to be *linearly specially unique (lsu)* in case (i) and *linearly specially isolated (lsi)* in case (ii);

(b) when $\mathfrak{F} = \mathrm{Div}^{1,\delta}_F$, $\widetilde{\Gamma}$ is said to be *algebraically specially unique (asu)* in case (i) and *algebraically specially isolated (asi)* in case (ii).

- When a section $\Gamma \subset F$ is asi, we will say that $\mathcal{F}$ is *rigidly specially presented (rsp)* as $\mathcal{F} \twoheadrightarrow\!\!\!\rightarrow L$ or by the sequence (2.2) corresponding to Γ. When Γ is ai (cf. Def. 2.2), we will say that $\mathcal{F}$ is *rigidly presented (rp)* via $\mathcal{F} \twoheadrightarrow\!\!\!\rightarrow L$ or (2.2).

For examples, c.f. e.g. § 7 below.

Lemma 2.11. *Let $\Gamma \subset F$ be a section corresponding to a sequence as in* (2.2)*. A section Γ', corresponding to $\mathcal{F} \twoheadrightarrow L'$, is s.t. $\Gamma \sim \Gamma'$ if and only if $L \cong L'$. In particular*
(a) $i(\Gamma) = i(\Gamma')$;
(b) Γ is lsu if and only if it is lsi if and only if it is li.

Proof. The first assertion follows from (2.6). Then, (a) and (b) are both clear. □

Proposition 2.12. *Let F be indecomposable and let $\Gamma \in \mathfrak{F} \subseteq \mathcal{S}^{1,\delta}_F$ be a section, where $\mathfrak{F}$ is an irreducible, projective scheme of dimension k. Assume:*
(a) $k \geqslant 1$, if $\mathfrak{F}$ is a linear system;
(b) either $k \geqslant 2$, or $k = 1$ and $\mathfrak{F}$ with base points, if $\mathfrak{F}$ is not linear.

Then, $\mathfrak{F}$ contains reducible unisecants $\widetilde{\Gamma}$ with

$$i(\widetilde{\Gamma}) \geqslant i(\Gamma). \tag{2.10}$$

Proof. If $k \geqslant 2$, let t be the unique integer such that $0 \leqslant k' := k - 2t \leqslant 1$. Let $f_1, \ldots, f_t$ be t general ρ-fibres of F. Since $k' \geqslant 0$, by imposing to the curves in $\mathfrak{F}$ to contain fixed general pairs of points on $f_1, \ldots, f_t$, we see that

$$\mathfrak{F}' := \mathfrak{F}\left(-\sum_{i=1}^{t} f_i\right) \subset \mathfrak{F}$$

is non-empty, all components of it have dimension k', and they all parametrize unisecants $\Gamma' \sim_{alg} \Gamma - \sum_{i=1}^t f_i$. Then $\mathfrak{F}$ contains reducible elements $\widetilde{\Gamma}$, and they verify (2.10) by upper-semicontinuity. This proves the assertion when $k \geqslant 2$.

So we are left with the case $k = 1$. Assume first that $\mathfrak{F}$ is a linear pencil. Since $\mathfrak{F} \subseteq |\mathcal{O}_F(\Gamma)|$, from the exact sequence $0 \to \mathcal{O}_F \to \mathcal{O}_F(\Gamma) \to \mathcal{O}_\Gamma(\Gamma) \to 0$, the line bundle $\mathcal{O}_\Gamma(\Gamma)$ is effective so $\Gamma^2 \geqslant 0$. Let $\mathrm{Bs}(\mathfrak{F})$ be the base locus of $\mathfrak{F}$. If $\Gamma^2 > 0$, take $p \in \mathrm{Bs}(\mathfrak{F})$. We can clearly split off the fibre through p with one condition, thus proving the result.

If $\Gamma^2 = 0$, $\mathfrak{F}$ is a base-point-free pencil. So F contains two disjoint sections and this implies that $\mathcal{F}$ is decomposable, a contradiction.

Finally, if $\mathfrak{F}$ is non-linear, then $\mathrm{Bs}(\mathfrak{F}) \neq \emptyset$ and we can argue as in the linear case with $\Gamma^2 > 0$. □

3. BRILL-NOETHER LOCI

As usual, $U_C(d)$ denotes the moduli space of (semi)stable, degree d, rank-two vector bundles on C. The subset $U^s_C(d) \subseteq U_C(d)$ parametrizing (isomorphism classes of) stable bundles, is an open subset. The points in $U_C{}^{ss}(d) := U_C(d) \setminus U^s_C(d)$ correspond to (S-equivalence classes of) *strictly semistable* bundles (cf. e.g. [41, 44]).

Proposition 3.1. *Let C be a smooth curve of genus $g \geqslant 1$ and let d be an integer.*
(i) If $d \geqslant 4g - 3$, then for any $[\mathcal{F}] \in U_C(d)$, one has $i(\mathcal{F}) = 0$.
(ii) If $g \geqslant 2$ and $d \geqslant 2g - 2$, for $[\mathcal{F}] \in U_C(d)$ general, one has $i(\mathcal{F}) = 0$.

Proof. For (i), see [36, Lemma 5.2]; for (ii) see [27, p. 100] or [2, Rem. 3]. □

Thus, from Proposition 3.1, Serre duality and invariance of stability under operations like tensoring with a line bundle or passing to the dual bundle, for $g \geqslant 2$ it makes sense to consider the proper sub-loci of $U_C(d)$ parametrizing classes $[\mathcal{F}]$ such that $i(\mathcal{F}) > 0$ for

$$2g - 2 \leqslant d \leqslant 4g - 4. \tag{3.1}$$

Definition 3.2. Given non-negative integers d, g and i, we set

$$k_i = d - 2g + 2 + i. \tag{3.2}$$

Given a curve C of genus g, we define

$$B_C^{k_i}(d) := \left\{ [\mathcal{F}] \in U_C(d) \,|\, h^0(\mathcal{F}) \geqslant k_i \right\} = \left\{ [\mathcal{F}] \in U_C(d) \,|\, h^1(\mathcal{F}) \geqslant i \right\}$$

which we call the k_i^{th}*–Brill-Noether locus.*

Remark 3.3. The Brill-Noether loci $B_C^{k_i}(d)$ have a natural structure of closed subschemes of $U_C(d)$:
(a) When d is odd, $U_C(d) = U_C^s(d)$, then $U_C(d)$ is a fine moduli space and the existence of a universal bundle on $C \times U_C(d)$ allows one to construct $B_C^{k_i}(d)$ as the degeneracy locus of a morphism between suitable vector bundles on $U_C(d)$ (see, e.g. [20, 30]). Accordingly, the *expected dimension* of $B_C^{k_i}(d)$ is $\max\{-1,\ \rho_d^{k_i}\}$, where

$$\rho_d^{k_i} := 4g - 3 - ik_i \tag{3.3}$$

is the *Brill-Noether number.* If $\emptyset \neq B_C^{k_i}(d) \neq U_C(d)$, then $B_C^{k_i+1}(d) \subseteq \mathrm{Sing}(B_C^{k_i}(d))$. Since any $[\mathcal{F}] \in B_C^{k_i}(d)$ is stable, it is a smooth point of $U_C(d)$ and $T_{[\mathcal{F}]}(U_C(d))$ can be identified with $H^0(\omega_C \otimes \mathcal{F} \otimes \mathcal{F}^\vee)^\vee$. If $[\mathcal{F}] \in B_C^{k_i}(d) \setminus B_C^{k_i+1}(d)$, the tangent space to $B_C^{k_i}(d)$ at $[\mathcal{F}]$ is the annihilator of the image of the cup–product, *Petri map* of $\mathcal{F}$ (see, e.g. [51])

$$P_{\mathcal{F}} : H^0(C, \mathcal{F}) \otimes H^0(C, \omega_C \otimes \mathcal{F}^\vee) \longrightarrow H^0(C, \omega_C \otimes \mathcal{F} \otimes \mathcal{F}^\vee). \tag{3.4}$$

If $[\mathcal{F}] \in B_C^{k_i}(d) \setminus B_C^{k_i+1}(d)$, then

$$\rho_d^{k_i} = h^1(C, \mathcal{F} \otimes \mathcal{F}^\vee) - h^0(C, \mathcal{F}) h^1(C, \mathcal{F})$$

and $B_C^{k_i}(d)$ is non–singular, of the expected dimension at $[\mathcal{F}]$ if and only if $P_{\mathcal{F}}$ is injective.
(b) When d is even, $U_C(d)$ is not a fine moduli space (because $U_C^{ss}(d) \neq \emptyset$). There is a suitable open, non-empty subscheme $\mathcal{Q}^{ss} \subset \mathcal{Q}$ of a certain Quot scheme $\mathcal{Q}$ defining $U_C(d)$ via the GIT-quotient sequence

$$0 \to PGL(q) \to \mathcal{Q}^{ss} \xrightarrow{\pi} U_C(d) \to 0$$

(cf. e.g. [46] for details); one can define $B_C^{k_i}(d)$ as the image via π of the degeneracy locus of a morphism between suitable vector bundles on $\mathcal{Q}^{ss}$. The fibres of π over strictly semistable bundle classes are not isomorphic to $PGL(q)$. It may happen for a component $\mathcal{B}$ of a Brill–Noether locus to be totally contained in $U_C^{ss}(d)$; in this case the lower bound $\rho_d^{k_i}$ for the expected dimension of $\mathcal{B}$ is no longer valid (cf. Corollary 6.9 below and [8, Remark 7.4]). The lower bound $\rho_d^{k_i}$ is still valid if $\mathcal{B} \cap U_C^s(d) \neq \emptyset$.

Definition 3.4. Assume $B_C^{k_i}(d) \neq \emptyset$. A component $\mathcal{B} \subseteq B_C^{k_i}(d)$ such that $\mathcal{B} \cap U_C^s(d) \neq \emptyset$ will be called *regular*, if $\dim(\mathcal{B}) = \rho_d^{k_i}$, *superabundant*, if $\dim(\mathcal{B}) > \rho_d^{k_i}$.

4. (SEMI)STABLE VECTOR BUNDLES AND EXTENSIONS

In this section we discuss how to easily produce special, (semi)stable vector bundles $\mathcal{F}$ as extensions of line bundles L and N as in (2.2). This is the same as considering vector bundles $\mathcal{F}$, with a sub-line bundle N s.t. $\mathcal{F} \otimes N^\vee$ has a nowhere vanishing section.

If $g = 2$, in the range (3.1) one has bundles $\mathcal{F}$ with slope $1 \leqslant \mu(\mathcal{F}) \leqslant 2$ on a hyperelliptic curve, which have been studied in [8, 9, 29, 31]. Thus, we will assume C non-hyperelliptic of genus $g \geqslant 3$, with d as in (3.1).

4.1. **Extensions and a result of Lange-Narashiman.** Let $\delta \leqslant d$ be a positive integer. Consider $L \in \mathrm{Pic}^{\delta}(C)$ and $N \in \mathrm{Pic}^{d-\delta}(C)$; $\mathrm{Ext}^1(L, N)$ parametrizes (strong) isomorphism classes of extensions (cf. [16, p. 31]). Any $u \in \mathrm{Ext}^1(L, N)$ gives rise to a degree d, rank-two vector bundle $\mathcal{F} = \mathcal{F}_u$ as in (2.2). In order to get $\mathcal{F}_u$ (semi)stable, a necessary condition is (cf. Remark 2.5)

$$2\delta - d \geqslant 0. \tag{4.1}$$

Therefore, by Riemann-Roch theorem, we have

$$m := \dim(\mathrm{Ext}^1(L, N)) = \begin{cases} 2\delta - d + g - 1 & \text{if } L \ncong N \\ g & \text{if } L \cong N. \end{cases} \tag{4.2}$$

Lemma 4.1. *Let $\mathcal{F}$ be a (semi)stable, special, rank-two vector bundle on C of degree $d \geqslant 2g - 2$. Then $\mathcal{F} = \mathcal{F}_u$, for a special, effective line bundle L on C, $N = \det(\mathcal{F}) \otimes L^\vee$ as in (2.2) and $u \in \mathrm{Ext}^1(L, N)$.*

Proof. By Serre duality, $i(\mathcal{F}) > 0$ gives a non-zero morphism $\mathcal{F} \xrightarrow{\sigma^\vee} \omega_C$. The line bundle $L := \mathrm{Im}(\sigma^\vee) \subseteq \omega_C$ is special. Set $\delta = \deg(L)$. Since $\mathcal{F}$ is (semi)stable, then (4.1) holds hence $\delta \geqslant \frac{d}{2} \geqslant g - 1$, therefore $\chi(L) \geqslant 0$, so L is effective. ☐

Remark 4.2. In the setting of Lemma 4.1, consider the scroll $F = \mathbb{P}(\mathcal{F})$ and let $\Gamma \subset F$ be the section corresponding to L, with $L \in \mathrm{Pic}^{\delta}(C)$ a special, effective quotient of minimal degree of $\mathcal{F}$. Suppose $\mathcal{F}$ indecomposable. From Proposition 2.12, one has

$$\Gamma \subset F \;\text{ is li with }\; a_F(\delta) \leqslant 1, \tag{4.3}$$

where $a_F(\delta)$ as in (2.9). Then $\mathcal{F}$ is rsp via L if $a_F(\delta) = 0$, and even rp if Γ is ai.

Fix L special, effective of degree δ and N of degree $d - \delta$, where d satisfies (3.1) and (4.1) (so $\deg(L) \geqslant \deg(N) \geqslant 0$). We fix once and for all the following notation:

$$\begin{array}{ll} j := h^1(L) > 0, & \ell := h^0(L) = \delta - g + 1 + j > 0, \\ r := h^1(N) \geqslant 0, & n := h^0(N) = d - \delta - g + 1 + r \geqslant 0. \end{array} \tag{4.4}$$

Any $u \in \mathrm{Ext}^1(L, N)$ gives rise to the following diagram

$$\begin{array}{ccccccc} (u): & 0 \to & N & \to & \mathcal{F}_u & \to & L & \to 0 \\ \deg & & d-\delta & & d & & \delta & \\ h^0 & & n & & & & \ell & \\ h^1 & & r & & & & j & \end{array} \tag{4.5}$$

Let

$$\partial_u : H^0(L) \to H^1(N)$$

be the *coboundary map* (simply denoted by ∂ if there is no danger of confusion) and let $\mathrm{cork}(\partial_u) := \dim(\mathrm{Coker}(\partial_u))$. Then

$$i(\mathcal{F}_u) = j + \mathrm{cork}(\partial_u).$$

As for (semi)stability of $\mathcal{F}_u$, information can be obtained by using [24, Prop. 1.1] (see Proposition 4.4 below) and the projection technique from [14] (see Theorem 5.4 below).

For the reader's convenience, we recall [24, Prop. 1.1] (cf. also [6, §'s 3, 4], [7, § 3]). Take $u \in \mathrm{Ext}^1(L, N)$. Tensor by $N^\vee$ and consider $\mathcal{E}_e := \mathcal{F}_u \otimes N^\vee$, which is an extension

$$(e): \quad 0 \to \mathcal{O}_C \to \mathcal{E}_e \to L \otimes N^\vee \to 0,$$

where $e \in \mathrm{Ext}^1(L \otimes N^\vee, \mathcal{O}_C)$. Then $\deg(\mathcal{E}_e) = 2\delta - d$. From (2.7), one has $s(\mathcal{F}_u) = s(\mathcal{E}_e)$ and, by Serre duality, u and e define the same point in

$$\mathbb{P} := \mathbb{P}(H^0(K_C + L - N)^\vee). \tag{4.6}$$

Remark 4.3. If $\deg(L - N) = 2\delta - d \geqslant 2$, then $\dim(\mathbb{P}) \geqslant g \geqslant 3$ and the map $\varphi := \varphi_{|K_C + L - N|} : C \to \mathbb{P}$ is a morphism. Set $X := \varphi(C) \subset \mathbb{P}$. For any positive integer h denote by $\mathrm{Sec}_h(X)$ the h^{th}*-secant variety of* X, defined as the closure of the union of all linear subspaces $\langle \varphi(D) \rangle \subset \mathbb{P}$, for all effective general divisors of degree h. One has

$$\dim(\mathrm{Sec}_h(X)) = \min\{\dim(\mathbb{P}), 2h - 1\}. \tag{4.7}$$

Proposition 4.4. *(see* [24, Prop. 1.1]*) Let* $2\delta - d \geqslant 2$. *For any integer*

$$\sigma \equiv 2\delta - d \pmod 2 \;\; \textit{and} \;\; 4 + d - 2\delta \leqslant \sigma \leqslant 2\delta - d,$$

one has

$$s(\mathcal{E}_e) \geqslant \sigma \Leftrightarrow e \notin \mathrm{Sec}_{\frac{1}{2}(2\delta - d + \sigma - 2)}(X).$$

To end this section, we remark that later on we will need the following technical result.

Lemma 4.5. *Let* L *and* N *be as in* (4.5) *and such that* $h^0(N - L) = 0$. *Take* $u, u' \in \mathrm{Ext}^1(L, N)$ *such that:*
(i) the corresponding rank-two vector bundles $\mathcal{F}_u$ *and* $\mathcal{F}_{u'}$ *are stable, and*
(ii) there exists an isomorphism φ

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\iota_1"] & \mathcal{F}_{u'} \arrow[r] \arrow[d, "\varphi"] & L \arrow[r] & 0 \\ 0 \arrow[r] & N \arrow[r, "\iota_2"] & \mathcal{F}_u \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

such that $\varphi \circ \iota_1 = \lambda \iota_2$, *for some* $\lambda \in \mathbb{C}^*$.

Then $\mathcal{F}_u = \mathcal{F}_{u'}$, *i.e.* u, u' *are proportional vectors in* $\mathrm{Ext}^1(L, N)$.

Proof. The proof is similar to that in [28, Lemma 1] so it can be left to the reader. □

5. STABLE BUNDLES AS EXTENSIONS OF LINE BUNDLES

In this section we start with line bundles L and N on C as in § 4, and consider rank–two vector bundles $\mathcal{F}$ on C arising as extensions as in (2.2). We give conditions under which $\mathcal{F}$ is stable, with a given speciality, and L is a quotient with suitable minimality properties.

5.1. **The case N non-special.** Here we focus on the case N non-special. Notation as in (4.4), (4.5), with therefore $r = 0$ (by the non-speciality assumption).

Theorem 5.1. *Let $j \geqslant 1$ and $g \geqslant 3$ be integers. Let C be of genus g with general moduli. Let δ and d be integers such that*

$$\delta \leqslant g - 1 + \frac{g}{j} - j, \tag{5.1}$$

$$\delta + g - 1 \leqslant d \leqslant 2\delta - 2. \tag{5.2}$$

Let $N \in \mathrm{Pic}^{d-\delta}(C)$ be general and $L \in W_{\delta}^{\delta-g+j}(C)$ be a smooth point (i.e. L of speciality j and so $h^0(L) = \ell$ as in (4.4)). Then, for $u \in \mathrm{Ext}^1(L, N)$ general, the corresponding rank-two vector bundles $\mathcal{F}_u$ is indecomposable with:
(i) $s(\mathcal{F}_u) = 2\delta - d$. In particular $2 \leqslant s(\mathcal{F}_u) \leqslant \frac{g}{j} - j$, hence $\mathcal{F}_u$ is also stable;
(ii) $i(\mathcal{F}_u) = j$;
(iii) L is a quotient of minimal degree of $\mathcal{F}_u$.

Remark 5.2. (1) Inequality in (5.1) is equivalent to $\rho(g, \ell - 1, \delta) \geqslant 0$, where $\rho(g, \ell - 1, \delta)$ is the Brill-Noether number as in §1 for line bundles $L \in \mathrm{Pic}^{\delta}(C)$ of speciality j; this is a necessary and sufficient condition for C of genus g with general moduli to admit such a line bundle L (cf. [1]). For what concerns (5.2), the upper-bound on d reads $2\delta - d \geqslant 2$, which is required to apply Proposition 4.4, whereas the lower-bound is equivalent to $\deg(N) = d - \delta \geqslant g - 1$, hence the general $N \in \mathrm{Pic}^{d-\delta}(C)$, for C general, is non-special.
(2) Notice that (5.2) implies $\delta \geqslant g + 1$. Thus, together with (5.1), one has $g \geqslant j^2 + 2j$ i.e. $1 \leqslant j \leqslant \sqrt{g+1} - 1$.

