LOCALIZATION OF CHARACTERISTIC CLASSES AND APPLICATIONS†.

TATSUO SUWA
Department of Mathematics, Hokkaido University
Sapporo 060-0810, Japan, e-mail: suwa@math.sci.hokudai.ac.jp

Notes written by Filippo Bracci
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1, 00133 Roma, Italy, e-mail:fbracci@mat.uniroma2.it

1. THE STOKES THEOREM AND THE DE RHAM THEOREM

1.1. **The Stokes Theorem.** Let M be a C^∞ manifold of dimension m. Let $U \subseteq M$ be an open subset of M. We indicate by $A^p(U)$ the vector space of complex-valued p-forms on U. More precisely, if TM is the tangent bundle of M and $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ is the complexified tangent bundle of M, a *complex-valued* p*-form* on U is a C^∞ section of the vector bundle $\Lambda^p(T^{\mathbb{C}}M)^*$ on U.

If U is a coordinate set with $\{x_1, \ldots, x_m\}$ local coordinates then a C^∞ section ω of $\Lambda^p(T^{\mathbb{C}}M)^*$ on U is given by

$$\omega = \sum_{1 \leq i_1 < \ldots < i_p \leq m} f_{i_1 \ldots i_p}(x) dx_{i_1} \wedge \ldots \wedge dx_{i_p},$$

for some complex-valued C^∞ functions $f_{i_1 \ldots i_p}$ defined on U.

A set $R \subset M$ is a *manifold of dimension* m *with* C^∞ *boundary* if Int$R$ is a m-dimensional manifold and for any $p \in \partial R$ there exists a open coordinate set U in M with local coordinates $\{x_1, \ldots, x_m\}$ such that $R \cap U = \{q \in U | x_1(q) \leq 0\}$ and $\partial R \cap U = \{q \in U | x_1(q) = 0\}$. Thus ∂R is a $(m-1)$-dimensional manifold. Moreover if M is oriented and the above system of local coordinates $\{x_1, \ldots, x_m\}$ is positive, we give ∂R the orientation coming from declaring $\{x_2, \ldots, x_m\}$ to be a positive system of local coordinates for ∂R.

Theorem 1.1.1 (Stokes)**.** *Suppose M is oriented. Let $R \subset M$ be an m-dimensional manifold with C^∞ boundary and let $i : R \to M$ be the inclusion map. Let $\omega \in A^{m-1}(M)$. Then*

$$\int_R d\omega = \int_{\partial R} i^* \omega.$$

1.2. **The de Rham cohomology.** The exterior derivative d of forms gives rise to a cohomological complex, the *de Rham complex*:

$$\ldots \to A^{p-1}(M) \stackrel{d^{p-1}}{\to} A^p(M) \stackrel{d^p}{\to} A^{p+1}(M) \to \ldots$$

Let us define the group of p-*closed* forms as $Z^p(M) := \mathrm{Ker} d^p$ and the group of p-*exact* forms as $B^p(M) := \mathrm{Im} d^{p-1}$. Since $d^p \circ d^{p-1} = 0$ then $B^p(M) \subseteq Z^p(M)$ and we can define the quotient group $H^p_d(M) := Z^p(M)/B^p(M)$ which is called the p-*th de Rham cohomology* of M. If $\omega \in Z^p(M)$, we denote by $[\omega] \in H^p_d(M)$ its image by the canonical projection $Z^p(M) \to H^p_d(M)$.

Note that if M is connected then $B^0(M) = \{0\}$ and $Z^0(M)$ contains only constant functions, therefore $H^0(M) \simeq \mathbb{C}$. Also we have the well known

† These are notes of a course given by Prof. Tatsuo Suwa on September 2001 at Università di Roma "Tor Vergata", supported by Progetto MURST di Rilevante Interesse Nazionale *Proprietà geometriche delle varietà reali e complesse*.

Lemma 1.2.1 (Poincaré). *The de Rham cohomology* $H^p_d(\mathbb{R}^m) = 0$ *for* $p \geq 1$.

More generally, let $H^p(M, \mathbb{C}) = H_p(M, \mathbb{C})^*$ where $H_p(M, \mathbb{C})$ is the p-th homology with complex coefficients on M, for instance, consider the singular homology with $\mathbb{C}$-coefficients on M. Then

Theorem 1.2.2 (de Rham). $H^p_d(M) \simeq H^p(M, \mathbb{C})$, *the isomorphism being given by*

$$[\omega] \mapsto \left(\sigma \mapsto \int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega \right),$$

where Δ^p *is the standard* p*-simplex and* $\sigma : \Delta^p \to M$ *is any singular simplex.*

Suppose M is oriented and compact (and $\partial M = \emptyset$). Let $\omega \in B^m(M)$. Therefore there exists $\theta \in A^{m-1}(M)$ such that $d\theta = \omega$. By Stokes' theorem it follows

$$\int_M \omega = \int_M d\theta = \int_{\partial M} \theta = 0,$$

since the boundary of M is empty. Thus the operator $\int_M : A^m(M) \to \mathbb{C}$ induces a well-defined operator, called *integration*

$$\begin{aligned} \int_M :& H^m_d(M) \to \mathbb{C}, \\ & [\omega] \mapsto \int_M \omega. \end{aligned}$$

2. The Čech-de Rham cohomology

The Čech-de Rham cohomology is defined for any coverings of a manifold M but for simplicity here we only consider a covering of M given by only two open sets.

2.1. Čech-de Rham cohomology.

Let M be a C^∞ manifold of dimension m. Let $\mathcal{U} := \{U_0, U_1\}$ be an open covering of M. Let $U_{01} := U_0 \cap U_1$. Define a vector space $A^p(\mathcal{U})$ as follows:

$$A^p(\mathcal{U}) := A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

Therefore an element $\sigma \in A^p(\mathcal{U})$ is given by a triple $\sigma = (\sigma_0, \sigma_1, \sigma_{01})$ such that σ_0 is a p-form on U_0, σ_1 is a p-form on U_1 and σ_{01} is a $(p-1)$-form on U_{01}.

Let us define the following operator D:

$$\begin{aligned} D :& A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U}) := A^{p+1}(U_0) \oplus A^{p+1}(U_1) \oplus A^p(U_{01}) \\ & \sigma = (\sigma_0, \sigma_1, \sigma_{01}) \mapsto (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}). \end{aligned}$$

One can check that $D \circ D = 0$. This allows to define a cohomological complex, *the Čech-de Rham complex*:

$$\ldots \to A^{p-1}(\mathcal{U}) \overset{D^{p-1}}{\to} A^p(\mathcal{U}) \overset{D^p}{\to} A^{p+1}(\mathcal{U}) \to \ldots$$

Let $Z^p_D(\mathcal{U}) := \mathrm{Ker} D^p$, $B^p_D(\mathcal{U}) := \mathrm{Im} D^{p-1}$. Since $D \circ D = 0$ the group $B^p_D(\mathcal{U}) \subseteq Z^p_D(\mathcal{U})$. thus we can define the quotient group

$$H^p_D(\mathcal{U}) := Z^p_D(\mathcal{U}) / B^p_D(\mathcal{U}),$$

called the *p-th Čech-de Rham cohomology with respect to* $\mathcal{U}$.

The canonical projection $Z^p_D(\mathcal{U}) \to H^p_D(\mathcal{U})$ is denoted by $\sigma \mapsto [\sigma]$.

Theorem 2.1.1. *The map*

$$\begin{aligned} A^p(M) &\to A^p(\mathcal{U}) \\ \omega &\mapsto (\omega, \omega, 0) \end{aligned}$$

induces an isomorphism

$$\alpha : H^p_d(M) \xrightarrow{\sim} H^p_D(\mathcal{U}). \tag{2.1}$$

Proof. First we have to show that α is well-defined. That is to say we have to show that

(1) if $d\omega = 0$ then $D(\omega, \omega, 0) = 0$ and
(2) if $\omega = d\theta$ for some $\theta \in A^{p-1}(M)$ then $(\omega, \omega, 0) = D\tau$ for some $\tau \in A^{p-1}(\mathcal{U})$.

The first one is immediate. As for the second define $\tau := (\theta, \theta, 0)$ and check that this does the job.

Now we have to prove that α is surjective. Let $\sigma := (\sigma_0, \sigma_1, \sigma_{01})$ be such that $D\sigma = 0$. Let $\{\rho_0, \rho_1\}$ be a partition of unity subordinated to the covering $\mathcal{U}$, *i.e.*, ρ_j is a C^∞ function on M with support in U_j ($j = 1, 2$) and $\rho_1(p) + \rho_2(p) = 1$ for any $p \in M$. Define $\omega := \rho_0\sigma_0 + \rho_1\sigma_1 - d\rho_0 \wedge \sigma_{01}$. Clearly $\omega \in A^p(M)$. Since $d\sigma_j = 0$ on U_j ($j = 0, 1$) and $\sigma_1 = \sigma_0 + d\sigma_{01}$ on U_{01} it is easy to see that $d\omega = 0$ on M. Moreover $[(\omega, \omega, 0)] = [\sigma]$, *i.e.*, $(\omega, \omega, 0) = \sigma + D\theta$ for some $\theta = (\theta_0, \theta_1, \theta_{01}) \in A^{p-1}(\mathcal{U})$. To see this we first show that there exists a $(p-1)$-form θ_0 on U_0 such that $\omega = \sigma_0 + d\theta_0$ on U_0. On $U_0 - U_1$ the function $\rho_0 \equiv 1$ and $\rho_1 \equiv 0$ and $\omega = \sigma_0$. On $U_0 \cap U_1$ it follows from $D\sigma = 0$ that

$$\omega = \rho_0\sigma_0 + \rho_1(\sigma_0 + d\sigma_{01}) - d\rho_0 \wedge \sigma_{01} = \sigma_0 + \rho_1 d\sigma_{01} + d\rho_1 \wedge \sigma_{01} = \sigma_0 + d(\rho_1\sigma_{01}).$$

Therefore once we define $\theta_0 := \rho_1\sigma_{01} \in A^{p-1}(U_0)$, we are done. Similarly one can check that $\theta_1 := -\rho_0\sigma_{01}$ works for $\omega = \sigma_1 + d\theta_1$ on U_1. Finally if we take $\theta_{01} = 0$, $\theta = (\theta_0, \theta_1, 0)$ does the job.

