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RESIDUES FOR HOLOMORPHIC FOLIATIONS OF SINGULAR PAIRS

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Abstract. Let X be a (possibly singular) subvariety of a complex manifold M and Y a subvariety of X. We assume that Y is the intersection locus of X with a submanifold $P \subset M$ and this intersection is generically transverse. For such a pair (X, Y), we prove a generalization of the classical Camacho-Sad residue theorem, in case there exists a holomorphic foliation $\mathcal{F}$ of X leaving Y invariant. Also, we compute explicitly the residues at isolated singular points.

Introduction

The classical Camacho-Sad residue (or index) theorem [CS] states that if X is a two dimensional complex manifold, $Y \subset X$ a non-singular compact complex curve invariant by a holomorphic foliation $\mathcal{F}$ of X, then the first Chern class $c_1(N_{Y,X})$ of the normal bundle $N_{Y,X}$ of Y in X localizes at the singularities of $\mathcal{F}$ in Y. That is to say, to each point p in $Y \cap \mathrm{Sing}(\mathcal{F})$ one can associate a complex number $\mathrm{Res}(\mathcal{F}, Y; p)$, called the *residue* of $\mathcal{F}$ at p relative to Y, depending only on the behavior of $\mathcal{F}$ near p, such that $\sum \mathrm{Res}(\mathcal{F}, Y; p) = \int_Y c_1(N_{Y,X})$. C. Camacho and P. Sad used their theorem to settle a problem raised by Poincaré on the existence of separatrices for germs of holomorphic foliations in $\mathbb{C}^2$. On the other hand the Camacho-Sad theorem can be seen as an obstruction to the existence of foliations having a given curve as invariant. Especially in this optic, the Camacho-Sad theorem has been generalized by several authors. Just to name a few, A. Lins Neto [Li] for the case of a singular curve Y in $X = \mathbb{CP}^2$, the second named author [Su2] for the general case Y singular, and D. Lehmann and the second named author [LS] for the case X is a complex manifold of arbitrary dimension and Y is an arbitrary (co)dimensional strongly locally complete intersection in X (we refer to [Su3] for a detailed history). Very recently, a paper by V. Cavalier, D. Lehmann and M. Soares

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[CLS] generalizes the Camacho-Sad theorem to the case Y is an arbitrary singular subvariety of a complex manifold X.

The basic idea in all these papers (which is however rather hidden in the first three) is that a holomorphic foliation $\mathcal{F}$ provides a vanishing of certain characteristic classes away from the singular locus of Y and $\mathcal{F}$. Provided that there exists a "good extension" of $N_{Y,X}$ on such a singular locus, the Čech-de Rham theory produces localization at some cohomological level and Poincaré and Alexander duality give then the residue theorems. Thus, taking this machinery for granted, to have a Camacho-Sad type theorem one has to face two problems: find an extension ("natural" to some extent) of the normal bundle of Y in X to all the singularities of Y, and then find some "good action" of $\mathcal{F}$ on such extension in order to get some vanishing theorems. According to this picture, in order to solve problems in discrete dynamics, the first named author together with M. Abate and F. Tovena (see [Ab], [Br], [BT], [ABT]) developed a way to obtain generalizations of the Camacho-Sad theorem in case the foliation $\mathcal{F}$ is replaced by a holomorphic map $f : X \to X$ pointwise fixing Y.

In all the previous works the ambient manifold X is supposed to be non-singular, which allows to have natural extensions of $N_{Y,X}$. The next step would be then to allow some singularities for X. This is not, however, just a merely technical game. Indeed, in some dynamical problems one has to face a singular ambient space. In the case of foliations (see [Ca]), this does not cause really a serious problem, for one can always resolve the singularities and pull back the foliation to a non-singular ambient; however, in the case of holomorphic maps this is no longer possible. Thus in [BS], for answering a question of discrete dynamics, the authors were forced to define residues for a singular X of dimension two and proved a Camacho-Sad type theorem in that case. The aim of this paper is to give a general version of the residue theorem introduced in [BS] with no restrictions on the dimension of X and (co)dimension of Y.

The setting is as follows. We let X be an analytic variety in a complex manifold M and Y a subvariety of X. We assume that Y is presented as the intersection of X with a submanifold $P \subset M$, in such a way that the intersection is generically transverse and $\dim P + \dim X = \dim M + \dim Y$. We call such a pair (X, Y) an *adequate singular pair* (see Definition 1.2). Many examples of algebraic varieties come up this way. Note that if (X, Y) is an adequate singular pair, the bundle $N_{Y,X}$, defined only on the non-singular part of Y in X, has a natural extension given by $N_{P,M}|_Y$. If $\mathcal{F}$ is a holomorphic foliation of X (actually, we do not need it to be defined outside X as required in [BS]) leaving Y invariant, we prove a localization theorem for $\varphi(N_{P,M}|_Y)$ near the singularities of Y and of $\mathcal{F}$ in Y, where φ is a homogeneous symmetric polynomial of an appropriate degree (see Theorem 2.1). This can be seen as a natural generalization of the Camacho-Sad theorem for the case X is singular. We also compute explicitly the residues at isolated singular points in the top degree case (see Theorem 3.1).

In the first section of the paper we recall some basic facts about holomorphic foliations and introduce the main objects of our study, *i.e.*, adequate singular pairs; in particular we discuss of their singularities. In the second section we derive our main residues theorem. In the third section we provide a computation of residues at isolated points.

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1. Preliminaries

Let M be a connected complex manifold. The symbol $\mathcal{C}^\infty_M$ will denote the sheaf of C^∞-functions on M, while $\mathcal{O}_M$ is the sheaf of holomorphic functions. If E is a (complex) vector bundle on M we denote by $\mathcal{C}^\infty(E)$ the sheaf of C^∞-sections of E while we reserve the script symbol $\mathcal{E} = \mathcal{O}_M(E)$ to denote the sheaf of *holomorphic sections* of E. Moreover we denote by TM the holomorphic tangent bundle of M and by $\Theta_M = \mathcal{O}_M(TM)$ the sheaf of germs of holomorphic vectors fields on M.

1.1 Foliations.

Definition 1.1. A *(singular) foliation* of M is a coherent subsheaf $\mathcal{F} \subset \Theta_M$ which is involutive, i.e., for any $p \in M$,

$$[\mathcal{F}_p, \mathcal{F}_p] \subseteq \mathcal{F}_p.$$

Let $\mathcal{Q} := \Theta_M/\mathcal{F}$ be the quotient sheaf. The singular set $\mathrm{Sing}(\mathcal{F})$ of a foliation is defined as

$$\mathrm{Sing}(\mathcal{F}) = \{\, p \in M : \mathcal{Q}_p \ \text{ is not } \ \mathcal{O}_{M,p}\text{-free}\,\}.$$

Note that $\mathrm{Sing}(\mathcal{F})$ is a closed subvariety of M. The *dimension* of $\mathcal{F}$ is the rank of $\mathcal{F}_p$ at some (and hence any) point $p \in M \setminus \mathrm{Sing}(\mathcal{F})$.

