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ON VALIRON'S THEOREM

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Abstract. This is a survey on Valiron's Theorem about the convergence properties of orbits of analytic self-maps of the disk of hyperbolic type and related questions in one and several variables.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let ϕ be an analytic function defined on $\mathbb{D}$. If $|\phi(z)| < 1$ for $|z| < 1$, then ϕ is a self-map of the disk $\mathbb{D}$ and one can iterate by letting $\phi_n = \phi \circ \cdots \phi$, n times. The natural question that arises is *given a point $z_0 \in \mathbb{D}$, what can be said about its orbit $z_n = \phi_n(z_0)$, as $n = 1, 2, 3, \dots$?* In this survey we will describe a theorem of Valiron which relates to this question and describe the multidimensional setting.

1.1. **Schwarz's Lemma.** One of the very first results one encounters in function theory is Schwarz's Lemma, which can be proved using the maximum principle.

Lemma 1 (Schwarz's Lemma)**.** *Suppose ϕ is an analytic self-map of $\mathbb{D}$. If, moreover, $\phi(0) = 0$ then*

(1) $|\phi(z)| \le |z|$ *for all* $z \in \mathbb{D}$.
(2) $|\phi(z_0)| = |z_0|$ *for some* $z_0 \neq 0$ *if and only if* ϕ *is a rotation.*
(3) $|\phi'(0)| \le 1$ *and* $|\phi'(0)| = 1$ *if and only if* ϕ *is a rotation.*

The proof is based on the fact that the function $\phi(z)/z$ is analytic and bounded by 1.

Geometrically, Schwarz's Lemma says that for every $0 < r < 1$:

$$\phi(r\mathbb{D}) \subset r\mathbb{D},$$

and from the proof one deduces more precisely that, except for rotations, for every $0 < r_0 < 1$ there exists $0 < s_0 < 1$ such that for $0 < r < r_0$,

$$\phi(r\mathbb{D}) \subset s_0 r\mathbb{D}. \tag{1.1}$$

The maximum principle and Schwarz's Lemma can be used to show that the automorphisms of $\mathbb{D}$ are of the form

$$\gamma(z) = c\frac{z-a}{1-\bar{a}z}$$

for some constants $|c| = 1$ and $|a| < 1$. All the automorphisms γ send the point $a \in \mathbb{D}$ to 0, and we write γ_a when the constant c equals 1. Using these automorphisms one can transfer the system of disks $r\mathbb{D}$ $(0 < r < 1)$ around any given point $a \in \mathbb{D}$:

$$\Delta(a, r) = \gamma_a^{-1}(r\mathbb{D}).$$

These are called pseudo-hyperbolic disks of radius r at a. Because linear fractional transformations map circles to circles, $\Delta(a, r)$ is an Euclidean disk, but a is not the Euclidean center (actually a is further from the origin). With this notation, a simple use of the γ_z and $\gamma_{\phi(z)}$ shows that given any analytic self-map ϕ of $\mathbb{D}$, and for any $z \in \mathbb{D}$, we always have,

$$\phi(\Delta(z, r)) \subset \Delta(\phi(z), r) \tag{1.2}$$

for all $0 < r < 1$. Moreover, by continuity and compactness, given a compact set $K = \{|z| \leq t\}$ for some $0 < t < 1$, and given a radius $0 < r_0 < 1$ there exists a constant $s_0 < 1$ so that uniformly for $z \in K$ and for $0 < r < r_0$:

$$\phi(\Delta(z, r)) \subset \Delta(\phi(z), s_0 r). \tag{1.3}$$

This can also be worded in terms of the pseudo-hyperbolic distance

$$d(z, w) = |\gamma_z(w)| = \left|\frac{z - w}{1 - \bar{w}z}\right| \qquad \text{for } z, w \in \mathbb{D}.$$

Although we call it distance, $d(z, w)$ does not satisfy the triangle inequality, yet it almost does for small distances because of the formula:

$$d(z, w) \leq \frac{d(z, \zeta) + d(\zeta, w)}{1 + d(z, \zeta)d(\zeta, w)}.$$

An actual distance is obtained by letting

$$\rho(z, w) = \log \frac{1 + d(z, w)}{1 - d(z, w)}.$$

This is the hyperbolic distance of $\mathbb{D}$.

1.2. **One fixed point in $\mathbb{D}$.** If a self-map of the disk fixes two points, conjugating it using an automorphism and using part 2. of Schwarz's Lemma 1, one proves that it is actually the identity map. Thus every other self-map of the disk can fix at most one point in $\mathbb{D}$.

If a self-map fixes exactly one point in $\mathbb{D}$ it is called of *elliptic type*. In this case, the map can be conjugated by automorphisms so that the fixed point is the origin, hence the power series expansion there is:

$$\phi(z) = \lambda z + O(z^2)$$

where $\lambda = \phi'(0) \in \overline{\mathbb{D}}$. Three subcases arise: when $|\lambda| = 1$, the map ϕ is a rotation; if $0 < |\lambda| < 1$ the fixed point is called *attractive*, if $\lambda = 0$ it is *superattractive*. The behavior of single orbits is well understood in all these cases.

For instance, in the attractive case, it is clear from (1.1) that every orbit tends to zero. Moreover, by Kœnigs Theorem (which is proved using property (1.3)), there is a one-to-one analytic map σ defined near 0 with $\sigma(0) = 0$ and $\sigma'(0) = 1$ such that

$$\sigma \circ \phi(z) = \lambda\sigma(z) \tag{1.4}$$

near 0, i.e one can change coordinates holomorphically so that ϕ becomes linear, and because of the condition on $\sigma'(0)$ being equal to 1, the orbits $z_n = \phi_n(z_0)$ asymptotically approach the corresponding orbit $\lambda^n\sigma(z_0)$. It is worth notice that σ can be extended (not univalently in general) to all $\mathbb{D}$ in such a way that (1.4) still holds.

In the superattractive case, the orbits tend fast to the origin. Even in this case it is possible to perform an holomorphic change of variables near the origin in such a way that ϕ assumes a simpler form. Namely, if $\phi(z) = O(z^k)$ then by Böttcher's Theorem there exists a one-to-one analytic map σ defined near 0 so that $\sigma(0) = 0$, $\sigma'(0) = 1$ and

$$\sigma \circ \phi(z) = \sigma(z)^k \tag{1.5}$$

near 0. In this case however the map σ cannot be in general well defined on all $\mathbb{D}$. For all these matters we refer the interested reader to [CG92].

1.3. **No fixed points in $\mathbb{D}$.** Assume that the self-map ϕ fixes no point in $\mathbb{D}$. Then either ϕ is an automorphism of $\mathbb{D}$ in which case it is an isometry for the hyperbolic distance, or, by Schwarz's Lemma, ϕ is a strict contraction, i.e.

$$d(\phi(z), \phi(w)) < d(z, w).$$

for all $z, w \in \mathbb{D}$. If ϕ is an automorphism then it can be conjugated to one of two maps: either multiplication by $T > 1$ on the upper half-plane $\mathbb{H} = \{\operatorname{Im} z > 0\}$ (*hyperbolic automorphism*) or translation by $b > 0$ on $\mathbb{H}$ (*parabolic automorphism*).

We will see that ϕ can also be classified as hyperbolic or parabolic when it is not an automorphism. However, even though self-maps of the disk with no fixed points do try to imitate the behavior of the automorphisms in the long run, this is only true to varying degrees and the situation is much more complicated, especially in the parabolic case. The main topic of this survey is to describe self-maps ϕ of hyperbolic type.

