RESIDUES FOR HOLOMORPHIC
FOLIATIONS OF SINGULAR PAIRS

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ABSTRACT. Let $X$ be a (possibly singular) subvariety of a complex manifold $M$ and $Y$ a subvariety of $X$. We assume that $Y$ is the intersection locus of $X$ with a submanifold $P \subset M$ and this intersection is generically transverse. For such a pair $(X,Y)$, we prove a generalization of the classical Camacho-Sad residue theorem, in case there exists a holomorphic foliation $F$ of $X$ leaving $Y$ invariant. Also, we compute explicitly the residues at isolated singular points.

Introduction

The classical Camacho-Sad residue (or index) theorem [CS] states that if $X$ is a two dimensional complex manifold, $Y \subset X$ a non-singular compact complex curve invariant by a holomorphic foliation $F$ of $X$, then the first Chern class $c_1(N_{Y,X})$ of the normal bundle $N_{Y,X}$ of $Y$ in $X$ localizes at the singularities of $F$ in $Y$. That is to say, to each point $p$ in $Y \cap \text{Sing}(F)$ one can associate a complex number $\text{Res}(F; Y, p)$, called the residue of $F$ at $p$ relative to $Y$, depending only on the behavior of $F$ near $p$, such that $\sum \text{Res}(F; Y, p) = \int_Y c_1(N_{Y,X})$. C. Camacho and P. Sad used their theorem to settle a problem raised by Poincaré on the existence of separatrices for germs of holomorphic foliations in $\mathbb{C}^2$. On the other hand the Camacho-Sad theorem can be seen as an obstruction to the existence of foliations having a given curve as invariant. Especially in this optic, the Camacho-Sad theorem has been generalized by several authors. Just to name a few, A. Lins Neto [Li] for the case of a singular curve $Y$ in $X = \mathbb{CP}^2$, the second named author [Su2] for the general case $Y$ singular, and D. Lehmann and the second named author [LS] for the case $X$ is a complex manifold of arbitrary dimension and $Y$ is an arbitrary (co)dimensional strongly locally complete intersection in $X$ (we refer to [Su3] for a detailed history). Very recently, a paper by V. Cavalier, D. Lehmann and M. Soares

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[CLS] generalizes the Camacho-Sad theorem to the case $Y$ is an arbitrary singular subvariety of a complex manifold $X$.

The basic idea in all these papers (which is however rather hidden in the first three) is that a holomorphic foliation $\mathcal{F}$ provides a vanishing of certain characteristic classes away from the singular locus of $Y$ and $\mathcal{F}$. Provided that there exists a “good extension” of $N_{Y,X}$ on such a singular locus, the Čech-de Rham theory produces localization at some cohomological level and Poincaré and Alexander duality give then the residue theorems. Thus, taking this machinery for granted, to have a Camacho-Sad type theorem one has to face two problems: find an extension (“natural” to some extent) of the normal bundle of $Y$ in $X$ to all the singularities of $Y$, and then find some “good action” of $\mathcal{F}$ on such extension in order to get some vanishing theorems. According to this picture, in order to solve problems in discrete dynamics, the first named author together with M. Abate and F. Tovena (see [Ab], [Br], [BT], [ABT]) developed a way to obtain generalizations of the Camacho-Sad theorem in case the foliation $\mathcal{F}$ is replaced by a holomorphic map $f : X \to X$ pointwise fixing $Y$.

In all the previous works the ambient manifold $X$ is supposed to be non-singular, which allows to have natural extensions of $N_{Y,X}$. The next step would be then to allow some singularities for $X$. This is not, however, just a merely technical game. Indeed, in some dynamical problems one has to face a singular ambient space. In the case of foliations (see [Ca]), this does not cause really a serious problem, for one can always resolve the singularities and pull back the foliation to a non-singular ambient; however, in the case of holomorphic maps this is no longer possible. Thus in [BS], for answering a question of discrete dynamics, the authors were forced to define residues for a singular $X$ of dimension two and proved a Camacho-Sad type theorem in that case. The aim of this paper is to give a general version of the residue theorem introduced in [BS] with no restrictions on the dimension of $X$ and (co)dimension of $Y$.

The setting is as follows. We let $X$ be an analytic variety in a complex manifold $M$ and $Y$ a subvariety of $X$. We assume that $Y$ is presented as the intersection of $X$ with a submanifold $P \subset M$, in such a way that the intersection is generically transverse and $\dim P + \dim X = \dim M + \dim Y$. We call such a pair $(X, Y)$ an adequate singular pair (see Definition 1.2). Many examples of algebraic varieties come up this way. Note that if $(X, Y)$ is an adequate singular pair, the bundle $N_{Y,X}$, defined only on the non-singular part of $Y$ in $X$, has a natural extension given by $N_{P,M|Y}$. If $\mathcal{F}$ is a holomorphic foliation of $X$ (actually, we do not need it to be defined outside $X$ as required in [BS]) leaving $Y$ invariant, we prove a localization theorem for $\varphi(N_{P,M|Y})$ near the singularities of $Y$ and of $\mathcal{F}$ in $Y$, where $\varphi$ is a homogeneous symmetric polynomial of an appropriate degree (see Theorem 2.1). This can be seen as a natural generalization of the Camacho-Sad theorem for the case $X$ is singular. We also compute explicitly the residues at isolated singular
points in the top degree case (see Theorem 3.1).

In the first section of the paper we recall some basic facts about holomorphic foliations and introduce the main objects of our study, i.e., adequate singular pairs; in particular we discuss of their singularities. In the second section we derive our main residues theorem. In the third section we provide a computation of residues at isolated points.