Remarks 1. If $\mathcal{Q}_p$ is $\mathcal{O}_{M,p}$-free, then so is $\mathcal{F}_p$. Therefore on $M^0 = M \setminus \mathrm{Sing}(\mathcal{F})$ there exists a holomorphic vector bundle $F \subset TM^0$ whose germs of holomorphic sections form $\mathcal{F}|_{M^0}$.

2. Sometimes the definition of foliation requires that $\mathcal{F}$ be also *reduced* (or *full*). This is a technical condition meaning that, for any open set $U \subset M$ and any section $s \in \Gamma(U, \Theta_M)$, if $s_p \in \mathcal{F}_p$ for all $p \in U \setminus \mathrm{Sing}(\mathcal{F})$, then actually s is in $\Gamma(U, \mathcal{F})$. However there is a canonical way to obtain a reduced foliation from a non-reduced one (see [BB], [Su1]).

Let X be a possibly singular subvariety in M. We denote by $\mathrm{Sing}(X)$ the singular set of X and by $X' = X \setminus \mathrm{Sing}(X)$ the non-singular part. In the following we need holomorphic foliations on X. First we need to define the sheaf of holomorphic "tangent vectors" Θ_X to X. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. This is defined as $\mathcal{O}_X = \mathcal{O}_M/\mathcal{I}_X$, where $\mathcal{I}_X \subset \mathcal{O}_M$ is the ideal sheaf of germs of holomorphic functions identically vanishing on X. Then Θ_X is defined as $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ with Ω_X being given by the following exact sequence of $\mathcal{O}_X$-modules:

$$\mathcal{I}_X/\mathcal{I}_X^2 \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow 0,$$

where $\Omega_M = \mathcal{O}_M(T^*M)$ and the first morphism is given by $[f] \mapsto df \otimes 1$. Note that on X', $\Theta_X = \mathcal{O}_X(TX')$. Note also that Θ_X acts on $\mathcal{O}_X$ as derivations, as in the case of non-singular base spaces. A *holomorphic foliation* on X is a coherent subsheaf $\mathcal{F} \subset \Theta_X$ such that $\mathcal{F}|_{X'}$ is a holomorphic foliation. We set $\mathrm{Sing}(\mathcal{F}) = \mathrm{Sing}(\mathcal{F}|_{X'}) \cup \mathrm{Sing}(X)$.

First examples of holomorphic foliations on X come from restriction of foliations of M. Namely, let $\mathcal{F}$ be a holomorphic foliation of M. We say that X is $\mathcal{F}$-invariant if every vector v in $\mathcal{F}$ leaves the ideal $\mathcal{I}_X$ invariant, which is equivalent to saying that v is tangent to X'. We denote by $\mathcal{F}|_X$ the image of $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_X$ in $\Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_X$. If X is $\mathcal{F}$-invariant, then actually $\mathcal{F}|_X \subset \Theta_X$ and if, moreover, $\dim(\mathrm{Sing}(\mathcal{F}) \cap X) < \dim X$, then $\mathcal{F}|_X$ is a foliation of X of the same dimension as $\mathcal{F}$ (cf. [Su3, Ch.VI, 6]).

Other natural examples of foliations on X come from the case X is pointwise fixed by a (nontrivial) holomorphic self-map of M (see [ABT]).

1.2 Adequate singular pairs.

Let M be a complex manifold of dimension m and let $P \subset M$ be a complex submanifold of dimension r. We denote by $N_{P,M}$ the normal bundle of P in M defined by the following exact sequence of holomorphic vector bundles:

$$0 \longrightarrow TP \longrightarrow TM|_P \longrightarrow N_{P,M} \longrightarrow 0.$$

Assume X is a complex submanifold of M of dimension n which intersects P along a submanifold $Y \subset M$ of dimension $n+r-m$ and such intersection is everywhere transversal. Then it is easy to see that the normal bundle $N_{Y,X} = N_{P,M}|_Y$. This apparently harmless observation will allow us to obtain extensions of the Camacho-Sad index theorem even in the case X and Y are singular. To make things precise we need some works. We begin with a definition.

Definition 1.2. Let M be a complex manifold of dimension m. Let $X \subset M$ be an analytic variety of pure dimension $n > 0$, $Y \subset X$ a subvariety of pure dimension $l > 0$. We say that (X, Y) is an *adequate singular pair* in M if there exists a submanifold $P \subset M$ of dimension $r = m + l - n$ such that

(1) $Y = X \cap P$ (set-theoretically),
(2) $\dim(\mathrm{Sing}(X) \cap P) < l$,
(3) X' intersects P generically transversally.

Remark Assume M, X, P, Y are as in Definition 1.2 and $Y \cap X'$ is connected. If conditions (1) and (2) are satisfied but condition (3) is not then X' intersects nontransversally P everywhere. Indeed, let $\mathcal{N} \subset Y \cap X'$ be defined as follows: $x \in \mathcal{N}$ if there exists an open neighborhood $U \subset Y \cap X'$ of x such that X intersects nontransversally P at every $p \in \mathcal{N}$. Clearly $\mathcal{N}$ is open and closed in $X' \cap Y$. Therefore condition (3) means exactly $\mathcal{N} = \emptyset$ on each connected component of $Y \cap X'$.

Example 1.1. Let X be an n-dimensional algebraic subvariety of $\mathbb{CP}^m$ and $P \subset \mathbb{CP}^m$ an r-dimensional general linear subspace (with $r - m + n > 0$). Then $(X, X \cap P)$ is an adequate singular pair in $\mathbb{CP}^m$.

Example 1.2. Let $X_1 \subset \mathbb{C}^m$ be a germ of a singular variety of dimension n at the origin O. Assume that the singularity of X_1 at O is isolated. Blow up the point O and let X be the strict transform of X_1 and P the exceptional divisor. Let $\tilde{Y} = P \cap X$, and let Y be a connected component of $\tilde{Y}$. If $\dim(\mathrm{Sing}(X) \cap P) < \dim Y$ then by the previous remark, either X' intersects nontransversally P everywhere on a connected component of $Y \cap X'$ or (X, Y) is an adequate singular pair in the blow up of $\mathbb{C}^m$ at O.

We examine more closely the singularities of adequate singular pairs. We set $Y' = (Y \setminus \mathrm{Sing}(Y)) \cap X'$ and

$$Y^{nt} = \{\, q \in X' \cap P : \dim(T_q X' + T_q P) < n + r - l\}. \tag{1.1}$$

Thus Y^{nt} is the set where X' and P are not transverse. In [BS] the following lemma is proved for $n = 2$ and $r = m - 1$.