Proof of Theorem 5.1. By Remark 5.2-(1) N is non-special, so (ii) holds. Moreover, always by Remark 5.2-(1), we can use Proposition 4.4 with $\sigma := 2\delta - d$. One has $\dim(\mathbb{P}) = 2\delta - d + g - 2$. From (4.7), we have

$$\dim\left(\mathrm{Sec}_{\frac{1}{2}(2(2\delta-d)-2)}(X)\right) = \min\{\dim(\mathbb{P}), 2(2\delta - d) - 3\} = 2(2\delta - d) - 3,$$

since (5.1) and (5.2) yield $2\delta - d \leqslant \frac{g}{j} - j$ which implies $2(2\delta - d) - 3 < 2\delta - d + g - 2$. In particular, $\dim\left(\mathrm{Sec}_{\frac{1}{2}(2(2\delta-d)-2)}(X)\right) < \dim(\mathbb{P})$. From Proposition 4.4, $u \in \mathrm{Ext}^1(L, N)$ general is such that $s(\mathcal{F}_u) \geqslant 2\delta - d$. If Γ is the section corresponding to $\mathcal{F}_u \twoheadrightarrow L$, one has $\Gamma^2 = 2\delta - d$ as in (2.3) so $s(\mathcal{F}_u) = 2\delta - d$, hence (i) and (iii) are also proved. ☐

Corollary 5.3. *Assumptions as in Theorem 5.1. Let Γ be the section of $F_u = \mathbb{P}(\mathcal{F}_u)$ corresponding to $\mathcal{F}_u \twoheadrightarrow L$. Then Γ is of minimal degree. In particular, Γ is li and $0 \leqslant \dim(\mathrm{Div}_{F_u}^{1,\delta}) \leqslant 1$.*

Proof. The fact that Γ is of minimal degree follows from (i) of Theorem 5.1. The rest is a consequence of minimality and Proposition 2.12. ☐

Theorem 5.4. *Let $j \geqslant 1$ and $g \geqslant 3$ be integers. Let C be of genus g with general moduli. Let δ and d be integers such that (5.1) holds and moreover*

$$\delta + g + 3 \leqslant d \leqslant 2\delta. \tag{5.3}$$

Let $N \in \mathrm{Pic}^{d-\delta}(C)$ and $L \in W_{\delta}^{\delta-g+j}(C)$ be general points. Then, for any $u \in \mathrm{Ext}^1(L, N)$, the corresponding rank-two vector bundle $\mathcal{F}_u$ is very-ample, with $i(\mathcal{F}_u) = j$. Moreover, for $u \in \mathrm{Ext}^1(L, N)$ general
(i) L is the quotient of minimal degree of $\mathcal{F}_u$, thus

$$0 \leqslant s(\mathcal{F}_u) = 2\delta - d \leqslant \frac{g - 4j - j^2}{j},$$

so $\mathcal{F}_u$ is stable when $2\delta - d > 0$, strictly semistable when $d = 2\delta$;
(ii) if Γ is the section of $F_u = \mathbb{P}(\mathcal{F}_u)$ corresponding to $\mathcal{F}_u \twoheadrightarrow L$, then $\mathrm{Div}_{F_u}^{1,\delta} = \{\Gamma\}$ and $\mathcal{F}_u$ is rp via L.

Proof. The proof is as in [14, Theorem 2.1], and it works also in the case $d = 2\delta$, not considered there. ☐

Remark 5.5. (1) The lower-bound in (5.3) reads $\deg(N) = d - \delta \geqslant g + 3$, hence $N \in \mathrm{Pic}^{d-\delta}(C)$ general is non-special and $\delta \geqslant g + 3$.
(2) From (5.1) and $\delta \geqslant g + 3$, one has $g \geqslant j^2 + 4j$ i.e. $1 \leqslant j \leqslant \sqrt{g+4} - 2$.
(3) The bounds on d in (5.2) and (5.3) are in general slightly worse than those in Theorem 0.1 (cf. [14, Remark 1.7]). For j close to the upper-bound (see Remark 5.2-(2), respectively (2) above), the difference is of the order of $\sqrt{g}$. However our approach gives the additional information of the description of vector bundles in irreducible components of $B_C^{k_j}(d)$ (see also § 7) simply as line bundle extensions, with no use of either limit linear series or degeneration techniques.

5.2. **The case N special.** In this section $N \in \mathrm{Pic}^{d-\delta}(C)$ is assumed to be special. Hence, in (4.4), we have $\ell, j, r > 0$ whereas $n \geqslant 0$ (according to the fact that N is either effective or not). For any integer $t > 0$, consider

$$\mathcal{W}_t := \left\{u \in \mathrm{Ext}^1(L,N) \mid \ \mathrm{cork}(\partial_u) \geqslant t\right\} \subseteq \mathrm{Ext}^1(L,N), \tag{5.4}$$

which has a natural structure of determinantal scheme; as such, $\mathcal{W}_t$ has *expected codimension*

$$c(l,r,t) := t(\ell - r + t) \tag{5.5}$$

As in (4.2), put $m = \dim(\mathrm{Ext}^1(L,N))$. Thus, if $m > 0$ and $\mathcal{W}_t \neq \emptyset$, then any irreducible component $\Lambda_t \subseteq \mathcal{W}_t$ is such that

$$\dim(\Lambda_t) \geqslant \min\{m, m - c(\ell,r,t)\}, \tag{5.6}$$

where the right-hand-side is the *expected dimension* of $\mathcal{W}_t$. These loci have been considered also in [7, § 2, 3], [33, § 6, 8] for low genus and for vector bundles with canonical determinant.

Remark 5.6. One has $\dim(\mathrm{Ker}(\partial_u)) = 1 + \dim(\langle\Gamma\rangle)$, where Γ is the section corresponding to the quotient $\mathcal{F} \twoheadrightarrow L$. Note that $\dim(\langle\Gamma\rangle) = -1$ if and only if $\dim(\mathrm{Ker}(\partial_u)) = 0$, i.e. $H^0(\mathcal{F}) = H^0(N)$. This happens if and only if Γ is a fixed component of $|\mathcal{O}_F(1)|$, i.e. if and only if the image of the map $\Phi_{|\mathcal{O}_F(1)|}$ has dimension smaller than 2. If $d \geqslant 2g-2$ and this happens, then one must have $n \geqslant i \geqslant j \geqslant 1$, where $i = i(\mathcal{F})$.

Remark 5.7. The coboundary map ∂_u can be interpreted in terms of multiplication maps among global sections of suitable line bundles on C. Indeed, consider $r \geqslant t$ and $\ell \geqslant \max\{1, r-t\}$. Denote by

$$\cup : H^0(L) \otimes H^1(N-L) \longrightarrow H^1(N),$$

the cup-product: for any $u \in H^1(N-L) \cong \mathrm{Ext}^1(L,N)$, one has $\partial_u(-) = - \cup u$. By Serre duality, the consideration of $\cup$ is equivalent to the one of the multiplication map

$$\mu := \mu_{L, K_C - N} : H^0(L) \otimes H^0(K_C - N) \to H^0(K_C + L - N) \tag{5.7}$$

(when $N = L$, μ coincides with $\mu_0(L)$ as in (1.1)). For any subspace $W \subseteq H^0(K_C - N)$, we set

$$\mu_W := \mu|_W : H^0(L) \otimes W \to H^0(K_C + L - N). \tag{5.8}$$

Imposing $\mathrm{cork}(\partial_u) \geqslant t$ is equivalent to

$$V_t := \mathrm{Im}(\partial_u)^{\perp} \subset H^0(K_C - N)$$

having dimension at least t. Therefore

$$\mathcal{W}_t = \left\{u \in H^0(K_C + L - N)^{\vee} \mid \exists\, V_t \subseteq H^0(K_C - N), \text{ s.t. } \dim(V_t) \geqslant t \text{ and } \mathrm{Im}(\mu_{V_t}) \subseteq \{u = 0\}\right\},$$

where $\{u = 0\} \subset H^0(K + L - N)$ is the hyperplane defined by $u \in H^0(K_C + L - N)^{\vee}$.

Theorem 5.8. *Let C be a smooth curve of genus $g \geqslant 3$. Let*

$$r \geqslant 1, \ \ell \geqslant \max\{1, r-1\}, \ m \geqslant \ell + 1$$

be integers as in (4.2), (4.4). *Then (cf.* (5.4),(5.5)*):*
(i) $c(\ell, r, 1) = \ell - r + 1 \geqslant 0$*;*
(ii) $\mathcal{W}_1$ *is irreducible of the expected dimension* $\dim(\mathcal{W}_1) = m - c(\ell, r, 1) \geqslant r$*. In particular* $\mathcal{W}_1 = \mathrm{Ext}^1(L,N)$ *if and only if* $\ell = r - 1$.

Proof. Part (i) and $m - c(\ell, r, 1) \geqslant r$ are obvious. Let us prove (ii). Since $\ell, r \geqslant 1$, both L and $K_C - N$ are effective. One has an inclusion

$$0 \to L \to K_C + L - N$$

obtained by tensoring by L the injection $\mathcal{O}_C \hookrightarrow K_C - N$ given by a given non–zero section of $K_C - N$. Thus, for any $V_1 \in \mathbb{P}(H^0(K_C - N))$, μ_{V_1} as in (5.8) is injective. Since $m \geqslant \ell + 1$, one has $\dim(\mathrm{Im}(\mu_{V_1})) = \ell \leqslant m - 1$, i.e. $\mathrm{Im}(\mu_{V_1})$ is contained in some hyperplane of $H^0(K_C + L - N)$. Let

$$\Sigma := \left\{\sigma := H^0(L) \otimes V_1^{\sigma} \subseteq H^0(L) \otimes H^0(K_C - N) \ \mid \ V_1^{\sigma} \in \mathbb{P}(H^0(K_C - N))\right\}.$$

Thus $\Sigma \cong \mathbb{P}(H^0(K_C - N))$, so it is irreducible of dimension $r - 1$. Since $\mathbb{P}(H^0(K_C + L - N)^{\vee}) = \mathbb{P}$ as in (4.6), we can define the incidence variety

$$\mathcal{J} := \left\{(\sigma, \pi) \in \Sigma \times \mathbb{P} \mid \mu_{V_1^{\sigma}}(\sigma) \subseteq \pi\right\} \subset \Sigma \times \mathbb{P}.$$

Let

$$\Sigma \xleftarrow{pr_1} \mathcal{J} \xrightarrow{pr_2} \mathbb{P}$$

be the two projections. As we saw, pr_1 is surjective. In particular $\mathcal{J} \neq \emptyset$ and, for any $\sigma \in \Sigma$,

$$pr_1^{-1}(\sigma) = \left\{\pi \in \mathbb{P} \mid \mu_{V_1^{\sigma}}(\sigma) \subseteq \pi\right\} \cong |\mathcal{I}_{\widehat{\sigma}|\mathbb{P}^{\vee}}(1)|,$$

where $\widehat{\sigma} := \mathbb{P}(\mu_{V_1^\sigma}(\sigma))$. Since $\dim(\widehat{\sigma}) = \ell - 1$, then $\dim(pr_1^{-1}(\sigma)) = m - 1 - \ell \geqslant 0$.

This shows that $\mathcal{J}$ is irreducible and $\dim(\mathcal{J}) = m - 1 - c(\ell, r, 1) \leqslant m - 1$. Then, $\widehat{\mathcal{W}}_1 := \mathbb{P}(\mathcal{W}_1) = \overline{pr_2(\mathcal{J})}$. Recalling (5.6), $\mathcal{W}_1 \neq \emptyset$ is irreducible, of the expected dimension $m - c(\ell, r, 1)$. ☐

Corollary 5.9. *Assumptions as in Theorem 5.8. If* $\ell \geqslant r$, *then* $\mathcal{W}_1 \subsetneq \mathrm{Ext}^1(L, N)$ *and*
(i) for $u \in \mathrm{Ext}^1(L, N)$ *general,* ∂_u *is surjective, in which case the corresponding bundle* $\mathcal{F}_u$ *is special with* $i(\mathcal{F}_u) = h^1(L) = j$*;*
(ii) for $v \in \mathcal{W}_1$ *general,* $\mathrm{cork}(\partial_v) = 1$*, hence the corresponding bundle* $\mathcal{F}_v$ *is special with* $i(\mathcal{F}_v) = j + 1$.

5.2.1. **Surjective coboundary**. Take $0 \neq u \in \mathrm{Ext}^1(L, N)$ and assume ∂_u is surjective (from Corollary 5.9, this happens e.g. when $\ell \geqslant r$, $m \geqslant \ell + 1$ and u general).

Theorem 5.10. *Let* $j \geqslant 1$ *and* $g \geqslant 3$ *be integers. Let* C *be of genus* g *with general moduli. Let* δ *and* d *be integers such that* (5.1) *holds and moreover*

$$2g - 2 \leqslant d \leqslant 2\delta - g. \tag{5.9}$$

Let $L \in W_\delta^{\delta - g + j}(C)$ *be a smooth point and* $N \in \mathrm{Pic}^{d-\delta}(C)$ *be any point. Then, for* $u \in \mathrm{Ext}^1(L, N)$ *general, the corresponding bundle* $\mathcal{F}_u$ *is indecomposable with*
(i) speciality $i(\mathcal{F}_u) = j = h^1(L)$.
(ii) $s(\mathcal{F}_u) = g - \epsilon$, $\epsilon \in \{0, 1\}$ *such that* $\epsilon \equiv d + g \pmod 2$. *In particular,* $\mathcal{F}_u$ *is stable.*
(iii) The minimal degree of a quotient of $\mathcal{F}_u$ *is* $\frac{d+g-\epsilon}{2}$ *and* $1 - \epsilon \leqslant \dim\left(\mathrm{Div}_{F_u}^{1, \frac{d+g-\epsilon}{2}}\right) \leqslant 1$*;*
(iv) L *is a minimal degree quotient of* $\mathcal{F}_u$ *if and only if* $\epsilon = 0$ *and* $d = 2\delta - g$.

Remark 5.11. (1) From (5.9) we get $\delta \geqslant \frac{3}{2}g - 1$ hence from (5.1), $j \leqslant \frac{\sqrt{g^2 + 16g} - g}{4}$.
(2) Since L is special, then $\delta \leqslant 2g - 2$. Therefore, the upper-bound in (5.9) implies $d - \delta \leqslant \delta - g \leqslant g - 2$, i.e., any $N \in \mathrm{Pic}^{d-\delta}(C)$ is special too.
(3) The inequalities (5.1), (5.9) imply $\ell \geqslant r$, $m \geqslant \ell + 1$ as in the assumptions of Corollary 5.9. Indeed:
- $\ell \geqslant r$ reads

$$\delta \geqslant g - 1 + r - j. \tag{5.10}$$

Since $r = \delta - d + g - 1 + n$, then (5.10) is equivalent to $d \geqslant 2g - 2 - j + n$. Thus (5.10) holds by (5.9), if $n \leqslant 1$. If $n \geqslant 2$, C with general moduli implies $r \leqslant \frac{g}{n} \leqslant \frac{g}{2}$ so $g - 1 + r - j \leqslant \frac{3}{2}g - 1 - j$ and (5.10) holds because $\delta \geqslant \frac{3}{2}g - 1$.
- We have $d - \delta \leqslant \delta - g < \delta$ by (5.9). So from (4.2) we have $m = 2\delta - d + g - 1$. Thus $m \geqslant \ell + 1$ reads $d \leqslant \delta + 2g - 3 - j$. By (5.9), to prove this it suffices to prove $2\delta - g \leqslant \delta + 2g - 3 - j$. This in turn is a consequence of (5.1).

(4) Notice that, under hypotheses of Theorem 5.10, when $\epsilon = 1$ L is not of minimal degree: from (iii), one would have $d = 2\delta - g + 1$ which is out of range in (5.9). Indeed, if $d = 2\delta - g + 1$ and e.g. $\delta = 2g - 2$, then $d = 3g - 3$, $\deg(N) = d - \delta = g - 1$, thus if N is general, it is non-special, which is a case already considered in Theorem 5.1. From (1) above, to allow minimality for L also for $\epsilon = 1$, one should replace (5.1), (5.9) in the statement of Theorem 5.10 with the more annoying conditions $\delta \leqslant \min\{g - 1 + \frac{g}{j} - j, 2g - 3\}$ and $d \leqslant 2\delta - g + \epsilon$, respectively.

Proof of Theorem 5.10. By Remark 5.11-(2), N is special. Moreover, by Remark 5.11-(3) and Corollary 5.9, for $u \in \mathrm{Ext}^1(L, N)$ general, ∂_u is surjective. Hence (i) holds.

From the upper-bound in (5.9) and $g \geqslant 3$, we can apply Proposition 4.4 with the choice $\sigma := g - \epsilon$, i.e., the maximum for which $\sigma \equiv 2\delta - d \pmod 2$, $\sigma \leqslant 2\delta - d$ and one has a strict inclusion

$$\mathrm{Sec}_{\frac{1}{2}(2\delta - d + g - \epsilon - 2)}(X) \subset \mathbb{P},$$

from (4.2) and (4.7).

If $\epsilon = 0$, (ii) follows from Propositions 2.6, 4.4. Let $\Gamma \subset F_u$ be a section of minimal degree, which we denote by m_0. Then, $\Gamma^2 = 2m_0 - d = g$ (cf. (2.3) and Remark 2.5). In particular, $m_0 = \frac{d+g}{2}$ and

$$1 = \Gamma^2 - g + 1 \leqslant \chi(\mathcal{N}_{\Gamma/F_u}) \leqslant \dim\left(\mathrm{Div}_{F_u}^{1, \frac{d+g}{2}}\right) \leqslant 1,$$

where the upper-bound holds by the minimality condition (cf. proof of Proposition 2.12). This proves (iii) in this case.

When $\epsilon = 1$, by Propositions 2.6, 4.4, one has $g - 1 \leqslant s(\mathcal{F}_u) \leqslant g$ and, by parity, the leftmost equality holds. As above, part (iii) holds also for $\epsilon = 1$.

Finally, L is a minimal degree quotient if and only if $2\delta = d + g - \epsilon$ which by (5.9) is only possible for $\epsilon = 0$, proving (iv) (cf. Remark 5.11-(4)). $\square$

5.2.2. **Non-surjective coboundary**. From Corollary 5.9, when $\ell \geqslant r$ and $m \geqslant \ell+1$, for $v \in \mathcal{W}_1 \subsetneq \mathrm{Ext}^1(L, N)$ general, one has $\mathrm{cork}(\partial_v) = 1$.

Definition 5.12. Take $\ell \geqslant r \geqslant t \geqslant 1$ integers. Assume
(1) there exists an irreducible component $\Lambda_t \subseteq \mathcal{W}_t$ with the expected dimension $\dim(\Lambda_t) = m - c(\ell, r, t)$;
(2) for $v \in \Lambda_t$ general, $\mathrm{cork}(\partial_v) = t$.
Any such Λ_t is called a *good component* of $\mathcal{W}_t$.

By Theorem 5.8, $\Lambda_1 = \mathcal{W}_1$ is good. In § 5.3 we shall give sufficient conditions for goodness when $t \geqslant 2$.

With notation as in (4.6), for any $t \geqslant 1$ and any good component Λ_t, we set

$$\widehat{\Lambda}_t := \mathbb{P}(\Lambda_t) \subset \mathbb{P} \tag{5.11}$$

(cf. notation as in the proof of Theorem 5.8 for $\widehat{\mathcal{W}}_1$).

Theorem 5.13. *Let $g \geqslant 3$, $d \geqslant 2g-2$, $j \geqslant 1$, $\ell \geqslant r \geqslant t \geqslant 1$ be integers. Take $\epsilon \in \{0,1\}$ such that*

$$d + g - c(\ell, r, t) \equiv \epsilon \pmod 2.$$

Take $\delta = \ell + g - 1 - j$ and assume

$$g \geqslant c(\ell, r, t) + \epsilon, \tag{5.12}$$

$$g + r - j - 1 \leqslant \delta \leqslant g - 1 + \frac{g}{j} - j, \tag{5.13}$$

$$2\delta - d \geqslant \max\{2, g - c(\ell, r, t) - \epsilon\}. \tag{5.14}$$

Let C be a curve of genus g with general moduli. Let $L \in W^{\ell-1}_{\delta}(C)$ be a smooth point, $N \in \mathrm{Pic}^{d-\delta}(C)$ of speciality r. Then, for any good component Λ_t and for $v \in \Lambda_t$ general, the corresponding bundle $\mathcal{F}_v$ is
(i) special with $i(\mathcal{F}_v) = j + t = h^1(L) + t$;
(ii) $s(\mathcal{F}_v) \geqslant g - c(\ell, r, t) - \epsilon \geqslant 0$; in particular, when $g - c(\ell, r, t) > 0$, $\mathcal{F}_v$ is stable, hence indecomposable.
(iii) If $2\delta = d + g - c(\ell, r, t) - \epsilon$, then L is a quotient of minimal degree of $\mathcal{F}_v$.

Remark 5.14. (1) As before, the upper-bound on δ in (5.13) is equivalent to $\rho(g, \ell-1, \delta) \geqslant 0$ whereas the lower bound to $\ell \geqslant r$.
(2) Condition $2\delta - d \geqslant 2$ in (5.14) is as in Proposition 4.4; the other condition in (5.14) will be clear by reading the proof of Theorem 5.13.
(3) Arguing as in Remark 5.11-(3), one shows that $m \geqslant \ell + 1$.