It is left to show that α is injective, but this follows easily and we skip its proof. □

2.2. **Integration.** Suppose that the m-dimensional manifold M is *oriented* and *compact* and let $\mathcal{U} := \{U_0, U_1\}$ be a covering of M. Let $R_0, R_1 \subset M$ be two compact manifolds of dimension m with C^∞ boundary with the following properties:

(1) $R_j \subset U_j$ for $j = 0, 1$,
(2) $\mathrm{Int}R_0 \cap \mathrm{Int}R_1 = \emptyset$ and
(3) $R_1 \cup R_2 = M$.

Let $R_{01} := R_0 \cap R_1$ and give R_{01} the orientation coming from being the boundary of R_0, *i.e.*, $R_{01} = \partial R_0$. equivalently give R_{01} the opposite orientation coming from being the boundary of R_1, *i.e.*, $R_{01} = -\partial R_1$. Define the following integration operator:

$$\begin{aligned} \int_M :& A^m(\mathcal{U}) \to \mathbb{C} \\ \sigma = (\sigma_0, \sigma_1, \sigma_{01}) &\mapsto \int_M \sigma := \int_{R_0} \sigma_0 + \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01}. \end{aligned}$$

Lemma 2.2.1. *The operator $\int_M$ has the following properties:*

(1) *Let $\sigma \in A^m(\mathcal{U})$. If $D\sigma = 0$ then $\int_M \sigma$ is independent of $\{R_0, R_1\}$.*
(2) *Let $\sigma \in A^m(\mathcal{U})$. If $\sigma = D\tau$ for some $\tau \in A^{p-1}(\mathcal{U})$ then $\int_M \sigma = 0$.*

Proof. Apply The Stokes theorem. □

Corollary 2.2.2. *The operator $\int_M$ defined on $A^m(\mathcal{U})$ induces an operator*

$$\begin{aligned}\int_M :&H^m_D(\mathcal{U}) \to \mathbb{C}\\ &[\sigma] \mapsto \int_M \sigma.\end{aligned}$$

Note that if $\omega \in H^m_d(M)$ and $\alpha(\omega) \in H^m_D(\mathcal{U})$ then

$$\int_M \alpha(\omega) = \int_M \omega. \tag{2.2}$$

2.3. **Relative Čech-de Rham cohomology.** Let M be a m-dimensional manifold, $\mathcal{U} := \{U_0, U_1\}$ a covering of M. Let us define

$$A^p(\mathcal{U}, U_0) := \{\sigma = (\sigma_0, \sigma_1, \sigma_{01}) \in A^p(\mathcal{U}) | \sigma_0 = 0\}.$$

By the very definition of D, if $\sigma \in A^p(\mathcal{U}, U_0)$ then $D\sigma \in A^p(\mathcal{U}, U_o)$. This gives rise to another complex, called the *relative Čech-de Rham complex*:

$$\ldots \to A^{p-1}(\mathcal{U}, U_0) \stackrel{D^{p-1}}{\to} A^p(\mathcal{U}, U_0) \stackrel{D^p}{\to} A^{p+1}(\mathcal{U}, U_0) \to \ldots$$

Similarly to what we did before, we define the *p-th relative Čech-de Rham cohomology with respect to $(\mathcal{U}, U_0)$* as

$$H^p_D(\mathcal{U}, U_0) := \mathrm{Ker} D^p / \mathrm{Im} D^{p-1}.$$

The relative Čech-de Rham cohomology is indeed a topological invariant of M:

Lemma 2.3.1. *There is a natural isomorphism*

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, U_0; \mathbb{C}),$$

where $H^p(M, U_0; \mathbb{C})$ is the p-th group of the relative cohomology with complex coefficients.

2.4. **Integration.** Suppose M is an m-dimensional oriented manifold (not necessarily compact). Let $S \subset M$ be a compact subset of M. Let $U_0 := M - S$ and let U_1 be an open neighborhood of S. Let R_1 be a compact manifold of dimension m with C^∞ boundary such that $S \subset \mathrm{Int} R_1 \subset R_1 \subset U_1$. Let $R_0 := M - \mathrm{Int} R_1$. Note that $R_0 \subset U_0$. The integral operator $\int_M$ (which is not defined in general for $A^m(\mathcal{U})$ unless M is compact) is well defined on $A^p(\mathcal{U}, U_0)$:

$$\begin{aligned}\int_M :&A^m(\mathcal{U}, U_0) \to \mathbb{C}\\ &\sigma = (0, \sigma_1, \sigma_{01}) \mapsto \int_M \sigma := \int_{R_1} \sigma_1 + \int_{R_{01}} \sigma_{01},\end{aligned}$$

and induces an operator $\int_M : H^m_D(\mathcal{U}, U_0) \to \mathbb{C}$.

If M is compact and $j^* : H^m_D(\mathcal{U}, U_0) \to H^m_D(\mathcal{U})$ is the map induces by the injection, then for any $\sigma \in H^m_D(\mathcal{U}, U_0)$ it follows

$$\int_M j^*(\sigma) = \int_M \sigma. \tag{2.3}$$

Example 2.4.1. Let $M = \mathbb{R}^m$ and $S = \{0\}$. Then $U_0 = \mathbb{R}^m - \{0\} \simeq S^{m-1}$ and $U_1 = \mathbb{R}^m$. We first calculate $H^0_D(\mathcal{U}, U_0)$. If $\sigma \in A^0(\mathcal{U}, U_0)$ then $\sigma = (0, f, 0)$ for some smooth function defined on U_1. If $D\sigma = 0$ by definition $f \equiv 0$ and therefore $H^0_D(\mathcal{U}, U_0) = \{0\}$. For $p = 1$, let $\sigma \in A^1(\mathcal{U}, U_0)$. Then $\sigma = (0, \sigma_1, f)$ where σ_1 is a 1-form on U_1 and f is a C^∞ function on $U_0 \cap U_1$. If σ is a cocycle then $d\sigma_1 = 0$ on U_1 and $df = \sigma_1$ on $U_0 \cap U_1$. By the Poincaré lemma the first condition implies that $\sigma_1 = dg$ for some C^∞ function g on U_1 and the second condition implies that $f \equiv g + c$ for some $c \in \mathbb{C}$. Therefore f has a smooth extension—still denoted by f—at $\{0\}$ and $\sigma = (0, df, f) = D(0, f, 0)$. Hence every cocycle is a coboundary and $H^1_D(\mathcal{U}, U_0) = \{0\}$. For $p \geq 2$ the map

$$\begin{aligned} H^{p-1}_d(U_0) &\to H^p_D(\mathcal{U}, U_0) \\ [\omega] &\mapsto [(0, 0, -\omega)] \end{aligned}$$

is an isomorphism (we left the details to the reader) and therefore by the de Rham Theorem for $p \geq 2$ it follows

$$H^p_D(\mathcal{U}, U_0) \simeq H^{p-1}_d(U_0) \simeq H^{p-1}(S^{m-1}) = \begin{cases} \mathbb{C} \ \text{ for } \ p = m \\ 0 \text{ for } \ p = 2, \dots, m-1. \end{cases}$$

In particular, when $m = 2$, identifying $\mathbb{R}^2$ with $\mathbb{C} = \{z\}$, an explicit generator of $H^2_D(\mathcal{U}, U_0) \simeq H^1_d(U_0)$ is given by the Cauchy kernel $\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z}$.

3. Characteristic classes of complex vector bundles

In this section we are going to discuss the Chern-Weil theory adapted to the previously introduced Čech-de Rham cohomology.

3.1. **Connections on complex vector bundles.** Let M be a C^∞ manifold of dimension m and let $E \to M$ be a complex vector bundle of rank r. Let U be a open subset of M and let us indicate by $A^p(U, E)$ the vector space of p-forms on U with coefficients in E. In other words, $A^p(U, E)$ is the space of C^∞ sections of the bundle $\Lambda^p(T^{\mathbb{C}}M)^* \otimes E$ on U. This means that locally an element of $A^p(U, E)$ is given by

$$\sum_i \omega_i \otimes s_i,$$

where the ω_i's are p-th forms on U and the s_i's are C^∞ sections of the bundle E on U.

Note that $A^1(M, E)$ is the space of the C^∞ sections of the vector bundle $(T^{\mathbb{C}}M)^* \otimes E \simeq \mathrm{Hom}(T^{\mathbb{C}}M, E)$. Also $A^0(M)$ indicates the space of C^∞ functions on M and $A^0(M, E)$ the space of C^∞ sections of the bundle E.

Definition 3.1.1. A *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \to A^1(M, E)$$

satisfying the following *Leibniz rule*:

$$\nabla(fs) = df \otimes s + f\nabla(s),$$

for any $f \in A^0(M)$ and $s \in A^0(M, E)$.