We will proceed in stages. The first claim is that given a self-map of the disk there exists a point $\zeta \in \partial\mathbb{D}$ such that every orbit of ϕ converges to ζ. This allows one to change variables to the upper half-plane and send ζ to infinity. Computation usually become easier in this formulation, although it might still be useful to work in both models in view of the possible extentions to several complex variables. The point ζ is the famous *Denjoy-Wolff point* of the map ϕ. The second claim is that, like in the case of automorphisms, if ϕ (not an automorphism) is in the upper half-plane model with

Denjoy-Wolff point at infinity, then either $\operatorname{Im}\phi(z) > \operatorname{Im} z$ for all $z \in \mathbb{H}$ (parabolic case), or there is $T > 1$ such that $\operatorname{Im}\phi(z) > T \operatorname{Im} z$ for all $z \in \mathbb{H}$ (hyperbolic case). Notice that if ϕ is not an automorphism then the previous inequalities are strict at *every* point. Indeed, by a generalization of Schwarz's Lemma known as Julia's Lemma (see below), if there is equality at some point then there is equality everywhere and ϕ is an automorphism of either parabolic type (in the first) or hyperbolic type (in the second).

In more geometric terms, letting $H(s) = \{\operatorname{Im} z > s\}$, $\phi(H(s)) \subset H(Ts)$ for some $T \geq 1$. The half-planes $H(s)$ are called *horodisks* because in the disk model they correspond to Euclidean disks tangent to $\partial\mathbb{D}$ at the Denjoy-Wolff point.

We first observe that if ϕ is a self-map of $\mathbb{D}$ with no fixed points and ϕ is not an automorphism, then no iterate of ϕ can have fixed points in $\mathbb{D}$ either. In fact, suppose that $\phi_N(z_0) = z_0$ for some $N \geq 2$ and some $z_0 \in \mathbb{D}$. The the orbit of z_0 is periodic of period N and so are the steps $d_n = d(z_n, z_{n+1})$, which contradicts the fact that d_n is a strictly decreasing sequence by Schwarz's Lemma.

This can be used to show that any orbit z_n cannot accumulate anywhere in $\mathbb{D}$, i.e. must eventually escape any given compact set. In fact, suppose that a subsequence z_{n_k} tends to $p \in \mathbb{D}$. Find $0 < t < 1$ so that $|p| < t$ and let $K = \{|z| \leq t\}$, also let $s < 1$ and $0 < r_0 < 1$ be given as in (1.3). Eliminating finitely many terms, we can assume that $z_{n_k} \in K$ for all k. Choose a radius $0 < r_0 < 1$ close enough to 1 so that the pseudo-hyperbolic disk $D = \Delta(z_{n_1}, r_0)$ contains K. By (1.3) we have

$$\phi_{n_k}(D) \subset \Delta(z_{n_k}, s_0^k r_0),$$

and since s_0^k tends to zero and z_{n_k} tends to p, we must have for large enough k_0 that $\phi_{n_{k_0}}(\overline{D}) \subset D$. This implies that $\phi_{n_{k_0}}$ has a fixed point in D but we have ruled out fixed points for the iterates of ϕ.

The next step is to show that given an orbit z_n, not only $|z_n|$ tends to one but actually z_n tends to some $\zeta \in \partial\mathbb{D}$. For this we need a boundary consequence of Schwarz's Lemma known as Julia's Lemma.

1.4. **Julia's Lemma.** We will present a simplified version of Julia's Lemma which is more suitable to our purposes. First we use the Poisson kernel at $\zeta \in \partial\mathbb{D}$ to define the horodisks at ζ:

$$H(t) = \{z \in \mathbb{D} : \frac{1 - |z|^2}{|\zeta - z|^2} > \frac{1}{t}\}. \tag{1.6}$$

Note that $H(t)$ is decreasing as $t \downarrow 0$ and $\cap_{t\downarrow 0} H(t) = \emptyset$ while $\cup_{t\uparrow\infty} H(t) = \mathbb{D}$.

Lemma 2. *Let ϕ be an analytic self-map of $\mathbb{D}$ and let $p_k \in \mathbb{D}$ be a sequence of points tending to $\zeta \in \partial\mathbb{D}$. If $\phi(p_k)$ also tends to ζ and the ratio*

$$\frac{1-|\phi(p_k)|}{1-|p_k|} \longrightarrow d > 0,$$

as $k \to \infty$, then for all $t > 0$

$$\phi(H(t)) \subset H(dt). \tag{1.7}$$

The general version of Julia's Lemma allows for $\phi(p_n)$ to be tending to some other boundary point $\eta \in \partial\mathbb{D}$ and also does not assume $d > 0$, but deduces it. The proof of this lemma is obtained by applying Schwarz's Lemma in the form (1.2) to hyperbolic disks centered at p_n of larger and larger radius so that these disks tend to the horodisk $H(t)$. Note also that while (1.2) contracts the hyperbolic radius, when $d > 1$ equation (1.7) only requires for a smaller horodisk to be mapped into a larger one.

Now consider an orbit z_n. We have seen above that $|z_n|$ tends to 1. Choose a subsequence z_{n_k} such that

$$|z_{n_k+1}| = |\phi(z_{n_k})| \geq |z_{n_k|}$$

and further assume that z_{n_k} tends to some point $\zeta \in \partial\mathbb{D}$. Since $d(z_{n_k}, \phi(z_{n_k})) \leq d(z_0, z_1)$, $\phi(z_{n_k})$ also tends to ζ. Hence, we can apply Julia's Lemma, with $p_k = z_{n_k}$ and with $d \leq 1$, to find that $\phi(H(t)) \subset H(t)$. This immediately implies that the whole orbit z_n must tend to ζ. Moreover if we let

$$\alpha = \liminf_{z\to\zeta} \frac{1-|\phi(z)|}{1-|z|}$$

then by Julia's Lemma $\phi(H(t)) \subset H(\alpha t)$. We call α the *coefficient of dilatation* of ϕ at its Denjoy-Wolff point. It follows from what we said so far that $\alpha \leq 1$, and the map ϕ is said to be of *hyperbolic type* if $\alpha < 1$, while it is of *parabolic type* if $\alpha = 1$. It can also be shown that $\alpha > 0$ always.

As we mentioned above the terminology parabolic vs. hyperbolic is used because one wishes to show that these maps tend to imitate the corresponding parabolic vs. hyperbolic automorphisms. However, this is not always the case, especially in the parabolic case. What happens in the hyperbolic case is the content of this survey.

2. Self-maps of the disk of hyperbolic type

The hyperbolic automorphisms in the upper half-plane model are easy to describe. They are of the form

$$\tau(z) = Az + b$$

with $A > 1$ and $b \in \mathbb{R}$. The only two fixed points for τ are infinity and $-b/(A-1)$. The hyperbolic geodesic $L = \{\operatorname{Re} z = -b/(A-1);\ \operatorname{Im} z > 0\}$ is invariant (L is also known as the axis of the automorphism) and invariant is also every half line originating from the fixed point $-b/(A-1)$ which lays in $\mathbb{H}$. It is clear then that for every orbit z_n of τ, the following three properties hold: (1) the ratios z_{n+1}/z_n tend to A; (2) the sequence $\operatorname{Arg} z_n$ has (a) a limit in $(0,\pi)$ and a computation shows that it equals $\operatorname{Arg}(z_0 - b/(A-1))$, hence it is a harmonic function of z_0 and (b) by varying z_0 this limit takes every value in $(0,\pi)$; (3) the sequence z_n/A^n tends $z_0 - b/(A-1)$.