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1. Preliminaries

Let $M$ be a connected complex manifold. The symbol $\mathcal{C}_M^\infty$ will denote the sheaf of $C^\infty$-functions on $M$, while $\mathcal{O}_M$ is the sheaf of holomorphic functions. If $E$ is a (complex) vector bundle on $M$ we denote by $\mathcal{C}(E)$ the sheaf of $C^\infty$-sections of $E$ while we reserve the script symbol $\mathcal{E} = \mathcal{O}_M(E)$ to denote the sheaf of holomorphic sections of $E$. Moreover we denote by $TM$ the holomorphic tangent bundle of $M$ and by $\Theta_M = \mathcal{O}_M(TM)$ the sheaf of germs of holomorphic vectors fields on $M$.

1.1 Foliations.

**Definition 1.1.** A (singular) foliation of $M$ is a coherent subsheaf $\mathcal{F} \subset \Theta_M$ which is involutive, i.e., for any $p \in M$,

$$[\mathcal{F}_p, \mathcal{F}_p] \subseteq \mathcal{F}_p.$$

Let $Q := \Theta_M/\mathcal{F}$ be the quotient sheaf. The singular set $\text{Sing}(\mathcal{F})$ of a foliation is defined as

$$\text{Sing}(\mathcal{F}) = \{ p \in M : Q_p \text{ is not } \mathcal{O}_{M,p}\text{-free} \}.$$

Note that $\text{Sing}(\mathcal{F})$ is a closed subvariety of $M$. The *dimension* of $\mathcal{F}$ is the rank of $\mathcal{F}_p$ at some (and hence any) point $p \in M \setminus \text{Sing}(\mathcal{F})$.

**Remarks 1.** If $Q_p$ is $\mathcal{O}_{M,p}$-free, then so is $\mathcal{F}_p$. Therefore on $M^0 = M \setminus \text{Sing}(\mathcal{F})$ there exists a holomorphic vector bundle $F \subset TM^0$ whose germs of holomorphic sections form $\mathcal{F}|_{M^0}$.

2. Sometimes the definition of foliation requires that $\mathcal{F}$ be also reduced (or full). This is a technical condition meaning that, for any open set $U \subset M$ and any section $s \in \Gamma(U, \Theta_M)$, if $s_p \in \mathcal{F}_p$ for all $p \in U \setminus \text{Sing}(\mathcal{F})$, then actually $s$ is in $\Gamma(U, \mathcal{F})$. However there is a canonical way to obtain a reduced foliation from a non-reduced one (see [BB], [Su1]).
Let $X$ be a possibly singular subvariety in $M$. We denote by $\text{Sing}(X)$ the singular set of $X$ and by $X' = X \setminus \text{Sing}(X)$ the non-singular part. In the following we need holomorphic foliations on $X$. First we need to define the sheaf of holomorphic “tangent vectors” $\Theta_X$ to $X$. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$. This is defined as $\mathcal{O}_X = \mathcal{O}_M / I_X$, where $I_X \subset \mathcal{O}_M$ is the ideal sheaf of germs of holomorphic functions identically vanishing on $X$. Then $\Theta_X$ is defined as $\text{Hom}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ with $\Omega_X$ being given by the following exact sequence of $\mathcal{O}_X$-modules:

$$I_X/I_X^2 \to \Omega_M \otimes \mathcal{O}_M \to \Omega_X \to 0,$$

where $\Omega_M = \mathcal{O}_M(T^*M)$ and the first morphism is given by $[f] \mapsto df \otimes 1$. Note that on $X'$, $\Theta_X = \mathcal{O}_X(TX')$. Note also that $\Theta_X$ acts on $\mathcal{O}_X$ as derivations, as in the case of non-singular base spaces. A holomorphic foliation on $X$ is a coherent subsheaf $F \subset \Theta_X$ such that $F|_{X'}$ is a holomorphic foliation. We set $\text{Sing}(F) = \text{Sing}(F|_{X'}) \cup \text{Sing}(X)$.

First examples of holomorphic foliations on $X$ come from restriction of foliations of $M$. Namely, let $F$ be a holomorphic foliation of $M$. We say that $X$ is $F$-invariant if every vector $v$ in $F$ leaves the ideal $I_X$ invariant, which is equivalent to saying that $v$ is tangent to $X'$. We denote by $F|_X$ the image of $F \otimes \mathcal{O}_M \mathcal{O}_X$ in $\Theta_M \otimes \mathcal{O}_M \mathcal{O}_X$. If $X$ is $F$-invariant, then actually $F|_X \subset \Theta_X$ and if, moreover, $\dim(\text{Sing}(F) \cap X) < \dim X$, then $F|_X$ is a foliation of $X$ of the same dimension as $F$ (cf. [Su3, Ch.VI, 6]).

Other natural examples of foliations on $X$ come from the case $X$ is pointwise fixed by a (nontrivial) holomorphic self-map of $M$ (see [ABT]).

### 1.2 Adequate singular pairs.

Let $M$ be a complex manifold of dimension $m$ and let $P \subset M$ be a complex submanifold of dimension $r$. We denote by $N_{P,M}$ the normal bundle of $P$ in $M$ defined by the following exact sequence of holomorphic vector bundles:

$$0 \to TP \to TM|_P \to N_{P,M} \to 0.$$

Assume $X$ is a complex submanifold of $M$ of dimension $n$ which intersects $P$ along a submanifold $Y \subset M$ of dimension $n + r - m$ and such intersection is everywhere transversal. Then it is easy to see that the normal bundle $N_{Y,X} = N_{P,M}|_Y$. This apparently harmless observation will allow us to obtain extensions of the Camacho-Sad index theorem even in the case $X$ and $Y$ are singular. To make things precise we need some works. We begin with a definition.