Example 3.1.2. Let $E = M \times \mathbb{C}$. Then $A^p(M, E) = A^p(M)$ and $\nabla := d$ is a connection on E.

Now we recall some basic facts about connections coming out from the very definition.

Lemma 3.1.3. *A connection ∇ on E is a local operator,* i.e., *if $U \subseteq M$ is a open set and $s \in A^0(M,E)$ is such that $s_{|U} = 0$ then $\nabla s = 0$ on U.*

The previous lemma allows to restrict a connection ∇ for E to an open set $U \subset M$ giving rise to well-defined connection $\nabla_{|U}$ for $E_{|U}$.

Lemma 3.1.4. *Suppose $\nabla_1, \dots, \nabla_k$ are connections for E and $f_1, \dots, f_k$ are C^∞ functions on M such that $\sum_j f_j \equiv 1$ then $\sum_{j=1}^k f_j \nabla_j$ is a connection for E.*

Using partitions of unity subordinated to a trivializing covering of a vector bundle the previous Lemma as a very strong consequence:

Corollary 3.1.5. *Every complex vector bundle admits a connection.*

One may also “derive” forms of higher degree on E. For our purposes we only need to define

$$\nabla : A^1(M,E) \to A^2(M,E).$$

To do this we note that any element of $A^1(M,E)$ is a linear combination of elements of the form $\omega \otimes s$ for $\omega \in A^1(M)$ and $s \in A^0(M,E)$. Therefore we define

$$\nabla : \omega \otimes s \mapsto dw \otimes s - \omega \wedge \nabla s,$$

for $\omega \in A^1(M)$ and $s \in A^0(M,E)$ and extend for linearity to the other elements of $A^1(M,E)$.

Definition 3.1.6. The *curvature* of ∇ is $K := \nabla \circ \nabla$.

Note that $K \neq 0$ in general, and it actually measures how far from the trivial bundle E is.

It is easy to show that $K(fs) = fK(s)$ for any $f \in A^0(M)$ and $s \in A^0(M,E)$.

Let U be a open set of M trivializing E, *i.e.*, $E_{|U} \simeq U \times \mathbb{C}^r$. Let $s_1, \dots, s_r$ be r sections of E on U linearly independent at each point of U (just take for instance s_j to be the inverse image of the j-th element of a basis of $\mathbb{C}^r$ under the diffeomorphism $E_{|U} \simeq U \times \mathbb{C}^r$). The set $\mathbb{S} := (s_1, \dots, s_r)$ is called a *frame* for E on U.

Let $\mathbb{S} := (s_1, \dots, s_r)$ be a frame for E on U. Note that any section of E on U is a linear combination (with C^∞ coefficients) of $s_1, \dots, s_r$. Then we may write

$$\nabla s_i = \sum_{j=1}^r \theta_{ij} \otimes s_j,$$

for some 1-forms θ_{ij} defined on U. The matrix $\theta := (\theta_{ij})$ is called the *connection matrix* with respect to $\mathbb{S}$.

Similarly for the curvature K we may write

$$K(s_i) = \sum_{j=1}^r k_{ij} \otimes s_j,$$

where k_{ij} are 2-forms on U. The matrix $k = (k_{ij})$ is the *curvature matrix* with respect to $\mathbb{S}$.

From $K = \nabla \circ \nabla$ it follows

$$k_{ij} = d\theta_{ij} - \sum_{k=1}^r \theta_{ik} \wedge \theta_{kj}, \tag{3.1}$$

or, in matrix notation $k = d\theta - \theta \wedge \theta$.

Now let $\mathbb{S}' = (s'_1, \dots, s'_r)$ be a frame on a open set U' and suppose that $U \cap U' \neq \emptyset$. Then there exist C^∞ functions a_{ij} on $U \cap U'$ such that

$$s'_i = \sum_{j=1}^{r} a_{ij} s_j,$$

and the matrix $A = (a_{ij})$ is pointwise invertible in $U \cap U'$.

Proposition 3.1.7. *Let θ' be the connection matrix with respect to $\mathbb{S}'$ and k' the curvature matrix with respect to $\mathbb{S}'$. Then*

(1) $\theta' = dA \cdot A^{-1} + A\theta A^{-1}$,
(2) $k' = AkA^{-1}$.

The proof is left to the reader.

3.2. **Chern forms.** Let M be a C^∞ manifold, E a rank r complex vector bundle on M and ∇ a connection for E.

Let $\mathbb{S}$ be a frame for E on a open set U and let $k = (k_{ij})$ the curvature matrix with respect to $\mathbb{S}$. We define a $2i$-form $\sigma_i(k)$ on U by

$$\det(I + k) = 1 + \sigma_1(k) + \sigma_2(k) + \ldots + \sigma_r(k).$$

Note that since the k_{ij}'s are even forms and the wedge product of two even forms is symmetric then the determinant of $I + k$ is well defined. Also note that $\sigma_1(k) = \mathrm{tr}(k)$ and $\sigma_r(k) = \det k$.

If $\mathbb{S}'$ is a frame for E on U' with $U \cap U' \neq \emptyset$ and k' is the curvature matrix with respect to $\mathbb{S}'$ then $\sigma_i(k') = \sigma_i(k)$ on $U \cap U'$ by Proposition 3.1.7.2. Therefore we can patch together the forms $\sigma_i(k)$ defined on the open sets trivializing E in order to obtain a global $2i$-form, denoted by $\sigma_i(\nabla)$.

Definition 3.2.1. The $2i$-form

$$c_i(\nabla) := \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla),$$

is the *i-th Chern form*.

The normalization is chosen in such a way that the first Chern class of the hyperplane bundle on the projctive space is 1.

Lemma 3.2.2. *The Chern forms have the following properties:*

(1) *For any connection ∇ for E it holds $dc_i(\nabla) = 0$ for every i.*
(2) *If ∇, ∇' are two connections for E then there exists a $(2i-1)$-form $c_1(\nabla, \nabla')$—called the* Bott difference form*—such that*

$$dc_i(\nabla, \nabla') = c_i(\nabla') - c_i(\nabla).$$

The previous Lemma implies that $c_i(\nabla)$ defines a class $[c_i(\nabla)] \in H^{2i}_d(M)$ and that this class is independent of the connection ∇ (but depends only on E). Therefore we can define

$$c_i(E) := [c_i(\nabla)]$$

and called it the *i-th Chern class of E*.

Remark 3.2.3. One may define $c_i(E)$ by means of obstruction theory. Roughly speaking $c_i(E)$ is the first obstruction to constructing $r - i + 1$ global sections of E which are pointwise linearly independent.

Now we generalize the previous construction to symmetric polynomials. Let φ be a symmetric polynomial in the variables $x_1, x_2, \ldots, x_l$. Let σ_i be the i-th elementary symmetric function. Recall that the elementary symmetric functions are defined by

$$\prod_{j=1}^{l}(1+x_j) = 1 + (x_1 + x_2 + \ldots + x_l) + (x_1x_2 + \ldots) + \ldots$$
$$= \sigma_0(x_1, \ldots, x_l) + \sigma_1(x_1, \ldots, x_l) + \sigma_2(x_1, \ldots, x_l) + \ldots$$

By a well known result, every symmetric polynomial can be written as a polynomial in the variables $\sigma_1, \ldots, \sigma_l$. Therefore there exists a polynomial p such that $\varphi = p(\sigma_1, \ldots, \sigma_l)$. We define

$$\varphi(\nabla) := p(c_1(\nabla), c_2(\nabla), \ldots, c_l(\nabla)).$$

Since the $c_i(\nabla)$'s are closed forms, then $d\varphi(\nabla) = 0$. Moreover if ∇, ∇' are two connections on E there exists a form $\varphi(\nabla, \nabla')$ such that

$$\varphi(\nabla) - \varphi(\nabla') = d\varphi(\nabla, \nabla').$$

Thus $\varphi(\nabla)$ defines a class in $H_d^*(M)$ independent of ∇:

$$\varphi(E) := [\varphi(\nabla)] \in H_d^*(M).$$

3.3. **Characteristic classes in the Čech-de Rham cohomology.** Let M be an m-dimensional manifold and let $\mathcal{U} := \{U_0, U_1\}$ be an open covering of M. Let $E \to M$ be a rank r complex vector bundle on M. Let ∇_j be a connection for $E_{|U_j}$. Let $c_i(\nabla_*)$ be the element of $A^{2i}(\mathcal{U})$ given by

$$c_i(\nabla_*) := (c_i(\nabla_0), c_i(\nabla_1), c_i(\nabla_0, \nabla_1)),$$

where $c_i(\nabla_0, \nabla_1)$ is the Bott difference form of the restrictions to $U_0 \cap U_1$ of the connections ∇_0, ∇_1 for $E_{|U_0 \cap U_1}$. Then $Dc_i(\nabla_*) = 0$. Therefore this defines a class $[c_i(\nabla_*)] \in H_D^{2i}(\mathcal{U})$.

Theorem 3.3.1. *The class* $[c_i(\nabla_*)] \in H_D^{2i}(\mathcal{U})$ *corresponds to the Chern class* $c_i(E) \in H_d^{2i}(M)$ *under the isomorphism* (2.1).