Proof of Theorem 5.13. By (5.14) we may apply Proposition 4.4 choosing $\sigma := g - c(\ell, r, t) - \epsilon$, which is non-negative by (5.12). This is the maximum integer such that $\sigma \leqslant 2\delta - d$, $\sigma \equiv 2\delta - d \pmod 2$ and such that $\dim(\widehat{\Lambda}_t) > \dim(\mathrm{Sec}_{\frac{1}{2}(2\delta - d + \sigma - 2)}(X))$, as it follows from (5.14). Then, if $v \in \widehat{\Lambda}_t$ general, the assertions hold. $\square$

Remark 5.15. When N of degree $d - \delta$ is non-effective of speciality r, by Riemann-Roch $r = \delta - d + g - 1$. Thus, by (3.2), one simply has $c(\ell, r, t) = t\,(d - 2g + 2 + j + t)$, i.e. $c(\ell, r, t) = t\, k_{j+t}$, and conditions in Theorem 5.13 can be replaced by more explicit numerical conditions on d and δ

$$\delta \leqslant g - 1 + \frac{g}{j} - j \quad \text{and} \quad d \leqslant g - 1 - t + \min\left\{\delta, g - 1 - j + \frac{g - \epsilon}{t}\right\},$$

where $\epsilon \in \{0,1\}$ such that $d + g - tk_{t+j} \equiv \epsilon \pmod 2$, and

$$2\delta - d \geqslant \max\{2, g - tk_{t+j} - \epsilon\}.$$

Remark 5.16. When otherwise N, of degree $d - \delta$, is effective and of speciality r one gets $d \geqslant \delta + g - r$. Moreover, since its Brill-Noether number $\rho(g, n-1, d-\delta)$ has to be non negative (by the generality of C), one gets $d \leqslant \delta + g - 1 + \frac{g}{r} - r$. Thus, conditions in Theorem 5.13 can be replaced by numerical conditions

$$g - 1 - j + r \leqslant \delta \leqslant g - 1 - j + \min\left\{\frac{g}{j}, \frac{g - \epsilon}{t} + r - t - 1\right\},$$

$$g + \delta - r \leqslant d \leqslant \delta + g - 1 + \frac{g}{r} - r \quad \text{and} \quad 2\delta - d \geqslant \max\{2, g - c(\ell, r, t) - \epsilon\},$$

with ϵ and $c(\ell, r, t)$ as in Theorem 5.13.

5.3. **Existence of good components.** Recalling what defined in (4.2), (4.4), (5.4), (5.5) and in Remark 5.7, one has

Theorem 5.17. *Let C be a smooth curve of genus $g \geqslant 3$. Take integers m, ℓ, r and t and assume $\ell \geqslant r \geqslant t \geqslant 2$. Take any integer η such that*

$$0 \leqslant \eta \leqslant \min\{t(r-t), \ell(t-1)\} \quad \text{and} \quad m \geqslant \ell t + 1 - \eta. \tag{5.15}$$

Suppose, in addition, that the subvariety $\Sigma_\eta \subseteq \mathbb{G}(t, H^0(K_C - N)) := \mathbb{G}$, parametrizing $V_t \in \mathbb{G}$ s.t.

$$\dim(\mathrm{Ker}(\mu_{V_t})) \geqslant \eta, \tag{5.16}$$

has pure codimension η in $\mathbb{G}$ and that, for the general point V_t in any irreducible component of Σ_η, equality holds in (5.16)*. Then:*
(i) $c(\ell, r, t) > 0$;
(ii) $\emptyset \neq \mathcal{W}_t \subset \mathcal{W}_1 \subset \mathrm{Ext}^1(L, N)$, where all the inclusions are strict;
(iii) there exists a good component Λ_t of $\mathcal{W}_t$.

Proof. By (5.15) one has $m \geqslant \ell + 1$; moreover $\ell \geqslant r$ by assumption. Thus, from Corollary 5.9, $\emptyset \neq \mathcal{W}_1 \subset \mathrm{Ext}^1(L, N)$ and the inclusion is strict. By definition $\mathcal{W}_t \subset \mathcal{W}_1$, where the inclusion is strict by Corollary 5.9. A similar argument as in the proof of Theorem 5.8 applies. ☐

Corollary 5.18. *Let C be of genus $g \geqslant 3$ with general moduli. Let $j \geqslant 1$, $\ell \geqslant r \geqslant t \geqslant 2$ and $m \geqslant \ell t + 1$ be integers. Let $L \in W^{\delta-g+j}_{\delta}(C)$ be a smooth point.*

If $j \geqslant t$, $N \in \mathrm{Pic}^{d-\delta}(C)$ is general and $\ell \leqslant 2\delta - d$, then for $V_t \in \mathbb{G}(t, H^0(K_C - N))$ general, μ_{V_t} is injective. In particular, there exists a good component $\emptyset \neq \Lambda_t \subseteq \mathcal{W}_t$.

Proof. Set $h := 2\delta - d$ and let $N_0 := L - D \in \mathrm{Pic}^{d-\delta}(C)$, with $D = \sum_{i=1}^{h} p_i \in C^{(h)}$ general. Since $0 < \ell \leqslant h$, we have $h^0(N_0) = 0$. Thus, $N \in \mathrm{Pic}^{d-\delta}(C)$ general is also non-effective, so $h^1(N) = h^1(N_0)$.

Let μ be as in (5.7). To prove injectivity of μ_{V_t} as in (5.8) for N and V_t general, it suffices to prove a similar condition for

$$\mu^0 : H^0(L) \otimes H^0\left(K_C - L + D\right) \to H^0\left(K_C + D\right).$$

Consider

$$W := H^0(K_C - L) \subset H^0\left(K_C - L + D\right).$$

One has $\dim(W) = j$. We have the diagram

$$\begin{tikzcd}
H^0(L) \otimes W \cong H^0(L) \otimes H^0(K_C - L) \arrow[rr, "\mu^0_W"] \arrow[dr, "\mu_0(L)"] & & H^0\left(K_C + D\right) \\
& H^0(K_C) \arrow[ur, "\iota"] &
\end{tikzcd}$$

where $\mu^0_W = \mu^0|_{H^0(L) \otimes W}$, $\mu_0(L)$ is as in (1.1) and ι is the obvious inclusion.

By Gieseker–Petri theorem $\mu_0(L)$ is injective. By composition with ι, μ^0_W is also injective. Since by assumptions $t \leqslant j$, then for any $\widetilde{V}_t \in \mathbb{G}(t, W)$, $\mu^0_{\widetilde{V}_t}$ is also injective. By semicontinuity, for $N \in \mathrm{Pic}^{d-\delta}(C)$ and $V_t \in \mathbb{G}(t, H^0(K_C - N))$ general, μ_{V_t} is injective. Then, one can conclude by using Theorem 5.17. ☐

6. Parameter spaces

Let C be a projective curve of genus g with general moduli. Given a sequence as in (4.5), for brevity we set

$$\rho(L) := \rho(g, \ell - 1, \delta) \quad \text{and} \quad \rho(N) := \rho(g, n - 1, d - \delta),$$

$$W_L := \begin{cases} W^{\ell-1}_{\delta}(C) & \text{if } \rho(L) > 0 \\ \{L\} & \text{if } \rho(L) = 0 \end{cases} \quad \text{and} \quad W_N := \begin{cases} W^{n-1}_{d-\delta}(C) & \text{if } \rho(N) > 0 \\ \{N\} & \text{if } \rho(N) = 0 \end{cases}.$$

Both W_L and W_N are irreducible, generically smooth, of dimensions $\rho(L)$ and $\rho(N)$ (cf. [1, p. 214]). Let

$$\mathcal{N} \to C \times \mathrm{Pic}^{d-\delta}(C) \ \text{ and } \ \mathcal{L} \to C \times \mathrm{Pic}^{\delta}(C)$$

be Poincaré line-bundles. With an abuse of notation, we will denote by $\mathcal{L}$ (resp., by $\mathcal{N}$) also the restriction of Poincaré line-bundle to the Brill-Noether locus. Set

$$\mathcal{Y} := \mathrm{Pic}^{d-\delta}(C) \times W_L \quad \text{and} \quad \mathcal{Z} := W_N \times W_L \subset \mathcal{Y}.$$

They are both irreducible, of dimensions

$$\dim(\mathcal{Y}) = g + \rho(L) \quad \text{and} \quad \dim(\mathcal{Z}) = \rho(N) + \rho(L). \tag{6.1}$$

Consider the natural projections

$$\begin{tikzcd}
C \times \mathrm{Pic}^{d-\delta}(C) & C \times \mathcal{Y} \arrow[l, "pr_{1,2}"'] \arrow[r, "pr_{2,3}"] \arrow[d, "pr_{1,3}"] & \mathcal{Y} \\
 & C \times W_L & .
\end{tikzcd}$$

As in [1, p. 164-179]), we define

$$\mathfrak{E}_\delta := R^1(pr_{2,3})_* \left(pr_{1,2}^*(\mathcal{N}) \otimes pr_{1,3}^*(\mathcal{L}^\vee)\right),$$

depending on the choices of d and δ. By (4.2), when $2\delta-d \geqslant 1$, $\mathfrak{E}_\delta$ is a vector bundle of rank $m = 2\delta-d+g-1$ on $\mathcal{Y}$ whereas, when $d = 2\delta$, $\mathfrak{E}_\delta$ is a vector bundle of rank $g-1$ on $\mathcal{Y} \setminus \Delta_{\mathcal{Y}}$, where $\Delta_{\mathcal{Y}} = \{(M, M) \mid M \in W_L\} \cong W_L$. We set

$$\mathcal{U} := \begin{cases} \mathcal{Y} & \text{if } 2\delta - d \geqslant 1 \\ \mathcal{Y} \setminus \Delta_{\mathcal{Y}} & \text{if } d = 2\delta \end{cases} \qquad \text{and} \qquad \mathbb{P}(\mathfrak{E}_\delta) \xrightarrow{\gamma} \mathcal{U}, \tag{6.2}$$

where γ the projective bundle morphism: the γ-fibre of $y = (N, L) \in \mathcal{U}$ is $\mathbb{P}(\mathrm{Ext}^1(L, N)) = \mathbb{P}$, as in (4.6).

From (4.2) and (6.1), one has

$$\dim(\mathbb{P}(\mathfrak{E}_\delta)) = g + \rho(L) + m - 1 \qquad \text{and} \qquad \dim(\mathbb{P}(\mathfrak{E}_\delta)|_{\mathcal{Z}}) = \rho(N) + \rho(L) + m - 1. \tag{6.3}$$

Since (semi)stability is an open condition (cf. e.g. [44, Prop. 6-c, p. 17]), for any choice of integers g, d and δ satisfying numerical conditions as in the theorems and corollaries proved in §'s 5.1 and 5.2, there is an open, dense subset $\mathbb{P}(\mathfrak{E}_\delta)^0 \subseteq \mathbb{P}(\mathfrak{E}_\delta)$ and a morphism

$$\pi_{d,\delta} : \mathbb{P}(\mathfrak{E}_\delta)^0 \to U_C(d). \tag{6.4}$$

We set

$$\mathcal{V}_d^{\delta,j} := \mathrm{Im}(\pi_{d,\delta}) \quad \text{and} \quad \nu_d^{\delta,j} = \dim(\mathcal{V}_d^{\delta,j}). \tag{6.5}$$

6.1. **Non-special** N.

We will put ourselves in the hypotheses either of Theorem 5.1 or of Theorem 5.4. In either case, $d - \delta \geqslant g - 1$ so N can be taken general in $\mathrm{Pic}^{d-\delta}(C)$ and $\mathcal{V}_d^{\delta,j} \subseteq B_C^{k_j}(d)$, by what proved about (semi)stability.

6.1.1. **Case** $2\delta - d \geqslant 1$. In this case, by what proved in Theorems 5.1, 5.4, one has $\mathcal{V}_d^{\delta,j} \subseteq B_C^{k_j}(d) \cap U_C^s(d)$. Therefore any irreducible component of $B_C^{k_j}(d)$ intersected by $\mathcal{V}_d^{\delta,j}$ has at least dimension $\rho_d^{k_j}$ (cf. Remark 3.3 and Definition 3.4).

Proposition 6.1. *Assumptions as in Theorem 5.4, with* $2\delta - d \geqslant 1$. *Then, for any integers* j, δ, d *therein, there exists an irreducible component* $\mathcal{B} \subseteq B_C^{k_j}(d)$ *such that:*
(i) $\mathcal{V}_d^{\delta,j} \subseteq \mathcal{B}$*;*
(ii) $\mathcal{B}$ *is regular and generically smooth;*
(iii) for $[\mathcal{E}] \in \mathcal{B}$ *general,* $\mathcal{E}$ *is stable, with* $s(\mathcal{E}) \geqslant 2\delta - d$ *and* $i(\mathcal{E}) = j$*.*

Proof. Parts (i) and (iii) follow from Theorem 5.4 (note that the Segre invariant is lower-semicontinuous; cf. also [24, § 3]). Assertion (ii) follows from the fact that, for $[\mathcal{F}] \in \mathcal{V}_d^{\delta,j}$ general, the Petri map $P_{\mathcal{F}}$ is injective (cf. Remark 3.3 and [14, Lemma2.1]). □

Lemma 6.2. *In the hypotheses of Theorem 5.4, with* $2\delta - d \geqslant 1$, *the morphism* $\pi_{d,\delta}$ *is generically injective.*

Proof. Let $[\mathcal{F}] \in \mathcal{V}_d^{\delta,j}$ be general; then $\mathcal{F} = \mathcal{F}_u$, for $u \in \mathrm{Ext}^1(L, N)$ and $y = (N, L) \in \mathcal{U}$ general. Then

$$\pi_{d,\delta}^{-1}([\mathcal{F}_u]) = \left\{(N', L', u') \in \mathbb{P}(\mathfrak{E})^0 \mid \mathcal{F}_{u'} \cong \mathcal{F}_u\right\}.$$

Assume by contradiction there exists $(N', L', u') \neq (N, L, u)$ in $\pi_{d,\delta}^{-1}([\mathcal{F}_u])$. Then $N \otimes L \cong N' \otimes L'$.
(1) If $L \cong L' \in W_L$ then $N \cong N' \in \mathrm{Pic}^{d-\delta}(C)$. Thus, $u, u' \in \mathbb{P}$. Let $\varphi : \mathcal{F}_{u'} \to \mathcal{F}_u$ be the isomorphism between the two bundles. Since $\mathcal{F}_u$ is stable, then $u \neq u' \in \mathbb{P}$ (notation as in (4.6)) and we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\iota_1"] & \mathcal{F}_{u'} \arrow[r] \arrow[d, "\varphi"] & L \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota_2"] & \mathcal{F}_u \arrow[r] & L \arrow[r] & 0.
\end{tikzcd}$$

The maps $\varphi \circ \iota_1$ and ι_2 determine two non-zero sections $s_1 \neq s_2 \in H^0(\mathcal{F}_u \otimes N^\vee)$. They are linearly dependent, otherwise the section $\Gamma \subset F_u$, corresponding to $\mathcal{F}_u \twoheadrightarrow L$, would not be li (cf. (2.6) and Theorem 5.4-(ii)). So $s_1 = \lambda s_2$. But then Lemma 4.5 implies $u = u'$, a contradiction.
(2) If $L \ncong L' \in W_L$ (in particular, $\rho(L) > 0$), sections $\Gamma \neq \Gamma'$, corresponding respectively to $\mathcal{F}_u \twoheadrightarrow L$ and $\mathcal{F}_u \twoheadrightarrow L'$, would be such that $\Gamma \sim_{alg} \Gamma'$ on F_u, contradicting Theorem 5.4-(ii). □

Example 6.3. One can have loci $\mathcal{V}_d^{\delta,j}$ of the same dimension for different values of δ. For instance, let $j = 2$, $g \geqslant 18$ and $d = 2g + 9$ in Theorem 5.4. Then $g + 5 \leqslant \delta \leqslant g + 6$ are admissible values in (5.1) and for both of them one has $2\delta - d > 0$. Now $\rho(g, 7, g + 5) = g - 16 > g - 18 = \rho(g, 8, g + 6)$; by (6.5) and Lemma 6.2 one has $\nu_{2g+9}^{g+5,2} = \nu_{2g+9}^{g+6,2} = 3g - 17$.

Remark 6.4. From (3.3), (6.5) and Lemma 6.2, for varieties $\mathcal{V}_d^{\delta,j}$ as in Proposition 6.1 (i.e. defined by integers d, δ and j as in Theorems 5.1, 5.4, with $2\delta - d \geqslant 1$) one has

$$\nu_d^{\delta,j} - \rho_d^{k_j} = d(j-1) - \delta(j-2) - (g-1)(j+1). \tag{6.6}$$

(1) For $j = 1$, $\nu_d^{\delta,1} - \rho_d^{k_1} = \delta - 2g + 2 \leqslant 0$, since L special, and equality holds if and only if δ reaches the upper-bound in (5.1).
(2) Otherwise, for $j \geqslant 2$, using the upper-bound in (5.1) and the fact $d < 2\delta$, from (6.6) one gets

$$\nu_d^{\delta,j} - \rho_d^{k_j} < \delta j - gj - g + j + 1 \leqslant 1 - j^2 < 0.$$

Thus, $\mathcal{V}_d^{\delta,j}$ can never be dense in a regular component of $B_C^{k_j}(d)$ unless $j = 1$ and $\delta = 2g - 2$.

Corollary 6.5. *Let C be of genus $g \geqslant 5$ with general moduli. For any integer d s.t. $3g + 1 \leqslant d \leqslant 4g - 5$, the variety $\mathcal{V}_d^{2g-2,1}$ is dense in a regular, generically smooth component $\mathcal{B} \subseteq B_C^{k_1}(d)$. Moreover:*
(i) $[\mathcal{F}_u] \in \mathcal{V}_d^{2g-2,1}$ general is stable and comes from $u \in \mathrm{Ext}^1(\omega_C, N)$ general, with $N \in \mathrm{Pic}^{d-2g+2}(C)$ general. In particular, $i(\mathcal{F}_u) = 1$.
(ii) The minimal degree quotient of $\mathcal{F}_u$ is ω_C, so $s(\mathcal{F}_u) = 4g - 4 - d > 0$.
(iii) $\mathrm{Div}_{F_u}^{1,2g-2} = \{\Gamma\}$, where Γ is the section of $F_u = \mathbb{P}(\mathcal{F}_u)$ corresponding to $\mathcal{F}_u \twoheadrightarrow \omega_C$ (i.e. $\mathcal{F}_u$ is rp via ω_C).

Proof. It follows from Theorem 5.4, with $2\delta - d \geqslant 1$ and $j = 1$, from Proposition 6.1 and from Remark 6.4. ☐

Remark 6.6. Using Theorem 5.1 and Corollary 5.3, one can prove results similar to Proposition 6.1 and Corollary 6.5 with slightly different numerical bounds. As in Remark 6.4, $\mathcal{V}_d^{\delta,j}$ can never be dense in a regular component of $B_C^{k_j}(d)$, unless $\delta = 2g - 2$ and $j = 1$. The numerical bounds in this case are $3g - 3 \leqslant d \leqslant 4g - 6$ with $g \geqslant 3$, hence the cases not already covered by Corollary 6.5 are $3g - 3 \leqslant d \leqslant \min\{3g, 4g - 6\}$.

Corollary 6.7. *Let C be of genus $g \geqslant 6$ with general moduli. For any integer d s.t. $3g - 3 \leqslant d \leqslant 3g$, the variety $\mathcal{V}_d^{2g-2,1}$ is dense in a regular, generically smooth component $\mathcal{B} \subseteq B_C^{k_1}(d)$. Moreover:*
(i) $[\mathcal{F}_u] \in \mathcal{V}_d^{2g-2,1}$ general is stable and comes from $u \in \mathrm{Ext}^1(\omega_C, N)$ general, with $N \in \mathrm{Pic}^{d-2g+2}(C)$ general (so non-special). In particular, $i(\mathcal{F}_u) = 1$.
(ii) The minimal degree quotient of $\mathcal{F}_u$ is ω_C, thus $s(\mathcal{F}_u) = 4g - 4 - d \geqslant g - 4$.
(iii) $\mathrm{Div}_{F_u}^{1,2g-2} = \{\Gamma\}$, where Γ the section of $F_u = \mathbb{P}(\mathcal{F}_u)$ corresponding to $\mathcal{F}_u \twoheadrightarrow \omega_C$ (i.e. $\mathcal{F}_u$ is rp via ω_C).