**Definition 1.2.** Let $M$ be a complex manifold of dimension $m$. Let $X \subset M$ be an analytic variety of pure dimension $n > 0$, $Y \subset X$ a subvariety of pure dimension $l > 0$. We say that $(X,Y)$ is an **adequate singular pair** in $M$ if there exists a submanifold $P \subset M$ of dimension $r = m + l - n$ such that
(1) \( Y = X \cap P \) (set-theoretically),
(2) \( \dim(\Sing(X) \cap P) < l \),
(3) \( X' \) intersects \( P \) generically transversally.

**Remark** Assume \( M, X, P, Y \) are as in Definition 1.2 and \( Y \cap X' \) is connected. If conditions (1) and (2) are satisfied but condition (3) is not then \( X' \) intersects \( P \) everywhere. Indeed, let \( \mathcal{N} \subset Y \cap X' \) be defined as follows: \( x \in \mathcal{N} \) if there exists an open neighborhood \( U \subset Y \cap X' \) of \( x \) such that \( X \) intersects \( P \) at every \( p \in \mathcal{N} \). Clearly \( \mathcal{N} \) is open and closed in \( X' \cap Y \). Therefore condition (3) means exactly \( \mathcal{N} = \emptyset \) on each connected component of \( Y \cap X' \).

**Example 1.1.** Let \( X \) be an \( n \)-dimensional algebraic subvariety of \( \mathbb{C}P^m \) and \( P \subset \mathbb{C}P^m \) an \( r \)-dimensional general linear subspace (with \( r - m + n > 0 \)). Then \( (X, X \cap P) \) is an adequate singular pair in \( \mathbb{C}P^m \).

**Example 1.2.** Let \( X_1 \subset \mathbb{C}^m \) be a germ of a singular variety of dimension \( n \) at the origin \( O \). Assume that the singularity of \( X_1 \) at \( O \) is isolated. Blow up the point \( O \) and let \( X \) be the strict transform of \( X_1 \) and \( P \) the exceptional divisor. Let \( \tilde{Y} = P \cap X \), and let \( Y \) be a connected component of \( \tilde{Y} \). If \( \dim(\Sing(X) \cap P) < \dim Y \) then by the previous remark, either \( X' \) intersects \( P \) everywhere on a connected component of \( Y \cap X' \) or \( (X, Y) \) is an adequate singular pair in the blow up of \( \mathbb{C}^m \) at \( O \).

We examine more closely the singularities of adequate singular pairs. We set \( Y' = (Y \setminus \Sing(Y)) \cap X' \) and

\[
(1.1) \quad Y^{nt} = \{ q \in X' \cap P : \dim(T_qX' + T_qP) < n + r - l \}.
\]

Thus \( Y^{nt} \) is the set where \( X' \) and \( P \) are not transverse. In [BS] the following lemma is proved for \( n = 2 \) and \( r = m - 1 \).

**Lemma 1.1.** If \( (X, Y) \) is an adequate singular pair, then \( Y^{nt} = X' \cap \Sing(Y) \).

**Proof.** Clearly \( X' \cap \Sing(Y) \subset Y^{nt} \), for if \( X' \) is transverse to \( P \) at a point \( q \), then \( Y \) is non-singular at \( q \). Conversely, suppose \( q \in Y' \). We wish to show that \( X' \) intersects \( P \) at \( q \). Let \( (z_1, \ldots, z_m) \) be local coordinates on an open set \( U \subset M \) such that \( q \in U \), \( X \cap U = \{ z_{n+1} = \ldots = z_m = 0 \} \) and \( Y \cap U = \{ z_{l+1} = \ldots = z_m = 0 \} \), where we recall that \( l = n + r - m \geq 1 \). There exist holomorphic functions \( h_1, \ldots, h_{m-r} \) on \( U \) such that \( P \cap U = \{ h_1(z) = \ldots = h_{m-r}(z) = 0 \} \). Then the goal is to show that

\[
(1.2) \quad \det \left( \frac{\partial h_j}{\partial z_k} \right)(q) \neq 0,
\]
where, $1 \leq j \leq m - r$ and $l + 1 \leq k \leq n$. Let $\mathcal{H}_j := h_j|_X$, for $j = 1, \ldots, m - r$. Let $\mathcal{I}_Y \subset \mathcal{O}_X$ be the ideal sheaf of holomorphic functions on $X$ identically vanishing on $Y$. Note that $\mathcal{I}_Y$ nearby $q$ is generated in $\mathcal{O}_X$ by $z_{l+1}, \ldots, z_n$. Since $\mathcal{H}_j \in \mathcal{I}_{Y,q}$, there exist holomorphic germs $a_{jk} \in \mathcal{O}_X,q$ on $X$, for $j = 1, \ldots, m - r$ and $k = l+1, \ldots, n$, such that $h_j = \sum_{k=l+1}^{n} a_{jk} z_k$. Since $\det \left( \frac{\partial h_j}{\partial z_k} \right)(q) = \det \left( \frac{\partial \mathcal{H}_j}{\partial z_k} \right)(q)$ and the latter term is non-zero if and only if

$$\det(a_{jk})(q) \neq 0,$$

we are then left to prove (1.3). For any point $p \in U \cap Y$ let $\mathcal{G}_p \subset \mathcal{O}_{X,p}$ be the ideal generated by $\mathcal{H}_1, \ldots, \mathcal{H}_{m-r}$. Shrinking $U$ if necessary, we note that

$$\mathcal{I}_{Y,p} = \mathcal{G}_p$$

if and only if $\det(a_{jk})(p) \neq 0$ and thus (1.4) is equivalent to $X$ being transverse to $P$ at $p$. Therefore we are left to prove that (1.4) holds for $p = q$. Note also that, by the Hilbert Nullstellensatz, $\mathcal{I}_{Y,p} = \sqrt{\mathcal{G}_p}$, where $\sqrt{\mathcal{G}_p}$ denotes the radical of $\mathcal{G}_p$. We are thus left to show that $\mathcal{G}_q = \sqrt{\mathcal{G}_q}$.