4. Complex manifold and the Grothendieck residue

4.1. **Complex manifolds.** Let $U \subset \mathbb{C}^n$ be a open set. Recall that a map $f : U \to \mathbb{C}$ is called *holomorphic* if f can be expressed as the sum of a convergent power series in a neighborhood of each point of U. A *complex manifold* of dimension n is a topological space together with an atlas $\{U_\alpha, \varphi_\alpha\}$ such that $\varphi_\alpha(U_\alpha)$ is an open set of $\mathbb{C}^n$ for any α and the transiction functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are holomorphic.

Let M be a complex manifold of dimension n, and let $(z_1, \ldots, z_n)$ be local coordinates on $U \subset M$. Then $z_i = x_i + \sqrt{-1}y_i$ for $x_i, y_i \in \mathbb{R}$ and $i = 1, \ldots, n$ and the local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on U give rise to a structure of $2n$-dimensional real manifold on M.

Let us indicate by T_pM the real tangent space (of dimension $2n$) of M at $p \in M$, and by TM the real tangent bundle of M. On U with local coordinates $\{z_1 = x_1 + \sqrt{-1}y_1, \ldots, z_n = x_n + \sqrt{-1}y_n\}$, the vector space T_pM is spanned by $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$, evaluated at p. Let $T_p^{\mathbb{C}}M := T_pM \otimes \mathbb{C}$ be the complexified tangent space of M at $p \in M$ and let $T^{\mathbb{C}}M := TM \otimes \mathbb{C}$ be the complexified tangent bundle of M. Thus a basis for $T_p^{\mathbb{C}}M$ is given by $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}}\}$, where

$$\frac{\partial}{\partial z_i} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right),$$
$$\frac{\partial}{\partial \overline{z_i}} := \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

We indicate by $\mathbb{T}_pM := \text{span}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$ and by $\overline{\mathbb{T}}_pM := \text{span}\{\frac{\partial}{\partial \overline{z_1}}, \ldots, \frac{\partial}{\partial \overline{z_n}}\}$. Therefore

$$T_p^{\mathbb{C}}M = \mathbb{T}_pM \oplus \overline{\mathbb{T}}_pM.$$

The vector space $\mathbb{T}_pM$ is called the *holomorphic part* of $T_p^{\mathbb{C}}M$ and the vector space $\overline{\mathbb{T}}_pM$ is called the *anti-holomorphic part* of $T_p^{\mathbb{C}}M$. Note that $\mathbb{T}_pM$ is a n-dimensional complex vector space.

By the Cauchy-Riemmann equations this decomposition is independent of the local coordinates chosen. Thus we have a decomposition of the complex tangent bundle

$$T^{\mathbb{C}}M = \mathbb{T}M \oplus \overline{\mathbb{T}}M.$$

Note that $\mathbb{T}M$ has a natural structure of *holomorphic vector bundle*, *i.e.*, it has a system of holomorphic transiction functions which gives it a structure of complex manifold.

A C^∞ section of $\mathbb{T}M$ is locally given by

$$v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i},$$

with the f_i's being C^∞ complex valued functions.

Proposition 4.1.1. *There is a real isomorphism (as real bundles) $\mathbb{T}M \simeq TM$, locally given by*

$$v = \sum_{i=1}^{n} f_i \frac{\partial}{\partial z_i} \mapsto \sum_{i=1}^{n} \mathit{Re} f_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \mathit{Im} f_i \frac{\partial}{\partial y_i} \tag{4.1}$$

Proof. One can check easily that (4.1) is a real isomorphism. Then, using Cauchy-Riemann equations, one can show that it gives rise to a vector bundle isomorphism. □

Example 4.1.2. In $\mathbb{C} = \{z\}$ we have $z\frac{\partial}{\partial z} \mapsto x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $z^2\frac{\partial}{\partial z} \mapsto (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$.

We say that a section $v = \sum f_i \frac{\partial}{\partial z_i}$ of $\mathbb{T}M$ is a *holomorphic vector field* if the f_i's are holomorphic, *i.e.*, if v is a holomorphic section of the holomorphic vector bundle $\mathbb{T}M$.

Similarly we have a decomposition of the complexified cotangent bundle of M:

$$A^0(M) := (T^{\mathbb{C}}M)^* = \mathbb{T}M^* \oplus \overline{\mathbb{T}}M^*.$$

In local coordinates a basis for $\mathbb{T}M^*$ is given by $\{dz_1, \ldots, dz_n\}$ and a basis for $\overline{\mathbb{T}}M^*$ is given by $\{d\overline{z_1}, \ldots, d\overline{z_n}\}$, where

$$dz_i := dx_i + \sqrt{-1}dy_i$$
$$d\overline{z_i} := dx_i - \sqrt{-1}dy_i.$$

Accordingly we have a decomposition for forms of higher degree. That is to say, an r-form $\omega \in A^r(M)$ can be written as

$$\omega = \sum_{p+q=r} \omega^{p,q},$$

where $\omega^{p,q}$ is a *form of type* (p,q). In local coordinates

$$\omega^{p,q} = \sum f_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz_{i_1} \wedge \ldots \wedge dz_{i_n} \wedge d\overline{z}_{j_1} \wedge \ldots \wedge d\overline{z}_{j_n},$$

where the $f_{i_1,\ldots,i_n,j_1\ldots,j_n}$'s are C^∞ complex valued functions.

Note that $\mathbb{T}M^* \wedge \ldots \mathbb{T}M^*$ has a natural structure of holomorphic vector bundle.

We say that a form ω of type $(p,0)$ is *holomorphic* if locally $\omega = \sum f_{i_1,\ldots,i_p} dz_{i_1} \wedge \ldots \wedge dz_{i_n}$ with the $f_{i_1,\ldots,i_n}$'s holomorphic functions, that is, ω is a holomorphic section of the holomorphic bundle $\mathbb{T}M^* \wedge \ldots \wedge \mathbb{T}M^*$ (p-times).

4.2. **Grothendieck residue.** Let $U \subset \mathbb{C}^n$ be an open set containing the origin $O = (0,\ldots,0) \in \mathbb{C}^n$. Let $f_1,\ldots,f_n : U \to \mathbb{C}$ be holomorphic functions such that $\{p \in U | f_i(p) = 0, i = 1,\ldots,n\} = \{O\}$. Let $\omega = h dz_1 \wedge \ldots \wedge dz_n$ be a holomorphic n-form on U. The *Grothendieck residue* at O of ω with respect to $f_1,\ldots,f_n$ is given by

$$\mathrm{Res}_O \begin{bmatrix} \omega \\ f_1,\ldots,f_n \end{bmatrix} := \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where $\Gamma := \{p \in U : |f_i(p)| = \epsilon_i, i = 1,\ldots,n\}$ for $\epsilon_i > 0$ so small that Γ is compact. Note that for generic ϵ_i small enough, the set Γ is a compact real n-dimensional manifold and we orient it so that $d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) > 0$.

Example 4.2.1. Let $\omega = h dz$ with h holomorphic near 0, and let f be a holomorphic function in a neighborhood of 0 so that $f(p) = 0$ implies $p = 0$. Then

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f \end{bmatrix} = \frac{1}{2\pi\sqrt{-1}} \int_{\{|f|=\epsilon\}} \frac{h}{f} dz.$$

5. Localization of the top Chern class

Let M be a complex manifold of dimension n. Let E be a complex vector bundle of rank r over M. Let $s : M \to E$ be a non-vanishing section of E, *i.e.*, s is a C^∞ map from M to E such that $s(x) \in E_x$ and $s(x) \neq 0$ for any $x \in M$.

Definition 5.0.2. A connection ∇ for E is *s-trivial* if $\nabla s = 0$.

Given a non-vanishing section s of E it is always possible to define an s-trivial connection ∇ for E (simply define $\nabla s = 0$).

Proposition 5.0.3. *If ∇ is an s-trivial connection for E then $c_r(\nabla) \equiv 0$.*

Proof. Let $U \subset M$ be an open set such that $E_{|U} \simeq U \times \mathbb{C}^r$. Since $s \neq 0$ everywhere on M, we may take a fram $\mathbb{S} = (s_1,\ldots,s_r)$ on U so that $s_1 = s$. Then the connection and the curvature matrices of ∇ with respect to $\mathbb{S}$ are of the form

$$\begin{pmatrix} 0 \ldots 0 \\ \star \end{pmatrix}$$

and since $c_r(\nabla) = \det k$ up to a constant, then $c_r(\nabla) = 0$. □

5.1. **The case rank$E$ =dim$M$.** Suppose now that the rank r of E coincides with the dimension of M, *i.e.*, $r = n$.

Let S be a closed set in M and let s be a section of E non-vanishing on $M - S$. We want to compute $c_n(E)$. Let $U_0 := M - S$, let U_1 be a neighborhood of S in M and let $\mathcal{U} := \{U_0, U_1\}$. Let ∇_i be a connection for E on U_i, $i = 0, 1$. By Theorem 3.3.1 the Chern class $c_n(E)$ is represented in $H^{2n}_D(\mathcal{U})$ by the element $c_n(\nabla_*) \in A^{2n}(\mathcal{U})$ given by

$$c_n(\nabla_*) := (c_n(\nabla_0), c_n(\nabla_1), c_n(\nabla_0, \nabla_1)),$$

where $c_n(\nabla_0, \nabla_1)$ is the Bott difference form of ∇_0, ∇_1 on $U_0 \cap U_1$.

Remark 5.1.1. Since s is non-vanishing on U_0, one can assume ∇_0 is an s-trivial connection for E on U_0. Thus by Proposition 5.0.3 we have $c_n(\nabla_0) = 0$ and actually $c_n(\nabla_*) \in A^{2n}(\mathcal{U}, U_0)$. This defines a class in $H^{2n}_D(\mathcal{U}, U_0)$ which we denote by $c_n(E, s)$ and call the *localization of* $c_n(E)$ *with respect to* s.