Lemma 1.1. *If (X, Y) is an adequate singular pair, then $Y^{nt} = X' \cap Sing(Y)$.*

Proof. Clearly $X' \cap \mathrm{Sing}(Y) \subset Y^{nt}$, for if X' is transverse to P at a point q, then Y is non-singular at q. Conversely, suppose $q \in Y'$. We wish to show that X' intersects transversally P at q. Let $(z_1, \dots, z_m)$ be local coordinates on an open set $U \subset M$ such that $q \in U$, $X \cap U = \{z_{n+1} = \ldots = z_m = 0\}$ and $Y \cap U = \{z_{l+1} = \ldots = z_m = 0\}$, where we recall that $l = n + r - m \geq 1$. There exist holomorphic functions $h_1, \dots, h_{m-r}$ on U such that $P \cap U = \{h_1(z) = \ldots = h_{m-r}(z) = 0\}$. Then the goal is to show that

$$\det\left(\frac{\partial h_j}{\partial z_k}\right)(q) \neq 0, \tag{1.2}$$

where, $1 \leq j \leq m-r$ and $l+1 \leq k \leq n$. Let $\overline{h}_j := h_j|_X$, for $j = 1, \dots, m-r$. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf of holomorphic functions on X identically vanishing on Y. Note that $\mathcal{I}_Y$ nearby q is generated in $\mathcal{O}_X$ by $z_{l+1}, \dots, z_n$. Since $\overline{h}_j \in \mathcal{I}_{Y,q}$, there exist holomorphic germs $a_{jk} \in \mathcal{O}_{X,q}$ on X, for $j = 1, \dots, m-r$ and $k = l+1, \dots, n$, such that $\overline{h}_j = \sum_{k=l+1}^{n} a_{jk} z_k$. Since

$$\det\left(\frac{\partial h_j}{\partial z_k}\right)(q) = \det\left(\frac{\partial \overline{h}_j}{\partial z_k}\right)(q)$$

and the latter term is non-zero if and only if

$$\det(a_{jk})(q) \neq 0, \tag{1.3}$$

we are then left to prove (1.3). For any point $p \in U \cap Y$ let $\mathcal{G}_p \subset \mathcal{O}_{X,p}$ be the ideal generated by $\overline{h}_1, \dots, \overline{h}_{m-r}$. Shrinking U if necessary, we note that

$$\mathcal{I}_{Y,p} = \mathcal{G}_p \tag{1.4}$$

if and only if $\det(a_{jk})(p) \neq 0$ and thus (1.4) is equivalent to X being transverse to P at p. Therefore we are left to prove that (1.4) holds for $p = q$. Note also that, by the Hilbert Nullstellensatz, $\mathcal{I}_{Y,p} = \sqrt{\mathcal{G}_p}$, where $\sqrt{\mathcal{G}_p}$ denotes the radical of $\mathcal{G}_p$. We are thus left to show that $\mathcal{G}_q = \sqrt{\mathcal{G}_q}$.

Since Y is locally complete intersection in X it follows that $\mathcal{O}_{X,q}/\mathcal{G}_q$ is a Cohen-Macaulay ring. Thus if $\mathcal{O}_{X,q}/\mathcal{G}_q$ is not reduced (which corresponds to $\mathcal{G}_q \neq \sqrt{\mathcal{G}_q} = \mathcal{I}_{Y,q}$) then $\mathcal{O}_{X,p}/\mathcal{G}_p$ is not reduced (and then $\mathcal{G}_p \neq \mathcal{I}_{Y,p}$) for any point p in a suitable open neighborhood of q (see, *e.g.*, [Lo, p.50]). By (1.4) this means that X is non-transversal to P on an open set in Y, against our hypothesis (3). $\square$

As a corollary of Lemma 1.1 we have

Corollary 1.2. *The subvariety $Y^{nt} \subset Y$ is made of connected components isolated in Y.*

Remark Assume (X, Y) is an adequate singular pair in M, $Y = X \cap P$. Then $N_{P,M}|_Y$ coincides with $N_{Y,X}$ on the open set $U \subset Y$ where Y is nonsingular and X intersects P transversally. Let Z be another complex manifold and $z_0 \in Z$. Let $\tilde{M} = M \times Z$, $\tilde{X} = X \times \{z_0\}$, $\tilde{Y} = Y \times \{z_0\}$ and $\tilde{P} = P \times \{z_0\}$. Then $(\tilde{X}, \tilde{Y})$ satisfies hypotheses (1), (2) and (3) in Definition 1.2 but it is not an adequate singular pair according to our definition for $\dim \tilde{P} < \dim \tilde{M} + \dim \tilde{Y} - \dim \tilde{X}$. In particular note that $N_{\tilde{P},\tilde{M}}$ has rank greater than that of $N_{\tilde{Y}',\tilde{X}'}$ and thus $N_{\tilde{P},\tilde{M}}$ does not provide an extension of $N_{\tilde{Y},\tilde{X}}$. However, if $\pi_1 : \tilde{M} \to M$ is the natural projection, the bundle $N = \pi_1^*(N_{P,M})$, which can naturally be thought of as a subbundle of

$N_{\tilde{P},\tilde{M}}$, coincides with $N_{\tilde{Y},\tilde{X}}$ on $\pi_1^{-1}(U) \cap \tilde{Y}$. Therefore in such a case one can easily extend the results presented in this paper for the case of adequate singular pairs. We warmly thank the referee for pointing out this example.

In general if X, Y, P, M are as in Definition 1.2 except $r < m - n + l$, it does not seem natural to ask for the existence of a subbundle N of $N_{P,M}$ which extends $N_{Y',X'}$ in order to include the previous example in the definition of adequate singular pairs. Also, Lemma 1.1 seems to heavily rely on the property of local complete intersection coming from the hypothesis $r = m + n - l$.

2. Residue theorems for foliations of adequate singular pairs

In this section we state and prove our main theorem. We use the notations previously introduced about adequate singular pairs.