Since $Y$ is locally complete intersection in $X$ it follows that $\mathcal{O}_{X,q}/\mathcal{G}_q$ is a Cohen-Macaulay ring. Thus if $\mathcal{O}_{X,q}/\mathcal{G}_q$ is not reduced (which corresponds to $\mathcal{G}_q \neq \sqrt{\mathcal{G}_q} = \mathcal{I}_{Y,q}$) then $\mathcal{O}_{X,p}/\mathcal{G}_p$ is not reduced (and then $\mathcal{G}_p \neq \mathcal{I}_{Y,p}$) for any point $p$ in a suitable open neighborhood of $q$ (see, e.g., [Lo, p.50]). By (1.4) this means that $X$ is non-transversal to $P$ on an open set in $Y$, against our hypothesis (3). \(\square\)

As a corollary of Lemma 1.1 we have

**Corollary 1.2.** The subvariety $Y^{nt} \subset Y$ is made of connected components isolated in $Y$.

**Remark** Assume $(X,Y)$ is an adequate singular pair in $M$, $Y = X \cap P$. Then $N_{P,M}|Y$ coincides with $N_{Y,X}$ on the open set $U \subset Y$ where $Y$ is nonsingular and $X$ intersects $P$ transversally. Let $Z$ be another complex manifold and $z_0 \in Z$. Let $\tilde{M} = M \times Z$, $\tilde{X} = X \times \{z_0\}$, $\tilde{Y} = Y \times \{z_0\}$ and $\tilde{P} = P \times \{z_0\}$. Then $(\tilde{X}, \tilde{Y})$ satisfies hypotheses (1), (2) and (3) in Definition 1.2 but it is not an adequate singular pair according to our definition for $\dim \tilde{P} < \dim \tilde{M} + \dim \tilde{Y} - \dim \tilde{X}$. In particular note that $N_{\tilde{P},\tilde{M}}$ has rank greater than that of $N_{\tilde{Y},\tilde{X}}$, and thus $N_{\tilde{P},\tilde{M}}$ does not provide an extension of $N_{Y,X}$. However, if $\pi_1 : \tilde{M} \to M$ is the natural projection, the bundle $N = \pi_1^*(N_{P,M})$, which can naturally be thought of as a subbundle of
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\[ N_{P,M}, \text{ coincides with } N_{Y,X} \text{ on } \pi_1^{-1}(U) \cap \tilde{Y}. \] Therefore in such a case one can easily extend the results presented in this paper for the case of adequate singular pairs. We warmly thank the referee for pointing out this example.

In general if \( X, Y, P, M \) are as in Definition 1.2 except \( r < m - n + l \), it does not seem natural to ask for the existence of a subbundle \( N \) of \( N_{P,M} \) which extends \( N_{Y',X'} \) in order to include the previous example in the definition of adequate singular pairs. Also, Lemma 1.1 seems to heavily rely on the property of local complete intersection coming from the hypothesis \( r = m + n - l \).

2. Residue theorems for foliations of adequate singular pairs

In this section we state and prove our main theorem. We use the notations previously introduced about adequate singular pairs.

**Theorem 2.1.** Let \( (X, Y) \) be an adequate singular pair in \( M \) and let \( F \) be a foliation of \( X \) of dimension \( d \leq l \) which leaves \( Y \) invariant. Let \( \Sigma = (\text{Sing}(F) \cup \text{Sing}(Y)) \cap Y \) and assume that \( \dim \Sigma < l \). Let \( \Sigma = \cup_{\gamma} \Sigma_{\gamma} \) be the decomposition into connected components and let \( \iota_{\gamma} : \Sigma_{\gamma} \hookrightarrow Y \) denote the inclusion. Let \( \varphi \) be a symmetric homogeneous polynomial of degree \( t > l - d \). Then

(i) For each compact connected component \( \Sigma_{\gamma} \) there exists a class \( \text{Res}_\varphi(F, Y; \Sigma_{\gamma}) \in H_{2l-2t}(\Sigma_{\gamma}; \mathbb{C}) \), called “residue”, which depends only on the local behavior of \( F \) near \( \Sigma_{\gamma} \).

(ii) If \( Y \) is compact we have

\[
\sum_{\gamma} (\iota_{\gamma})_* \text{Res}_\varphi(F, Y; \Sigma_{\gamma}) = \varphi(N_{P,M}) \wedge [Y] \quad \text{in } H_{2l-2t}(Y; \mathbb{C}).
\]

**Remarks**

1. If \( (X, Y) \) is an adequate singular pair in \( M \), by the very definition, it follows that \( \dim(\text{Sing}(Y) \cup \text{Sing}(X)) \cap Y = \dim(Y^{nt} \cup \text{Sing}(X)) \cap Y < l \). Therefore the hypothesis that \( \dim \Sigma < l \) in Theorem 2.1 refers only to the singularities of the foliation \( F \).