Now suppose S is compact and let $\{S_\lambda\}_{\lambda \in J}$ be the set of connected components of S. For each $\lambda \in J$, let R_λ be a compact C^∞ manifold with boundary containing S_λ and such that $R_\lambda \cap R_\mu = \emptyset$ for $\lambda, \mu \in J$, $\lambda \neq \mu$. Let $R_{0\lambda} := -\partial R_\lambda$. Then

$$\int_M c_n(E, s) = \sum_{\lambda \in J} \left(\int_{R_\lambda} c_n(\nabla_1) + \int_{R_{0\lambda}} c_n(\nabla_0, \nabla_1) \right).$$

One can easily show that the addends on the right-hand side of the previous formula are independent of R_λ and therefore we may define

$$\mathrm{Res}_{c_n}(s, E, S_\lambda) := \int_{R_\lambda} c_n(\nabla_1) + \int_{R_{0\lambda}} c_n(\nabla_0, \nabla_1). \tag{5.1}$$

Proposition 5.1.2. *If M is compact*

$$\sum_{\lambda \in J} \mathit{Res}_{c_n}(s, E, S_\lambda) = \int_M c_n(E).$$

Proof. Apply formulas (2.2) and (2.3). □

5.2. **The calculation of the residue for a point and a holomorphic section.** Suppose E is a holomorphic vector bundle and s is a holomorphic section of E. We are going to calculate the residue $\mathrm{Res}_{c_n}(s, E, S_\lambda)$ when $S_\lambda = \{p\}$. In this situation we may take U_1 to be a trivializing set for E and also we may assume that $s(q) \neq 0$ for any $q \in U_1 - \{p\}$. Let $\mathbb{S} := (s_1, \ldots, s_n)$ be a frame for E on U_1. Then $s = \sum_{i=1}^n f_i s_i$ for some holomorphic functions f_i defined on U_1. Since p is an isolated zero of s, it follows that $\{p\} = \{q \in U_1 : f_i(q) = 0, i = 1, \ldots, n\}$.

Theorem 5.2.1. $\mathit{Res}_{c_n}(s, E, p) = \mathit{Res}_p \begin{bmatrix} df_1 \wedge \ldots \wedge df_n \\ f_1, \ldots, f_n \end{bmatrix}$.

Proof. We give the proof for $n = 1$ (for $n > 1$ one needs the Čech-de Rham cohomology theory for $(n+1)$-open sets). Thus $s = f s_1$ for some holomorphic function f on U_1.

Let R be a closed "disc" contained in U_1 and containing p. By definition

$$\mathrm{Res}_p(s, E, p) = \int_R c_1(\nabla_1) + \int_{-\partial R} c_1(\nabla_0, \nabla_1).$$

One may assume ∇_1 as an s_1-trivial connection on U_1, thus $c_1(\nabla_1) \equiv 0$. Therefore $\mathrm{Res}_p(s, E, p) = -\int_{\partial R} c_1(\nabla_0, \nabla_1)$.

Now we recall how the Bott difference form $c_1(\nabla_0, \nabla_1)$ is defined. Let $\tilde{E} := E \times \mathbb{R}$ be the trivial bundle over $M \times \mathbb{R}$, and let t be the coordinate on $\mathbb{R}$. Define a connection for $\tilde{E}$ on $U_{01} \times \mathbb{R}$ by $\tilde{\nabla} := (1-t)\nabla_0 + t\nabla_1$. Let $\pi : U_{01} \times [0,1] \to U_{01}$ be the canonical projection and let π_* be the *integration along the fibers of* π. That is to say, let ω be a 2-form on $U_{01} \times \mathbb{R}$ given locally by $\omega = dt \wedge H_1 + H_2$, with H_1 a 1-form and H_2 a 2-form given by a linear combination of terms of the form $dz_i \wedge dz_j$ (*i.e.*, not containing dt). Then $\pi_*(\omega) := \int_0^1 dt \wedge H_1$ gives rise to a 1-form on U_{01}. Thus we define

$$c_1(\nabla_0, \nabla_1) := \pi_* c_1(\tilde{\nabla}).$$

In our case, we let θ_i be the connection matrix of ∇_i, $i = 0, 1$ (on U_{01}) with respect to the frame s_1. Therefore $\theta_1 = 0$ but $\theta_0 \neq 0$ (it would be zero in the frame s). Thus the connection matrix for $\tilde{\nabla}$ is given by $\tilde{\theta} := (1-t)\theta_0$. Note that if $\theta'_0 = 0$ is the connection matrix for ∇_0 with respect to the frame s, since $s = f s_1$, then by Proposition 3.1.7.1 we have

$$0 = \theta'_0 = \theta_0 + \frac{df}{f},$$

from which it follows $\theta_0 = -\frac{df}{f}$. Hence $\tilde{\theta} = (1-t)\theta_0 = (t-1)\frac{df}{f}$ and by equation (3.1), if $\tilde{k}$ is the curvature matrix of $\tilde{\nabla}$, we have

$$\tilde{k} = d\tilde{\theta} - \tilde{\theta} \wedge \tilde{\theta} = dt \wedge \frac{df}{f}.$$

Thus

$$c_1(\tilde{\nabla}) := \frac{\sqrt{-1}}{2\pi} dt \wedge \frac{df}{f} \overset{\pi_*}{\to} \frac{\sqrt{-1}}{2\pi}\frac{df}{f} =: c_1(\nabla_0, \nabla_1).$$

Therefore

$$\mathrm{Res}_p(s, E, p) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial R} \frac{df}{f} = \mathrm{Res}_p \left[\frac{df}{f} \right],$$

as wanted. □

6. Indices of vector fields and residues of singular foliations

6.1. **The Poincaré-Hopf Theorem.** Let M be a C^∞ compact manifold of dimension n. Let v be a vector field on M with only isolated zeros. Let $p \in M$ and fix a coordinate open set with local coordinates $\{x_1, \ldots, x_n\}$ around p. In this open set $v = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ for some C^∞ functions a_i. There exists a small open ball U_ϵ (contained in the coordinate set) such that $p \in U_\epsilon$ and $v(q) \neq 0$ for every $q \in U_\epsilon - \{p\}$. Let $S_\epsilon = \partial U_\epsilon \simeq S^{n-1}$. Consider the map

$$\begin{aligned} \gamma_v : & S_\epsilon \to S^{n-1} \\ & q \mapsto \frac{(a_1(q), \ldots, a_n(q))}{\|(a_1(q), \ldots, a_n(q))\|}. \end{aligned}$$

We define the *Poincaré-Hopf index* of v at p as

$$PH(v, p) := \text{the mapping degree of } \gamma_v.$$

Note that $PH(v, p) = 0$ if $v(p) \neq 0$.

Example 6.1.1. If $v = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ then $P(v, 0) = 1$. If $v = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$ then $PH(v, 0) = 2$.

Theorem 6.1.2. *If v is a vector field on M with only isolated zeros $p_1, \dots, p_r$ then*

$$\sum_{i=1}^{r} PH(v, p_i) = \chi(M),$$

where $\chi(M)$ is the Euler number of M.

We want to give a sketch of the proof of such a theorem when M is complex and v is holomorphic using the "principle of localization of characteristic classes".

Suppose first that M is a compact complex n-dimensional manifold and v is a C^∞ vector fields. Recall that

$$\int_M c_n(\mathbb{T}M) = \chi(M). \tag{6.1}$$

Let $S = S(v)$ be the zero set of v. Suppose that S is compact and let $\{S_\lambda\}_{\lambda \in J}$ be the set of connected components of S. We localize $c_n(\mathbb{T}M)$ with respect to v as in Remark 5.1.1 to get a class $c_n(\mathbb{T}M, v) \in H_D^{2n}(\mathcal{U}, U_0)$ where $U_0 := M - S$, U_1 is an open neighborhood of S and $\mathcal{U} = \{U_0, U_1\}$. Then we may define residues $\mathrm{Res}_{c_n}(v, \mathbb{T}M, S_\lambda)$ as in (5.1). Thus by Proposition 5.1.2 and (6.1) we have

$$\sum_{\lambda \in J} \mathrm{Res}_{c_n}(v, \mathbb{T}M, S_\lambda) = \chi(M), \tag{6.2}$$

which generalizes the Poincaré-Hopf Theorem.

Now suppose $S_\lambda = \{p_\lambda\}$ for each $\lambda \in J$. Fix $p \in S$. Suppose v is a holomorphic vector field, given by $v = \sum_{i=1}^n a_i \frac{\partial}{\partial z_i}$ locally around p, for some holomorphic functions a_i. By Theorem 5.2.1 we have $\mathrm{Res}_{c_n}(v, \mathbb{T}M, p) = \mathrm{Res}_p \begin{bmatrix} da_1 \wedge \dots \wedge da_n \\ a_1, \dots, a_n \end{bmatrix}$ and a direct calculation shows that this last term is equal to $PH(v, p)$. This gives Theorem 6.1.2 in case v is holomorphic.

6.2. **Non singular Foliations and the Bott vanishing Theorem.** Let M be a complex manifold of dimension n.

Definition 6.2.1. A rank one holomorphic subbundle F of $\mathbb{T}M$ is a one dimensional *non-singular foliation*.