A quicker way to describe this dynamic would have been to notice that τ can be conjugated via a translation to the map $z \mapsto Az$. Back in the disk model the axis L is an arc of circle orthogonal to $\partial\mathbb{D}$, intersecting $\partial\mathbb{D}$ at 1 and at some other fixed point $p \in \partial\mathbb{D} \setminus \{1\}$. All arcs of circle interesecting $\overline{\mathbb{D}}$ in 1 and p are invariant for the automorphism and the three properties above become: (1) the ratios $(1-z_{n+1})/(1-z_n)$ tend to α; (2) the sequence $\operatorname{Arg}(1-z_n)$ has (a) a limit in $(-\pi/2,\pi/2)$ which is a harmonic function of z_0 and (b) this limit takes every value in $(-\pi/2,\pi/2)$; (3) the sequence $(1-z_n)/\alpha^n$ tends to a limit.

Assume now that ϕ is a self-map of the disk with Denjoy-Wolff point at 1 (without loss of generality) and coefficient of dilatation $\alpha < 1$. Or, equivalently, assume that Φ is a self-map of the upper half-plane $\mathbb{H}$ and $\Phi(z) = Az + p(z)$ with $\operatorname{Im} p(z) \geq 0$ and $A = 1/\alpha = \inf_{z\in\mathbb{H}} \frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z} > 1$. It is natural to ask if the three properties of hyperbolic automorphisms above are also shared by the orbits of Φ. Valiron shows that properties (1) and (2) (a) still hold, see [Va31] or Chapter VI of [Va54] (he doesn't seem to have considered property 2 (b)). Next we present a slighty different proof of his result.

2.1. **Property (1):** ***Given an orbit*** z_n ***of*** Φ***, the ratio*** $\frac{z_{n+1}}{z_n}$ ***tends to*** A**.** This property is intimately connected with the Julia-Carathéodory Theorem. We state it somewhat reworded, in the upper half-plane model.

Theorem 3 (Julia-Carathéodory)**.** *Let* Φ *be an analytic self map of* $\mathbb{H}$*. Let*

$$A = \inf_{z\in\mathbb{H}} \frac{\operatorname{Im}\Phi(z)}{\operatorname{Im} z}.$$

Then,

$$K\text{-}\lim_{z\to\infty} \frac{\Phi(z)}{z} = A. \tag{2.1}$$

For a proof see [Sh93] p. 66-69, which, as one might guess, is based on Schwarz's Lemma. By K-$\lim_{z\to\infty}$, "non-tangential limit", we mean that z tends to infinity in such a way that $|\operatorname{Arg} z - \pi/2| < \pi/2 - \delta$ for any given $\delta > 0$

In particular, when Φ is of hyperbolic type then (2.1) holds. Yet one cannot immediately deduce from it property (1) for the orbits of Φ since, in principle, z_n might tend to 1 tangentially.

Lemma 4. *Let Φ be a hyperbolic holomorphic self map of $\mathbb{H}$. Then any given orbit z_n satisfies*

$$|\operatorname{Arg} z_n - \pi/2| < \pi/2 - \delta$$

for some fixed $\delta > 0$ depending only on z_0.

Proof. Schwarz's Lemma imposes that z_{n+1} belongs to the pseudo-hyperbolic disk Δ centered at z_n of radius $d_0 = d(z_0, z_1)$, and the hyperbolic type imposes that $\operatorname{Im} z_{n+1} \geq A \operatorname{Im} z_n$ for some $A > 1$. So z_{n+1} is forced to land in the intersection (never empty!) of Δ with the half-plane $\{\operatorname{Im} z \geq A \operatorname{Im} z_n\}$. Applying a dilation $1/\operatorname{Im} z_n$ to this picture we see that $z_{n+1}/\operatorname{Im} z_n$ belongs to the intersection of a pseudo-hyperbolic disk of radius d_0, centered at some point with imaginary part equal to 1, and the half-plane $\{\operatorname{Im} z \geq A\}$. From this we deduce that

$$|\operatorname{Arg}(z_{n+1} - z_n) - \pi/2| \leq \pi/2 - \delta_0$$

for some $\delta_0 > 0$ which depends only on z_0. Now consider a sector $S(\delta) = \{|\operatorname{Arg} z - \pi/2| \leq \pi/2 - \delta\}$ and let R be the union of all the sectors $z + S(\delta_0)$ as z describes the segment $[-\overline{z_0}, z_0]$. It is clear that the orbit z_n never leaves the region R, and that R is contained in a larger sector $S(\delta_1)$ with $0 < \delta_1 < \delta_0$. □

Now that we know that every orbit stays confined in a non-tangential approach region, we can apply Julia-Carathéodory's theorem and obtain property (1) that z_{n+1}/z_n always tends to A.

2.2. **Property (2) (a):** ***Given an orbit z_n of Φ, the limit $\operatorname{Arg} z_n$ exists and is a harmonic function of z_0.*** Observe first that $\operatorname{Arg} z_n = \operatorname{Arg} \Phi_n(z_0)$ is a bounded harmonic function in z_0, so once the existence of the limit is established, harmonicity follows by Harnack's principle. We write $z_n = x_n + iy_n$. Property (1) can be written as $z_{n+1} = Az_n + o(1)z_n$, thus dividing by y_n we get

$$\frac{z_{n+1}}{y_n} = A\frac{z_n}{y_n} + o(1)\frac{z_n}{y_n}.$$

However, Lemma 4 implies that $z_n/y_n = x_n/y_n + i$ is bounded away from 0 and ∞. So, taking the imaginary part of both sides, we obtain

$$\frac{y_{n+1}}{y_n} = A + o(1). \tag{2.2}$$

Consider the automorphism of $\mathbb{H}$ that sends z_n back to i, i.e.

$$\tau_n(z) = \frac{z - x_n}{y_n}. \tag{2.3}$$

Then set

$$q_n = \tau_n(z_{n+1}) = \frac{x_{n+1} - x_n}{y_n} + i\frac{y_{n+1}}{y_n}. \tag{2.4}$$

It follows immediately from (2.2) that $\operatorname{Im} q_n \to A$ as n tends to infinity. Also, by conformal invariance, the sequence

$$d(i, q_n) = d(z_n, z_{n+1}) = d_n \geq \frac{1-A}{1+A} > 0$$

and is a decreasing sequence. Therefore it has a limit $d_\infty > 0$. Geometrically, if $\mathcal{C}$ is the boundary of the pseudo-hyperbolic disk $\Delta = \Delta(i, d_\infty)$, then $\mathcal{C}$ intersects the horizontal line $\{\operatorname{Im} z = A\}$ in one or two points, q^+ and q^-, which are the only points where the sequence q_n can accumulate. If q^+ and q^- happen to coincide then that is the limit of q_n. Moreover, if q^+ and q^- are distinct, then let

$$B = \max_{\zeta \in \mathcal{C}} \operatorname{Im} \zeta > A.$$

So one can choose n_0 so that $\operatorname{Im} q_n < B$ for $n \geq n_0$. Hence the tail $\{q_n\}_{n \geq n_0}$ cannot jump from q^+ to q^- because the whole sequence q_n stays in the complement of Δ. Therefore, we have shown that q_n always has a finite limit, which we call q_∞. For future use we note here that $q_\infty = b_\infty + iA$ where

$$b_\infty = \lim_{n \to \infty} \frac{x_{n+1} - x_n}{y_n}. \tag{2.5}$$

Now, since τ_n is a translation followed by a dilation the slope of the straight segment $[z_n, z_{n+1}]$ is the same as the slope of $[i, q_n]$, hence we get

$$\operatorname{Arg}(z_{n+1} - z_n) \longrightarrow \operatorname{Arg}(q_\infty - i).$$