Proof. We need to prove that $\pi_{d,2g-2}$ is generically injective. The proof of Lemma 6.2 shows that, for $[\mathcal{F}_u] \in \mathcal{V}_d^{2g-2,1}$ general, one has $\dim(\pi_{d,2g-2}^{-1}([\mathcal{F}_u])) \leqslant \dim(\mathrm{Div}_{F_u}^{1,2g-2})$. By construction of $\mathcal{V}_d^{2g-2,1}$ and by (2.3), $\mathcal{N}_{\Gamma/F_u} \cong K_C - N$. Since N is general of degree $d - 2g + 2$, one has $h^1(N) = 0$. From Remark 2.3 we conclude. To prove the injectivity of $P_{\mathcal{F}_u}$ one can argue as in [14, Lemma2.1] (we leave the easy details to the reader). ☐

6.1.2. **Case** $d = 2\delta$. Fom what proved in Theorem 5.4, for any integers $j \geqslant 1$, g and δ as in (5.1) and in Remark 5.5, we have

$$\mathcal{V}_{2\delta}^{\delta,j} \subseteq B_C^{k_j}(2\delta) \cap U_C^{ss}(2\delta).$$

Lemma 6.8. *The morphism $\mathbb{P}(\mathfrak{E}_\delta)^0 \xrightarrow{\pi_{2\delta,\delta}} U_C(d)$ contracts the γ-fibres, with γ as in (6.2). Thus, $\nu_{2\delta}^{\delta,j} \leqslant g + \rho(L)$.*

Proof. For any $y = (N, L) \in \mathcal{U}$, $\gamma^{-1}(y) \cong \mathbb{P}$ as in (4.6). For any $u \in \mathbb{P}$, one has $\mathrm{gr}(\mathcal{F}_u) = L \oplus N$, where $\mathrm{gr}(\mathcal{F}_u)$ is the graded object associated to $\mathcal{F}_u$ (cf. [44, Thm. 4]). Therefore, all elements in a γ-fibre determine S-equivalent bundles (cf. e.g. [30, 46]). This implies that $\pi_{d,\delta}$ contracts any γ-fibre. ☐

Corollary 6.9. *Let C be of genus $g \geqslant 5$ with general moduli. One has*

$$B_C^{k_1}(4g-4) = \overline{\mathcal{V}_{4g-4}^{2g-2,1}}.$$

Thus:
(i) $B_C^{k_1}(4g-4)$ is irreducible, of dimension $g < \rho_{4g-4}^{k_1} = 2g - 2$. In particular, it is birational to $\mathrm{Pic}^{2g-2}(C)$, with $B_C^{k_2}(4g-4) = \{[\omega_C \oplus \omega_C]\}$;
(ii) $[\mathcal{F}_u] \in B_C^{k_1}(4g-4)$ general comes from $u \in \mathrm{Ext}^1(\omega_C, N)$ general, with $N \in \mathrm{Pic}^{2g-2}(C)$ general. Hence, $i(\mathcal{F}_u) = 1$.
(iii) The minimal degree quotient of $\mathcal{F}_u$ is ω_C, thus $s(\mathcal{F}_u) = 0$ and $\mathcal{F}_u$ is strictly semistable.
(iv) $\mathrm{Div}_{F_u}^{1,2g-2} = \{\Gamma\}$, where Γ the section of $F_u = \mathbb{P}(\mathcal{F}_u)$ corresponding to $\mathcal{F}_u \twoheadrightarrow \omega_C$ (i.e. $\mathcal{F}_u$ is rp via ω_C).

Proof. From Theorem 5.4, the only case for $d = 4g-4$ is $j = 1$ and $\delta = 2g-2$. Since $d = 2\delta$, from (6.2) we have $\mathcal{U} \cong \mathrm{Pic}^{2g-2}(C) \setminus \{\omega_C\}$ and $\mathfrak{E}$ is a vector bundle of rank $g-1$ on $\mathcal{U}$. From Lemma 6.8, the moduli map $\pi_{d,2g-2}$ factors through a map from $\mathcal{U}$ to $B_C^{k_1}(4g-4)$, which is injective by Chern class reasons.

Next we prove that $B_C^{k_1}(4g-4)$ is irreducible. Consider $[\mathcal{F}]$ general in a component of $B_C^{k_1}(4g-4)$; it can be presented via an exact sequence as in (2.2), with L special and effective (cf. Lemma 4.1). Since $s(\mathcal{F}) = 0$, then $\deg(L) = 2g-2$, i.e. $L \cong \omega_C$. Thus, we are in the image of $\mathcal{U}$ to $B_C^{k_1}(4g-4)$.

The remaining assertions are easy to check and can be left to the reader. □

Corollary 6.9 has been proved already in [8, Theorems 7.2, 7.3 and Remark 7.4], via different techniques. Our proof is completely independent.

6.2. **Special N.** Under the numerical assumptions of Theorem 5.10, any $N \in \mathrm{Pic}^{d-\delta}(C)$ is special (cf. Remark 5.11-(2)) and, for $u \in \mathrm{Ext}^1(L,N)$ general, ∂_u is surjective (cf. Remark 5.11-(3)). Hence $i(\mathcal{F}_u) = h^1(L) = j$. We have:

Proposition 6.10. *Assumptions as in Theorem 5.10. For any integers j, δ and d therein, there exists an irreducible component $\mathcal{B} \subseteq B_C^{k_j}(d)$ such that:*
(i) $\mathcal{V}_d^{\delta,j} \subseteq \mathcal{B}$;
(ii) $\mathcal{B} \cap U_C^s(d) \neq \emptyset$;
(iii) For $[\mathcal{E}] \in \mathcal{B}$ general, $\mathcal{E}$ is stable, with $s(\mathcal{E}) \geqslant g-\epsilon$ and ϵ as in Theorem 5.10. The minimal degree quotients of $\mathcal{E}$ as well as the minimal degree sections of $\mathbb{P}(\mathcal{E})$ are as in (iii) and (iv) of Theorem 5.10. In particular, L is of minimal degree if and only if $d = 2\delta - g$.
(iv) If moreover $d \geqslant \delta + g - 3$ (so $\delta \geqslant 2g-3$), then $\mathcal{B}$ is also regular and generically smooth.

Proof. Assertions (i), (ii) and (iii) follow from Theorem 5.10, the map (6.4) and the fact that the Segre invariant is lower-semicontinuous (cf. e.g. [24, § 3]).

To prove (iv), we argue as in [14, Lemma 2.1]. Take $\mathcal{F}_0 = L \oplus N$, with $N \in \mathrm{Pic}^{d-\delta}(C)$ general. Then, N is non-effective and $1 \leqslant h^1(N) \leqslant 2$. The Petri map $P_{\mathcal{F}_0}$ decomposes as $\mu_0(L) \oplus \mu$, where $\mu_0(L)$ is the Petri map of L as in (1.1) and μ is as in (5.7). Since C has general moduli, $\mu_0(L)$ is injective (cf. [1, (1.7), p. 215]). The injectivity of μ is immediate when $h^1(N) = 1$ (cf. the proof of Theorem 5.8). When $h^1(N) = 2$, the generality of N implies that $|K_C - N|$ is a base-point-free pencil so the injectivity of μ follows from the base-point-free pencil trick, since $h^0(N - (K_C - L)) = 0$ (because $K_C - L$ is effective and N non-effective). By semicontinuity on the elements of $\mathrm{Ext}^1(L,N)$ and the fact that $\mathcal{V}_d^{\delta,j} \subseteq \mathcal{B}$, the Petri map $P_{\mathcal{E}}$ is injective. One concludes by Remark 3.3. □

Remark 6.11. Computing $\dim(\mathbb{P}(\mathfrak{E}_\delta)) - \rho_d^{k_j}$ one finds the right-hand-side of (6.6). Since $d < 2\delta$ (see (5.9)), as in Remark 6.4-(2) one sees that $\dim(\mathbb{P}(\mathfrak{E}_\delta)) - \rho_d^{k_j} < 0$, unless $j = 1$ and $\delta = 2g-2$, in which case $\dim(\mathbb{P}(\mathfrak{E}_\delta)) - \rho_d^{k_j} = 0$. As in Lemma 6.2, we see that $\pi_{d,2g-2}$ is generically injective. Thus, with notation as in (6.5), one has $\nu_d^{\delta,j} \geqslant \rho_d^{k_j}$ only if $j = 1$, $\delta = 2g-2$ and $N \in \mathrm{Pic}^{d-\delta}(C)$ is general, in which case $\nu_d^{2g-2,1} = \rho_d^{k_1}$.

Corollary 6.12. *Let C be of genus $g \geqslant 3$ with general moduli. For any integer d such that $2g-2 \leqslant d \leqslant 3g-4$, one has $\nu_d^{2g-2,1} = \rho_d^{k_1} = 6g-6-d$. Moreover:*
(i) $[\mathcal{F}_u] \in \mathcal{V}_d^{2g-2,1}$ general is stable and comes from $u \in \mathrm{Ext}^1(\omega_C, N)$ general, with $N \in \mathrm{Pic}^{d-2g+2}(C)$ general (hence special, non-effective). In particular, $i(\mathcal{F}_u) = 1$.
(ii) If $3g-5 \leqslant d \leqslant 3g-4$, then $\mathcal{V}_d^{2g-2,1}$ is dense in a regular, generically smooth component of $B_C^{k_1}(d)$.
(iii) $s(\mathcal{F}_u) = g - \epsilon$, with ϵ as in Theorem 5.10. Quotients of minimal degree of $\mathcal{F}_u$ (equivalently sections of minimal degree on $F_u = \mathbb{P}(\mathcal{F}_u)$) are those described in Theorem 5.10-(111) and (iv). In particular, they are li sections.
(iv) The canonical section $\Gamma \subset F_u$ is the only special section; it is lsu and asu but not ai. Moreover, it is of minimal degree only when $d = 3g-4$.
(v) $\mathcal{F}_u$ is rsp but not rp via ω_C.

Proof. (i), (ii) and (iii) follow from Theorem 5.10, Proposition 6.10 and Remarks 3.3, 6.11. Sections of minimal degree are li (see the proof of Proposition 2.12).

As for (iv) and (v), from Serre duality and the fact that $\mathcal{F}_u$ is of rank-two with $\det(\mathcal{F}_u) = \omega_C \otimes N$, one has

$$h^0(\mathcal{F}_u \otimes N^\vee) = h^1(\mathcal{F}_u^\vee \otimes \omega_C \otimes N) = h^1(\mathcal{F}_u). \tag{6.7}$$

Since $i(\mathcal{F}_u) = 1$, from (2.6) Γ is li. Since N is special and non-effective, from (2.3), $\mathrm{Div}_{F_u}^{1,2g-g}$ is smooth, of dimension $3g-3-d \geqslant 1$ at Γ. Thus, Γ is not ai but, since $W_{2g-2}^{g-1}(C) = \{\omega_C\}$, it is asu (see the proof of Proposition 2.12 and Remark 6.11). For the same reason, from Theorem 5.10-(iv), the only possibility for ω_C

to be a minimal quotient is $d = 3g - 4$. Finally, the fact that $\Gamma \subset F_u$ is the only special section follows from Remark 6.11. $\square$

Recall that, when $N \in \mathrm{Pic}^{d-\delta}(C)$ is special and $L \in W^{\delta-g+j}_{\delta}(C)$ is a smooth point, assumptions as in Theorem 5.8 imply that ∂_u is surjective for $u \in \mathrm{Ext}^1(L, N)$ general (cf. Corollary 5.9), and so $i(\mathcal{F}_u) = j$.

Therefore, to have $i(\mathcal{F}_u) > j$, we are forced to use degeneracy loci described in (5.4). To do this let $y = (N, L)$ be general in $\mathcal{U}$, respectively in $\mathcal{Z}$, when N is non-effective, respectively when it is effective (recall notation as in (6.2)). Set $\mathbb{P}(y) := \gamma^{-1}(y) \cong \mathbb{P}$. Take numerical assumptions as in Remark 5.15, respectively in Remark 5.16, according to N is respectively non-effective or effective.

With notation as in (5.11), for any good component $\widehat{\Lambda}_t(y) \subseteq \widehat{W}_t(y) \subset \mathbb{P}(y)$ we have

$$\emptyset \neq \widehat{\mathcal{W}}^{\mathrm{Tot}}_t \subset \mathbb{P}(\mathfrak{E}_\delta),$$

where a point in $\widehat{\mathcal{W}}^{\mathrm{Tot}}_t$ corresponds to the datum of a pair (y, u), with $y = (N, L)$ and $u \in \widehat{W}_t(y)$. Any irreducible component of $\widehat{\mathcal{W}}^{\mathrm{Tot}}_t$ has dimension at least $\dim(\mathbb{P}(\mathfrak{E}_\delta)) - c(\ell, r, t)$ (where $c(\ell, r, t)$ as in (5.5) and where $\dim(\mathbb{P}(\mathfrak{E}_\delta))$ as in (6.3)). From the generality of y, for any good component $\widehat{\Lambda}_t(y)$, we have an irreducible component

$$\widehat{\Lambda}^{\mathrm{Tot}}_t \subseteq \widehat{\mathcal{W}}^{\mathrm{Tot}}_t \subset \mathbb{P}(\mathfrak{E}_\delta)$$

such that

(i) $\widehat{\Lambda}^{\mathrm{Tot}}_t$ dominates $\mathcal{U}$ (resp., $\mathcal{Z}$);
(ii) $\dim(\widehat{\Lambda}^{\mathrm{Tot}}_t) = \dim(\mathbb{P}(\mathfrak{E}_\delta)) - c(\ell, r, t)$;
(iii) for $(y, u) \in \widehat{\Lambda}^{\mathrm{Tot}}_t$ general, $\mathrm{cork}(\partial_u) = t$;
(iv) if $\lambda := \gamma|_{\widehat{\Lambda}^{\mathrm{Tot}}_t}$, for y general one has $\lambda^{-1}(y) = \widehat{\Lambda}_t(y)$.

Definition 6.13. Any component $\widehat{\Lambda}^{\mathrm{Tot}}_t$ satisfying (i)-(iv) above will be called a *(total) good component* of $\widehat{\mathcal{W}}^{\mathrm{Tot}}_t$.

We set

$$\mathcal{V}^{\delta,j,t}_d := \mathrm{Im}\left(\pi_{d,\delta}|_{\widehat{\Lambda}^{\mathrm{Tot}}_t}\right) \subseteq B^{k_j+t}_C(d) \quad \text{and} \quad \nu^{\delta,j,t}_d := \dim(\mathcal{V}^{\delta,j,t}_d). \tag{6.8}$$

Two cases have to be discussed, according to the effectivity of N.

6.2.1. N **non-effective**. With assumptions as in Remark 5.15, N can be taken general in $\mathrm{Pic}^{d-\delta}(C)$; the general bundle in $\mathcal{V}^{\delta,j,t}_d$ is stable (by Theorem 5.13 and by the open nature of stability). For brevity sake, set

$$\varphi_0(\delta, j, t) := \dim(\widehat{\Lambda}^{\mathrm{Tot}}_t) - \rho^{k_j+t}_d = d(j-1) - \delta(j-2) - (g-1)(j+1) + jt, \tag{6.9}$$

which therefore takes into account the expected dimension of the general fibre of $\pi_{d,\delta}|_{\widehat{\Lambda}^{\mathrm{Tot}}_t}$ and the codimension of its image in a regular component of $B^{k_j+t}_C(d)$).

One has $\varphi_0(\delta, j, t) \geqslant \nu^{\delta,j,t}_d - \rho^{k_j+t}_d$ with equality if and only if $\pi_{d,\delta}|_{\widehat{\Lambda}^{\mathrm{Tot}}_t}$ is generically finite. Thus, from Remark 3.3, it is clear that $\mathcal{V}^{\delta,j,t}_d$ cannot fill up a dense subset of a component of $B^{k_j+t}_C(d)$ if $\varphi_0(\delta, j, t) < 0$; in other words, the negativity of $\varphi_0(\delta, j, t)$ gives numerical obstruction to describe the general point of a (regular) component of $B^{k_j+t}_C(d)$.

- For $j = 1$, one has

$$\varphi_0(\delta, 1, t) = \delta - 2g + 2 + t. \tag{6.10}$$

- When $j \geqslant 2$, from Remark 5.15 and arguing as in Remark 6.4, one gets

$$\varphi_0(\delta, j, t) \leqslant j(t - j). \tag{6.11}$$

Thus, $\mathcal{V}^{\delta,j,t}_d$ never fills up a dense subset of a component of $B^{k_j+t}_C(d)$ as soon as $j > t \geqslant 1$.

6.2.2. N **effective**. With assumptions as in Remark 5.16, N is general in $W_{\rho(N)}$. From the second equality in (6.3), for any $n \geqslant 1$, one puts

$$\varphi_n(\delta, j, t) := \dim(\widehat{\Lambda}^{\mathrm{Tot}}_t) - \rho^{k_j+t}_d = \varphi_0(\delta, j, t) - n(r - t), \tag{6.12}$$

where $\varphi_0(\delta, j, t)$ as in (6.9) above.

Remark 6.14. For a total good component $\widehat{\Lambda}^{\mathrm{Tot}}_t$ and for $(L, N, u) \in \widehat{\Lambda}^{\mathrm{Tot}}_t$ general, one has $n(r - t) = h^0(N)\ \mathrm{rk}(\partial_u)$. Hence, $n(r - t)$ is non-negative and it is zero if and only if $r = t$, i.e. ∂_u is the zero map. Therefore, $\varphi_n(\delta, j, t) \leqslant \varphi_0(\delta, j, t)$ and equality holds if and only if $r = t$. The possibility for a $\mathcal{V}^{\delta,j,t}_d$ to fill up a dense subset of a component of $B^{k_j+t}_C(d)$ can be discussed as in § 6.2.1.

By definition of $\nu_d^{\delta,j,t}$, it is clear that $\nu_d^{\delta,j,t} - \rho_d^{k_{j+t}} \leqslant \varphi_n(\delta, j, t)$, for any $n \geqslant 0$. Thus a necessary condition for $\nu_d^{\delta,j,t} \geqslant \rho_d^{k_{j+t}}$, i.e. for $\mathcal{V}_d^{\delta,j,t}$ to have at least the dimension of a regular component of $B_C^{k_{j+t}}(d)$, is $\varphi_n(\delta, j, t) \geqslant 0$.

Next proposition easily follows.

Proposition 6.15. *Assumptions as in Theorem 5.13 (more precisely, either as in Remark 5.15, when N is non-effective, or as in Remark 5.16, when N is effective). Then for any integers j, δ and d therein, there exists an irreducible component $\mathcal{B} \subseteq B_C^{k_{j+t}}(d)$ such that:*
(i) $\mathcal{V}_d^{\delta,j,t} \subseteq \mathcal{B}$*;*
(ii) For $[\mathcal{E}] \in \mathcal{B}$ *general,* $s(\mathcal{E}) \geqslant g - c(\ell, r, t) - \epsilon \geqslant 0$*, where* $c(\ell, r, t)$ *as in* (5.5) *and* $\epsilon \in \{0, 1\}$ *such that* $d + g - c(\ell, r, t) \equiv \epsilon \pmod 2$*;*
(iii) $\mathcal{B} \cap U_C^s(d) \neq \emptyset$*, if* $g - c(\ell, r, t) - \epsilon > 0$*.*

Remark 6.16. In order to estimate $\nu_d^{\delta,j,t}$, one has to estimate the dimension of the general fibre of the map $\pi_{d,\delta}$ restricted to a total good component $\widehat{\Lambda}_t^{\mathrm{Tot}}$. Thus, if for $[\mathcal{F}] \in \mathcal{V}_d^{\delta,j,t}$ general we put for simplicity $f_{\mathcal{F}} := \dim\left(\pi_{d,\delta}|^{-1}_{\widehat{\Lambda}_t^{Tot}}([\mathcal{F}])\right)$, a rough estimate is

$$f_{\mathcal{F}} \leqslant a_F(\delta), \tag{6.13}$$

where $F = \mathbb{P}(\mathcal{F})$ and $a_F(\delta)$ the dimension of the scheme of special unisecants of degree δ on F as in (2.9).