Theorem 2.1. *Let (X, Y) be an adequate singular pair in M and let $\mathcal{F}$ be a foliation of X of dimension $d \leq l$ which leaves Y invariant. Let $\Sigma = (Sing(\mathcal{F}) \cup Sing(Y)) \cap Y$ and assume that $\dim \Sigma < l$. Let $\Sigma = \cup_\gamma \Sigma_\gamma$ be the decomposition into connected components and let $\iota_\gamma : \Sigma_\gamma \hookrightarrow Y$ denote the inclusion. Let φ be a symmetric homogeneous polynomial of degree $t > l - d$. Then*
(i) *For each compact connected component Σ_γ there exists a class $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) \in H_{2l-2t}(\Sigma_\gamma; \mathbb{C})$, called "residue", which depends only on the local behavior of $\mathcal{F}$ near Σ_γ.*
(ii) *If Y is compact we have*

$$\sum_\gamma (\iota_\gamma)_* \mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) = \varphi(N_{P,M}) \frown [Y] \qquad \textit{in} \;\; H_{2l-2t}(Y; \mathbb{C}).$$

Remarks 1. If (X, Y) is an adequate singular pair in M, by the very definition, it follows that $\dim(\mathrm{Sing}(Y) \cup \mathrm{Sing}(X)) \cap Y = \dim(Y^{nt} \cup \mathrm{Sing}(X)) \cap Y < l$. Therefore the hypothesis that $\dim \Sigma < l$ in Theorem 2.1 refers only to the singularities of the foliation $\mathcal{F}$.
2. Let $\sigma_1, \ldots, \sigma_s$ denote the elementary symmetric functions in the s variables $X_1, \ldots, X_s$. If $\varphi \in \mathbb{C}[X_1, \ldots, X_s]$ is a symmetric homogeneous polynomial of degree t then there exists a unique polynomial $\tilde{\varphi}$ in the variables $\sigma_1, \ldots, \sigma_s$ such that $\varphi = \tilde{\varphi}(\sigma_1, \ldots, \sigma_s)$. If $c_j(N_{P,M}) \in H^{2j}(P; \mathbb{C})$ denotes the j-th Chern class of $N_{P,M}$ then one defines $\varphi(N_{P,M}) = \tilde{\varphi}(c_1(N_{P,M}), \ldots, c_s(N_{P,M})) \in H^{2t}(P; \mathbb{C})$.
3. The residues appearing in part (i) of Theorem 2.1 have often intrinsic relations with the "dynamics" of the foliation $\mathcal{F}$ (see [ABT], [BS], [Ca] and [CS] for more about this). The rest of this section is devoted to the proof of Theorem 2.1. In the next section we give an explicit expression for the residue $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma)$.

2.1 The proof of Theorem 2.1. The proof of this theorem is made in two steps. The first step consists in defining a "good connection" for the bundle $N_{P,M}$

on $Y \setminus \Sigma$ in such a way that its curvature is vanishing. The second step consists in using the Čech-de Rham theory to localize $\varphi(N_{P,M}|_Y)$ around Σ exploiting the previous constructed connection. This strategy is rather well known after the works of Lehmann [Le] and Lehmann and the second named author [LS]. However, for the reader convenience, we describe it in here.

2.1.1 The Bott vanishing theorem. Let $X^0 := X' \setminus \mathrm{Sing}(\mathcal{F})$ and $Y^0 := Y' \cap X^0$. Let $F \subset TX^0$ denote the holomorphic bundle associated to $\mathcal{F}$. One can define an operator $\Xi : \mathcal{C}^\infty(F|_{Y^0}) \times \mathcal{C}^\infty(N_{Y^0,X^0}) \to \mathcal{C}^\infty(N_{Y^0,X^0})$ called a *holomorphic action* of $F|_{Y^0}$ on N_{Y^0,X^0} as follows. For $u \in \mathcal{F}|_{Y^0}$ and $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$ the operator Ξ is defined as

$$\Xi(u,s) = \chi([\tilde{u}, \tilde{s}]|_Y), \tag{2.1}$$

where $\tilde{u} \in \mathcal{F}$ is such that $\tilde{u}|_Y = u$, $\tilde{s} \in \mathcal{C}^\infty(TX^0)$ and $\chi(\tilde{s}|_Y) = s$, where $\chi : TX^0|_Y \to N_{Y^0,X^0}$ is the canonical projection. One can prove that Ξ depends only on u and s and not on the extensions $\tilde{u}, \tilde{s}$ chosen to define it. Also one can extend naturally Ξ to $\mathcal{C}^\infty(F|_{Y^0})$, for $\mathcal{F}|_{Y^0}$ generates $\mathcal{C}^\infty(F|_{Y^0})$ as a $\mathcal{C}^\infty_{Y^0}$-module. The map Ξ satisfies the following properties:
(i) $\Xi([u,v],s) = \Xi(u, \Xi(v,s)) - \Xi(v, \Xi(u,s))$ for $u, v \in \mathcal{C}^\infty(F|_{Y^0})$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(ii) $\Xi(hu,s) = h\Xi(u,s)$ for $h \in \mathcal{C}^\infty_{Y^0}$, $u \in \mathcal{C}^\infty(F|_{Y^0})$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(iii) $\Xi(u,hs) = h\Xi(u,s) + u(h)s$ for $h \in \mathcal{C}^\infty_{Y^0}$, $u \in \mathcal{C}^\infty(F|_{Y^0})$, $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$,
(iv) $\Xi(u,s) \in \mathcal{O}_Y(N_{Y^0,X^0})$ for $u \in \Theta_{Y^0}$, $s \in \mathcal{O}_Y(N_{Y^0,X^0})$.

Properties (ii) and (iii) above say that the pair $(\Xi, F|_{Y^0})$ can be viewed as a *partial connection in the sense of Bott* for N_{Y^0,X^0}. Namely, Ξ defines a $\mathbb{C}$-linear map δ from $\mathcal{C}^\infty(F|_{Y^0})$ to $\mathcal{C}^\infty(F|^*_{Y^0} \otimes N_{Y^0,X^0})$ such that for $f \in \mathcal{C}^\infty_{Y^0}$ and $s \in \mathcal{C}^\infty(N_{Y^0,X^0})$

$$\delta(fs) = df \otimes s + f\delta s.$$

This latter can be extended (not uniquely) to a $(1,0)$-connection ∇_0 for N_{Y^0,X^0} such that

$$(\nabla_0)_u \cdot = \delta(u) = \Xi(u, \cdot)$$

for any $u \in \mathcal{C}^\infty(F|_{Y^0})$ (see [BB, p. 291]). We call a Ξ-*connection* any such connection ∇_0. For any symmetric homogeneous polynomial φ of degree $t > l - d$ and Ξ-connection ∇_0 for N_{Y^0,X^0}, we have the so-called "Bott vanishing theorem":

$$\varphi(\nabla_0) = 0. \tag{2.2}$$

Formula (2.2) follows from properties (i), (iv) of Ξ (see [BB, p. 295] or [Su3, Ch. II.9] for details).

2.1.2 Čech-de Rham theory and localization. Let U_0 be a tubular neighborhood of Y^0 in P, and let $\rho : U_0 \to Y^0$ be the C^∞-retraction. Since $N_{P,M}|_{Y^0} =$

N_{Y^0,X^0}, we can take ∇_0 a Ξ-connection for $N_{P,M}|_{Y^0}$ on Y^0 as explained in 2.1.1 and consider the connection $\rho^*(\nabla_0)$ for $\rho^*(N_{P,M})$ on U_0. Note that $\rho^*(N_{P,M}|_{Y^0})$ is C^∞-equivalent to $N_{P,M}$ on U_0. With some abuse of notation we simply denote $\rho^*(N_{P,M}|_{Y^0})$ by $N_{P,M}$ and also we denote by ∇_0 its connection $\rho^*(\nabla_0)$.