Fix $\epsilon > 0$ and consider the angular sector

$$S_\epsilon = \{z \in \mathbb{H} : |\operatorname{Arg} z - \operatorname{Arg}(q_\infty - i)| < \epsilon\}.$$

Then, there exists $n_0 = n_0(\epsilon)$ such that for $n \geq n_0$, z_n belongs to the shifted sector $z_{n_0} + S_\epsilon$. Letting n tend to infinity we get

$$\operatorname{Arg}(q_\infty - i) - \epsilon \leq \liminf_{n \to \infty} \operatorname{Arg} z_n \leq \limsup_{n \to \infty} \operatorname{Arg} z_n \leq \operatorname{Arg}(q_\infty - i) + \epsilon.$$

This is geometrically clear but can also be seen from the formula

$$\operatorname{Arg}(z) = \operatorname{Arg}(z - z_{n_0}) + \operatorname{Arg}\left(1 + \frac{z_{n_0}}{z - z_{n_0}}\right) = \operatorname{Arg}(z - z_{n_0}) + o(1)$$

as z tends to infinity. Finally, since ϵ was arbitrary we obtain

$$\lim_{n \to \infty} \operatorname{Arg} z_n = \operatorname{Arg}(q_\infty - i).$$

2.3. **Property (2) (b):** ***For every angle*** $\theta \in (0, \pi)$ ***one can find an orbit*** z_n ***of*** Φ ***such that*** $\theta(z_0) = \lim_{n \to \infty} \operatorname{Arg} z_n$ ***is equal to*** θ***.*** This property is best established by constructing a conjugation (change of variables) in the spirit of Kœnigs' Theorem in the elliptic case, see (1.4). The existence of such a conjugation is by itself very interesting, and, after the original work of Valiron, many others authors deal with such a problem. Valiron finds a map σ such that

$$\sigma \circ \Phi = A\sigma \tag{2.6}$$

by showing that the normalized sequence of iterates $\Phi_n(z)/|\Phi_n(z_0)|$ converges uniformly to it. We will use a slightly different normalization suggested by Pommerenke in [Po79] which has been found useful in other situations, namely for the parabolic case [Po79], and for backward iterates [PC00] and [PC02]. We also recall the work by Cowen [Co81], where a different approach, based on the uniformization theorem, is used.

The strategy is to renormalize the iterates of Φ using the automorphisms τ_n introduced in (2.3), i.e., choose an orbit $z_n = x_n + iy_n$ and then study the convergence of the sequence $\sigma_n = \tau_n \circ \Phi_n$. Observe that $\sigma_n(z_0) = i$ for all $n = 1, 2, 3, \ldots$, and since $\sigma_n(z_1) = q_n$ as in (2.4) we also have

$$\lim_{n\to\infty} \sigma_n(z_1) = b_\infty + iA.$$

In particular, every normal sublimit of σ_n is a non-constant analytic function.

We first claim that $d(\sigma_n, \sigma_{n+1})$ tends to 0 as n tends to infinity. By Schwarz's Lemma,

$$d(\sigma_n(z), i) = d(\sigma_n(z), \sigma_n(z_0)) \leq d(z, i).$$

So $\sigma_n(z)$ stays in a compact subset of $\mathbb{H}$ and since

$$\sigma_{n+1}(z) = (\tau_{n+1} \circ \Phi \circ \tau_n^{-1})(\sigma_n(z)),$$

it will be enough to show that the sequence $\psi_n = \tau_{n+1} \circ \Phi \circ \tau_n^{-1}$ converges uniformly on compact subsets of $\mathbb{H}$ to the identity. Write

$$\psi_n(z) = \frac{\Phi(x_n + zy_n) - x_{n+1}}{y_{n+1}} = z\frac{y_n}{y_{n+1}}\frac{\Phi(x_n + zy_n)}{x_n + zy_n} + \frac{x_n}{y_n}\frac{y_n}{y_{n+1}}\frac{\Phi(x_n + zy_n)}{x_n + zy_n} - \frac{x_{n+1}}{y_{n+1}}.$$

For fixed z the sequence $x_n + zy_n$ tends to infinity non-tangentially, so we can apply Julia-Carathéodory's Theorem 3, and using the fact that $x_n/y_n = \cot \operatorname{Arg} z_n$ has a limt, we obtain that $\psi_n(z)$ tends to z.

This implies that if σ_N is a subsequence converging to a normal sublimit σ, then σ_{N+1} will tend to σ as well. Therefore, since

$$\sigma_n \circ \Phi = (\tau_n \circ \tau_{n+1}^{-1}) \circ \sigma_{n+1}$$

for all n and since

$$\tau_n \circ \tau_{n+1}^{-1}(z) = \frac{x_{n+1} - x_n}{y_n} + z\frac{y_{n+1}}{y_n} \to b_\infty + Az$$

by (2.5) and (2.2), we obtain that every sublimit σ must satisfy the functional equation

$$\sigma \circ \Phi = A\sigma + b_\infty. \tag{2.7}$$

Finally, since

$$d(i, \sigma_n(z)) = d(\tau_n \circ \Phi_n(z_0), \tau_n \circ \Phi_n(z)) = d(\Phi_n(z_0), \Phi_n(z))$$

is a decreasing sequence, it must converge to $d(i, \sigma(z))$. Hence any other normal sublimit $\tilde{\sigma}$ must satisfy $d(i, \tilde{\sigma}(z)) = d(i, \sigma(z))$ for all $z \in \mathbb{H}$, i.e., $\tilde{\sigma}$ can only differ from σ by an automorphism of $\mathbb{H}$ which fixes i. However, writing $T(z) = Az + b_\infty$, equation (2.7) can be iterated to $\sigma \circ \Phi_n = T_n \circ \sigma$, hence we get that $\sigma(z_n) = T_n(i)$ which is a sequence tending to infinity. In particular, $\tilde{\sigma}$ can only differ from σ by an automorphism of $\mathbb{H}$ which fixes i and infinity, but this can only be the identity.

In conclusion, we have shown that given an orbit z_n of Φ one can renormalize the iterates of Φ with some automorphisms τ_n of $\mathbb{H}$ built from z_n so that $\tau_n \circ \Phi_n$ converges uniformly on compact subsets of $\mathbb{H}$ to a function σ which satisfies the functional equation

$$\sigma \circ \Phi = T \circ \sigma = A\sigma + b_\infty \tag{2.8}$$

where $T(z) = Az + b_\infty$ and b_∞ is a real number depending continuously on z_0. In fact, if $\theta(z_0) = \lim_{n\to\infty} \operatorname{Arg} \Phi_n(z_0)$, see Property 2 (a) above, then

$$b_\infty = (A - 1)\cot(\theta(z_0)). \tag{2.9}$$

Writing $\hat{\sigma} = \sigma + b_\infty/(A - 1)$, one sees that $\hat{\sigma}$ satisfies (2.6), and a computation using (2.9) shows that actually $\hat{\sigma}$ and Valiron's conjugation are the same function. Yet, one may ask: how many solutions do (2.6) and (2.8) have? Also, is it possible to choose z_0 so that in (2.8) the coefficient b_∞ becomes 0? Namely, we still haven't established Property 2 (b).