Remark 6.17. Assume $j = 1$ in Proposition 6.15.
(1) When $N \in \mathrm{Pic}^{d-\delta}(C)$ is general, assumptions as in Remark 5.15 give $\delta \leqslant 2g - 2$ and N non-effective, for any $t \geqslant 1$. The only case to consider is therefore $\varphi_0(\delta, 1, t)$. A necessary condition for $\mathcal{V}_d^{\delta,1,t}$ to have dimension at least $\rho_d^{k_{j+t}}$ is $\varphi_0(\delta, 1, t) \geqslant 0$, i.e. $\delta \geqslant 2g - 2 - t$ (cf. (6.10)). Thus:

- when $\delta = 2g - 2 - t$, then $L = \omega_C(-D_t)$, with $D_t \in C^{(t)}$, $t < g$, imposing independent conditions to $|\omega_C|$. Since $\varphi_0(\delta, 1, t) = 0$, for $D_t \in C^{(t)}$ general, the estimate (6.13) and a parameter count suggest that for $[\mathcal{F}] \in \mathcal{V}_d^{2g-2-t,1,t}$ general one has $a_F(2g - 2 - t) = 0$, i.e. $\mathcal{F}$ is rsp via $\omega_C(-D_t)$, and $\mathcal{B} = \overline{\mathcal{V}_d^{\delta,1,t}}$ is regular.
- to the opposite, when $\delta = 2g - 2$, then $L = \omega_C$. Let $[\mathcal{F}] \in \mathcal{V}_d^{2g-2,1,t}$ be general, and let $\Gamma \subset F = \mathbb{P}(\mathcal{F})$ be the canonical section corresponding to $\mathcal{F} \twoheadrightarrow \omega_C$. By definition of $\mathcal{V}_d^{2g-2,1,t}$, $\mathcal{F} = \mathcal{F}_v$ for $v \in \Lambda_t \subset \mathrm{Ext}^1(N, \omega_C)$ general in a good component. By (2.6) and (6.7), one has $\dim(|\mathcal{O}_F(\Gamma)|) = t$. Thus, $[\mathcal{F}] \in \mathcal{V}_d^{2g-2,1,t}$ general is not rsp via ω_C, since the general fibre of $\pi_{d,2g-2}|_{\widehat{\Lambda}_t^{\mathrm{Tot}}}$ has dimension at least t.

It is therefore natural to expect that the component $\mathcal{B}$ in Proposition 6.15 is such that

$$\mathcal{B} = \overline{\mathcal{V}_d^{2g-2,1,t}} = \overline{\mathcal{V}_d^{2g-3,1,t}} = \cdots = \overline{\mathcal{V}_d^{2g-2-t,1,t}},$$

where $[\mathcal{F}] \in \mathcal{B}$ general is rsp only when $[\mathcal{F}]$ is considered as element in $\mathcal{V}_d^{2g-2-t,1,t}$.
(2) One may expect something similar when $j = 1$ and N effective general in $W_{\rho(N)}$. In this case, $\varphi_n(\delta, 1, t) \geqslant 0$ gives $\delta \geqslant 2g - 2 + rn - t(n + 1)$ whereas, from the first line of bounds on δ in Remark 5.16, we get $\delta \leqslant \min\{2g - 2, g - 2 + r - t + \frac{g-\epsilon}{t}\}$. A necessary condition for $\nu_d^{\delta,j,t} \geqslant \rho_d^{k_{j+t}}$ is therefore

$$rn - t(n + 1) < 0, \tag{6.14}$$

otherwise either L would be non special, contradicting Lemma 4.1, or $L \cong \omega_C$, so $\mathcal{F}_v$ would be not rsp as in (1) above. In the next section, we will discuss these questions.

7. Low speciality, canonical determinant

In this section we apply results in §'s 5.1, 5.2 and 6 to describe Brill-Noether loci of vector bundles with canonical determinant and Brill-Noether loci of vector bundles of fixed degree d and low speciality $i \leqslant 3$ on a curve C with general moduli. In particular, the more general analysis discussed in the previous sections allows us to determine rigidly specially presentation of the general point of irreducible components arising from constructions in § 6,

From now on, for any integers $g \geqslant 3$, $i \geqslant 1$ and $2g - 2 \leqslant d \leqslant 4g - 4$, we will set

$$\widetilde{B_C^{k_i}}(d) := \begin{cases} B_C^{k_i}(d) & \text{if either } d \text{ odd or } d = 4g-4 \\ B_C^{k_i}(d) \cap U_C^s(d) & \text{otherwise} \end{cases} \tag{7.1}$$

7.1. **Vector bundles with canonical determinant.** Given an integer d and any $\xi \in \mathrm{Pic}^d(C)$, there exists the *moduli space of (semi)stable, rank-two vector bundles with fixed determinant* ξ. Following [32, 33], we denote it by $M_C(2,\xi)$ (sometimes a different notation is used, see e.g. [44, 6, 7, 54, 39, 55, 5, 26]).

The scheme $M_C(2,\xi)$ is defined as the fibre over $\xi \in \mathrm{Pic}^d(C)$ of the *determinantal map*

$$U_C(d) \xrightarrow{\det} \mathrm{Pic}^d(C). \tag{7.2}$$

For any $\xi \in \mathrm{Pic}^d(C)$, $M_C(2,\xi)$ is smooth, irreducible, of dimension $3g-3$ (cf. [35, 44]).

Brill-Noether loci can be considered in $M_C(2,\xi)$. Recent results for arbitrary ξ are given in [39, 40, 25]. A case which has been particularly studied (for its connections with Fano varieties) is $M_C(2,\omega_C)$. Seminal papers on the subject are [7, 32]; other important results are contained in [54, 55, 23, 26]. If $[\mathcal{F}] \in M_C(2,\omega_C)$, Serre duality gives

$$i(\mathcal{F}) = k_i(\mathcal{F}) := k. \tag{7.3}$$

For $[\mathcal{F}] \in M_C(2,\omega_C) \subset U_C(2g-2)$, the Petri map $P_{\mathcal{F}}$ in (3.4) splits as $P_{\mathcal{F}} = \lambda_{\mathcal{F}} \oplus \mu_{\mathcal{F}}$, where

$$\lambda_{\mathcal{F}} : \bigwedge^2 H^0(\mathcal{F}) \to H^0(\omega_C) \quad \text{and} \quad \mu_{\mathcal{F}} : \mathrm{Sym}^2(H^0(\mathcal{F})) \to H^0(\mathrm{Sym}^2(\mathcal{F}));$$

the latter is called the *symmetric Petri map.*

For $[\mathcal{F}] \in M_C(2,\omega_C)$ general, one has $k = 0$ (cf. [33, § 4], after formula (4.3)). For any $k \geqslant 1$, one sets

$$M_C^k(2,\omega_C) := \{[\mathcal{F}] \in M_C(2,\omega_C) \mid h^0(\mathcal{F}) = h^1(\mathcal{F}) \geqslant k\}$$

which is called the k^{th}-*Brill-Noether locus* in $M_C(2,\omega_C)$. In analogy with (7.1), we set

$$\widetilde{M_C^k}(2,\omega_C) := M_C^k(2,\omega_C) \cap U_C^s(2g-2).$$

By [33, Prop. 1.4], [7, § 2] and (7.3), one has

$$\mathrm{expcodim}_{M_C(2,\omega_C)}(\widetilde{M_C^k}(2,\omega_C)) = \frac{k(k+1)}{2} \leqslant k^2 = i(\mathcal{F})k_i(\mathcal{F}).$$

Similarly to $\widetilde{B_C^{k_i}}(d)$, if $[\mathcal{F}] \in \widetilde{M_C^k}(2,\omega_C)$, then $\widetilde{M_C^k}(2,\omega_C)$ is smooth and regular (i.e. of the expected dimension) at $[\mathcal{F}]$ if and only if $\mu_{\mathcal{F}}$ is injective (see [7, 32, 33]).

Several basic questions on $\widetilde{M_C^k}(2,\omega_C)$, like non-emptiness, irreducibility, etc., are still open. A description of these bundles in terms of extensions (as we do here) is available only for some k on C general of genus $g \leqslant 12$ (cf. [33, § 4], [7]). Further existence results are contained in [52, 26]. On the other hand, if one assumes $[\mathcal{F}] \in M_C^k(2,\omega_C)$, injectivity of $\mu_{\mathcal{F}}$ on C general of genus $g \geqslant 1$ has been proved in [54] (cf. [5] for $k < 6$ with a different approach).

7.2. **Case** $i = 1$**.** In this case $\rho_d^{k_1} = 6g-6-d$. Using notation and results as in § 6, we get:

Theorem 7.1. *Let C be of genus $g \geqslant 5$, with general moduli. For any integer d s.t. $2g-2 \leqslant d \leqslant 4g-4$,*

$$\widetilde{B_C^{k_1}}(d) = \overline{\mathcal{V}_d^{2g-2,1}},$$

as in Corollaries 6.5, 6.7, 6.9 and 6.12. In particular,

(i) $\widetilde{B_C^{k_1}}(d)$ is non-empty, irreducible. For $2g-2 \leqslant d \leqslant 4g-5$ it is regular, whereas $\dim(\widetilde{B_C^{k_1}}(4g-4)) = g < \rho_{4g-4}^{k_1} = 2g-2$.

(ii) For $3g-5 \leqslant d \leqslant 4g-4$, $\widetilde{B_C^{k_1}}(d)$ is generically smooth.

(iii) $[\mathcal{F}] \in \widetilde{B_C^{k_1}}(d)$ general is stable for $2g-2 \leqslant d \leqslant 4g-5$, and strictly semistable for $d = 4g-4$, fitting in a (unique) sequence

$$0 \to N \to \mathcal{F} \to \omega_C \to 0,$$

where $N \in \mathrm{Pic}^{d-2g+2}(C)$ is general, the coboundary map is surjective and $i(\mathcal{F}) = 1$.

(iv) For $3g-4 \leqslant d \leqslant 4g-4$ and $[\mathcal{F}] \in \widetilde{B_C^{k_1}}(d)$ general, one has $s(\mathcal{F}) = 4g-4-d$, the quotient of minimal degree being ω_C. The section $\Gamma \subset F = \mathbb{P}(\mathcal{F})$ corresponding to $\mathcal{F} \twoheadrightarrow \omega_C$ is the only special section of F. Moreover:

- *for $d \geqslant 3g-3$, Γ is ai,*
- *for $d = 3g-4$, Γ is lsu and asu but not ai.*

(v) For $2g-2 \leqslant d \leqslant 3g-5$ and $[\mathcal{F}] \in \widetilde{B_C^{k_1}}(d)$ general, one has $s(\mathcal{F}) = g-\epsilon$, with $\epsilon \in \{0,1\}$ such that $d+g \equiv \epsilon$ (mod 2)*. The section $\Gamma \subset F$ is the only special section; it is asu but not ai. Moreover, Γ is not of minimal degree; indeed:*

- *when $d+g$ is even, minimal degree sections of F are li sections of degree $\frac{d+g}{2}$ s.t.* $\dim(\mathrm{Div}_F^{1,\frac{d+g}{2}}) = 1$*;*

- *when $d+g$ is odd, minimal degree sections are li of degree* $\frac{d+g-1}{2}$ *and* $\dim(\mathrm{Div}_F^{1,\frac{d+g-1}{2}}) \leqslant 1$.

(vi) In particular, for $2g-2 \leqslant d \leqslant 4g-4$, $[\mathcal{F}] \in \widetilde{B_C^{k_1}}(d)$ general is rp via ω_C.

Proof. All the assertions, except the irreducibility, follow from Corollaries 6.5, 6.7, 6.9 and 6.12. For $d = 4g-4$ irreducibility has been proved in Corollary 6.9. Thus, we focus on cases $2g-2 \leqslant d \leqslant 4g-5$.

Let us consider an irreducible component $\mathcal{B} \subseteq \widetilde{B_C^{k_1}}(d)$. From Lemma 4.1, $[\mathcal{F}] \in \mathcal{B}$ general is as in (2.2), with $h^1(L) = j \geqslant 1$ and L of minimal degree among special, effective quotient line bundles. Moreover $\dim(\mathcal{B}) \geqslant \rho_d^{k_1}$ (cf. Remark 3.3). Two cases have to be considered.

(1) If $i(\mathcal{F}) = 1$, then $j = 1$ (notation as in (4.4), (4.5)) and $\partial : H^0(L) \to H^1(N)$ is surjective. In particular $\ell \geqslant r$. If $r = 0$ then we are in cases of Corollaries 6.5, 6.7, and $\mathcal{B} = \overline{\mathcal{V}_d^{2g-2,1}}$. If $r > 0$, as in Remark 6.11 one has

$$0 \leqslant \dim(\mathbb{P}(\mathfrak{E}_\delta)) - \dim(\mathcal{B}) \leqslant \delta - 2g + 2$$

(cf. (6.6)). Hence $\delta = 2g-2$ and $\mathcal{B} = \overline{\mathcal{V}_d^{2g-2,1}}$ as in Corollary 6.12.

(2) Assume $i(\mathcal{F}) = i > 1$. As in Remarks 6.4, 6.6, 6.11 one has $L = \omega_C$. Thus $i > 1$ forces $r \geqslant \mathrm{cork}(\partial) = i-1 > 0$. Recalling (4.2) and (4.6), one has $\dim(\mathbb{P}) = 5g-6-d$. Therefore, $\mathcal{B}$ must be regular and $\mathcal{F}$ corresponds to the general point of $\mathrm{Ext}^1(\omega_C, N)$, with $N \in \mathrm{Pic}^{d-2g+2}(C)$ general, so non-effective. In particular, one has $2g-2 \leqslant d \leqslant 3g-4$ and $r = 3g-3-d$. On the other hand, since

$$\ell = g, \; 1 \leqslant r \leqslant g-1, \; 2g-1 \leqslant m = 5g-5-d \leqslant 3g-3$$

we are in the hypotheses of Corollary 5.9, hence $\mathrm{cork}(\partial) = 0$, a contradiction. □

Remark 7.2. (1) Theorem 7.1 gives alternative proofs of results in [45, 27, 8] for the rank-two case. It provides in addition a description of the general point of $\widetilde{B_C^{k_1}}(d) \cong \widetilde{B_C^1}(4g-4-d)$, for any $2g-2 \leqslant d \leqslant 4g-4$. The same description is given in [3], with a different approach, i.e. using *general negative elementary transformations* as in [27]. In terms of scrolls of speciality 1, partial classification are given also in [18, Theorem 3.9].

(2) As a consequence of Theorem 7.1, one observes that the Segre invariant s does not stay constant on a component of the Brill-Noether locus. For example, the general element of $\widetilde{B_C^{k_1}}(4g-7)$ has $s = 3$ and $i = 1$; on the other hand, in Theorem 5.4, we constructed vector bundles in $\mathcal{V}_{4g-7}^{2g-3,1} \subset \widetilde{B_C^{k_1}}(4g-7)$ with $s = i = 1$. The minimal special quotient of the latter vector bundles is the canonical bundle minus a point, whereas for the general vector bundle in $\widetilde{B_C^{k_1}}(4g-7)$ is the canonical bundle.

(3) From the proof of Theorem 7.1, for $d \leqslant 4g-3$, the map $\pi_{d,2g-2}$ is birational onto $\widetilde{B_C^{k_1}}(d) = B_C^{k_1}(d)$, i.e. $B_C^{k_1}(d)$ is uniruled.

Theorem 7.3. *Let C be of genus $g \geqslant 5$, with general moduli. Then $\widetilde{M_C^1}(2, \omega_C) \neq \emptyset$. Moreover, there exists an irreducible component which is*

- (i) *generically smooth*
- (ii) *regular (i.e. of dimension $3g-4$), and*
- (iii) *its general point $[\mathcal{F}_u]$ comes from $u \in \mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C))$ general. In particular, $s(\mathcal{F}_u) = g - \epsilon$, where $\epsilon \in \{0, 1\}$ such that $g \equiv \epsilon \pmod 2$.*

Proof. Take $u \in \mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C))$ general. With notation as in (4.4), (4.5) one has

$$\ell = r = g, \text{ and } m = 3g-3 \geqslant \ell + 1.$$

Thus, from Corollary 5.9 and from (7.3), $h^0(\mathcal{F}_u) = h^1(\mathcal{F}_u) = 1$. From (4.2), $\dim(\mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C)) = 3g-4$. Thus, $\mathcal{F}_u$ stable with $s(\mathcal{F}_u) = g - \epsilon$ follows from Proposition 4.4, with $\sigma = g - \epsilon$. This shows that $\widetilde{M_C^1}(2, \omega_C) \neq \emptyset$.

Since $\bigwedge^2 H^0(\mathcal{F}_u) = (0)$, $\mu_{\mathcal{F}_u}$ is injective if and only if $P_{\mathcal{F}_u}$ is. On the other hand, one has $H^0(\mathcal{F}_u) \otimes H^0(\omega_C \otimes \mathcal{F}_u^\vee) \cong \mathbb{C}$. Therefore, one needs to show that $P_{\mathcal{F}_u}$ is not the zero-map. This follows by limit of $P_{\mathcal{F}_u}$ when u tends to 0, so that $\mathcal{F}_0 = \mathcal{O}_C \oplus \omega_C$: then the limit of $P_{\mathcal{F}_u}$ is the map $H^0(\mathcal{O}_C) \otimes H^0(\mathcal{O}_C) \to H^0(\mathcal{O}_C)$.

To get (i)-(iii) at once, one observes that $\pi_{2g-2,2g-2}|_{\mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C))}$ is generically injective, since the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{F}_u \to \omega_C \to 0$$

is unique: indeed, the surjection $\mathcal{F}_u \twoheadrightarrow \omega_C$ is unique and $h^0(\mathcal{F}_u) = 1$ (cf. (2.6) and computations as in (6.7)), moreover, by Lemma 4.5, two general vector bundles in $\mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C))$ cannot be isomorphic. □

Remark 7.4. (1) For a similar description, cf. [7]. Generic smoothness for components of $\widetilde{M_C^1}(2, \omega_C)$ follows also from results in [54, 5].

(2) From Theorem 7.1, $[\mathcal{F}] \in \widetilde{B_C^{k_1}}(2g-2)$ general fits in a sequence $0 \to \eta \to \mathcal{F} \to \omega_C \to 0$, with $\eta \in \mathrm{Pic}^0(C)$ general. Hence the map $\widetilde{d} := \det|_{\widetilde{B_C^{k_1}}(2g-2)}$ is dominant. Since $\mathrm{Pic}^{2g-2}(C)$ and $\widetilde{B_C^{k_1}}(2g-2)$ are irreducible and generically smooth, then $\widetilde{d}^{-1}(\eta) = \widetilde{M_C^1}(2, \omega_C \otimes \eta)$ is equidimensional and each component is generically smooth. Theorem 7.3 yields that in this situation each component of $\widetilde{M_C^1}(2, \omega_C \otimes \eta)$ has dimension $3g-4$ (equal to the expected dimension). This agrees with [39, Theorem 1.1].

7.3. Case $i = 2$.

In this case, $\rho_d^{k_2} = 8g - 11 - 2d$.

Theorem 7.5. *Let C be of genus $g \geqslant 3$, with general moduli. For any integer d s.t. $2g-2 \leqslant d \leqslant 3g-6$, one has $\widetilde{B_C^{k_2}}(d) \neq \emptyset$.*
(i) $\overline{\mathcal{V}_d^{2g-3,1,1}}$ is the unique component of $\widetilde{B_C^{k_2}}(d)$, whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 2$. Moreover, $\overline{\mathcal{V}_d^{2g-3,1,1}} = \overline{\mathcal{V}_d^{2g-2,1,1}}$ and it is a regular component of $\widetilde{B_C^{k_2}}(d)$.
(ii) For $[\mathcal{F}] \in \overline{\mathcal{V}_d^{2g-3,1,1}}$ general, one has $s(\mathcal{F}) \geqslant 3g-4-d-\epsilon > 0$ and $\mathcal{F}$ fits in an exact sequence

$$0 \to N \to \mathcal{F} \to \omega_C(-p) \to 0,$$

with

- *$p \in C$ general,*
- *$N \in \mathrm{Pic}^{d-2g+3}(C)$ general, and*
- *$\mathcal{F} = \mathcal{F}_v$ and v is general in the good locus $\mathcal{W}_1 \subset \mathrm{Ext}^1(\omega_C(-p), N)$.*

(iii) A section $\Gamma \subset F$, corresponding to a quotient $\mathcal{F} \twoheadrightarrow \omega_C(-p)$, is not of minimal degree. However, it is of minimal degree among special sections and it is asi but not ai (i.e. $\mathcal{F}$ is rsp but not rp via $\omega_C(-p)$).
(iv) For $g \geqslant 13$ and $2g+6 \leqslant d \leqslant 3g-7$, $\overline{\mathcal{V}_d^{2g-3,1,1}}$ is generically smooth.

Proof. Once part (i) has been proved, parts (ii)–(iii) follow from Theorem 5.13 and Proposition 6.15, with $\delta = 2g-3$, $j = t = 1$, whereas part (iv) follows from Proposition 6.1-(ii), with $j = 2$.

The proof of part (i) consists of four steps.

Step 1. In this step, we show that if $\mathcal{B}$ is an irreducible component of $\widetilde{B_C^{k_2}}(d)$ such that, for $[\mathcal{F}] \in \mathcal{B}$ general, $i(\mathcal{F}) = 2$, then $\mathcal{B}$ comes from a total good component $\widehat{\mathcal{W}}_1^{\mathrm{Tot}} \subseteq \mathbb{P}(\mathfrak{E}_\delta)$, for some δ (cf. Thm. 5.8 and Def. 5.12).