We set $\Sigma := (\mathrm{Sing}(\mathcal{F}) \cup \mathrm{Sing}(Y)) \cap Y$, which is a subvariety of dimension strictly less than l in Y by hypothesis. Let U_1 be an open neighborhood of Σ in P such that U_1 is union of disjoint open sets $U_{1,\gamma}$ each of them containing exactly one connected component—say Σ_γ—of Σ and such that $U_{1,\gamma}$ is a regular neighborhood of Σ_γ for every γ (this is possible by Corollary 1.2). Moreover we may assume that $U_0 \cup U_1$ is a regular neighborhood of Y in P. On U_1 we choose an arbitrary connection ∇_1 for $N_{P,M}$. Let $H^*_D(\mathcal{U})$ be the Čech-de Rham cohomology associated to the covering $\mathcal{U} = \{U_0, U_1\}$. We recall briefly how this is defined (see, *e.g.*, [Su3, Ch.II,3, Ch.IV,2, Ch.VI,4] for details). For any $p > 0$ define the $\mathbb{C}$-vector space $A^p(\mathcal{U}) = A^p(U_0) \oplus A^p(U_0) \oplus A^{p-1}(U_0 \cap U_1)$, where $A^p(V)$ is the space of p-forms on the open set $V \subset P$. Thus an element $\alpha \in A^p(\mathcal{U})$ is a triple $(\alpha_0, \alpha_1, \alpha_{01})$ such that $\alpha_j \in \mathcal{C}^\infty(\wedge^p T^*P; U_j)$ for $j = 0, 1$ and α_{01} is a $p-1$ form on $U_0 \cap U_1$. One defines an operator $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ given by $D\alpha = D(\alpha_0, \alpha_1, \alpha_{01}) = (d\alpha_0, d\alpha_1, \alpha_1 - \alpha_0 - d\alpha_{01})$. It can be checked that $D \circ D = 0$ and thus $\{A^*(\mathcal{U}), D\}$ is a complex, called the *Čech-de Rham complex*. Its cohomology $H^*_D(\mathcal{U})$ is called the *Čech-de Rham cohomology*. One feature of this cohomology is that the natural map $A^p(U_0 \cup U_1) \ni \omega \mapsto (\omega|_{U_0}, \omega|_{U_1}, 0) \in A^p(\mathcal{U})$ induces an isomorphism $H^p(U_0 \cup U_1; \mathbb{C}) \simeq H^p_D(\mathcal{U})$, and therefore, since $U_0 \cup U_1$ is a regular neighborhood of Y, an isomorphism $H^p(Y; \mathbb{C}) \simeq H^p_D(\mathcal{U})$. Also, one can consider the sub-complex $A^*(\mathcal{U}, U_0) = A^*(U_1) \oplus A^{*-1}(U_0 \cap U_1)$. Namely, an element $\alpha = (\alpha_0, \alpha_1, \alpha_{01}) \in A^*(\mathcal{U}, U_0)$ if and only if $\alpha_0 \equiv 0$. Its cohomology $H^*_D(\mathcal{U}, U_0)$ is the *relative Čech-de Rham cohomology*. It holds $H^*_D(\mathcal{U}, U_0) \simeq H^*(Y, Y \setminus \Sigma; \mathbb{C})$. If E is a complex vector bundle on $U_0 \cup U_1$ and ∇^j is a connection for $E|_{U_j}$, $j = 0, 1$, there exists a $(2t-1)$-form $\varphi(\nabla^0, \nabla^1)$ on $U_0 \cap U_1$, called the *Bott difference form*, such that $d\varphi(\nabla^0, \nabla^1) = \varphi(\nabla^1) - \varphi(\nabla^0)$ and the class of $(\varphi(\nabla^0), \varphi(\nabla^1), \varphi(\nabla^0, \nabla^1))$ in $H^{2t}(\mathcal{U})$ corresponds to $\varphi(E) \in H^{2t}(U_0 \cup U_1; \mathbb{C})$.

In our situation, from (2.1), it follows that $\varphi(N_{P,M}) \in H^{2t}(U_0 \cup U_1; \mathbb{C})$ is represented in $H^{2t}_D(\mathcal{U})$ by the cocycle

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)) = (0, \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)). \tag{2.2}$$

Therefore $[\varphi(\nabla_*)] \in H^{2t}_D(\mathcal{U}, U_0)$. Then we have a first localization of $\varphi(N_{P,M}|_Y) \in H^{2t}(Y; \mathbb{C})$ at cohomology level, which we denote by $\varphi_\Sigma(N_{P,M}) \in H^{2t}(Y, Y \setminus \Sigma; \mathbb{C})$. Since $H^{2t}(Y, Y \setminus \Sigma; \mathbb{C}) = \bigoplus_\gamma H^{2t}(Y, Y \setminus \Sigma_\gamma; \mathbb{C})$ we can also write $\varphi_\Sigma(N_{P,M}) = \sum_\gamma \varphi_{\Sigma_\gamma}(N_{P,M})$. If Σ_γ is compact we can consider the *Alexander homomorphism*

$$A_\gamma : H^*(Y, Y \setminus \Sigma_\gamma; \mathbb{C}) \to H_{2l-*}(\Sigma_\gamma; \mathbb{C}).$$

Alexander homomorphism is defined as follows. Let $\tilde{R}_{1,\gamma}$ be a compact real $2r$-dimensional manifold with C^∞ boundary in $U_{1,\gamma}$ such that Σ_γ is contained in the interior of $\tilde{R}_{1,\gamma}$ and such that the boundary $\partial\tilde{R}_{1,\gamma}$ is transverse to Y. We set $R_{1,\gamma} = \tilde{R}_{1,\gamma} \cap Y$. If $\theta \in H^p(Y, Y \setminus \Sigma_\gamma; \mathbb{C})$, it is represented by a couple $(\theta_1, \theta_{01}) \in A^p(\mathcal{U}, U_0)$, with θ_1 a p-form on $U_{1,\gamma}$ and θ_{01} a $(p-1)$-form on $U_{1,\gamma} \cap U_0$. Then $A_\gamma(\theta)$ is represented by a $(2l-p)$-cycle C in Σ_γ such that for any closed $(2l-p)$-form τ_1 on $U_{1,\gamma}$ one has

$$\int_C \tau_1 = \int_{R_{1,\gamma}} \theta_1 \wedge \tau_1 - \int_{\partial R_{1,\gamma}} \theta_{01} \wedge \tau_1. \tag{2.3}$$

The image of $A_\gamma(\varphi_{\Sigma_\gamma}(N_{P,M})) \in H_{2l-2t}(\Sigma_\gamma; \mathbb{C})$ is denoted by $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma)$. Let $\iota_\gamma : \Sigma_\gamma \hookrightarrow Y$ denote the inclusion and $(\iota_\gamma)_*$ the morphism induced in homology by ι_γ. If Y is compact, we have

$$\sum_\gamma (\iota_\gamma)_* \mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma) = \mathrm{Poi}(\varphi(N_{P,M}|_Y)),$$

where $\mathrm{Poi} : H^*(Y; \mathbb{C}) \to H_{2l-*}(Y; \mathbb{C})$ denotes the Poincaré homomorphism, which is given by the cap product with the fundamental cycle $[Y]$ of Y. Theorem 2.1 is thus proved.