2. Let \( \sigma_1, \ldots, \sigma_s \) denote the elementary symmetric functions in the \( s \) variables \( X_1, \ldots, X_s \). If \( \varphi \in \mathbb{C}[X_1, \ldots, X_s] \) is a symmetric homogeneous polynomial of degree \( t \) then there exists a unique polynomial \( \tilde{\varphi} \) in the variables \( \sigma_1, \ldots, \sigma_s \) such that \( \varphi = \tilde{\varphi}(\sigma_1, \ldots, \sigma_s) \). If \( c_j(N_{P,M}) \in H^{2j}(P; \mathbb{C}) \) denotes the \( j \)-th Chern class of \( N_{P,M} \) then one defines \( \varphi(N_{P,M}) = \tilde{\varphi}(c_1(N_{P,M}), \ldots, c_s(N_{P,M})) \in H^{2t}(P; \mathbb{C}) \).

3. The residues appearing in part (ii) of Theorem 2.1 have often intrinsic relations with the “dynamics” of the foliation \( F \) (see [ABT], [BS], [Ca] and [CS] for more about this). The rest of this section is devoted to the proof of Theorem 2.1. In the next section we give an explicit expression for the residue \( \text{Res}_\varphi(F, Y; \Sigma_{\gamma}) \).

2.1 The proof of Theorem 2.1. The proof of this theorem is made in two steps. The first step consists in defining a “good connection” for the bundle \( N_{P,M} \).
on \( Y \setminus \Sigma \) in such a way that its curvature is vanishing. The second step consists in using the Čech-de Rham theory to localize \( \varphi(N_{P,M}|_{Y}) \) around \( \Sigma \) exploiting the previous constructed connection. This strategy is rather well known after the works of Lehmann [Le] and Lehmann and the second named author [LS]. However, for the reader convenience, we describe it in here.

### 2.1.1 The Bott vanishing theorem

Let \( X^0 := X' \setminus \text{Sing}(\mathcal{F}) \) and \( Y^0 := Y' \cap X^0 \). Let \( F \subset TX^0 \) denote the holomorphic bundle associated to \( \mathcal{F} \). One can define an operator \( \Xi : C^\infty(F|_{Y^0}) \times C^\infty(N_{Y^0,X^0}) \to C^\infty(N_{Y^0,X^0}) \) called a holomorphic action of \( F|_{Y^0} \) on \( N_{Y^0,X^0} \) as follows. For \( u \in \mathcal{F}|_{Y^0} \) and \( s \in C^\infty(N_{Y^0,X^0}) \) the operator \( \Xi \) is defined as

\[
\Xi(u,s) = \chi([\tilde{u}, \tilde{s}]|_{Y}),
\]

where \( \tilde{u} \in \mathcal{F} \) is such that \( \tilde{u}|_{Y} = u \), \( \tilde{s} \in C^\infty(TX^0) \) and \( \chi(\tilde{s}|_{Y}) = s \), where \( \chi : TX^0|_{Y} \to N_{Y^0,X^0} \) is the canonical projection. One can prove that \( \Xi \) depends only on \( u \) and \( s \) and not on the extensions \( \tilde{u}, \tilde{s} \) chosen to define it. Also one can extend naturally \( \Xi \) to \( C^\infty(F|_{Y^0}) \), for \( \mathcal{F}|_{Y^0} \) generates \( C^\infty(F|_{Y^0}) \) as a \( C^\infty_{Y^0} \)-module. The map \( \Xi \) satisfies the following properties:

1. \( \Xi([u,v],s) = \Xi(u,\Xi(v,s)) - \Xi(v,\Xi(u,s)) \) for \( u, v \in C^\infty(F|_{Y^0}), s \in C^\infty(N_{Y^0,X^0}) \),
2. \( \Xi(hu,s) = h\Xi(u,s) \) for \( h \in C^\infty_{Y^0}, u \in C^\infty(F|_{Y^0}), s \in C^\infty(N_{Y^0,X^0}) \),
3. \( \Xi(u,hs) = h\Xi(u,s) + u(h)s \) for \( h \in C^\infty_{Y^0}, u \in C^\infty(F|_{Y^0}), s \in C^\infty(N_{Y^0,X^0}) \),
4. \( \Xi(u,s) \in \mathcal{O}_Y(N_{Y^0,X^0}) \) for \( u \in \Theta_{Y^0}, s \in \mathcal{O}_Y(N_{Y^0,X^0}) \).

Properties (ii) and (iii) above say that the pair \( (\Xi,F|_{Y^0}) \) can be viewed as a partial connection in the sense of Bott for \( N_{Y^0,X^0} \). Namely, \( \Xi \) defines a \( C^\infty \)-linear map \( \delta \) from \( C^\infty(F|_{Y^0}) \) to \( C^\infty(F|_{Y^0} \otimes N_{Y^0,X^0}) \) such that for \( f \in C^\infty_{Y^0} \) and \( s \in C^\infty(N_{Y^0,X^0}) \)

\[
\delta(fs) = df \otimes s + f\delta s.
\]

This latter can be extended (not uniquely) to a \( (1,0) \)-connection \( \nabla_0 \) for \( N_{Y^0,X^0} \) such that

\[
(\nabla_0)_{u} \cdot = \delta(u) = \Xi(u, \cdot)
\]

for any \( u \in C^\infty(F|_{Y^0}) \) (see [BB, p. 291]). We call a \( \Xi \)-connection any such connection \( \nabla_0 \). For any symmetric homogeneous polynomial \( \varphi \) of degree \( t > l - d \) and \( \Xi \)-connection \( \nabla_0 \) for \( N_{Y^0,X^0} \), we have the so-called “Bott vanishing theorem”:

\[
\varphi(\nabla_0) = 0.
\]

Formula (2.2) follows from properties (i), (iv) of \( \Xi \) (see [BB, p. 295] or [Su3, Ch. II.9] for details).