Indeed, let (2.2) be a special presentation of $\mathcal{F}$ with L a quotient of minimal degree. Then, from Remarks 6.4, 6.6, 6.11, one has $h^1(L) = j = 1$ as $\mathcal{B}$ is a component. Hence, with notation as in (4.4), (4.5) and (5.4), one must have $t = 1$ and $\ell \geqslant r-1$. Moreover, $d \leqslant 3g-6$, $\delta \geqslant g-1$ and $j = 1$ imply that $m = 2\delta - d + g - 1 \geqslant \delta - g + 3 = \ell + 1$ (recall notation as in (4.2)). Therefore, we can apply Theorem 5.8, finishing the proof of this step.

Step 2. In this step we determine which of the constructed loci either $\mathcal{V}_d^{\delta,j}$, as in (6.5), or $\mathcal{V}_d^{\delta,j,t}$, as in (6.8), has general point $[\mathcal{F}]$ such that $i(\mathcal{F}) = 2$ and dimension at least $\rho_d^{k_2} = 8g - 11 - 2d$ (hence, it can be conjecturally dense in a component of $\widetilde{B_C^{k_2}}(d)$). We will prove that this only happens for $2g-3 \leqslant \delta \leqslant 2g-2$ and $j = t = 1$. Moreover, we will show that the presentation of $\mathcal{F}$ is specially rigid only if $\delta = 2g-3$.

Let $\mathcal{V}$ be any such locus, and let $\mathcal{F}$ be its general point which is presented as in (2.2) with special quotient L. As in Step 1, one finds $j = 1$ hence N has to be special and $t = 1$, so $\mathcal{V}$ has to be necessarily of the form $\mathcal{V}_d^{\delta,1,1}$ (i.e. loci of the form $\mathcal{V}_d^{\delta,j}$ are excluded). We have two cases to consider: (a) N non-effective, (b) N effective.

Case (a). As in Remark 6.17-(1), recall that a necessary condition for $\dim(\mathcal{V}_d^{\delta,1,1}) = \nu_d^{\delta,1,1} \geqslant \rho_d^{k_2}$ is $\varphi_0(\delta,1,1) \geqslant 0$, i.e. $2g-3 \leqslant \delta \leqslant 2g-2$.

In case $\delta = 2g-2$, $[\mathcal{F}] \in \mathcal{V}_d^{2g-2,1,1}$ general is not rsp via ω_C (it follows from the fact that $i = 2$ and computation as in (6.7)).

In case $\delta = 2g-3$, the hypotheses $2g-2 \leqslant d \leqslant 3g-6$ ensure stability for $\mathcal{F}$ (cf. Theorems 5.8, 5.13 and Proposition 6.15). By definition, $\mathcal{V}_d^{2g-3,1,1} = \mathrm{Im}(\pi_{d,2g-3}|_{\widehat{\mathcal{W}}_1^{\mathrm{Tot}}})$, where $\widehat{\mathcal{W}}_1^{\mathrm{Tot}} \subset \mathbb{P}(\mathfrak{E}_{2g-3})$ is the good locus for $t = 1$. To accomplish the proof, we need to show that the fibre of $\pi_{d,2g-3}|_{\widehat{\mathcal{W}}_1^{\mathrm{Tot}}}$ over $[\mathcal{F}] \in \mathcal{V}_d^{2g-3,1,1}$ general is finite. As in (6.13) in Remark 6.16, it suffices to prove the following:

Claim 7.6. *$a_F(2g-3) = 0$.*

Proof of the Claim. Assume by contradiction this is not zero. Since $\mathcal{F}$ is stable, hence unsplit, from $\varphi_0(2g-3,1,1) = 0$ and Remark 4.2, $a_F(2g-3)$ must be 1 (cf. Proposition 2.12).

Let $\mathfrak{F}$ be the corresponding one-dimensional family of sections of $F = \mathbb{P}(\mathcal{F})$, which has positive self-intersection, since $\mathcal{F}$ is stable. From Proposition 2.12 and Step 1, the system $\mathfrak{F}$ cannot be contained in a linear system, otherwise we would have sections of degree lower than $2g-3$.

Thus, from the proof of Proposition 2.12, there is an open, dense subset $C^0 \subset C$ such that, for any $q \in C^0$, one has $\mathcal{F} = \mathcal{F}_v$ with $v = v_q \in \mathrm{Ext}^1(\omega_C(-q), N_q))$, where $\{N_q\}_{q \in C^0}$ is a 1-dimensional family of non-isomorphic line bundles of degree $d-2g+3$, whose general member is general in $\mathrm{Pic}^{d-2g+3}(C)$. Let $\Gamma_q \subset F_v$ be the section corresponding to $\mathcal{F}_v \twoheadrightarrow \omega_C(-q)$; so the one-dimensional family is $\mathfrak{F} = \{\Gamma_q\}_{q \in C^0}$.

We set $\widetilde{\Gamma}_q := \Gamma_q + f_q$, for $q \in C^0$. From (2.1), $\widetilde{\Gamma}_q$ corresponds to $\mathcal{F}_v \twoheadrightarrow \omega_C(-q) \oplus \mathcal{O}_q$, whose kernel we denote by N'_q. Then $\widetilde{\mathfrak{F}} = \{\widetilde{\Gamma}_q\}_{q \in C^0}$ is a one-dimensional family of unisecants of F_v of degree $2g-2$ and speciality 1 (cf. (2.8)). For $h, q \in C^0$, we have

$$c_1(N'_h) = \det(\mathcal{F}_v) \otimes \omega_C^\vee = c_1(N'_q).$$

Therefore, from (2.6), $\widetilde{\mathfrak{F}}$ is contained in a linear system $|\mathcal{O}_{F_v}(\Gamma)|$. By Bertini's theorem, the general member of $|\mathcal{O}_{F_v}(\Gamma)|$ is a section of degree $2g-2$. In particular, $\dim(|\mathcal{O}_{F_v}(\Gamma)|) \geqslant 2$.

If L_Γ is the corresponding quotient line bundle, since $\Gamma \sim \widetilde{\Gamma}_q$, then $c_1(L_\Gamma) = \omega_C$, i.e. Γ is a canonical section. This is a contradiction: indeed, if M_{ω_C} is the kernel of the surjection $\mathcal{F}_v \twoheadrightarrow \omega_C$, we have (cf. (6.7))

$$2 \leqslant \dim(|\mathcal{O}_{F_v}(\Gamma)|) = h^0(\mathcal{F}_v \otimes M_{\omega_C}^\vee) - 1 = i(\mathcal{F}_v) - 1 = 1.$$

$\square$

Case (b). As in Remark 6.17-(2), a necessary condition for $\nu_d^{\delta,1,1} \geqslant \rho_d^{k_2}$ is (6.14), i.e. $nr - n - 1 < 0$. Since $n, r \geqslant 1$, the only possibility is $r = t = 1$. Taking into account (6.10), (6.12) and Proposition 6.15, one has $\varphi_n(\delta, 1, 1) = \varphi_0(\delta, 1, 1) = \delta - 2g + 3 \geqslant 0$. In any case we would have $d \geqslant 3g - 5$, which is out of our range. Thus, case (b) cannot occur.

Step 3. In this step we prove that $\overline{\mathcal{V}_d^{2g-3,1,1}}$ is actually a component of $\widetilde{B_C^{k_2}}(d)$.

Let $\mathcal{B} \subseteq \widetilde{B_C^{k_2}}(d)$ be a component containing $\overline{\mathcal{V}_d^{2g-3,1,1}}$ and let $[\mathcal{F}] \in \mathcal{B}$ general. By semicontinuity, $\mathcal{F}$ has speciality $i = 2$. It has also a special presentation as in (2.2), with $2g-3 \leqslant \deg(L) = \delta \leqslant 2g-2$. Since C has general moduli, then $h^1(L) = j = 1$ so the corank of the coboundary map is $t = 1$. If $\delta = 2g-3$, from Step 2 we are done.

Assume therefore $\delta = 2g-2$, so $L = \omega_C$. Notice that:*(i)* $r = h^1(N) \leqslant g$; *(ii)* $m = \dim(\mathrm{Ext}^1(N, \omega_C)) \geqslant g+1$. Indeed, (i) is trivial if N is effective. On the other hand, if $h^0(N) = 0$, then $h^1(N) = 3g-3-d < g$ since $d \geqslant 2g-2$. As for (ii), $m = 5g-5-d$ (cf. (4.2)), hence (ii) follows since $d \leqslant 3g-6$. So we are in position to apply Theorem 5.8 and Corollary 5.9, which yield that $\widehat{\mathcal{W}}_1^{\mathrm{Tot}} \subseteq \mathbb{P}(\mathfrak{E}_{2g-2})$ is irreducible and good. Hence $\dim(\widehat{\mathcal{W}}_1^{\mathrm{Tot}}) \leqslant 8g-10-2d$ (equality holds when N is general, i.e. non effective).

On the other hand, $\mathcal{B}$ is the image of $\widehat{\mathcal{W}}_1^{\mathrm{Tot}}$ via $\pi_{d,2g-2}$ (cf. Step 1) and the general fibre of this map has dimension at least 1 because $h^1(\mathcal{F}) = 2$ (cf. (2.6) and computation as in (6.7)). Thus

$$8g-11-2d \geqslant \dim(\mathcal{B}) \geqslant \dim(\overline{\mathcal{V}_d^{2g-3,1,1}}) = 8g-11-2d = \rho_d^{k_2}.$$

This proves that $\mathcal{B} = \overline{\mathcal{V}_d^{2g-3,1,1}}$ is a regular component.

The previous argument also shows that $\overline{\mathcal{V}_d^{2g-3,1,1}} = \overline{\mathcal{V}_d^{2g-2,1,1}}$ (cf. Remark 6.17) and that the dimension of the general fibre of $\pi_{d,2g-2}|_{\widehat{\mathcal{W}}_1^{\mathrm{Tot}}}$ onto $\mathcal{V}_d^{2g-2,1,1}$ has exactly dimension 1 (actually, it is a $\mathbb{P}^1$, cf. Lemma 2.11).

Step 4. Assume we have a component $\mathcal{B} \subseteq \widetilde{B_C^{k_2}}(d)$, whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 2$. From Step 1, $[\mathcal{F}] \in \mathcal{B}$ general can be specially presented as in (2.2), with $h^1(L) = j = 1$, so N is special. The same discussion as in Steps 2 and 3 shows that $\mathcal{B} = \overline{\mathcal{V}_d^{2g-3,1,1}}$. $\square$

Remark 7.7. (i) For $[\mathcal{F}] \in \overline{\mathcal{V}_d^{2g-3,1,1}}$ general, one has $\ell = g-1$, $r = 3g-4-d$ and $m = 5g-7-d$ (cf. (4.2), (4.4)). So $\ell \geqslant r+1$ (because $d \geqslant 2g-2$), moreover $m \geqslant \ell + g$ (because $d \leqslant 3g-6$). Note that the inequality $m \geqslant \ell + 1$ is necessary to ensure $\emptyset \neq \mathcal{W}_1^{Tot} \subset \mathbb{P}(\mathfrak{E}_{2g-2})$ (see the proof of Theorem 5.8).

(ii) Step 4 of Theorem 7.5 shows that, if $\mathcal{B}$ is a component of $\widetilde{B_C^{k_2}}(d)$, different from $\overline{\mathcal{V}_d^{2g-3,1,1}}$, then $\mathcal{B}$ is a component of $\widetilde{B_C^{k_i}}(d)$, for some $i \geqslant 3$, and as such it is not regular. Otherwise, we would have

$$8g-11-2d \leqslant \dim(\mathcal{B}) = 4g-3-i(d-2g+2+i) \leqslant 10-3d-16,$$

i.e $d \leqslant 2g-7$ which is out of our range for d.

Remark 7.8. (1) Take $\widetilde{\Gamma}_p$ as in the proof of Claim 7.6. Then, $\mathcal{N}_{\widetilde{\Gamma}_p/F_v}$ is non special on the (reducible) unisecant $\widetilde{\Gamma}_p$. Indeed, $\omega_{\widetilde{\Gamma}_p} \otimes \mathcal{N}^{\vee}_{\widetilde{\Gamma}_p/F_v}|_{\Gamma} \cong \mathcal{O}_{\Gamma}(K_F)$ whereas $\omega_{\widetilde{\Gamma}_p} \otimes \mathcal{N}^{\vee}_{\widetilde{\Gamma}_p/F_v}|_{f_p} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. Thus $\widetilde{\Gamma}_p \in \mathrm{Div}^{1,2g-2}_{F_v}$ is a smooth point. Moreover, $h^0(\mathcal{N}_{\widetilde{\Gamma}_p/F_v}) = 3g-2-d \geqslant 2$ for $d \leqslant 3g-4$. From the generality of v in the good locus $\mathcal{W}_1$, (2.6) and from computation as in (6.7), one has that $\widetilde{\Gamma}_p \subset F_v$ is a (reducible) unisecant, moving in a complete linear pencil of special unisecants whose general member is a canonical section, and $\widetilde{\Gamma}_p$ is algebraically equivalent on F_v to non-special sections of degree $2g-2$.

As soon as $d \leqslant 3g-6$, there are in $\mathrm{Div}^{1,2g-2}_{F_v}$ unisecants containing two general fibres (cf. Proposition 2.12) hence the ruled surface F_v has (non special) sections of degree smaller than $2g-3$.
(2) Take $N \in \mathrm{Pic}^k(C)$ general with $0 < k \leqslant g-2$. Since N is special, non-effective, from Corollary 5.9 and Remark 5.15, $v \in \Lambda_1 \subset \mathrm{Ext}^1(\omega_C, N)$ general determines $\mathcal{F} := \mathcal{F}_v$ stable, with $i(\mathcal{F}) = 2$. If Γ denotes the canonical section corresponding to $\mathcal{F} \to \omega_C$, from (2.6) one has $\dim(|\mathcal{O}_F(\Gamma)|) = 1$ and all unisecants in this linear pencil are special (cf. Lemma 2.11). Since F is indecomposable, $|\mathcal{O}_F(\Gamma)|$ has base-points (cf. the proof of Proposition 2.12, from which we keep the notation). Thus, $\mathcal{F}$ is rsp via $\omega_C(-p)$, for $p = \rho(q)$ and $q \in F$ a base point of the pencil (recall Remark 6.17).

Remark 7.9. In [50, 51] the locus $\widetilde{B^2_C}(b)$ is studied, for $g \geqslant 2$ and $3 \leqslant b \leqslant 2g-1$. It is proved there with different arguments that, when C has general moduli, then $\widetilde{B^2_C}(b)$ is not empty, irreducible, regular (with $\rho^2_b = 2b-3$), generically smooth and $[\mathcal{E}] \in \widetilde{B^2_C}(b)$ general is stable, with $h^0(\mathcal{E}) = 2$, fitting in a sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to L \to 0. \tag{7.4}$$

Considering the natural isomorphism $\widetilde{B^2_C}(b) \cong \widetilde{B^{k_2}_C}(4g-4-b)$, when $d := 4g-4-b$ is as in Theorem 7.5 we recover Teixidor's results (without irreducibility) via a different approach. Thus, Teixidor's results and our analysis imply that, for any $2g-2 \leqslant d \leqslant 4g-7$, $\widetilde{B^{k_2}_C}(d) = \overline{\mathcal{V}^{2g-3,1,1}_d}$. Theorem 7.5 provides in addition the rigidly special presentation of the general element of $\widetilde{B^{k_2}_C}(d)$.

Theorem 7.10. *Let C be of genus $g \geqslant 3$, with general moduli. Then, $\widetilde{M^2_C}(2, \omega_C) \neq \emptyset$ and irreducible. Moreover, it is regular (i.e. of dimension $3g-6$), and its general point $[\mathcal{F}_v]$ fits into an exact sequence*

$$0 \to \mathcal{O}_C(p) \to \mathcal{F}_v \to \omega_C(-p) \to 0,$$

where

- *$p \in C$ is general, and*
- *$v \in \Lambda_1 = \mathcal{W}_1 \subset \mathrm{Ext}^1(\omega_C(-p), \mathcal{O}_C(p))$ is general.*

Proof. Irreducibility follows from [40, Thm. 1.3]. With notation as in (4.4), (4.5), one has

$$\ell = r = g-1, \; m = h^1(2p - K_C) = 3g-5 \geqslant g = \ell + 1;$$

from Corollary 5.9, $\mathcal{W}_1$ is good and $v \in \mathcal{W}_1$ general is such that $\mathrm{cork}(\partial_v) = 1$. In particular, $\dim(\mathcal{W}_1) = 3g-6$.

Stability of $\mathcal{F}_v$, with $1 < s(\mathcal{F}_v) = \sigma < g$, follows from Proposition 4.4. Finally one uses the same approach as in Claim 7.6 to deduce that $\pi_{2g-2,2g-3}|_{\mathcal{W}_1^{\mathrm{Tot}}}$ is generically finite (cf. (6.4)), since $\mathcal{F}_v$ is rsp via $\omega_C(-p)$. ☐

Generic smoothness of the components of $\widetilde{M^2_C}(2, \omega_C)$ follows from results in [54, 5]. Theorem 7.10 can be interpreted in the setting of [7] as saying that, for a curve C of general moduli of genus $g \geqslant 3$, $\mathbb{P}(\mathrm{Ext}^1(\omega_C, \mathcal{O}_C))$ is not contained in the divisor D_1 considered in that paper.

7.4. **Case** $i = 3$.

One has $\rho^{k_3}_d = 10g - 18 - 3d$. We have the following:

Theorem 7.11. *Let C be of genus $g \geqslant 8$, with general moduli. For any integer d s.t. $2g-2 \leqslant d \leqslant \frac{5}{2}g - 6$, one has $\widetilde{B^{k_3}_C}(d) \neq \emptyset$. Moreover:*
(i) $\overline{\mathcal{V}^{2g-4,1,2}_d}$ is the unique component of $\widetilde{B^{k_3}_C}(d)$ of types either (6.5) or (6.8), whose general point corresponds to a vector bundle $\mathcal{F}$ with $i(\mathcal{F}) = 3$. Furthermore, it is regular and $\overline{\mathcal{V}^{2g-4,1,2}_d} = \overline{\mathcal{V}^{2g-3,1,2}_d} = \overline{\mathcal{V}^{2g-2,1,2}_d}$.
(ii) For $[\mathcal{F}] \in \overline{\mathcal{V}^{2g-4,1,2}_d}$ general, one has $s(\mathcal{F}) \geqslant 5g - 10 - 2d - \epsilon \geqslant 2 - \epsilon$ and $\mathcal{F}$ fits into an exact sequence

$$0 \to N \to \mathcal{F} \to \omega_C(-D_2) \to 0,$$

where

- *$D_2 \in C^{(2)}$ is general,*
- *$N \in \mathrm{Pic}^{d-2g+4}(C)$ is general (special, non-effective),*

- *$\mathcal{F} = \mathcal{F}_v$ with v general in a good component $\Lambda_2 \subset \mathrm{Ext}^1(N, \omega_C(-D_2))$ (cf. Definition 5.12).*

(iii) Any section $\Gamma \subset F = \mathbb{P}(\mathcal{F})$, corresponding to a quotient $\mathcal{F} \to\!\!\!\!\to \omega_C(-D_2)$, is not of minimal degree. However, it is minimal among special sections of F; moreover, Γ is asi but not ai (i.e., $\mathcal{F}$ is rsp via $\omega_C(-D_2)$).

Proof. As in Theorem 7.5, once (i) has been proved, parts (ii)-(iii) follow from results proved in previous sections. Precisely, by definition of $\mathcal{V}_d^{2g-4,1,2}$ one has $L = \omega_C(-D_2)$, with $D_2 \in C^{(2)}$, $t = 2$ and $N \in \mathrm{Pic}^{d-2g+4}(C)$ of speciality $r \geqslant 2$. From regularity of the component, Proposition 6.15 and (6.9), (6.10), (6.12) give

$$0 = \nu_d^{2g-4,1,2} - \rho_d^{k_3} \leqslant \dim(\widehat{\Lambda}_2^{\mathrm{Tot}}) - \rho_d^{k_3} = \varphi_n(2g-4,1,2) = \varphi_0(2g-4,1,2) - n(r-2) = -n(r-2).$$

Thus, $n(r-2) = 0$. This implies that the general fibre of $\pi_{d,2g-4}|_{\widehat{\Lambda}_2^{\mathrm{Tot}}}$ is finite, i.e. $[\mathcal{F}_v] \in \mathcal{V}_d^{2g-4,1,2}$ general is rsp via $\omega_C(-D_2)$ (correspondingly $\Gamma \subset F_v = \mathbb{P}(\mathcal{F}_v)$ is asi as in (iii)).

Since $n(r-2) = 0$, then either $n = 0$ or $r = 2$. The latter case cannot occur otherwise we would have $n = d - 3g + 7 \leqslant -\frac{g}{2} + 1 < 0$, by the assumptions on d.