Semi-conformality of σ. All the previous questions can be answered if we can show that the conjugating map σ that we have found in (2.8) has the property of being *semi-conformal* at infinity. Without loss of generality we can work with $\hat{\sigma}$ instead of σ. Thus we want to show that $\text{K-lim}_{z\to\infty} \hat{\sigma}(z) = \infty$ and that

$$\text{K-lim}_{z\to\infty} \operatorname{Arg} \frac{\hat{\sigma}(z)}{z} = 0. \tag{2.10}$$

To this end we introduce the functions

$$g_n = A^{-n} \hat{\sigma} \circ \tau_n^{-1} - \frac{b_\infty}{A - 1}$$

which are self-maps of $\mathbb{H}$ (we follow the same argument as in Section 2 of [PC00]). Notice that, since $\hat{\sigma}(z_0) = \sigma(z_0) + b_\infty/(A-1) = i + b_\infty/(A-1)$, then

$$g_n(i) = A^{-n}\hat{\sigma}(z_n) - \frac{b_\infty}{A-1} = i.$$

Also if q_n is defined as in (2.4), then

$$g_n(q_n) = A\hat{\sigma}(z_0) = Ai + b_\infty = q_\infty.$$

So any normal sublimit g of the sequence g_n must fix i and also q_∞ (since $q_n \to q_\infty$). Thus by Schwarz's lemma, g is the identity, i.e., $g_n(z) \to z$ uniformly on compact subsets of $\mathbb{H}$. Now let K be a compact subset of $\mathbb{H}$. As $n \to \infty$,

$$d\left(A^n\left(z + \frac{b_\infty}{A-1}\right), \hat{\sigma}(x_n + zy_n)\right) = d(z, g_n(z)) \to 0$$

uniformly for $z \in K$, hence

$$\operatorname{Arg}\hat{\sigma}(x_n + zy_n) - \operatorname{Arg}\left(z + \frac{b_\infty}{A-1}\right) \to 0.$$

But by (2.9) we also have

$$\operatorname{Arg}(x_n + y_n z) = \operatorname{Arg}(z + \frac{x_n}{y_n}) \to \operatorname{Arg}(z + \cot\theta(z_0)) = \operatorname{Arg}\left(z + \frac{b_\infty}{A-1}\right).$$

Hence,

$$\operatorname{Arg}\hat{\sigma}(x_n + zy_n) - \operatorname{Arg}(x_n + y_n z) \to 0.$$

By choosing K to be a hyperbolic disk of larger and larger radius we see that the union of the sets $x_n + y_n K$ eventually covers sectors of larger and larger opening. We have proved the semi-conformality of $\hat{\sigma}$, and thus of σ.

Now that we know that $\hat{\sigma}$ is semi-conformal, iterating (2.6), which is satisfied by $\hat{\sigma}$, we get $\hat{\sigma} \circ \Phi_n = A^n\hat{\sigma}$ and evaluating at z_0 we obtain $\hat{\sigma}(z_n) = A^n\hat{\sigma}(z_0)$. Applying (2.10) to z_n we see that $\operatorname{Arg} A^n\hat{\sigma}(z_0) - \operatorname{Arg} z_n$ tends to zero. In other words, $\theta(z_0) = \operatorname{Arg}\hat{\sigma}(z_0)$. It then remains to show that by varying z_0, $\operatorname{Arg}\hat{\sigma}(z_0)$ takes on every value in $(0, \pi)$. This follows at once from the semiconformality as well and it is explained in Lemma 5 below. For a proof of this lemma see Section 5 of [PC00], and also [Co81].

Lemma 5. *Suppose $\hat{\sigma}$ is an analytic self-map of $\mathbb{H}$ which has non-tangential limit infinity at infinity and is semi-conformal, i.e., (2.10) holds for $\hat{\sigma}$. Then there is a simply-connected region Ω in $\mathbb{H}$ with an* inner-tangent at infinity, *i.e. for every $\alpha \in (0, \pi/2)$ there is $R > 0$ so that*

$$\{|\operatorname{Arg} z - \pi/2| < \alpha;\ |z| > R\} \subset \Omega,$$

with the property that $\hat{\sigma}$ restricted to Ω is one-to-one and $\hat{\sigma}(\Omega)$ also has an inner tangent at infinity.

The previous lemma in particular allows to select a simply-connected region $\Omega \subset \mathbb{H}$, called a *fundamental set* for ϕ, so that

(1) The map ϕ is one-to-one on Ω.
(2) The set Ω is *fundamental* for ϕ, in the sense that $\phi(\Omega) \subseteq \Omega$ and for any compact subset $K \subset\subset \mathbb{H}$ there exists $N = N(K)$ so that $\phi_n(K) \subset \Omega$ for all $n > N$.
(3) The map $\hat{\sigma}$ is one-to-one on Ω.
(4) The set $\hat{\sigma}(\Omega)$ is fundamental for the hyperbolic automorphism $\zeta \mapsto A\zeta$.

The explicit knowledge of the set Ω (and the intertwining map σ) coincides with the knowledge of the analytic and dynamical properties of ϕ. One could say that the dynamical properties of ϕ are read by means of the geometrical properties of the couple (σ, Ω). For instance, ϕ is one-to-one on $\mathbb{H}$ if and only if $\Omega = \mathbb{H}$ if and only if σ is one-to-one on $\mathbb{H}$.

To go back to our questions, we are left to deal with the uniqueness of the map σ. We have

Proposition 6 (Uniqueness of conjugation)**.** *Suppose σ is an analytic self-map of $\mathbb{H}$ which satisfies the functional equation (2.6). Then σ has non-tangential limit ∞ at ∞, it is semi-conformal at ∞ (*i.e. (2.10) *holds for σ). Moreover, every other self-map of $\mathbb{H}$ satisfying (2.6) is a positive constant multiple of σ.*

Oddly enough, even if everyone would swear that all the solution built by Valiron [Va31], Pommerenke [Po79], Cowen [Co81] and Bourdon-Shapiro [BS97] coincide, it seems that no one proved this explicitly. In case the map σ is known to fix ∞ as non-tangential limit and to be semi-conformal at ∞, the proof can be done directly (see the proof of Theorem 1.2 of [PC00]). Here we present a different proof which is based on the existence of an intertwining map semi-conformal at ∞ and a theorem on the commutator of hyperbolic automorphisms due to Heins [He41].

The Proof of Proposition 6. Let us first define the following sets of holomorphic mappings:

$$\mathcal{C}_A := \{F : \mathbb{H} \to \mathbb{C} \text{ holomorphic } | F(Az) = AF(z) \ \ \forall z \in \mathbb{H}\},$$

$$\mathcal{S} := \{\sigma : \mathbb{H} \to \mathbb{C} \text{ holomorphic } | \sigma \circ \phi = A\sigma\}.$$

The set $\mathcal{C}_A$ is thus formed by holomorphic maps which commute with the linear fractional map (hyperbolic automorphism of $\mathbb{H}$) $\zeta \mapsto A\zeta$; while the set $\mathcal{S}$ is made of all solutions of the functional equation (2.6). Notice that for the moment we are not restricting ourselves to *self-maps* of $\mathbb{H}$. The two sets are essentially the same as the following lemma shows (see also Lemma 4 in [Co81]). As a matter of notation, we let σ_V be the Valiron intertwining mapping constructed before.