3. Computation of residues.

In this section we compute the residues given by Theorem (2.1) in some special cases and compare them with the ones obtained by [Le] and [LS] when X is non-singular. We retain the notation introduced in the previous sections.

First we write the general expression for $\mathrm{Res}(\mathcal{F}, Y; \Sigma_\gamma)$. Let $\mathcal{U} = \{U_0, U_1\}$, $\tilde{R}_{1,\gamma}$, $R_{1,\gamma}$ be as in section 2. By (2.3) it follows that $\mathrm{Res}_\varphi(\mathcal{F}, Y; \Sigma_\gamma)$ is represented by a $(2l-2t)$-cycle C in Σ_γ such that for any closed $(2l-2t)$-form τ_1 on $U_{1,\gamma}$ one has

$$\int_C \tau_1 = \int_{R_{1,\gamma}} \varphi(\nabla_1) \wedge \tau_1 - \int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1) \wedge \tau_1, \tag{3.1}$$

where ∇_0, ∇_1 and $\varphi(\nabla_0, \nabla_1)$ are as in (2.2). In particular if $t = l$, then the residue is a complex number.

Our aim is to compute explicitly the residue in case $t = l$, $d = 1$ and $\Sigma_\gamma = \{q_\gamma\}$. Moreover we assume $\mathcal{F}_{q_\gamma}$ is generated on $\mathcal{O}_{X,q_\gamma}$ by a single element of Θ_{X,q_γ}.

Since Σ_γ is reduced to one point, we can take $\tilde{U}_{1,\gamma}$ an open neighborhood of q_γ with local coordinates $\{z_1, \dots, z_m\}$ centered at q_γ and such that $P \cap \tilde{U}_{1,\gamma} =$

$\{z_{r+1} = \ldots = z_m = 0\}$. Moreover we may assume that on $U_{1,\gamma} := \tilde{U}_{1,\gamma} \cap P$ the bundle $N_{P,M}|_{U_{1,\gamma}}$ is holomorphically trivial and both X and Y are non-singular on $\tilde{U}_{1,\gamma} \setminus \{q_\gamma\}$. Let $\sigma : TM|_P \longrightarrow N_{P,M}$ be the canonical projection. Let $\delta_j := \sigma(\frac{\partial}{\partial z_j})$ for $j = r+1, \ldots, m$. Then $\{\delta_{r+1}, \ldots, \delta_m\}$ is a holomorphic base frame of $N_{P,M}$ on $U_{1,\gamma}$.

We can take ∇_1 to be the trivial connection for $N_{P,M}$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$. Thus $\varphi(\nabla_1) = 0$ and

$$\mathrm{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = -\int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1). \tag{3.2}$$

Now we find "good coordinates" to express the Bott difference form. First, by the local parameterization theorem, we can find an open set $\tilde{U}_0$ in M containing $Y^0 \cap \tilde{U}_{1,\gamma}$ and holomorphic functions $h_1, \ldots, h_l \in \mathcal{O}_M(\tilde{U}_{1,\gamma})$ such that, if $j : X \hookrightarrow M$, then

$$j^*(dh_1 \wedge \ldots \wedge dh_l \wedge dz_{r+1} \wedge \ldots \wedge dz_m|_{\tilde{U}_0 \cap X}) \neq 0. \tag{3.3}$$

This means that $\{h_1, \ldots, h_l, z_{r+1}, \ldots, z_m\}$ can be thought of as local coordinates of X^0 and $\{h_1, \ldots, h_l\}$ as local coordinates of Y^0 on $\tilde{U}_0$.

¿From the coherence of $\mathcal{F}$ and since $\mathcal{F}_{q_\gamma}$ is generated by only one element of Θ_{X,q_γ}, up to shrink $\tilde{U}_{1,\gamma}$ we can assume that on $\tilde{U}_{1,\gamma} \cap X$ the foliation $\mathcal{F}$ is generated by the holomorphic vector field $\xi \in \Theta_X$. With a slight abuse of notation, by (3.3), we can write ξ on $\tilde{U}_0 \cap X$ as

$$\xi = \sum_{j=r+1}^{m} \xi(z_j) \frac{\partial}{\partial z_j} + \sum_{i=1}^{l} \xi(h_i) \frac{\partial}{\partial h_i}, \tag{3.4}$$

where, for $\xi(z_j)$ and $\xi(h_i)$ to make sense, one should think of z_j and h_i as elements of $\mathcal{O}_X$ via the surjection $\mathcal{O}_M \to \mathcal{O}_X$.

Remark 3.1. *In case $\mathcal{F}$ comes from the restriction of a foliation of M given in $\tilde{U}_{1,\gamma}$ by a vector field $\hat{\xi}$, then $\mathcal{F}$ on $X^0 \cap \tilde{U}_{1,\gamma}$ is given by*

$$\xi = \hat{\xi}|_{X^0} = \sum_{j=r+1}^{m} \hat{\xi}(z_j)|_X \frac{\partial}{\partial z_j} + \sum_{i=1}^{l} \hat{\xi}(h_i)|_X \frac{\partial}{\partial h_i}. \tag{3.5}$$

Note that, since q_γ is an isolated singularity, $\xi(p) \neq 0$ for $p \in X^0 \cap \tilde{U}_{1,\gamma}$, and thus

$$\bigcap_{j=r+1}^{m} \{p \in \tilde{U}_0 | \xi(z_j)(p) = 0\} \bigcap_{i=1}^{l} \{p \in \tilde{U}_0 | \xi(h_i)(p) = 0\} = \emptyset.$$