Thus $n = 0$ and $r = 3g-5-d$. Moreover, from (4.2), (5.5), one has $m = 5g-10-d$ and $c(\ell, r, 2) = 2d+10-4g$, so a good component $\Lambda_2 \subset \mathbb{P}(\mathrm{Ext}^1(\omega_C(-D_2), N))$ has dimension $9g - 20 - 3d$. If we add up to this quantity g, for the parameters of N, we get $10g - 20 - 3d$. Thus, regularity forces D_2 to be general in $C^{(2)}$.

Now, $\mathcal{N}_{\Gamma/F_v} \cong K_C - D_2 - N$ (cf. (2.3)) so $h^i(\mathcal{N}_{\Gamma/F_v}) = h^{1-i}(N + D_2)$ for $0 \leqslant i \leqslant 1$. By the assumptions on d, $\deg(N + D_2) = d - 2g + 6 \leqslant \frac{g}{2}$, thus generality of N implies that $N + D_2$ is also general, so $h^0(N + D_2) = 0$ and $h^1(N + D_2) = 3g - 7 - d \geqslant \frac{g}{2} - 1$. This implies that Γ is not ai and not of minimal degree among quotient line bundles of $\mathcal{F}$.

Numerical conditions of Theorem 5.13 (see also Remark 5.15) are satisfied for $j = 1$, $t = 2$ and $\delta = 2g - 4$, under the assumptions $d \leqslant \frac{5}{2}g - 6$.

Finally, the fact that Γ is of minimal degree among special, quotient line bundles of $\mathcal{F}$ follows from the proof of part (i) below, which consists of the following steps.

Step 1. In this step we determine which of the loci of the form $\mathcal{V}_d^{\delta,j}$, as in (6.5), or $\mathcal{V}_d^{\delta,j,t}$, as in (6.8):

(a) has the general point $[\mathcal{F}]$ with $i(\mathcal{F}) = 3$,
(b) is the image, via $\pi_{d,\delta}$, of a parameter space in $\mathbb{P}(\mathfrak{E}_\delta)$ of dimension at least $\rho_d^{k_3} = 10g - 18 - 3d$.

Let $\mathcal{V}$ be such locus and use notation as in (4.4), (4.5). From Remarks 6.4, 6.6, 6.11, conditions (a) and (b) are both satisfied only if the presentation of $[\mathcal{F}] \in \mathcal{V}$ general as in (2.2) with L special and effective, is such that $N \in \mathrm{Pic}^{d-\delta}(C)$ is special and the coboundary map $\partial : H^0(L) \to H^1(N)$ is not surjective. Possibilities are:
(i) $j = 1$ and $t = \mathrm{cork}(\partial) = 2$;
(ii) $j = 2$ and $t = 1$.

In any event, one has $\ell \geqslant r$ (in particular, we will be in position to apply Theorems 5.8, 5.17; cf. e.g the proof of Claim 7.13 below). Indeed:

- in case *(i)*, the only possibilities for $\ell < r$ are $r - 2 \leqslant \ell \leqslant r - 1$. Then, $\dim(\mathbb{P}(\mathrm{Ext}^1(L, N)) = 2\delta - d + g - 2$, $\rho(L) = g - (\delta - g + 2)$, $\rho(N) = g - rn$, so the number of parameters is $\delta + 4(g-1) - d - rn < \rho_d^{k_3}$ since $d \leqslant \frac{5}{2}g - 6$.
- in case *(ii)*, the only possibility for $\ell < r$ is $\ell = r - 1$. The same argument as above applies, the only difference is that $\rho(L) = g - 2(\delta - g + 3)$.

Since $\ell \geqslant r$, we see that case *(ii)* cannot occur by (6.11) and (6.12). Thus, we focus on $\mathcal{V}_d^{\delta,1,2}$, investigating for which δ it satisfies (b). We will prove that this only happens for $2g - 4 \leqslant \delta \leqslant 2g - 2$.

We have two cases: (1) N effective, (2) N non-effective. We will show that only case (2) occurs.

Case (1). When N is effective, from Remark 6.17, a necessary condition for (b) to hold is (6.14), which reads $(r-2)n - 2 < 0$. This gives $2 \leqslant r \leqslant 3$, since $r \geqslant t = 2$. We can apply Theorem 5.13 (more precisely, Remark 5.16): the first line of bounds on δ in Remark 5.16 gives $\delta \leqslant \frac{3g-\epsilon}{2} + r - 4$. In particular, one must have $\delta \leqslant \frac{3g-\epsilon}{2} - 1$.

On the other hand, another necessary condition for (b) to hold is $\varphi_n(\delta, 1, 2) \geqslant 0$ (cf. Proposition 6.15). From Remark 6.14, $\varphi_n(\delta, 1, 2) \leqslant \varphi_0(\delta, 1, 2) = \delta - 2g + 4$ (cf. (6.10)) and $\varphi_0(\delta, 1, 2) \geqslant 0$ gives $\delta \geqslant 2g - 4$ which contradicts $\delta \leqslant \frac{3g-\epsilon}{2} - 1$, since $g \geqslant 8$.

Case (2). When N is non-effective, we apply Theorem 5.13 (more precisely, Remark 5.15), with $j = 1$ and $t = 2$. By the same argument as in case (1), we see that $2g - 4 \leqslant \delta \leqslant 2g - 2$.

Step 2. In this step we prove that the loci $\mathcal{V}_d^{\delta,1,2}$, with $2g - 4 \leqslant \delta \leqslant 2g - 2$, are not empty. Precisely, we will exhibit components $\mathcal{V}_d^{\delta,1,2}$ which are the image, via $\pi_{d,\delta}$, of a total good component $\widehat{\Lambda}_2^{\mathrm{Tot}} \subset \mathbb{P}(\mathfrak{E}_\delta)$, of dimension $\rho_d^{k_3} + \delta - 2g + 4$ (cf. Definition 6.13 and (6.8)).

We will treat only the case $\delta = 2g - 4$, i.e. $L = \omega_C(-D_2)$, with $D_2 \in C^{(2)}$, since the cases $L = \omega_C$, $\omega_C(-p)$ can be dealt with similar arguments and can be left to the reader.

Claim 7.12. *Let $N \in \mathrm{Pic}^{d-2g+4}(C)$ be general. For $V_2 \in \mathbb{G}(2, H^0(K_C - N))$ general, the map μ_{V_2} as in* (5.8) *is injective.*

Proof of Claim 7.12. The general $V_2 \in \mathbb{G}(2, H^0(K_C - N))$ determines a base point free linear pencil on C. Indeed, $h^0(K_C - N) = 3g - 5 - d \geqslant 5$. Take $\sigma_1, \sigma_2 \in H^0(K_C - N)$ general sections. If $p \in C$ is such that $\sigma_i(p) = 0$, for $i = 1, 2$, by the generality of the sections we would have $p \in \mathrm{Bs}(|K_C - N|)$ so $h^0(N + p) = 1$. This is a contradiction because N is general and $\deg(N) < g - 1$. The injectivity of μ_{V_2} follows from the base-point-free pencil trick: indeed, $\mathrm{Ker}(\mu_{V_2}) \cong H^0(N(-D_2))$ which is zero since N is non-effective. ☐

Claim 7.13. *Let $N \in \mathrm{Pic}^{d-2g+4}(C)$ and $D_2 \in C^{(2)}$ be general. Then, there exists a unique good component $\Lambda_2 \subset \mathrm{Ext}^1(\omega_C(-D_2), N)$ whose general point v is such that $\mathrm{Coker}(\partial_v)^\vee$ is general in $\mathbb{G}(2, H^0(K_C - N))$ (cf. Remark 5.7).*

Proof of Claim 7.13. With notation as in (4.2), (4.4), we have $\ell = g - 2$, $m = 5g - 9 - d$ and $r = 3g - 5 - d$. Then assumptions on d and g imply

$$m \geqslant 2\ell + 1 \ \text{ and } \ \ell \geqslant r \geqslant t = 2 \tag{7.5}$$

(cf. Step 1 for $\ell \geqslant r$). From (7.5) and Claim 7.12, we are in position to apply Theorem 5.17, with $\eta = 0$ and $\Sigma_\eta = \mathbb{G}(2, H^0(K_C - N))$.

This yields the existence of a good component $\Lambda_2 \subseteq \mathcal{W}_2 \subset \mathrm{Ext}^1(\omega_C(-D_2), N)$. Actually, Λ_2 is the only good component whose general point v gives $\mathrm{Coker}(\partial_v)^\vee = V_2$ general in $\mathbb{G}(2, H^0(K_C - N))$.

Indeed any component of $\mathcal{W}_2$, whose general point v is such $\dim(\mathrm{Coker}(\partial_v)) = 2$, is obtained in the following way (cf. the proofs of Theorems 5.8, 5.17):

- take any $\Sigma \subseteq \mathbb{G}(2, H^0(K_C - N))$ irreducible, of codimension $\eta \geqslant 0$;
- for V_2 general in Σ, consider $H^0(\omega_C(-D_2)) \otimes V_2$ and the map μ_{V_2} as in (5.8);
- let $\tau := \dim(\mathrm{Ker}(\mu_{V_2})) \geqslant 0$ and $\mathbb{P} := \mathbb{P}(\mathrm{Ext}^1(\omega_C(-D_2), N)) = \mathbb{P}(H^0(2K_C - D_2 - N)^\vee)$ (cf. (4.6));
- consider the incidence variety
$$\mathcal{J}_\Sigma := \{(\sigma, \pi) \in \Sigma \times \mathbb{P} \mid \mathrm{Im}(\mu_{V_2}) \subset \pi\}\,.$$
Since $m \geqslant 2\ell + 1 - \tau$, one has $\mathcal{J}_\Sigma \neq \emptyset$ (cf. the proofs of Theorems 5.8, 5.17);
- consider the projections $\Sigma \xleftarrow{pr_1} \mathcal{J}_\Sigma \xrightarrow{pr_2} \mathbb{P}$;
- the fibre of pr_1 over any point V_2 in the image is $\{\pi \in \mathbb{P} \,|\, \mathrm{Im}(\mu_{V_2}) \subset \pi\}$, i.e. it is isomorphic to the linear system of hyperplanes of $\mathbb{P}$ passing through the linear subspace $\mathbb{P}(\mathrm{Im}(\mu_{V_2}))$. For $V_2 \in \Sigma$ general, this fibre is irreducible of dimension $m - 1 - 2\ell + \tau = 3g - 6 - d + \tau$. In particular, there exists a unique component $\mathcal{J} \subseteq \mathcal{J}_\Sigma$ dominating Σ via pr_1;
- since $r = 3g - 5 - d$, one has
$$\dim(\mathcal{J}) = 9g - 20 - 3d + \tau - \eta = \mathrm{expdim}(\widehat{\mathcal{W}}_2) + \tau - \eta,$$

where $\widehat{\mathcal{W}}_2 = \mathbb{P}(\mathcal{W}_2) \subset \mathbb{P}$ (notation as in the proof of Theorem 5.8). By construction, $pr_2(\mathcal{J}) \subseteq \widehat{\mathcal{W}}_2$. Moreover, if ϵ denotes the dimension of the general fibre of $pr_2|_{\mathcal{J}}$, then $pr_2(\mathcal{J})$ can fill up a component $\mathcal{X}$ of $\widehat{\mathcal{W}}_2$ only if $\tau - \eta - \epsilon \geqslant 0$: the component $\mathcal{X}$ is good when equality holds.

When $\Sigma = \mathbb{G}(2, H^0(K_C - N))$, then $\eta = 0$ and, by Claim 7.12, also $\tau = 0$. Since $\mathcal{J} \subseteq \mathcal{J}_{\mathbb{G}(2,H^0(K_C-N))}$ is the unique component dominating $\mathbb{G}(2, H^0(K_C - N))$, then $pr_2(\mathcal{J})$ fills up a component $\widehat{\Lambda}_2$ of $\widehat{\mathcal{W}}_2$, i.e. $\epsilon = 0$. Thus $\widehat{\Lambda}_2$ is good. ☐

By the generality of $N \in \mathrm{Pic}^{d-2g+4}(C)$ and of $D_2 \in C^{(2)}$, Claim 7.13 ensures the existence of a total good component $\widehat{\Lambda}_2^{\mathrm{Tot}} \subset \mathbb{P}(\mathfrak{E}_{2g-4})$.

For $2g - 4 \leqslant \delta \leqslant 2g - 2$, we will denote by V_d^δ the total good component we constructed in this step. To ease notation, we will denote by $\mathcal{V}_d^\delta$ its image in $\widetilde{B_C^{k_3}}(d)$ via $\pi_{d,\delta}$, which is a $\mathcal{V}_d^{\delta,1,2}$ as in Step 1.

Step 3. In this step, we prove that $\mathcal{V}_d^{2g-2}$ has dimension $\rho_d^{k_3}$.

From Step 2, one has $\dim(V_d^{2g-2}) = \rho_d^{k_3} + 2 = 10g - 16 - 3d$. We want to show that the general fibre of $\pi_{d,2g-2}|_{V_d^{2g-2}}$ has dimension two. To do this, we use similar arguments as in the proof of Lemma 6.2.

Let $[\mathcal{F}] \in \mathcal{V}_d^{2g-2}$ be general; by Step 2, $\mathcal{F} = \mathcal{F}_u$, for $u \in \widehat{\Lambda}_2 \subset \mathbb{P}(H^0(2K_C - N)^\vee)$ general and $N \in \mathrm{Pic}^{d-2g+2}(C)$ general, where $\widehat{\Lambda}_2 = pr_2(\mathcal{J})$ and $\mathcal{J} \subset \mathbb{G}(2, H^0(K_C - N)) \times \mathbb{P}(H^0(2K_C - N)^\vee)$ the unique component dominating $\mathbb{G}(2, H^0(K_C - N))$ (cf. the proof of Claim 7.13). Then

$$(\pi_{d,2g-2}|_{V_d^{2g-2}})^{-1}([\mathcal{F}_u]) = \left\{(N', \omega_C, u') \in V_d^{2g-2} \mid \mathcal{F}_{u'} \cong \mathcal{F}_u\right\}.$$

In particular, one has $N \cong N'$ so $u, u' \in \widehat{\Lambda}_2 \subset \mathbb{P}(H^0(2K_C - N)^\vee)$.

Let $\varphi : \mathcal{F}_{u'} \overset{\cong}{\to} \mathcal{F}_u$ be the isomorphism between the two bundles and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\iota_1"] & \mathcal{F}_{u'} \arrow[r] \arrow[d, "\varphi"] & \omega_C \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r, "\iota_2"] & \mathcal{F}_u \arrow[r] & \omega_C \arrow[r] & 0.
\end{tikzcd}$$

If $u = u'$, then $\varphi = \lambda \in \mathbb{C}^*$ (since $\mathcal{F}_u$ is simple) and the maps $\lambda\iota_1$ and ι_2 determine two non-zero sections $s_1 \neq s_2 \in H^0(\mathcal{F}_u \otimes N^\vee)$. Similar computation as in (6.7) shows that $h^0(\mathcal{F}_u \otimes N^\vee) = i(\mathcal{F}_u) = 3$, since $u \in \widehat{\Lambda}_2$ general. Therefore, if $\Gamma \subset F_u$ denotes the section corresponding to $\mathcal{F}_u \twoheadrightarrow\!\!\!\to \omega_C$, $(\pi_{d,2g-2}|_{\mathcal{V}_d^{2g-2}})^{-1}([\mathcal{F}_u])$ contains a $\mathbb{P}^2$ isomorphic to $|\mathcal{O}_{F_u}(\Gamma)|$ (cf. (2.6) and Lemma 2.11).

The case $u \neq u'$ cannot occur. Indeed, for any inclusion ι_1 as above, there exist an inclusion ι_2 and a $\lambda = \lambda(\iota_1, \iota_2) \in \mathbb{C}^*$ such that $\varphi \circ \iota_1 = \lambda\iota_2$, otherwise we would have $\dim(|\mathcal{O}_{F_u}(\Gamma)|) > 2$, a contradiction. One concludes by Lemma 4.5.

In conclusion, the general fibre of $\pi_{d,2g-2}|_{\mathcal{V}_d^{2g-2}}$ has dimension two (actually, this fibre is a $\mathbb{P}^2$).

Step 4. In this step we prove that $\overline{\mathcal{V}_d^{2g-2}} = \overline{\mathcal{V}_d^{2g-3}} = \overline{\mathcal{V}_d^{2g-4}} := \overline{\mathcal{V}}$. In particular, the presentation of $[\mathcal{F}] \in \overline{\mathcal{V}}$ general will be specially rigid only for $\delta = 2g-4$.

From Step 2 one has $\dim(V_d^\delta) = \rho_d^{k_3} + \delta - 2g + 4$, for $2g - 4 \leqslant \delta \leqslant 2g - 2$. Moreover, the general element of V_d^δ can be identified with a pair (F, Γ), where $F = \mathbb{P}(\mathcal{F})$, $\Gamma \subset F$ a section corresponding to $\mathcal{F} \twoheadrightarrow\!\!\!\to \omega_C(-D)$, where $D \in C^{(2g-2-\delta)}$ and, for $\delta = 2g-2$, one has $D = 0$ and $\dim(|\mathcal{O}_F(\Gamma)|) = 2$.

We will now prove that there exist dominant, rational maps:

(a) $r_1 : V_d^{2g-2} \times C \dashrightarrow V_d^{2g-3}$, such that $r_1((F,\Gamma), p) = (F, \Gamma_p)$, where $\Gamma_p \subset F$ is a section corresponding to $\mathcal{F} \twoheadrightarrow\!\!\!\to \omega_C(-p)$,

(b) $r_2 : \widetilde{V}_d^{2g-2} \dashrightarrow V_d^{2g-4}$, where $\widetilde{V}_d^{2g-2}$ is a finite cover $\varphi : \widetilde{V}_d^{2g-2} \to V_d^{2g-2}$ endowed with a rational map $\psi : \widetilde{V}_d^{2g-2} \dashrightarrow C^{(2)}$: if $\xi \in \widetilde{V}_d^{2g-2}$ is general and $\varphi(\xi) := (F, \Gamma)$, then $r_2(\xi) = (F, \Gamma')$, with Γ' a section corresponding to $\mathcal{F} \twoheadrightarrow\!\!\!\to \omega_C(-\psi(\xi))$.

The existence of these maps clearly proves that $\overline{\mathcal{V}_d^{2g-2}} = \overline{\mathcal{V}_d^{2g-3}} = \overline{\mathcal{V}_d^{2g-4}}$.

(a) Take (F, Γ) general in V_d^{2g-2} and $p \in C$ general. Then, the restriction map

$$\mathbb{C}^3 \cong H^0(\mathcal{O}_F(\Gamma)) \to H^0(\mathcal{O}_{f_p}(\Gamma)) \cong \mathbb{C}^2$$

is surjective, because the general member of $|\mathcal{O}_F(\Gamma)|$ is irreducible. Hence there is a unique $\Gamma_p \in |\mathcal{O}_F(\Gamma - f_p)|$.

We claim that Γ_p is irreducible, i.e. it is a section. If not, Γ_p would be a section plus a number $n \geqslant 1$ of fibres. As we saw, $n \leqslant 1$ (cf. Step 1) so $n = 1$. This determines an automorphism of C and, since C has general moduli, this automorphism must be the identity. This is impossible because the map $\Phi_\Gamma : F \dashrightarrow \mathbb{P}^2$, given by $|\mathcal{O}_F(\Gamma)|$, is dominant hence it is ramified only in codimension one.

In conclusion, Γ_p corresponds to $\mathcal{F} \twoheadrightarrow\!\!\!\to \omega_C(-p)$ and (F, Γ_p) belongs to V_d^{2g-3}, and this defines r_1. The proof that (F, Γ_p) belongs to V_d^{2g-3} is postponed for a moment (cf. Claim 7.14).

(b) Given (F, Γ) general in V_d^{2g-2}, we can consider the map Φ_Γ as in Case (a). Since Φ_Γ maps the rulings of F to lines, it determines a morphism $\Psi : C \to C' \subset (\mathbb{P}^2)^\vee$. From Step 1, no (scheme-theoretical) fibre of Ψ can have length bigger than two. Therefore, since C has general moduli, $\Psi : C \to C'$ is birational and moreover, since $g \geqslant 8$, C' has a certain number n of double points, corresponding to curves of type $\Gamma_D + f_D$, with $D \in C^{(2)}$ fibre of Ψ over a double point of C'.

Then the general point ξ of $\widetilde{V}_d^{2g-2}$ corresponds to a triple (F, Γ, D) (with $D \in C^{(2)}$ as above), the pair (F, Γ_D) belongs to V_d^{2g-4} (cf. Claim 7.14) and $r_2(F, \Gamma, D) = (F, \Gamma_D)$, $\psi(F, \Gamma, D) = D$.