Lemma 7. *There is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{C}_A$ given by:*

$$\mathcal{C}_A \ni F \mapsto F \circ \sigma_V \in \mathcal{S}.$$

Proof. Let $F \in \mathcal{C}_A$. Let us denote by $\Phi(z) = Az$. Then

$$(F \circ \sigma_V) \circ \phi = F \circ (\sigma_V \circ \phi) = F \circ \Phi \circ \sigma_V = \Phi \circ (F \circ \sigma_V).$$

On the other hand if $\sigma \in \mathcal{S}$, since σ_V is univalent on Ω (the fundamental set constructed before), one can define a holomorphic map $\tilde{F}$ on $\sigma_V(\Omega)$ by

$$\tilde{F}(\sigma_V(x)) := \sigma \circ \sigma_V^{-1}(x).$$

Since $A(\sigma_V(\Omega)) \subseteq \sigma_V(\Omega)$, on $\sigma_V(\Omega)$ we have

$$\tilde{F} \circ \Phi = \sigma \circ \sigma_V^{-1} \circ \Phi = \sigma \circ \phi \circ \sigma_V^{-1} = \Phi \circ \sigma \circ \sigma_V^{-1} = \Phi \circ \tilde{F}. \tag{2.11}$$

Then one can extend $\tilde{F}$ to all of $\mathbb{H}$ as follows:

$$F(z) = A^{-n}\tilde{F}(A^n z) \quad \text{for } z \in \mathbb{H} \text{ and } n \in \mathbb{N} \text{ such that } A^n z \in \sigma_C(V).$$

The map F is well defined, *i.e.*, it is independent of $n \in \mathbb{N}$ by (2.11). Moreover $F \in \mathcal{C}_A$ and $\sigma = F \circ \sigma_V$. □

Now we can complete the proof of Proposition 6 as follows. Let $\sigma \in \mathcal{S}$ be such that $\sigma(\mathbb{H}) \subseteq \mathbb{H}$. From Lemma 7 it follows that $\sigma = F \circ \sigma_C$ for some $F : \mathbb{H} \to \mathbb{C}$ such that $F(Aw) = AF(w)$. If $F(\mathbb{H}) \subseteq \mathbb{H}$, by a theorem of Heins [He41], we must have $F(w) = \mu w$ for some $\mu \in \mathbb{R}^+$ and therefore $\sigma = \mu\sigma_V$, which in particular proves that σ has fixed point ∞ and it is semi-conformal at ∞. We are thus left to prove that if $F(\sigma_V(\mathbb{H})) \subseteq \mathbb{H}$ then actually $F(\mathbb{H}) \subseteq \mathbb{H}$. Assume this is not the case. Then there exists $w_0 \in \mathbb{H}$ such that $\operatorname{Im} F(w_0) \leq 0$. Since $\sigma_V(\mathbb{H})$ is fundamental for $w \mapsto Aw$, it follows that there exists $n \in \mathbb{N}$ such that $A^n w_0 \in \sigma_V(\mathbb{H})$. But then

$$\operatorname{Im} F(A^n w_0) = \operatorname{Im} A^n F(w_0) \leq 0,$$

meaning that $F \circ \sigma_V(\mathbb{H}) \not\subset \mathbb{H}$ against our hypothesis.

Remark 8. More generally, arguing as in Proposition 4 of [Co81] one can prove that for any $\sigma \in \mathcal{S}$ (no restriction on the image $\sigma(\mathbb{H})$) there exists a holomorphic map $g : \{\zeta \in \mathbb{C} : |\log|\zeta|| < \pi^2/\log A\} \to \mathbb{C}$ such that σ is given by $w \mapsto \sigma_V(w) \cdot g(\exp(2\pi i \log \sigma_V(w)/\log A))$. Thus Proposition 6 says that if $\sigma(\mathbb{H}) \subseteq \mathbb{H}$ then g is a real positive constant.

2.4. **Property 3:** ***The ratios z_n/A^n do not always converge.*** This property is connected to the conformality at infinity of Valiron's conjugation. In fact, let σ be the limit of $\tau_n \circ \Phi_n$, and without loss of generality assume that $b_\infty = 0$ so that σ satisfies (2.6). Let $\alpha = \inf_{z \in \mathbb{H}} \operatorname{Im} \sigma(z)/\operatorname{Im} z$. There are two possibilities: either

α is 0 or it is positive. In either case, since z_n approaches infinity non-tangentially, Julia-Carathéodory's Theorem 3 applies, so that

$$\frac{\sigma(z_n)}{z_n} = \frac{A^n}{z_n}\sigma(z_0) \to \alpha$$

When $\alpha > 0$ it is costumary to say that σ has a finite angular derivative at infinity. Valiron gives a couple of necessary and sufficient conditions for this to happen, which are quite tautological. Bourdon and Shapiro [BS97] show that if Φ extends analytically near infinity then $\alpha > 0$. Arguing as in [BG03] one can state the Bourdon-Shapiro theorem as follows:

Theorem 9. *Suppose Φ is an analytic self-map of $\mathbb{H}$ such that $\Phi(z) = Az + \Gamma(z)$, with $A > 1$, and there exist $M, \epsilon > 0$ such that $|\Gamma(z)| \leq M|z|^{1-\epsilon}$ for all $z \in \mathbb{H}$. Then the conjugating map σ has a finite angular derivative at infinity.*

The question of the convergence of the ratio z_n/A^n is strictly related to that of the existence of fixed points for intertwining mappings $\sigma : \mathbb{H} \to \mathbb{H}$. Indeed, assume that σ_V has finite angular derivative at infinity, say $\alpha > 0$. Then $\sigma_\lambda := \lambda\sigma_V$ for $\lambda > 1/\alpha$ is a holomorphic self-map of $\mathbb{H}$ such that it has non-tangential limit ∞ at ∞, and $\sigma'_\lambda(\infty) = \lambda\alpha > 1$. Therefore ∞ is the Denjoy-Wolff point of σ_λ for all $\lambda > 1/\alpha$. In particular σ_λ has no fixed points in $\mathbb{H}$. Therefore, *the ratio z_n/A^n is convergent if and only if there exists one—and hence infinitely many—intertwining maps σ with Denjoy-Wolff point at ∞*. One is thus forced to study the following curve T:

$$T : \mathbb{R}^+ \ni t \mapsto H(t\sigma_V) \in \overline{\mathbb{H}} \cup \{\infty\},$$

where for a holomorphic self-map $f \neq Id$ of $\mathbb{H}$, $H(f)$ is the so-called *Heins map*, defined to be the (unique) fixed point of f in $\mathbb{H}$ if f has fixed points, or the Denjoy-Wolff point of f in case f has no fixed points in $\mathbb{H}$. The map H is easily seen to be continuous on the subset of the complex Banach space $H^\infty(\mathbb{H})$ given by functions with range in $\overline{\mathbb{H}}$, and it can be shown that it is holomorphic on the open set given by functions whose image is relatively compact in $\mathbb{H}$ (see [Br02]). Therefore the curve $t \to T(t)$ is a continuous curve in $\overline{\mathbb{H}}$ that can be continuously extended to $[0, \infty)$ as $T(0) = 0$ (the geometric meaning is that the constant function $z \mapsto 0$ is a solution of (2.6)). Moreover it is analytic at a point t_0 whenever $T(t_0) \in \mathbb{H}$. The question on the ratio z_n/A^n can be stated in terms of T as follows: *the ratio z_n/A^n is convergent if and only if the curve T reaches infinity in a finite time, namely if and only if there exists $t_0 \in (0, +\infty)$ such that $T(t_0) = \infty$* (and then $T(t) = \infty$ for $t > t_0$). The curve T reads the geometrical properties of ϕ. For instance it is easy to see that if ϕ is such that $\lim_{z \to p} |\phi(z)| < 1$ for all $p \in \partial\mathbb{H} \setminus \{\infty\}$, then $T(t) \in \mathbb{H} \cup \{\infty\}$ for all $t \in (0, +\infty)$, and in particular T is analytic in its interior. With a slightly more subtle argument on commuting mappings (using Behan's lemma, see, *e.g.*, [Ab89]) one can show that if $T(t) = 0$ for some $t \in (0, +\infty)$ then f cannot fix 0 in the sense of non-tangential limits. It would be interesting to pursue a systematic study of the relations between properties of ϕ and properties of T.