Let $\mathfrak{M}$ be the $(m-r)$-square matrix with entries $\frac{\partial \xi(z_j)}{\partial z_k}$ for $j,k = r+1,\dots,m$. Recall that for a symmetric homogeneous polynomial $\varphi = \sum c_k \sigma_k^{j_k}$ with σ_k elementary symmetric functions, $c_k \in \mathbb{C}$ and $j_k \in \mathbb{N}$, one can define $\varphi(\mathfrak{M}) := \sum c_k \sigma_k^{j_k}(\mathfrak{M})$ where $\sigma_k(\mathfrak{M})$ are defined by means of the following relation

$$\det(I + t\mathfrak{M}) = 1 + t\sigma_1(\mathfrak{M}) + \ldots + t^{m-r}\sigma_{m-r}(\mathfrak{M}).$$

With the above notation we have

Theorem 3.1. *Let $d = 1$, $l = t$ and $\Sigma_\gamma = \{q_\gamma\}$. Assume that $\mathcal{F}$ is generated by $\xi \in \Theta_X$ near q_γ. Then there exists a small $\epsilon > 0$ such that*

$$\mathrm{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^l \int_\Gamma \frac{\varphi(\mathfrak{M}) dh_1 \wedge \ldots \wedge dh_l}{\xi(h_1)\cdots\xi(h_l)},$$

where $\Gamma = \{\, p \in U_0 \cap Y : |\xi(h_i)(p)| = \epsilon,\ i = 1,\dots,l\}$ is a real l-cycle oriented so that $d(\arg\xi(h_1) \wedge \ldots \wedge d(\arg\xi(h_l)) \geq 0$.

The proof of Theorem 3.1 for $l = 1$, $n = 2$ can be inferred from that of [BT, eq. (2.7)]. We also explicitly note that once appropriate coordinates are introduced as before the argument is similar to the one in [LS, Thm. 1']. However for the sake of clearness we give here a proof of Theorem 3.1 for $l = 2$. For $l > 2$ the argument is the same and is left to the reader.

Proof of Theorem 3.1 for $l = 2$. Let $U = \tilde{U}_0 \cap Y$. Let $W_j = \{\, p \in U : \xi(h_j)(p) \neq 0\,\}$, $j = 1, 2$. Note that $W_1 \cup W_2 = U$. Let $\mathcal{W} = \{W_1, W_2\}$. Let $\{A^*(\mathcal{W}), D\}$ be the Čech-de Rham complex associated to $\mathcal{W}$ as defined in section 2.1.2. Recall that an element $\alpha \in A^k(\mathcal{W})$ is made of a triple $(\alpha_1, \alpha_2, \alpha_{12})$ with α_j a differential form of Y of degree k on W_j $(j = 1, 2)$ and α_{12} a differential form of Y of degree $k-1$ on $W_1 \cap W_2$. The differential D of $A^*(\mathcal{W})$ is given by

$$D(\alpha_1, \alpha_2, \alpha_{12}) = (d\alpha_1, d\alpha_2, \alpha_2 - \alpha_1 - d\alpha_{12}).$$

One can define a linear operator $\int_{\partial R_{1,\gamma}} : A^3(\mathcal{W}) \to \mathbb{C}$ as follows. Let $T_1 = \{\, p \in \partial R_{1,\gamma} : |\xi(h_1)(p)| \geq |\xi(h_2)(p)|\,\}$ with positive orientation, $T_2 = \{\, p \in \partial R_{1,\gamma} : |\xi(h_2)(p)| \geq |\xi(h_1)(p)|\,\}$ with positive orientation and Γ defined as in Theorem 3.1. Then

$$\int_{\partial R_{1,\gamma}} (\alpha_1, \alpha_2, \alpha_{12}) := \int_{T_1} \alpha_1 + \int_{T_2} \alpha_2 + \int_\Gamma \alpha_{12}.$$

It is easy to show that $\int_{\partial R_{1,\gamma}} \circ D = 0$.

Recall that $\varphi(\nabla_0, \nabla_1)$ is a $(2t-1)$-form of P on $U_0 \cap U_1$ and we need to integrate it on $\partial R_{1,\gamma} \subset Y$. Thus we can consider the restriction $\varphi(\nabla_0, \nabla_1)|_Y$, denoted by $\kappa(\nabla_0, \nabla_1)$. Thus $\kappa(\nabla_0, \nabla_1)$ is a $(2l-1)$-form (that is a 3-form) on $Y \cap U$. Hence

$$(\kappa(\nabla_0, \nabla_1)|_{W_1}, \kappa(\nabla_0, \nabla_1)|_{W_2}, 0)$$

defines an element $\varpi \in A^3(\mathcal{W})$. Note that, since

$$d\kappa(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0) = 0 - 0 = 0,$$

$D\varpi = 0$. On the other hand on $W_1 \cap W_2$ we have the l-form (that is a 2-form) $\varphi(\mathfrak{M}) \frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}$. We can consider the element $\phi \in A^3(\mathcal{W})$ given by

$$\phi = (0, 0, (2\pi\sqrt{-1})^{-2} \varphi(\mathfrak{M}) \frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}).$$

Since the form $\varphi(\mathfrak{M}) \frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}$ is holomorphic on $W_1 \cap W_2$ it follows that $D\phi = 0$. The aim is now to show that there exists $\tau \in A^2(\mathcal{W})$ such that

$$\phi - \varpi = D\tau. \tag{3.6}$$

Suppose we proved (3.6). Then $\int_{\partial R_{1,\gamma}} \varpi = \int_{\partial R_{1,\gamma}} \phi$ and the result follows.

Thus we are left to prove (3.6). On W_1, since $\xi(h_1) \neq 0$, a basis of TY is given by $\{\xi|_Y, \frac{\partial}{\partial h_2}\}$. Similarly, on W_2, $\{\xi|_Y, \frac{\partial}{\partial h_1}\}$ is a basis of TY. On W_i, $i = 1, 2$, we define the following Ξ-connection ∇^i of type $(1, 0)$:

$$\nabla^i_{\xi|_Y} \delta_j = \Xi(\xi|_Y, \delta_j),$$
$$\nabla^i_{\frac{\partial}{\partial h_k}} \delta_j = 0,$$

for $k = i+1 \mod (2)$, $j = r+1, \dots, m$ (here we recall that Ξ is defined by (2.1) and $\{\delta_{r+1}, \dots, \delta_m\}$ is the holomorphic frame for $N_{P,M}$ on $\tilde{U}_{1,\gamma}$ fixed at the beginning of this section). Recall that for $k \leq 2l = 4$ connections $\tilde{\nabla}_j$'s for a vector bundle E over some open set of Y, one can define a $2l - k + 1$-form $\varphi(\tilde{\nabla}_1, \dots, \tilde{\nabla}_k)$ (see, *e.g.*, [Su3,p.69]), such that

$$\sum_{a=1}^{k} (-1)^{a-1} \varphi(\tilde{\nabla}_1, \dots, \widehat{\tilde{\nabla}_a}, \dots, \tilde{\nabla}_k) + (-1)^{k-1} d\varphi(\tilde{\nabla}_1, \dots, \tilde{\nabla}_k) = 0. \tag{3.7}$$