Claim 7.14. *With the above notation, (F, Γ_p) belongs to V_d^{2g-3} and (F, Γ_D) belongs to V_d^{2g-4}.*

Proof of Claim 7.14. We prove the claim for (F, Γ_D), since the proof is similar in the other case. Take (F, Γ) general in V_d^{2g-2}; this determines a sequence

$$0 \to N \to \mathcal{F} \to \omega_C \to 0, \tag{7.6}$$

where N is general of degree $d - 2g + 2$ and the corresponding extension is general in the unique (good) component $\widehat{\Lambda}_2 \subset \mathbb{P}(\mathrm{Ext}^1(\omega_C, N))$ dominating $\mathbb{G}(2, H^0(K_C - N))$ (cf. the proof of Step 2); thus, if ∂ is the coboundary map, then $\mathrm{Coker}(\partial)$ is a general two-dimensional quotient of $H^1(N)$.

On the other hand, (F, Γ_D) determines a sequence

$$0 \to N(D) \to \mathcal{F} \to \omega_C(-D) \to 0.$$

Since $\deg(N(D)) = d - 2g + 4 \leqslant \frac{g}{2} - 2 < g - 1$ and $N(D)$ is general of its degree, then $h^0(N(D)) = 0$. In view of

$$0 \to N \to N(D) \to \mathcal{O}_D \to 0,$$

one has the exact sequence

$$0 \to H^0(\mathcal{O}_D) \cong \mathbb{C}^2 \to H^1(N) \xrightarrow{\alpha} H^1(N(D)) \to 0.$$

The existence of the unisecant $\Gamma_D + f_D$ on F gives rise to the sequence

$$0 \to N \to \mathcal{F} \to \omega_C(-D) \oplus \mathcal{O}_D \to 0 \tag{7.7}$$

(cf. (2.1)). This sequence corresponds to an element $\xi \in \mathrm{Ext}^1(\omega_C(-D) \oplus \mathcal{O}_D, N)$, which by Serre duality, is isomorphic to $H^0(\mathcal{O}_D)^\vee \oplus \mathrm{Ext}^1(\omega_C(-D), N)$ (cf. [21, Prop. III.6.7, Thm. III.7.6]). So $\xi = (\sigma, \eta)$, with $\sigma \in H^0(\mathcal{O}_D)^\vee$and $\eta \in \mathrm{Ext}^1(\omega_C(-D), N) \cong H^1(N(D) \otimes \omega_C^\vee)$.

We have the following diagram

$$\begin{tikzcd}
H^0(\mathcal{O}_D) \oplus H^0(\omega_C(-D)) \arrow[r, "\partial_0"] & H^1(N) \arrow[d, "\alpha"] \\
H^0(\omega_C(-D)) \arrow[u] \arrow[r, "\partial'"] & H^1(N(D)) \arrow[d] \\
0 \arrow[u] & 0
\end{tikzcd}$$

where ∂_0, ∂' are the coboundary maps. The action of ξ on $\{0\} \oplus H^0(\omega_C(-D))$ coincides with the action of η on $H^0(\omega_C(-D))$ via cup-product. This yields an isomorphism $\mathrm{Coker}(\partial') \xrightarrow{\cong} \mathrm{Coker}(\partial_0)$.

Notice that (7.7) can be seen as a limit of (7.6). Since $\mathrm{Coker}(\partial)$ is a general two-dimensional quotient of $H^1(N)$, then also $\mathrm{Coker}(\partial_0)$ is general. The above argument implies that $\mathrm{Coker}(\partial')$ is also general, proving the assertion (cf. the proof of Claim 7.13). ☐

Finally, to prove that r_1, r_2 are dominant, it suffices to prove the following:

Claim 7.15. *The general fibre of r_i has dimension two, for $1 \leqslant i \leqslant 2$.*

Proof of Claim 7.15. It suffices to prove that there are fibres of dimension two. For r_1, take (F, Γ, p) general in $V_d^{2g-2} \times C$. The fibre of r_1 containing this triple consists of all triples (F, Γ', p), with $\Gamma' \in |\mathcal{O}_F(\Gamma)|$ so it has dimension two since $i(\mathcal{F}) = 3$ (cf. computation as in (6.7)). The same argument works for r_2. ☐

Step 5. In this step we prove that $\overline{\mathcal{V}}$ is an irreducible component of $\widetilde{B_C^{k_3}}(d)$.

Claim 7.16. *Let $(F, \Gamma_D) \in V_d^{2g-2-i}$ be general, with $1 \leqslant i \leqslant 2$ and $D \in C^{(i)}$. Then $|\Gamma_D + f_D|$ has dimension two, its general member Γ is smooth and it corresponds to a sequence $0 \to N(-D) \to \mathcal{F} \to \omega_C \to 0$. Consequently, the pair (F, Γ_D) is in the image of r_i.*

Proof of Claim 7.16. Given the first part of the statement, the conclusion is clear. To prove the first part, note that the existence of $\Gamma \subset F$ gives an exact sequence

$$0 \to N \to \mathcal{F} \to \omega_C(-D) \to 0, \tag{7.8}$$

hence $h^0(\mathcal{O}_F(\Gamma_D + f_D)) = h^0(\mathcal{F} \otimes N^\vee(D)) = h^0(\mathcal{F} \otimes \omega_C \otimes \det(\mathcal{F})^\vee) = h^1(\mathcal{F}) = 3$ (cf. (2.6)). This implies the assertion. ☐

Let now $\mathcal{B} \subseteq \widetilde{B_C^{k_3}}(d)$ be a component containing $\overline{\mathcal{V}}$. From Step 4, $[\mathcal{F}] \in \mathcal{B}$ general has speciality $i = 3$ and a special presentation as in (2.2) with L of minimal degree δ. Thus $2g - 4 \leqslant \delta \leqslant 2g - 2$, since the Segre invariant is lower semi-continuous (cf. Remark 2.5 and also [24, § 3]).

By Claim 7.16, $2g - 3 \leqslant \delta \leqslant 2g - 2$ does not occur under the minimality assumption on L. Indeed, in both cases we have a two-dimensional linear system $|\Gamma|$, whose general member is a section, corresponding to a surjection $\mathcal{F} \twoheadrightarrow \omega_C$ and we proved that there would be curves in this linear system containing two rulings.

If $\delta = 2g - 4$, we have an exact sequence as in (7.8). By specializing to a general point of $\overline{\mathcal{V}} = \overline{\mathcal{V}^{2g-4,1,2}}$, because of Claim 7.16, we see that in (7.8) one has $h^0(N) = 0$. Hence, $h^1(N)$ is constant. Since for the general element of $\overline{\mathcal{V}}$, $\mathrm{Ker}(\mu_{V_2}) = (0)$ the same happens for the general element of $\mathcal{B}$ i.e., with notation as in the proof of Claim 7.13, τ is constant equal to zero. Therefore, also $\eta = \epsilon = 0$ for the general point of $\mathcal{B}$ (see l.c.), which implies the assertion. ☐

With our approach, we cannot conclude that $\overline{\mathcal{V}_d^{2g-4,1,2}}$ in Theorem 7.11 is the unique regular component, whose general point $[\mathcal{F}]$ is such that $i(\mathcal{F}) = 3$, because we do not know if $\Lambda_2 \subset \mathcal{W}_2$ is the only good component when $N \in \mathrm{Pic}^{d-2g+4}(C)$ and $D_2 \in C^{(2)}$ general. However, results in [47, 51] imply that $\widetilde{B_C^{k_3}}(d)$ is irreducible for $d \leqslant \frac{10}{3}g - 7$, though they say nothing on rigid special presentation of the general element. Putting all together, we have:

Corollary 7.17. *Under the assumptions of Theorem 7.11, one has* $\widetilde{B^{k_3}_C}(d) = \overline{\mathcal{V}^{2g-4,1,2}_d}$.

Remark 7.18. (1) Theorem 5.4, for $j = 3$, shows the existence of elements of $\widetilde{B^{k_3}_C}(d)$ with injective Petri map in the range $g \geqslant 21$, $g + 3 \leqslant \delta \leqslant \frac{4}{3}g - 4$, $2g + 6 \leqslant d \leqslant \frac{8}{3}g - 9$. This gives a proof, alternative to the one in [51], of generic smoothness of $\widetilde{B^{k_3}_C}(d)$ in the above range.

(2) If $\frac{5}{2}g - 5 \leqslant d \leqslant \frac{10}{3}g - 7$, $\widetilde{B^{k_3}_C}(d)$ is, as we saw, irreducible but in general it is no longer true that it is determined by a (total) good component. To see this, we consider a specific example.

Take $\widetilde{B^{k_3}_C}(3g-4)$, which is non-empty, irreducible, generically smooth, of dimension $g-6$ and $[\mathcal{F}] \in \widetilde{B^{k_3}_C}(3g-4)$ general is such that $i(\mathcal{F}) = 3$ by [47, 51]. By Lemma 4.1, $\mathcal{F}$ can be rigidly presented as in (2.2), where $L \in W^{\delta-g+j}_\delta(C)$ and $1 \leqslant j \leqslant 3$.

The cases $j = 2$, 3 cannot occur: the stability condition (4.1) imposes $\delta > \frac{3}{2}g - 2$, but if $j = 3$, $\rho(L) \geqslant 0$ forces $\delta \leqslant \frac{4}{3}g - 3$ whereas if $j = 2$, $\rho(L) \geqslant 0$ implies $\delta \leqslant \frac{3}{2}g - 3$; in both cases we get a contradiction.

The only possible case is therefore $j = 1$, so the corank of the coboundary map is $t = 2$, which implies that N is of speciality $r \geqslant 2$. Since $\chi(N) = 2g - 3 - \delta$, the case N non-effective would give $\delta > 2g - 3$, i.e. $L \cong \omega_C$. But in this case, $a_F(2g - 2) \geqslant 2$ (usual computations as in (6.7)) against the rigidity assumptions.

Therefore N must be effective, with $n = h^0(N) = 2g - 3 - \delta + r$. We want to show that the hypotheses of Corollary 5.9 hold. Assume by contradiction $\ell < r$; then

$$\delta < g - 2 + r. \tag{7.9}$$

From stability $3g - 4 < 2\delta < 2g - 4 + 2r$, i.e. $g - 2r < 0$. Since C has general moduli, one has $\rho(N) \geqslant 0$, hence $h^0(N) = 1$. So $d - \delta = g - r$ and (7.9) yields $d = \delta + d - \delta < 2g - 2$ a contradiction. Thus, $\ell \geqslant r$.

Now, from (4.2), $m = 2\delta - 2g + 3$ since N is not isomorphic to L. Thus, $m \geqslant \ell + 1$: this is equivalent to $\delta \geqslant g$, which holds by stability.

In conclusion, by Corollary 5.9, $\mathcal{W}^{\rm Tot}_1$ is irreducible, of the expected dimension. Assume that $\widehat{\mathcal{W}}^{\rm Tot}_1$ contains a total good component $\widehat{\Lambda}^{\rm Tot}_2$, whose image via $\pi_{3g-4,\delta}$ is $\widetilde{B^{k_3}_C}(3g-4)$. Thus, $r \geqslant 2$. On the other hand $r = 2$ cannot occur since

$$c(\ell, 2, 2) = 2(\delta - g + 2) > 2\delta - 2g + 2 = m - 1 = \dim(\mathbb{P}(\mathrm{Ext}^1(L, N))),$$

a contradiction. Therefore, one has $r \geqslant 3$.

From the second equality in (6.3)

$$\dim(\mathbb{P}(\mathfrak{E}_\delta)|_{\mathcal{Z}}) = (r + 1)\delta - (2r - 1)g - r(r - 3)$$

and the codimension of $\widehat{\Lambda}^{\rm Tot}_2$ is

$$c(\ell, r, 2) = 2(\delta - g + 4 - r).$$

Set $a := a_{F_v}(\delta)$. From Remark 4.2 we can assume $a \leqslant 1$. Therefore,

$$\dim(\mathrm{Im}(\pi_{d,\delta}|_{\widehat{\Lambda}^{\rm Tot}_2})) = g - 6$$

gives $(r - 1)\delta = 2rg - 2g + r^2 - 5r + 2 + a$, i.e.

$$\delta = 2g + r - 4 + \frac{a - 2}{r - 1}.$$

This yields a contradiction. Indeed, since $0 \leqslant a \leqslant 1$, $r \geqslant 3$ and δ is an integer, the only possibility is $r = 3$, $a = 0$, $\delta = 2g - 2$ which we already saw to contradict the rigidity assumption.

Theorem 7.19. *Let C be of genus $g \geqslant 4$, with general moduli. Then, $\widetilde{M^3_C}(2, \omega_C) \neq \emptyset$. Moreover, there exists an irreducible component which is regular (i.e. of dimension $3g - 9$), whose general point $[\mathcal{F}]$ fits in a sequence*

$$0 \to \mathcal{O}_C(p + q) \to \mathcal{F} \to \omega_C(-p - q) \to 0, \tag{7.10}$$

where

- *$p + q \in C^{(2)}$ general, and*
- *$\mathcal{F} = \mathcal{F}_v$ with $v \in \Lambda \subset \mathcal{W}_2 \subset \mathrm{Ext}^1(\omega_C(-p), \mathcal{O}_C(p))$ general in Λ, which is a component of $\mathcal{W}_2$ of dimension $3g - 10$ (hence, not good).*

Proof. With notation as in (4.4), (4.5), for $\mathcal{F} = \mathcal{F}_v$ as in (7.10), we have

$$\ell = r = g - 2,\ t = 2,\ m = h^1(2p + 2q - K_C) = 3g - 7.$$

Consider the map (notation as in (5.7) and (5.8))

$$\mu : H^0(\omega_C(-p - q)) \otimes H^0(\omega_C(-p - q)) \to H^0(\omega_C^{\otimes 2}(-2p - 2q)).$$

For $V_2 \in \mathbb{G}(2, H^0(\omega_C(-p-q)))$ general, μ_{V_2} has kernel of dimension 1 (cf. computations as in Claim 7.13). Arguing as in the proofs of Theorem 5.17 and Claim 7.13, there is a component $\Lambda \subset \mathcal{W}_2 \subset \mathrm{Ext}^1(\omega_C(-p-q), \mathcal{O}_C(p+q))$ (dominating $\mathbb{G}(2, H^0(\omega_C(-p-q))$, hence not good) of dimension $3g-10$.

Stability of $\mathcal{F}$ follows from Proposition 4.4. This shows that $\widetilde{M^3_C}(2, \omega_C) \neq \emptyset$. Regularity and generic smoothness follow from the injectivity of the symmetric Petri map as in [54, 5].

The fact that $[\mathcal{F}]$ general has a presentation as in (7.10) follows from an obvious parameter computation. ☐

By [23, § 4.3], $\widetilde{M^3_C}(2, \omega_C)$ contains a unique regular component; Theorem 7.19 provides in addition a rigidly special presentation of its general element.

7.5. **A conjecture for $i \geqslant 4$.** For any integers $i \geqslant 4$ and d as in Theorems 5.1, 5.4, 5.10, 5.13, one has $\widetilde{B^{k_i}_C}(d) \neq \emptyset$. In particular, when d is as in Theorem 5.4, with $j = i$, one deduces that $\widetilde{B^{k_i}_C}(d)$ contains a regular, generically smooth component. This gives existence results in the same flavour as Theorem 0.1.

One may wish to give a special, rigid presentation of the general point of all components of $\widetilde{B^{k_i}_C}(d)$. The following less ambitious conjecture is inspired by the results in this paper.

Conjecture 7.20. *Let $i \geqslant 4$ and $g > i^2 - i + 1$ be integers. Let C be of genus g, with general moduli. Let d be an integer such that*

$$2g-2 \leqslant d < 2g-2-i+\frac{g-\epsilon}{i-1},$$

with $\epsilon \in \{0,1\}$ such that $d + g - (i-1)k_i \equiv \epsilon \pmod 2$. Then, there exists an irreducible, regular component $\mathcal{B} \subseteq \widetilde{B^{k_i}_C}(d)$, s.t.

$$\mathcal{B} = \overline{\mathcal{V}_d^{2g-1-i,1,i-1}}.$$

In particular, $[\mathcal{F}] \in \mathcal{B}$ general is stable, with $i(\mathcal{F}) = i$, $s(\mathcal{F}) \geqslant g-(i-1)k_i - \epsilon > 0$ and it is rigidly specially presented as

$$0 \to N \to \mathcal{F} \to \omega_C(-D_{i-1}) \to 0,$$

where

- *$D \in C^{(i-1)}$ is general,*
- *$N \in \mathrm{Pic}^{d-2g+1+i}(C)$ is general (i.e. special, non-effective),*
- *$\mathcal{F} = \mathcal{F}_v$ with $v \in \Lambda_{i-1} \subset \mathrm{Ext}^1(N, \omega_C(-D))$ general in a good component;*

The bounds on g and d in Conjecture 7.20 ensure the following:

1. $\rho_d^{k_i} \geqslant 0$ which is equivalent to $d \leqslant 2g-2-i+\frac{4g-3}{i}$ (cf. (3.3)).
2. $N \in \mathrm{Pic}^{d-2g+1+i}(C)$ general is special, non-effective; indeed $r = 3g-2-i-d > 0$.
3. $r \geqslant i-1 = \mathrm{cork}(\partial_v)$, which is equivalent to $d \leqslant 3g-1-2i$.
4. There are no obstructions for a good component $\Lambda_{i-1} \subset \mathrm{Ext}^1(\omega_C(-D), N)$ to exist; indeed, one has
$$\dim(\mathbb{P}) = m-1 = 5g-4-2i-d$$
from (4.2), and from (5.5), Remark 5.15, we have
$$c(\ell, r, t) = (i-1)k_i = (i-1)(d-2g+2+i),$$
since $t = i-1$ and N non-effective. Therefore
$$\dim(\Lambda_{i-1}) = m-1-c(\ell, r, t) = 3g-2-i-i(d-2g+2+i)$$
is non-negative as soon as $d \leqslant 2g-3-i+\frac{3g-2}{i}$.
5. From Remark 5.15, $v \in \Lambda_{i-1}$ general is such that $s(\mathcal{F}_v) \geqslant g-(i-1)k_i-\epsilon$, which is positive because of the upper-bound on d. Thus $\mathcal{F}_v$ is stable.
6. The interval $2g-2 \leqslant d < 2g-2-i+\frac{g-\epsilon}{i-1}$ is not empty.

From Remark 4.2 and (6.9), to prove Conjecture 7.20 it would suffice to prove the following two facts:

(a) *For $i \geqslant 4$, $D \in C^{(i-1)}$ general and $L = \omega_C(-D)$, there exists a good component*

$$\Lambda_{i-1} \subseteq \mathcal{W}_{i-1} \subset \mathrm{Ext}^1(\omega_C(-D), N).$$

(b) *For $v \in \Lambda_{i-1}$ general, $\mathcal{F}_v$ is rsp via $\omega_C(-D)$.*

Concerning (a), notice that for no $V_{i-1} \in \mathbb{G}(i-1, H^0(K_C - N))$ the map $\mu_{V_{i-1}}$ can be injective. Indeed, $d \geqslant 2g-2$ and $i \geqslant 4$ imply

$$\dim(H^0(K_C - D) \otimes V_{i-1}) = (i-1)g - (i-1)^2 > 5g - 3 - d - 2i = h^0(2K_C - N - D).$$

Hence, according to Theorem 5.17, one should find an irreducible subvariety $\Sigma_\eta \subset \mathbb{G}(i-1, H^0(K_C - D))$ of codimension $\eta := d + (i-6)g + 7 - (i-2)^2$ such that $\dim(\mathrm{Ker}(\mu_{V_{i-1}})) = \eta$ for $V_{i-1} \in \Sigma_\eta$ general.

Concerning (b), the minimality assumption implies $a_{F_v}(2g-1-i) \leqslant 1$, for $v \in \Lambda_{i-1}$ general and $F_v = \mathbb{P}(\mathcal{F}_v)$. To prove rigidity, one has to show that $a_{F_v}(2g-1-i) = 0$. This is equivalent to prove a regularity statement for a Severi variety of nodal curves on F_v. Indeed, for any section Γ_D corresponding to a quotient $\mathcal{F}_v \twoheadrightarrow\!\!\!\to \omega_C(-D)$ as above, the linear system $|\Gamma_D + f_D|$ has dimension $i-1$, it is independent on D and its general member Γ is a section corresponding to a quotient $\mathcal{F}_v \twoheadrightarrow\!\!\!\to \omega_C$. The curve $\Gamma_D + f_D$ belongs to the Severi variety of $(i-1)$-nodal curves in $|\Gamma|$. So rigidity is equivalent to show that this Severi variety has the expected dimension zero. Proving this is equivalent to prove that D, considered as a divisor on Γ_D, imposes independent condition to $|\Gamma|$. Unfortunately, the known results on regularity of Severi varieties (see [38, 48, 49]) do not apply in this situation.

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