3. Several complex variables

We fix $N = 2, 3, 4, \ldots$ and $\mathbb{B} = \mathbb{B}^N = \{z \in \mathbb{C}^N : \|z\| < 1\}$, where

$$\|z\|^2 = (z, z) \qquad \text{and} \qquad (z, w) = \sum_{j=1}^{N} z^j \overline{w^j}.$$

Let ϕ be a self-map of $\mathbb{B}$. As in the disk case we can say that ϕ is of *elliptic type* if it fixes at least one point in $\mathbb{B}$ (however, now, ϕ could fix more than just one-point and not be the identity). We are interested in the case when ϕ has no fixed points in $\mathbb{B}$. The Denjoy-Wolff Theorem still hold (see [Ab89] Theorem 2.2.31), namely, the iterates of ϕ converge to one point on $\partial\mathbb{B}$. By conjugating with a unitary map we can assume without loss of generality that this special point is $e_1 = (1, 0, \ldots, 0)$. Once again maps with no fixed points in $\mathbb{B}$ will be divided into hyperbolic and parabolic type, but before we can do this we need to introduce a few tools.

3.1. **A special automorphism.** For $a \in \mathbb{B}$, we define the projections

$$P_a(z) = \frac{(z, a)}{(a, a)} a \qquad \text{and} \qquad Q_a(z) = z - P_a(z).$$

Then we let

$$\gamma_a(z) = \frac{P_a(z) + s_a Q_a(z) - a}{1 - (z, a)} \tag{3.1}$$

where $s_a = \sqrt{1 - \|a\|^2}$, and so that $\gamma_a(a) = 0$. It is well-known that γ_a is an automorphism of $\mathbb{B}$.

We define the pseudo-hyperbolic distance between two points $a, b \in \mathbb{B}$ as

$$d(a, b) = \|\gamma_a(b)\| < 1.$$

Schwarz's Lemma ([Ab89] Thm. 2.2.12) and [Ab89] Cor. 2.2.2, imply as in the disk that $d(\phi(a), \phi(b)) \leq d(a, b)$. Another quantity which is decreased by self-maps of the ball is

$$Q(a, b) = \frac{|1 - (a, b)|^2}{(1 - \|a\|^2)(1 - \|b\|^2)}, \tag{3.2}$$

i.e., $Q(\phi(a), \phi(b)) \leq Q(a, b)$ ([Ab89] Prop 2.2.17)

3.2. **Hyperbolic versus parabolic.** We will again consider the orbit of the origin $z_n = \phi_n(0)$, thus $z_n \to e_1$. It follows that one can extract a subsequence z_N with the property that $\|z_{N+1}\| \geq \|z_N\|$. Hence

$$c = \liminf_{z \to e_1} \frac{1 - \|\phi(z)\|}{1 - \|z\|} \leq 1$$

so by Julia's Lemma ([Ab89] Thm. 2.2.21)

$$\frac{|1 - (\phi(z), e_1)|^2}{1 - \|\phi(z)\|^2} \leq c \frac{|1 - (z, e_1)|^2}{1 - \|z\|^2},$$

and in particular,

$$\frac{|1 - z_{n+1}^1|^2}{1 - \|z_{n+1}\|^2} \leq c \frac{|1 - z_n^1|^2}{1 - \|z_n\|^2}. \tag{3.3}$$

We say ϕ is of *hyperbolic type* if $c < 1$, and of *parabolic type* if $c = 1$. The quantity $c = c(\phi)$ is called the coefficient of dilatation of ϕ. It is clear that

$$c(\phi_n) = [c(\phi)]^n. \tag{3.4}$$

In the sequel we will assume that ϕ is a self-map of the ball of hyperbolic type. First we describe the automorphisms of hyperbolic type.

3.3. **Automorphisms of the ball of hyperbolic type.** As in the one-dimensional case it is best to move to an "upper half-plane" model. It turns out that $\mathbb{B}$ is biholomorphic to the domain

$$\mathbb{H}^N = \{w = (w^1, w') \in \mathbb{C}^N : \operatorname{Im} w^1 > \|w'\|^2\}$$

via a map very similar to the classical Caley transform. Given $a \in \mathbb{H}^N$ with $\operatorname{Im} a^1 - \|a'\|^2 > 1$ there is an automorphism of hyperbolic type Ψ_a which sends the point $\iota = (i, 0')$ to a. We first build the inverse of such mapping. Consider the translation (we refer to [Ab89] p. 155 for these automorphisms of $\mathbb{H}^N$.)

$$h_b(w) = (w^1 + b^1 + 2i\langle w', b'\rangle, w' + b') \tag{3.5}$$

where $b = (-\operatorname{Re} a^1 + i\|a'\|^2, -a') \in \partial\mathbb{H}^N$. Then

$$h_b(a) = (i(\operatorname{Im} a^1 - \|a'\|^2), 0').$$

Now consider the non-isotropic dilation

$$\delta_A(w) = \left(Aw^1, \sqrt{A}w'\right) \tag{3.6}$$

where $A = \operatorname{Im} a^1 - \|a'\|^2$. The automorphism $\Phi_a = \delta_{1/A} \circ h_b$ sends a to ι,

$$\Phi_a(w) = \left(\frac{w^1 - \operatorname{Re} a^1 + i\|a'\|^2 - 2i\langle w', a'\rangle}{A}, \frac{w' - a'}{\sqrt{A}}\right).$$

The inverse is

$$\Psi_a(z) = \left(Az^1 + \operatorname{Re} a^1 + i\|a'\|^2 + 2i\sqrt{A}\langle z', a'\rangle, \sqrt{A}z' + a'\right). \tag{3.7}$$

More generally, given a unitary transformation U of $\mathbb{C}^{N-1}$ one can consider the automorphism

$$\Psi(z) = \left(Az^1 + \operatorname{Re} a^1 + i\|a'\|^2 + 2i\sqrt{A}\langle U(z'), a'\rangle, \sqrt{A}U(z') + a'\right). \tag{3.8}$$

Varying a with $A = \operatorname{Im} a^1 - \|a'\|^2 > 1$ and U as above, the automorphisms Ψ describe all possible hyperbolic automorphisms of $\mathbb{H}^N$ with infinity as attracting fixed point.

3.4. **Self-maps of the ball of hyperbolic type.** Let Φ be a holomorphic self-map of $\mathbb{H}^N$ without fixed points in $\mathbb{H}^N$, such that its Denjoy-Wolff point is ∞, and of hyperbolic type.

Following the lead of the one-dimensional case the following open problem arises:

Open Problem 10. How closely are the orbits of Φ trying to imitate the behavior of the orbits of a corresponding hyperbolic automorphism Ψ as in (3.8)?

The automorphism Ψ in (3.8) fixes exactly two points: infinity and the point $c \in \partial\mathbb{H}^N$. To see this first solve

$$\sqrt{A}U(c') + a' = c'.$$

Taking U^{-1} and dividing by $\sqrt{A}$, one gets

$$(I - \frac{1}{\sqrt{A}}U^{-1})(c') = -\frac{1}{\sqrt{A}}U^{-1}(a')$$

which is invertible. Now solve

$$Ac^1 + \operatorname{Re} a^1 + i\|a'\|^2 + 2i\sqrt{A}\langle U(c'), a'\rangle = c^1$$

using the fact that $\sqrt{A}U(c') = (I - (\sqrt{A}U)^{-1})^{-1}(a')$.