### 2.1.2 Čech-de Rham theory and localization

Let \( U_0 \) be a tubular neighborhood of \( Y^0 \) in \( P \), and let \( \rho : U_0 \to Y^0 \) be the \( C^\infty \)-retraction. Since \( N_{P,M}|_{Y^0} = \)
\(N_{Y_0,X_0}\), we can take \(\nabla_0\) a \(\Xi\)-connection for \(N_{P,M}|_{Y^0}\) on \(Y^0\) as explained in 2.1.1 and consider the connection \(\rho^*(\nabla_0)\) for \(\rho^*(N_{P,M})\) on \(U_0\). Note that \(\rho^*(N_{P,M}|_{Y^0})\) is \(C^\infty\)-equivalent to \(N_{P,M}\) on \(U_0\). With some abuse of notation we simply denote \(\rho^*(N_{P,M}|_{Y^0})\) by \(N_{P,M}\) and also we denote by \(\nabla_0\) its connection \(\rho^*(\nabla_0)\).

We set \(\Sigma := (\text{Sing}(\mathcal{F}) \cup \text{Sing}(\mathcal{Y})) \cap Y\), which is a subvariety of dimension strictly less than \(l\) in \(Y\) by hypothesis. Let \(U_1\) be an open neighborhood of \(\Sigma\) in \(P\) such that \(U_1\) is union of disjoint open sets \(U_{1,\gamma}\) each of them containing exactly one connected component—say \(\Sigma_\gamma\)—of \(\Sigma\) and such that \(U_{1,\gamma}\) is a regular neighborhood of \(\Sigma_\gamma\) (this is possible by Corollary 1.2). Moreover we may assume that \(U_0 \cup U_1\) is a regular neighborhood of \(Y\) in \(P\). In our situation, from (2.1), it follows that \(\phi|_{\Sigma_\gamma}\) at cohomology level, which we denote by \(\phi|_{\Sigma_\gamma}\) for \(\phi\) in \(\Sigma_\gamma\). It holds

\[\phi(\nabla_0) = (\phi(\nabla_0), \phi(\nabla_1), \phi(\nabla_0, \nabla_1)) = (\nabla_0, \phi(\nabla_1), \phi(\nabla_0, \nabla_1)).\]

Therefore \(\phi(\nabla_\ast) \in H^2_{\mathcal{D}}(\mathcal{U}, U_0)\). Then we have a first localization of \(\phi(N_{P,M}|_{Y}) \in H^{2l}(Y; \mathbb{C})\) at cohomology level, which we denote by \(\phi|_{\Sigma_\gamma}(N_{P,M}) \in H^{2l}(Y \setminus \Sigma; \mathbb{C})\). Since \(H^{2l}(Y, Y \setminus \Sigma; \mathbb{C}) = \bigoplus_{\gamma} H^{2l}(Y, Y \setminus \Sigma_\gamma; \mathbb{C})\) we can also write \(\phi|_{\Sigma_\gamma}(N_{P,M}) = \sum_{\gamma} \phi|_{\Sigma_\gamma}(N_{P,M})\). If \(\Sigma_\gamma\) is compact we can consider the Alexander homomorphism

\[A_\gamma : H^*(Y, Y \setminus \Sigma_\gamma; \mathbb{C}) \to H_{2l-\ast}(\Sigma_\gamma; \mathbb{C}).\]
Alexander homomorphism is defined as follows. Let \( \tilde{R}_{1,\gamma} \) be a compact real 2-dimensional manifold with \( C^\infty \) boundary in \( U_{1,\gamma} \) such that \( \Sigma_\gamma \) is contained in the interior of \( \tilde{R}_{1,\gamma} \) and such that the boundary \( \partial \tilde{R}_{1,\gamma} \) is transverse to \( Y \). We set \( R_{1,\gamma} = \tilde{R}_{1,\gamma} \cap Y \). If \( \theta \in H^p(Y, Y \setminus \Sigma_\gamma; \mathbb{C}) \), it is represented by a couple \( (\theta_1, \theta_{01}) \in A^p(U_{1,\gamma}) \), with \( \theta_1 \) a \( p \)-form on \( U_{1,\gamma} \) and \( \theta_{01} \) a \( (p-1) \)-form on \( U_{1,\gamma} \cap U_0 \). Then \( A_\gamma(\theta) \) is represented by a \( (2l-p) \)-cycle \( C \) in \( \Sigma_\gamma \) such that for any closed \( (2l-p) \)-form \( \tau_1 \) on \( U_{1,\gamma} \) one has

\[
(2.3) \quad \int_C \tau_1 = \int_{R_{1,\gamma}} \theta_1 \wedge \tau_1 - \int_{\partial R_{1,\gamma}} \theta_{01} \wedge \tau_1.
\]

The image of \( A_\gamma(\varphi_{\Sigma_\gamma}(N_{P,M})) \in H_{2l-2t}(\Sigma_\gamma; \mathbb{C}) \) is denoted by \( \text{Res}_{\varphi}(\mathcal{F}, Y; \Sigma_\gamma) \). Let \( t_\gamma : \Sigma_\gamma \hookrightarrow Y \) denote the inclusion and \( (t_\gamma)_* \) the morphism induced in homology by \( t_\gamma \). If \( Y \) is compact, we have

\[
\sum_\gamma (t_\gamma)_* \text{Res}_{\varphi}(\mathcal{F}, Y; \Sigma_\gamma) = \text{Poi}(\varphi(N_{P,M}|_Y)),
\]

where \( \text{Poi} : H^*(Y; \mathbb{C}) \rightarrow H_{2l-*}(Y; \mathbb{C}) \) denotes the Poincaré homomorphism, which is given by the cap product with the fundamental cycle \([Y]\) of \( Y \). Theorem 2.1 is thus proved.