Moreover, if all $\tilde{\nabla}_1, \dots, \tilde{\nabla}_k$ are α-connection with respect to some holomorphic action α on E, then $\varphi(\tilde{\nabla}_1, \dots, \tilde{\nabla}_k) = 0$ (see, *e.g.*, [Su3, Ch.II,Thm. 9.11]). Going back to our situation, we define

$$\tau = (\varphi(\nabla_0, \nabla_1, \nabla^1), \varphi(\nabla_0, \nabla_1, \nabla^2), \varphi(\nabla_0, \nabla_1, \nabla^1, \nabla^2)),$$

where we recall that ∇_0 is the original Ξ-connection for $N_{Y^0,X^0} = N_{P,M}|_{Y^0}$ on $U_0 \cap Y$ and ∇_1 is the trivial connection for $N_{P,M}$ on $U_{1,\gamma}$ with respect to the frame $\{\delta_{r+1},\dots,\delta_m\}$ (and here we consider its restriction to Y). Now we calculate $D\tau$. By (3.7) we get

$$d\varphi(\nabla_0,\nabla_1,\nabla^1) = -\varphi(\nabla_1,\nabla^1) + \varphi(\nabla_0,\nabla^1) - \varphi(\nabla_0,\nabla_1) = -\varphi(\nabla_0,\nabla_1),$$

for ∇_1, ∇^1 are both α_1-connections with respect to the trivial action given by $\alpha_1(\frac{\partial}{\partial z_1},\delta_j) = 0$, and ∇_0, ∇^1 are both Ξ-connections. Similarly we get

$$d\varphi(\nabla_0,\nabla_1,\nabla^2) = -\varphi(\nabla_0,\nabla_1).$$

Finally by (3.7) and since $\nabla_0, \nabla^1, \nabla^2$ are all Ξ-connections

$$\begin{aligned}\varphi(\nabla_0,\nabla_1,\nabla^2) &- \varphi(\nabla_0,\nabla_1,\nabla^1) - d\varphi(\nabla_0,\nabla_1,\nabla^1,\nabla^2)\\ &= -\varphi(\nabla_1,\nabla^1,\nabla^2) + \varphi(\nabla_0,\nabla^1,\nabla^2) = -\varphi(\nabla_1,\nabla^1,\nabla^2).\end{aligned}$$

Thus

$$D\tau = (-\varphi(\nabla_0,\nabla_1), -\varphi(\nabla_0,\nabla_1), -\varphi(\nabla_1,\nabla^1,\nabla^2)).$$

Therefore (3.6) will follow as soon as we show that

$$-\varphi(\nabla_1,\nabla^1,\nabla^2) = (2\pi\sqrt{-1})^{-2}\varphi(\mathfrak{M})\frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}. \tag{3.8}$$

To prove (3.6) we first calculate the connection matrices $\theta_1, \theta^1, \theta^2$ of $\nabla_1, \nabla^1, \nabla^2$ with respect to the frame $\{\delta_{r+1},\dots,\delta_m\}$. Clearly $\theta_1 \equiv 0$. As for θ^1 (and similarly for θ^2) we have

$$\Xi(\xi|_Y,\delta_j) = \nabla^1_{\xi|_Y}\delta_j = \nabla^1_{\xi(h_1)\frac{\partial}{\partial h_1}+\xi(h_2)\frac{\partial}{\partial h_2}}\delta_j = \xi(h_1)\nabla^1_{\frac{\partial}{\partial h_1}}\delta_j. \tag{3.9}$$

Now, by the very definition of Ξ (see (2.1) and (3.4)) we have

$$\begin{aligned}\Xi(\xi|_Y,\delta_j) &= \chi([\xi,\frac{\partial}{\partial z_j}]|Y) = \chi([\sum_{k=r+1}^m \xi(z_k)\frac{\partial}{\partial z_j} + \sum_{i=1}^2 \xi(h_i)\frac{\partial}{\partial h_i},\frac{\partial}{\partial z_j}]|_Y)\\ &= -\sum_{k=r+1}^m \frac{\partial\xi(z_k)}{\partial z_j}|_Y\delta_k. \end{aligned}\tag{3.10}$$

¿From (3.9) and (3.10) we find

$$\theta^i = -\frac{dh_i}{\xi(h_i)}\mathfrak{M}, \quad i = 1,2.$$

Now let $\tilde{\nabla}$ denote the connection for the bundle $TY \times \mathbb{R}^2$ over $W_1 \cap W_2 \times \mathbb{R}^2$ defined by $\tilde{\nabla} = (1 - \sum_{k=1}^2 t_k)\nabla_1 + \sum_{j=1}^2 t_j \nabla^j$. The connection matrix of $\tilde{\nabla}$ is then given by $\tilde{\theta} = -\sum_{i=1}^2 t_i/\xi(h_i)\mathfrak{M}\, dh_i$ and the curvature matrix $\tilde{K}$ is

$$\tilde{K} = -\sum_{i=1}^{2} dt_i \wedge \frac{dh_i}{\xi(h_i)}\mathfrak{M} + \text{terms not containing } dt_i.$$

Let Δ^2 be the standard 2-simplex in $\mathbb{R}^2$ and denote by $\beta : Y \times \Delta^2 \to Y$ the projection. By the very definition $\varphi(\nabla_1, \nabla^1, \nabla^2)$ is given by $(2\pi\sqrt{-1})^{-2}\beta_*(\varphi(\tilde{K}))$, where $\beta_* : \Omega^*(Y \times \Delta^2) \to \Omega^{*-2}(Y)$ denotes the integration along the fibers. From the expression of $\tilde{K}$ formula (3.8) follows and we are done. $\square$

Final Remarks 1. When $\{q_\gamma\}$ is such that $q_\gamma \in X'$, then one can take $h_1, \ldots, h_l$ to be part of local coordinates in an open (in M) neighborhood of q_γ. Thus in such a case Theorem (3.1) reduced to [LS, Thm.2], which indeed is the classical Camacho-Sad formula for $l = 1$, $n = 2$.
2. When $l = n - 1$, instead of a foliation $\mathcal{F}$ of X one can consider a holomorphic self-map $f : X \to X$ which pointwise fixes Y. Generically (see [ABT]) this allows to define a one-dimensional foliation $\mathcal{F}_f$ of Y and a holomorphic action Ξ_f of $\mathcal{F}_f$ on $N_{Y',X'}$ outside some "singularities" of f on Y. Arguing as in section 2 one has a residue theorem for this case as well, which generalizes [BS, Thm. 2.2] where this result was achieved for $l = 1, n = 2$ and under the assumption that f were a holomorphic self-map of all the ambient M pointwise fixing P as well.

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