Let F be a one-dimensional non-singular foliation on M. A (one dimensional) complex manifold $L \subset M$ is called a *leaf* of F if $\mathbb{T}_p L = F_p$ for every $p \in L$. Note that by Frobenius' Theorem each point of M is contained in a unique leaf and that actually the leaves of F form a partition of M.

Definition 6.2.2. A holomorphic vector bundle E is an F*-bundle* if there exists a $\mathbb{C}$-linear map α, called *a holomorphic action*,

$$\alpha : A^0(M, F) \times A^0(M, E) \to A^0(M, E)$$

with the following properties:

(1) $\alpha([u, v]), s) = \alpha(u, \alpha(v, s)) - \alpha(v, \alpha(u, s))$ for any $u, v \in A^0(M, F)$ and $s \in A^0(M, E)$,
(2) $\alpha(fu, s) = f\alpha(u, s)$ for any $f \in A^0(M)$, $u \in A^0(M, F)$, $s \in A^0(M, E)$,
(3) $\alpha(u, fs) = u(f)s + f\alpha(u, s)$ for any $f \in A^0(M)$, $u \in A^0(M, F)$, $s \in A^0(M, E)$,
(4) if $u \in A^0(M, F), s \in A^0(M, E)$ are holomorphic then $\alpha(u, s)$ is holomorphic.

Let E be a holomorphic vector bundle on M and let ∇ be a connection for E. We say that ∇ is of *type* $(1, 0)$ if the connection matrix of ∇ in every (holomorphic) frame of E has only $(1, 0)$-forms as entries.

Definition 6.2.3. Suppose E is an F-bundle. A connection for E is an F-*connection* if

(1) $(\nabla s)(u) = \alpha(u,s)$ for any $u \in A^0(M,F)$ and $s \in A^0(M,E)$,
(2) ∇ is of type $(1,0)$.

If E is an F-bundle it is always possible to define an F-connection for E (define $\nabla_u s = \alpha(u,s)$ for $u \in A^0(M,F)$, $s \in A^0(M,E)$ and extend it).

Theorem 6.2.4 (Bott vanishing Theorem)**.** *Let M be a complex n-dimensional manifold. Let E be a holomorphic vector bundle on M, F a one-dimensional non-singular foliation on M and suppose E is an F-bundle. Let ∇ be an F-connection for E. For any symmetric homogeneous polynomial φ of degree n it follows $\varphi(\nabla) \equiv 0$.*

In case F is a non-singular foliation of dimension p (*i.e.*, F is an involutive rank p holomorphic subbundle of $\mathbb{T}M$) then the Bott vanishing theorem holds for symmetric homogeneous polynomial of degree $> n-p$.

Sketch of the proof of Theorem 6.2.4. Let r be the rank of E. In a neighborhood of each point we may choose local coordinates $\{z_1,\ldots,z_n\}$ so that F is generated by $\frac{\partial}{\partial z_1}$. From the very definition of holomorphic action we may find a local frame $\mathbb{S} = (s_1,\ldots,s_r)$ of E so that $\alpha(\frac{\partial}{\partial z_1}, s_i) = 0$ for any i. Thus $\nabla_{\frac{\partial}{\partial z_1}} = 0$. Let $\theta = (\theta_{ij})$ be the connection matrix of ∇ in the frame $\mathbb{S}$. Since ∇ is of type $(1,0)$ then the θ_{ij}'s are $(1,0)$-forms and $\theta_{ij}(\frac{\partial}{\partial z_1}) = 0$ for $i,j = 1,\ldots,r$. This means that each θ_{ij} is of the form $\sum_{l=2}^n f_l dz_l$ for some C^∞ functions f_l. Hence each entry of the curvature matrix k of ∇ is of the form $\sum_{l=2}^n \eta_l \wedge dz_l$ for some 1-forms η_l. Therefore since φ has degree n, it follows $\varphi(k) = 0$. □

6.3. **The Baum-Bott Residue Theorem.**

Definition 6.3.1. A one dimensional *singular foliation* $\mathcal{F}$ on M is determined by the following data:

(1) a open covering $\{U_\alpha\}$ of M,
(2) a family of holomorphic vector fields $\{v_\alpha\}$ such that v_α is defined on U_α and
(3) a family of holomorphic non-vanishing functions $\{f_{\alpha\beta}\}$ such that $f_{\alpha\beta}$ is defined on $U_\alpha \cap U_\beta$ (once non-empty),

with the property that, if $U_\alpha \cap U_\beta \neq \emptyset$ then $v_\beta = f_{\alpha\beta} v_\alpha$.

Let $S(v_\alpha) := \{p \in U_\alpha | v_\alpha(p) = 0\}$. Since $v_\beta = f_{\alpha\beta} v_\alpha$ then $S(v_\alpha) \cap U_\beta = S(v_\beta) \cap U_\alpha$ and thus we may consider the *singular set* $S = S(\mathcal{F})$ of $\mathcal{F}$ defined as

$$S = S(\mathcal{F}) := \{p \in M | \exists \alpha : v_\alpha(p) = 0\}.$$

Let $M_0 := M - S$.

The functions $\{f_{\alpha\beta}\}$ satisfy the cocycle conditions relative to the covering $\{U_\alpha\}$ and therefore they define a rank one holomorphic vector bundle F, called the *tangent bundle of* $\mathcal{F}$. We have the vector bundle homomorphism

$$i : F \to \mathbb{T}M.$$

A section f of F is given locally on U_α by a C^∞ function f_α and on $U_\alpha \cap U_\beta \neq \emptyset$ it holds $f_\alpha = f_{\alpha\beta} f_\beta$. The map i is locally defined as

$$i : (f_\alpha) \mapsto f_\alpha v_\alpha.$$

Since on $U_\alpha \cap U_\beta$ it holds $f_\alpha = f_{\alpha\beta} f_\beta$ and $v_\alpha = f_{\beta\alpha} v_\beta$, one can easily check that i is well defined.

Note that the map i is injective only on M_0. Thus $F_0 := i(F_{|M_0})$ is a rank one holomorphic subbundle of $\mathbb{T}M_{|M_0}$ and therefore $\mathcal{F}$ gives rise to a non-singular foliation on M_0. We define the quotient vector bundle $N_{F_0} := \mathbb{T}M_0/F_0$ on M_0 and call it the *normal bundle* of the foliation $\mathcal{F}$.

Remark 6.3.2. One may consider the sheaf of holomorphic sections of F—still denoted by $\mathcal{F}$—and the sheaf Θ_M of holomorphic sections of $\mathbb{T}M$. Thus one gets an injective map $\mathcal{F} \hookrightarrow \Theta_M$ (basically because the singular set $S(\mathcal{F})$ has at least codimension 1 in M), and one may consider the exact sequence of sheaves

$$0 \to \mathcal{F} \to \Theta_M \to \mathcal{N}_{\mathcal{F}} \to 0,$$

where $\mathcal{N}_{\mathcal{F}}$ is the quotient sheaf. However $\mathcal{N}_{\mathcal{F}}$ is not locally free (actually it is locally free only on M_0) and therefore it is not the sheaf of section of a vector bundle (if one restricts $\mathcal{N}_{\mathcal{F}}$ to M_0 then one get the sheaf of holomorphic sections of N_{F_0}).

By the very definition, on M_0 we have the following exact sequence of vector bundles:

$$0 \to F_0 \xrightarrow{\varpi} \mathbb{T}M_0 \xrightarrow{\pi} N_{F_0} \to 0. \tag{6.3}$$

One can check the following lemma.

Lemma 6.3.3. *The vector bundle N_{F_0} is an F_0-bundle with the action*

$$\begin{aligned} A^0(M_0, F_0) \times A^0(M_0, N_{F_0}) &\to A^0(M_0, N_{F_0}) \\ (u, \pi(w)) &\mapsto \pi([u, w]). \end{aligned}$$

We define the *virtual normal bundle $\nu_{\mathcal{F}}$ of $\mathcal{F}$* as

$$\nu_{\mathcal{F}} := \mathbb{T}M - F,$$

in the sense of K-theory.

If E is a complex vector bundle on M, the *total Chern class* of E is by definition

$$c(E) := 1 + c_1(E) + \ldots + c_n(E),$$

which is an invertible element in $H^*(M, \mathbb{C})$, the inverse being given by expanding $1/c(E)$ (note that this is actually a finite sum since $H^p(M, \mathbb{C}) = 0$ for $p > n$).

For the virtual normal bundle $\nu_{\mathcal{F}}$ the total Chern class is defined as

$$c(\nu_{\mathcal{F}}) := \frac{c(\mathbb{T}M)}{c(F)},$$

and the i-th Chern class $c_i(\nu_{\mathcal{F}})$ is the component of $c(\nu_{\mathcal{F}})$ in $H^{2i}(M, \mathbb{C})$. Thus if φ is a symmetric polynomial we define $\varphi(\nu_{\mathcal{F}})$ as a polynomial in the Chern classes of $\nu_{\mathcal{F}}$.

Let φ be a symmetric homogeneous polynomial of degree n. Thus

$$\varphi(\nu_{\mathcal{F}}) = \sum_i \varphi_i(\mathbb{T}M)\psi_i(F)$$

where the $\varphi_i(\mathbb{T}M)$'s are polynomials in $c_i(\mathbb{T}M)$ and the $\psi_i(F)$'s are polynomials in $c_i(F)$.