3. Computation of residues.

In this section we compute the residues given by Theorem (2.1) in some special cases and compare them with the ones obtained by [Le] and [LS] when \( X \) is non-singular. We retain the notation introduced in the previous sections.

First we write the general expression for \( \text{Res}(\mathcal{F}, Y; \Sigma_\gamma) \). Let \( \mathcal{U} = \{U_0, U_1\} \), \( \tilde{R}_{1,\gamma}, R_{1,\gamma} \) be as in section 2. By (2.3) it follows that \( \text{Res}_{\varphi}(\mathcal{F}, Y; \Sigma_\gamma) \) is represented by a \( (2l-2t) \)-cycle \( C \) in \( \Sigma_\gamma \) such that for any closed \( (2l-2t) \)-form \( \tau_1 \) on \( U_{1,\gamma} \) one has

\[
(3.1) \quad \int_C \tau_1 = \int_{R_{1,\gamma}} \varphi(\nabla_1) \wedge \tau_1 - \int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1) \wedge \tau_1,
\]

where \( \nabla_0, \nabla_1 \) and \( \varphi(\nabla_0, \nabla_1) \) are as in (2.2). In particular if \( t = l \), then the residue is a complex number.

Our aim is to compute explicitly the residue in case \( t = l, \ d = 1 \) and \( \Sigma_\gamma = \{q_\gamma\} \). Moreover we assume \( \mathcal{F}_{q_\gamma} \) is generated on \( \mathcal{O}_{X,q_\gamma} \) by a single element of \( \Theta_{X,q_\gamma} \).

Since \( \Sigma_\gamma \) is reduced to one point, we can take \( \tilde{U}_{1,\gamma} \) an open neighborhood of \( q_\gamma \) with local coordinates \( \{z_1, \ldots, z_m\} \) centered at \( q_\gamma \) and such that \( P \cap \tilde{U}_{1,\gamma} = \)
\{z_{r+1} = \ldots = z_m = 0\}$. Moreover we may assume that on $U_{1,\gamma} := \tilde{U}_{1,\gamma} \cap P$ the bundle $N_{P,M}|_{U_{1,\gamma}}$ is holomorphically trivial and both $X$ and $Y$ are non-singular on $\tilde{U}_{1,\gamma} \setminus \{q_\gamma\}$. Let $\sigma : TM|_P \rightarrow N_{P,M}$ be the canonical projection. Let $\delta_j := \sigma(\frac{\partial}{\partial z_j})$ for $j = r+1, \ldots, m$. Then $\{\delta_{r+1}, \ldots, \delta_m\}$ is a holomorphic base frame of $N_{P,M}$ on $U_{1,\gamma}$.

We can take $\nabla_1$ to be the trivial connection for $N_{P,M}$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$. Thus $\varphi(\nabla_1) = 0$ and

$$\text{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = -\int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1).$$

Now we find “good coordinates” to express the Bott difference form. First, by the local parameterization theorem, we can find an open set $\tilde{U}_0$ in $M$ containing $Y^0 \cap \tilde{U}_{1,\gamma}$ and holomorphic functions $h_1, \ldots, h_l \in \mathcal{O}_M(\tilde{U}_{1,\gamma})$ such that, if $j : X \hookrightarrow M$, then

$$(3.3) \quad j^*(dh_1 \wedge \ldots \wedge dh_l \wedge dz_{r+1} \wedge \ldots \wedge dz_m|_{\tilde{U}_0 \cap X}) \neq 0.$$ This means that $\{h_1, \ldots, h_l, z_{r+1}, \ldots, z_m\}$ can be thought of as local coordinates of $X^0$ and $\{h_1, \ldots, h_l\}$ as local coordinates of $Y^0$ on $\tilde{U}_0$.

From the coherence of $\mathcal{F}$ and since $\mathcal{F}_{q_\gamma}$ is generated by only one element of $\Theta_{X,q_\gamma}$, up to shrink $\tilde{U}_{1,\gamma}$ we can assume that on $\tilde{U}_{1,\gamma} \cap X$ the foliation $\mathcal{F}$ is generated by the holomorphic vector field $\xi \in \Theta_X$. With a slight abuse of notation, by (3.3), we can write $\xi$ on $\tilde{U}_0 \cap X$ as

$$\text{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = -\int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1).$$

(3.4) $$\xi = \sum_{j=r+1}^m \xi(z_j) \frac{\partial}{\partial z_j} + \sum_{i=1}^l \xi(h_i) \frac{\partial}{\partial h_i},$$

where, for $\xi(z_j)$ and $\xi(h_i)$ to make sense, one should think of $z_j$ and $h_i$ as elements of $\mathcal{O}_X$ via the surjection $\mathcal{O}_M \twoheadrightarrow \mathcal{O}_X$.