Therefore using an appropriate translation as in (3.5), the map Ψ can be conjugated to an automorphism whose fixed points are 0 and ∞, i.e. to

$$\Psi_0(z) = (Az^1, \sqrt{A}U(z')). \tag{3.9}$$

Moreover, by linear algebra Ψ_0 can be further conjugated via a unitary matrix so that U becomes diagonal.

Our open problem can be rephrased as

Open Problem 11. Given a holomorphic self-map Φ of $\mathbb{H}^N$ without fixed points in $\mathbb{H}^N$, such that its Denjoy-Wolff point is ∞, and which is of hyperbolic type with dilation coefficient $A > 1$, does there exist a unitary trasformation U of $\mathbb{C}^{N-1}$ and a conjugation σ (also a self-map of $\mathbb{H}^N$) such that

$$\sigma \circ \Phi = \Psi_0 \circ \sigma$$

where Ψ_0 is as in (3.9), and so that σ has some degree of regularity at infinity to be determined (something along the lines of semi-conformality)?

In a recent preprint [BG03] the result of Bourdon and Shapiro has been generalized to several complex variable, *i.e.*, Valiron's conjugation is established under some smoothness assumptions at infinity for Φ.

What we can show is a partial answer to Open Problem 10 which resembles Lemma 4 in the one-dimensional case. Namely, we can show that the orbits of Φ always remain in a Korányi approach region at infinity, see definition below (the fact that the orbits of Ψ_0 remain in a Korányi approach region at infinity can easily be verified).

3.5. **Korányi approach of the orbits.** Back in the ball setting, let ϕ be a holomorphic self-map of $\mathbb{B}$ without fixed points in $\mathbb{B}$, such that its Denjoy-Wolff point is e_1, and which is of hyperbolic type with dilation coefficient $c < 1$.

Given a parameter $M > 0$ the Korányi regions at e_1 of amplitude M are the sets

$$K(R) = \left\{ z \in \mathbb{B} : \frac{|1 - z^1|}{1 - \|z\|^2} < R \right\}.$$

We need a preliminary result.

Claim 12. *If $c < 3 - \sqrt{8}$, then the orbit $z_n = \phi_n(0)$ tends to e_1 while staying in a Korányi approach region, i.e.,*

$$L_n = \frac{|1 - (z_n, e_1)|}{1 - \|z_n\|^2} = \frac{|1 - z_n^1|}{1 - \|z_n\|^2} \leq M < \infty$$

for some constant $M < \infty$.

Assuming Claim 12 for the moment, we show the Korányi approach of the orbit $z_n = \phi_n(0)$. Using (3.4), we can find an integer N large enough so that $c^N < 3 - \sqrt{8}$, and Claim 12 implies that z_{kN}, $k = 1, 2, 3 \cdots$, stays in a Korányi region. However, for $j = 1, \ldots, N-1$, $d(z_{kN+j}, z_{kN}) \leq d(0, z_j)$, by Schwarz's Lemma. Hence, since the hyperbolic neighborhood of a Korányi region is still a Korányi region, we find that the whole orbit z_n remains in a Korányi region. Moreover, by the same argument, any orbit $\phi_n(z_0)$ has the same property.

Proof Claim 12: We rewrite (3.3) as

$$S_n = \left| \frac{1 - z_{n+1}^1}{1 - z_n^1} \right| \leq c \frac{L_n}{L_{n+1}}. \tag{3.10}$$

Recalling the definition and monotonicity property of $Q(a, b)$ given in (3.2), we see that $Q(z_n, z_{n+1})$ is decreasing and thus

$$Q(z_n, z_{n+1}) = \frac{|1 - (z_n, z_{n+1})|^2}{(1 - \|z_n\|^2)(1 - \|z_{n+1}\|^2)} \leq Q(z_0, z_1) = \frac{1}{1 - \|z_1\|^2} < \infty. \tag{3.11}$$

Notice that

$$1 - (z_n, z_{n+1}) = (e_1 - z_n, e_1) + (e_1, e_1 - z_{n+1}) - (e_1 - z_n, e_1 - z_{n+1}).$$

Therefore,

$$|1-(z_n,z_{n+1})| \geq |(1-z_n^1)+(1-\overline{z_{n+1}^1})| - \|e_1-z_n\|\|e_1-z_{n+1}\|.$$

Expanding the square, we have

$$\frac{\|e_1-z_n\|^2}{1-\|z_n\|^2} = 2\frac{1-\operatorname{Re} z_n^1}{1-\|z_n\|^2} - 1 \leq 2L_n.$$

So, after a square root and the triangle inequality, (3.11) becomes

$$\frac{|(1-z_n^1)+(1-\overline{z_{n+1}^1})|}{\sqrt{1-\|z_n\|^2}\sqrt{1-\|z_{n+1}\|^2}} - 2\sqrt{L_nL_{n+1}} \leq \sqrt{Q(z_0,z_1)}. \tag{3.12}$$

Now suppose that

$$\limsup_{n\to\infty} L_n = +\infty.$$

Then one can find a subsequence L_N such that $L_N \leq L_{N+1}$ and $L_N \to +\infty$. By (3.10), $\limsup_{N\to\infty} S_N \leq c$. On the other hand, dividing by $\sqrt{L_NL_{N+1}}$ and letting N tend to infinity in (3.12), we also have

$$\limsup_{N\to\infty} \frac{|(1-z_n^1)+(1-\overline{z_{n+1}^1})|}{\sqrt{|1-z_N^1|}\sqrt{|1-z_{N+1}^1|}} \leq 2.$$

Squaring both sides and reorganizing

$$\limsup_{N\to\infty} |1-S_N||1-\frac{1}{S_N}| \leq 4.$$

So if S is a sublimit of S_N it must satisfy $S \leq c$ and

$$(1-S)^2 \leq 4S,$$

i.e., $3-\sqrt{8} \leq S \leq 3+\sqrt{8}$. In particular, if c happens to be less than $3-\sqrt{8}$, then no sublimit of S_N can exists and therefore L_n remains bounded. □

3.6. **Conclusion.** The problem one encounters after this claim is established is that one would like to use the Julia-Carathéodory Theorem for self-maps of the ball. Such result exists, see [Ab89] Theorem (2.2.29), however in order to use it one would need a much more restrictive approach for the orbit of ϕ: "special and restricted". Of course, even the orbits of the automorphism Ψ_0 do not have this property in general, but there is always one orbit that does. Our hope is to be able to produce at least one orbit of ϕ that has a special and restricted approach and then renormalize ϕ using this orbit.

There is a different approach which seems to bypass the unitary matrix U of Open Problem 11.

Open Problem 13. Given a holomorphic self-map Φ of $\mathbb{H}^N$ without fixed points in $\mathbb{H}^N$, such that its Denjoy-Wolff point is ∞, and which is of hyperbolic type with dilation coefficient $A > 1$, does there exist a conjugation $\eta : \mathbb{H}^N \longrightarrow \mathbb{H}$ such that

$$\eta \circ \Phi = A\eta,$$

and so that σ has some degree of regularity at infinity to be determined (something along the lines of semi-conformality)?

Of course if one can find σ which solves Open Problem 11 then $\eta = \pi^1 \circ \sigma$, where $\pi^1(z^1, z') = z^1$, will solve Open Problem 13.

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