Let U_1 be a neighborhood of S in M, $U_0 := M_0$ and $\mathcal{U} := \{U_0, U_1\}$. Let ∇_i^M be a connection for $\mathbb{T}M$ on U_i, $i = 0, 1$, let ∇_i^F be a connection for F on U_i, $i = 0, 1$ and let ∇ be an F_0-connection for N_{F_0} on U_0 in such a way that the triple $(\nabla_0^F, \nabla_0^M, \nabla)$ is *compatible* with the exact sequence 6.3. This means that $\nabla_0^M \circ \varpi = (1 \otimes \varpi) \circ \nabla_0^F$ and $\nabla_0 \circ \pi = (1 \otimes \pi) \circ \nabla_0^M$. Such a triple can be constructed starting from an F_0-connection ∇ for N_{F_0} (see Lemma 6.3.3), defining a connection compatible to ∇ for a complement of $\varpi(F)$ in $\mathbb{T}M$, extending this to all $\mathbb{T}M$ and finally defining a connection for F compatible with the previous ones.

Let

$$\begin{aligned} \varphi(\nabla_1^\bullet) &:= \sum \varphi_i(\nabla_1^M)\psi_i(\nabla_1^F), \\ \varphi(\nabla_0^\bullet) &:= \sum \varphi_i(\nabla_0^M)\psi_i(\nabla_0^F). \end{aligned}$$

As in Lemma 3.2.2, one can show—and for this the compatibility condition does not play any role—that $d\varphi(\nabla_i^\bullet) = 0$, $i = 0, 1$ and there exists a $(2n-1)$-form $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ such that

$$\varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet) = d\varphi(\nabla_0^\bullet, \nabla_1^\bullet).$$

Thus $[(\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet))] \in H_D^{2n}(\mathcal{U})$ and one can check that it corresponds to $\varphi(\nu_{\mathcal{F}})$ under the isomorphism (2.1).

Now by the compatibility condition $\varphi(\nabla_0^\bullet) = \varphi(\nabla)$ and by the Bott vanishing Theorem 6.2.4 $\varphi(\nabla) = 0$. Therefore $\varphi(\nu_{\mathcal{F}})$ is represented by $(0, \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)) \in A^{2n}(\mathcal{U}, U_0)$, whose class $\varphi(\nu_{\mathcal{F}}, \mathcal{F}) \in H_D^{2n}(\mathcal{U}, M_0)$ is called the *localization of $\varphi(\nu_{\mathcal{F}})$ with respect to $\mathcal{F}$.*

For each compact connected component S_λ of $S(\mathcal{F})$ one can define the *Baum-Bott residue* $\mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}, S_\lambda)$ similarly to what we did in (5.1). If M is compact, by (2.2) and (2.3), we have the *Baum-Bott residue formula*:

$$\sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, \nu_{\mathcal{F}}, S_\lambda) = \int_M \varphi(\nu_{\mathcal{F}}).$$

Remark 6.3.4.
(1) If $S_\lambda = \{p\}$ the Bott residue is again expressed in terms of a Grothendieck residue.
(2) If $\mathcal{F}$ is generated by a global vector field v then $\varphi(\nu_{\mathcal{F}}) = \varphi(\mathbb{T}M)$. Indeed in this case F has a global non-vanishing section ($x \mapsto v(x)$) and therefore it is trivial. Hence $c(F) = 1$ and the result follows from the very definition of total Chern class. Also, if $\varphi = c_n$, we recover the Poincaré-Hopf theorem.

6.4. **Residues relative to invariant submanifolds.** Let W be a $(n+k)$-dimensional complex manifold and let $M \subset W$ be a complex submanifold of dimension n. Suppose $\mathcal{F}$ is a one dimensional singular foliation on W. Assume that $\mathcal{F}_x \subset \mathbb{T}M_x$ for any $x \in M$, *i.e.*, M is *invariant* by $\mathcal{F}$. Thus $\mathcal{F}$ induces a singular foliation $\mathcal{F}_M$ on M. Let $S := S(\mathcal{F}_M) := S(\mathcal{F}) \cap M$, $M_0 := M - S$. Let F be the tangent bundle of $\mathcal{F}$. Let $F_{M_0} := F_{|M_0}$ and let $N_{M_0} := TW_{|M_0}/\mathbb{T}M_0$ be the normal bundle of M_0 (this coincides with the restriction of the normal bundle of M, $N_M := \mathbb{T}W_{|M}/TM$, to M_0). Thus we have the following exact sequence of vector bundles:

$$0 \to TM_0 \to TW_{|M_0} \xrightarrow{\eta} N_{M_0} \to 0.$$

Lemma 6.4.1. *There exists a holomorphic action of F_{M_0} on N_{M_0} given by*

$$\begin{aligned} A^0(M_0, F_{M_0}) \times A^0(M_0, N_{M_0}) &\to A^0(M_0, N_{M_0}) \\ (u, v) &\mapsto \alpha(u, v) := \eta([\tilde{u}, \tilde{w}]_{|M_0}), \end{aligned}$$

where $\tilde{u}, \tilde{w}$ are any vector fields on W such that $\tilde{u}_{|M_0} = u$ and $\eta(\tilde{w})_{|M_0} = w$.

Let φ be a symmetric homogeneous polynomial of degree n. Let U_1 be an open neighborhood of S and $\mathcal{U} := \{M_0, U_1\}$. Let ∇_0 be an F_0-connection for N_M on M_0. By the Bott vanishing Theorem 6.2.4, $\varphi(\nabla_0) = 0$. Let ∇_1 be a connection for N_M on U_1. The isomorphism 2.1 allows to represent $\varphi(N_M)$ by the class $[(0, \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)] \in H_D^{2n}(\mathcal{U}, M_0)$, the *localization of φ with respect to $\mathcal{F}$,* denoted by $\varphi(N_M, \mathcal{F})$.

If a connected component S_λ of S is compact then one may define the residue $\mathrm{Res}_\varphi(\mathcal{F}, N_M, S_\lambda)$ (see (5.1)). If M is compact (as in Proposition 5.1.2) the following *generalized Camacho-Sad formula* holds:

$$\sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, N_M, S_\lambda) = \int_M \varphi(N_M).$$

Now suppose $S_\lambda = \{p\}$ and let $\{z_1, \ldots, z_{n+k}\}$ be local coordinates on an open set $U \subset W$ containing p in such a way that $M := \{q \in U | z_{n+1}(q) = \ldots = z_{n+k}(q) = 0\}$. Suppose $\mathcal{F}$ is generated by $\tilde{v} := \sum_{i=1}^{n+k} a_i \frac{\partial}{\partial z_i}$ on U. Since M is invariant for $\mathcal{F}$ then $\tilde{v}$ is tangent to M, *i.e.*,

$$a_{n+i} = \sum_{j=1}^{k} c_{ij} z_{n+j}, \quad \text{for } \ i = 1, \ldots, k.$$

For a $k \times k$-matrix A we define $\sigma_i(A)$ to be given by the following relation:

$$\det(I + tA) = 1 + t\sigma_1(A) + \ldots + t^k \sigma_k(A).$$

Thus, if $\varphi = p(\sigma_1, \sigma_2, \ldots)$, then $\varphi(A) := p(\sigma_1(A), \sigma_2(A), \ldots)$.

Let $C = (c_{ij})$. It is possible to show that

$$\mathrm{Res}_\varphi(\mathcal{F}, N_M, p) = \mathrm{Res}_p \begin{bmatrix} \varphi(C) dz_1 \wedge \ldots \wedge dz_n \\ a_1, \ldots, a_n \end{bmatrix}.$$

7. Characteristic classes on singular varieties

7.1. **Locally complete intersection.** Let W be a $(n+k)$-dimensional complex manifold.

Definition 7.1.1. A closed set $V \subset W$ is a *variety* in W (or *subvariety* or *analytic set*) if for any $p \in V$ there exist an open neighborhood $\tilde{U}$ of p in W and $f_1, \ldots, f_r$ holomorphic functions on $\tilde{U}$ such that

$$V \cap \tilde{U} = \{q \in \tilde{U} | f_1(q) = \ldots = f_r(q) = 0\}.$$

The functions $(f_1, \ldots, f_r)$ are called a set of *defining functions* for V.

A point $p \in V$ is *regular* if there exists a set of defining functions $(f_1, \ldots, f_r)$ around p such that $\mathrm{rank} \frac{\partial(f_1, \ldots, f_r)}{\partial(z_1, \ldots, z_{n+k})} = r$.

By the implicit function theorem, if $p \in V$ is a regular point then V is (in a neighborhood of p) a $(n+k-r)$-dimensional complex manifold.

A point $p \in V$ which is not a regular point is called a *singular point*. We denote by $\mathrm{Sing}(V)$ the set of singular points of V and by $V' := V - \mathrm{Sing}(V)$. Note that V' is a complex submanifold of W (possibly not connected). We define the dimension of V to be the maximum of the dimensions of the connected components of V'. We say that V is *pure dimension* if all the connected components of V' have the same dimension. The variety V is said *irreducible* if V' is connected.

Example 7.1.2. Let $W = \mathbb{C}^2$ with coordinates $\{z_1, z_2\}$.

(1) The function $f(z_1, z_2) := z_1 z_2$ defines a subvariety V such that $\mathrm{Sing}(V) = \{(0,0)\}$. To see that $(0,0)$ is a singular point one can show that for any 3-dimensional sphere S^3 centered at $(0,0)$ the intersection $V \cap S^3$ consists of two connected components.

(2) The function $f(z_1, z_2) := z_1^3 - z_2^2$ defines a subvariety V with an isolated singularity at $(0,0)$. Note that in this case for any 3-dimensional sphere S^3 the intersection $V \cap S^3$ is connected.

If $W = \mathbb{CP}^{n+k}$ and V is a variety given (globally) as the zero set of a finite number of homogeneous polynomials, then V is said a *projective algebraic variety*. A theorem of Chow says that any compact variety in $\mathbb{CP}^{n+k}$ is algebraic.

Definition 7.1.3. Let V be a closed set in $\mathbb{C}^{n+k}$ and $\tilde{U} \subset \mathbb{C}^{n+k}$ an open set containing V. We say that V is *complete intersection* if there exists a set of defining functions $(h_1, \ldots, h_k)$ for V on $\tilde{U}$ such that $dh_1 \wedge \ldots \wedge dh_k \not\equiv 0$ on V.

If $V \subset \tilde{U} \subseteq \mathbb{C}^{n+k}$ is a complete intersection variety defined by $h_1, \dots, h_k$ then $\mathrm{Sing}(V) = \{q \in V : dh_1 \wedge \dots \wedge dh_k(q) = 0\}$. Also note that V' is a complex manifold of dimension n. Thus one could equivalently define a complete intersection variety $V \in \tilde{U} \subseteq \mathbb{C}^{n+k}$ as an n-dimensional variety with a set of k (independent) defining functions on $\tilde{U}$.

Definition 7.1.4. Let W be a $(n+k)$-dimensional complex manifold. A variety $V \subset W$ is a *local complete intersection (L.C.I.) defined by a section* if there exist a rank k holomorphic vector bundle N over W and a holomorphic section $s : W \to N$ such that for any $p \in V$ there exist a open set $\tilde{U}$ in W, $p \in \tilde{U}$, and a (holomorphic) frame $(s_1, \dots, s_k)$ for N so that, if $s = \sum_{i=1}^k h_i s_i$, then $V \cap \tilde{U}$ is a complete intersection defined by $(h_1, \dots, h_k)$.

If V is a L.C.I. defined by a section $s : W \to N$ then the normal vector bundle $N_{V'} := \mathbb{T}W_{|V'}/TV'$ on V' coincides with the restriction $N_{|V'}$ of N to V'. Thus N extends the normal vector bundle of V to W.

7.2. **Grothendieck residue relative to a subvariety.** Let V be a subvariety of $\mathbb{C}^{n+k}$ of pure dimension n contained in an open set $\tilde{U} \subset \mathbb{C}^{n+k}$ and suppose it has an isolated singularity at the origin O. Let $f_1, \dots, f_n$ be holomorphic functions on $\tilde{U}$ such that $V \cap \{f_1 = \dots = f_n = 0\} = \{O\}$. Let ω be a holomorphic n-form. Then we define the *Grothendieck residue* as

$$\mathrm{Res}_O \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V := \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where $\Gamma := \{z \in \tilde{U} : |f_i(z)| = \epsilon_i, i = 1, \dots, n\} \cap V$, for small $\epsilon_i > 0$, oriented so that $d(\arg f_1) \wedge \dots \wedge d(\arg f_n) > 0$. Note that for generic small $\epsilon_i > 0$ the n-cycle Γ is a submanifold of V.

If V is a complete intersection defined by $(h_1, \dots, h_k)$ on $\tilde{U}$ then by the *projection formula* we have

$$\mathrm{Res}_O \begin{bmatrix} \omega \\ f_1, \dots, f_n \end{bmatrix}_V = \mathrm{Res}_O \begin{bmatrix} \omega \wedge dh_1 \wedge \dots \wedge dh_k \\ f_1, \dots, f_n, h_1, \dots, h_k \end{bmatrix}. \tag{7.1}$$

7.3. **Residues on normal bundles.** Let W be a $(n+k)$-dimensional complex manifold, V a L.C.I. defined by a section of the rank k holomorphic vector bundle N over W. Let $\mathcal{F}$ be a one dimensional holomorphic foliation on W leaving V invariant, *i.e.*, the vectors in $\mathcal{F}_{|V'}$ are tangent to V'. We want to compute $\varphi(N)$ for a symmetric homogeneous polynomial φ of degree n.

Let $\mathcal{F}_V$ be the foliation on V' induced by $\mathcal{F}$. Let $S := S(\mathcal{F}, V) := (S(\mathcal{F}) \cap V) \cup \mathrm{Sing}(V))$. Let $V_0 := V - S \subset V'$. Let $\tilde{U}_0$ be a tubular neighborhood of V_0, $\rho : \tilde{U}_0 \to V_0$ the C^∞ retraction and let $\tilde{U}_1$ be an open neighborhood of S in W. Let $\mathcal{U} := \{\tilde{U}_0, \tilde{U}_1\}$, $\tilde{U} := \tilde{U}_0 \cup \tilde{U}_1$.

Let F be the one dimensional tangent bundle to $\mathcal{F}$, $F_0 := F_{|V_0}$. By Lemma 6.4.1 there is a holomorphic action of F_0 on N_{V_0}. Let ∇ be an F_0-connection for N_{V_0}. Let ∇_1 be a connection for N on $\tilde{U}_1$ and let $\nabla_0 := \rho^*(\nabla)$ be a connection for N on $\tilde{U}_0$. By (2.1) the class $\varphi(N_{|\tilde{U}})$ is represented by $\varphi(\nabla_*) := (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)) \in A^{2n}(\mathcal{U})$. By the Bott vanishing Theorem 6.2.4 it follows that $\varphi(\nabla_0) = \rho^*(\varphi(\nabla)) = 0$ and actually $\varphi(\nabla_*) \in A^{2n}(\mathcal{U}, \tilde{U}_0)$.

Now suppose S is a compact. For each connected component S_λ of S let $\tilde{R}_\lambda$ be a $2(n+k)$-dimensional compact C^∞ manifold with boundary such that S_λ is contained in the interior of $\tilde{R}_\lambda$, $\tilde{R}_\lambda \cap S = S_\lambda$ and $\partial\tilde{R}_\lambda$ is transverse to V at $V \cap \tilde{R}_\lambda$. Let $R_\lambda := \tilde{R}_\lambda \cap V$ and $R_{0\lambda} := -\partial R_\lambda$. Define

$$\mathrm{Res}_\varphi(\mathcal{F}, N_V, S_\lambda) := \int_{R_\lambda} \varphi(\nabla_1) + \int_{R_{0\lambda}} \varphi(\nabla_0, \nabla_1).$$

One can easily show that the number is independent of the choice of $R_\lambda, R_{0\lambda}$. Thus

$$\int_V \varphi(\nabla_*) = \sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, N_V, S_\lambda).$$

Also, if V is compact, by formulas similar to (2.2), (2.3), one gets a *general Camacho-Sad type formula*:

$$\int_V \varphi(N) = \sum_\lambda \mathrm{Res}_\varphi(\mathcal{F}, N_V, S_\lambda).$$

Suppose $S_\lambda = \{p\}$. Let $\mathcal{F}$ be generated by the vector field $\tilde{v}$ near p and let $(h_1, \ldots, h_k)$ be a set of defining functions for V near p. Since V is $\mathcal{F}$-invariant then for $i = 1, \ldots, k$

$$\tilde{v}(h_i) = \sum_{j=1}^{k} c_{ij} h_j,$$

for some holomorphic functions c_{ij}. Let $C = (c_{ij})$.

Lemma 7.3.1 (Existence of good local coordinates)**.** *There exists a local system of coordinates* $\{z_1, \ldots, z_{n+k}\}$ *near* p *such that, if* $\tilde{v} = \sum_{i=1}^{n+k} a_i \frac{\partial}{\partial z_i}$, *then* $\{a_1 = \ldots = a_n = 0\} \cap V = \{p\}$.

In a local coordinates system as in Lemma 7.3.1 by (7.1) we have

$$\mathrm{Res}_\varphi(\mathcal{F}, N_V, p) = \mathrm{Res}_p \begin{bmatrix} \varphi(C) dz_1 \wedge \ldots \wedge dz_n \\ a_1, \ldots, a_n \end{bmatrix}_V = \mathrm{Res}_p \begin{bmatrix} \varphi(C) dz_1 \wedge \ldots \wedge dz_n \wedge dh_1 \ldots \wedge dh_k \\ a_1, \ldots, a_n, h_1, \ldots, h_k \end{bmatrix}.$$

REFERENCES

1. P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Diff. Geom. 7 (1972), 279-342.
2. R. Bott and L. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics 82, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
3. W. Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.
4. M. Greenberg and J. Harper, *Algebraic topology: a first course*, Addison-Wisley, 1981.
5. P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley and Sons, New York, 1978.
6. V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, Englewood Cliffs, 1974.
7. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, 1965.
8. R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.
9. F. Hirzebruch, *Topological methods in algebraic geometry*, Springer-Verlag, New York, 1966.
10. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, I, II*, Wiley Classics Library, John Wiley and Sons, New York, 1996.
11. K. Kodaira, *Complex Manifolds and Deformation of complex structures*, Springer-Verlag, New York, 1986.
12. E. Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. Lecture Note Series 77, Cambridge Univ. Press, Cambridge, London, 1984.
13. Y. Matsushima, *Differential manifolds*, Marcel Dekker, New York, 1972.
14. J. Milnor, *singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton University Press, Princeton, 1968.
15. J. Milnor and J. Stasheff, *Characteristic classes*, Ann. of Math. Studies 76, Princeton University Press, Princeton, 1974.
16. N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.
17. T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations*, Actualités Mathématiques, Hermann, Paris, 1998.