**Remark 3.1.** In case $\mathcal{F}$ comes from the restriction of a foliation of $M$ given in $\tilde{U}_{1,\gamma}$ by a vector field $\xi$, then $\mathcal{F}$ on $X^0 \cap \tilde{U}_{1,\gamma}$ is given by

$$\text{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = -\int_{\partial R_{1,\gamma}} \varphi(\nabla_0, \nabla_1).$$

(3.5) $$\xi = \xi|_{X^0} = \sum_{j=r+1}^m \xi(z_j)|_X \frac{\partial}{\partial z_j} + \sum_{i=1}^l \xi(h_i)|_X \frac{\partial}{\partial h_i}.$$ Note that, since $q_\gamma$ is an isolated singularity, $\xi(p) \neq 0$ for $p \in X^0 \cap \tilde{U}_{1,\gamma}$, and thus

$$\bigcap_{j=r+1}^m \{p \in \tilde{U}_0|\xi(z_j)(p) = 0\} \cap_{i=1}^l \{p \in \tilde{U}_0|\xi(h_i)(p) = 0\} = \emptyset.$$
Let $\mathfrak{M}$ be the $(m - r)$-square matrix with entries $\frac{\partial^2 \varphi}{\partial z_j \partial z_k}$ for $j, k = r + 1, \ldots, m$. Recall that for a symmetric homogeneous polynomial $\varphi = \sum c_k \sigma_k^j$ with $\sigma_k$ elementary symmetric functions, $c_k \in \mathbb{C}$ and $j_k \in \mathbb{N}$, one can define $\varphi(\mathfrak{M}) := \sum c_k \sigma_k^j(\mathfrak{M})$ where $\sigma_k(\mathfrak{M})$ are defined by means of the following relation

$$\det(I + t\mathfrak{M}) = 1 + t\sigma_1(\mathfrak{M}) + \ldots + t^{m-r}\sigma_{m-r}(\mathfrak{M}).$$

With the above notation we have

**Theorem 3.1.** Let $d = 1$, $l = t$ and $\Sigma_\gamma = \{q_\gamma\}$. Assume that $\mathcal{F}$ is generated by $\xi \in \Theta_X$ near $q_\gamma$. Then there exists a small $\epsilon > 0$ such that

$$\text{Res}_\varphi(\mathcal{F}, Y; \{q_\gamma\}) = \left(\frac{1}{2\pi i}\right)^l \int_\Gamma \frac{\varphi(\mathfrak{M})d\xi_1 \wedge \ldots \wedge d\xi_l}{\xi(h_1) \cdots \xi(h_l)},$$

where $\Gamma = \{p \in U_0 \cap Y : |\xi(h_i)(p)| = \epsilon, \ i = 1, \ldots, l\}$ is a real $l$-cycle oriented so that $d(\arg \xi(h_1) \wedge \ldots \wedge d(\arg \xi(h_1)) \geq 0$.

The proof of Theorem 3.1 for $l = 1, n = 2$ can be inferred from that of [BT, eq. (2.7)]. We also explicitly note that once appropriate coordinates are introduced as before the argument is similar to the one in [LS, Thm. 1']. However for the sake of clearness we give here a proof of Theorem 3.1 for $l = 2$. For $l > 2$ the argument is the same and is left to the reader.

**Proof of Theorem 3.1 for $l = 2$.** Let $U = \tilde{U}_0 \cap Y$. Let $W_j = \{p \in U : |\xi(h_j)(p)| \neq 0\}$, $j = 1, 2$. Note that $W_1 \cup W_2 = U$. Let $\mathcal{W} = \{W_1, W_2\}$. Let $\{A^k(\mathcal{W}), D\}$ be the Čech-de Rham complex associated to $\mathcal{W}$ as defined in section 2.1.2. Recall that an element $\alpha \in A^k(\mathcal{W})$ is made of a triple $(\alpha_1, \alpha_2, \alpha_12)$ with $\alpha_j$ a differential form of $Y$ of degree $k$ on $W_j$ ($j = 1, 2$) and $\alpha_12$ a differential form of $Y$ of degree $k - 1$ on $W_1 \cap W_2$. The differential $D$ of $A^*(\mathcal{W})$ is given by

$$D(\alpha_1, \alpha_2, \alpha_12) = (d\alpha_1, d\alpha_2, \alpha_1 - \alpha_12).$$

One can define a linear operator $\int_{\partial R_{1, \gamma}} : A^3(\mathcal{W}) \to \mathbb{C}$ as follows. Let $T_1 = \{p \in \partial R_{1, \gamma} : |\xi(h_1)(p)| \geq |\xi(h_2)(p)|\}$ with positive orientation, $T_2 = \{p \in \partial R_{1, \gamma} : |\xi(h_2)(p)| \geq |\xi(h_1)(p)|\}$ with positive orientation and $\Gamma$ defined as in Theorem 3.1. Then

$$\int_{\partial R_{1, \gamma}}(\alpha_1, \alpha_2, \alpha_12) := \int_{T_1} \alpha_1 + \int_{T_2} \alpha_2 + \int_{\Gamma} \alpha_12.$$

It is easy to show that $\int_{\partial R_{1, \gamma}} \circ D = 0$.

Recall that $\varphi(\nabla_0, \nabla_1)$ is a $(2t - 1)$-form of $P$ on $U_0 \cap U_1$ and we need to integrate it on $\partial R_{1, \gamma} \subset Y$. Thus we can consider the restriction $\varphi(\nabla_0, \nabla_1)|_Y$,
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denoted by $\kappa(\nabla_0, \nabla_1)$. Thus $\kappa(\nabla_0, \nabla_1)$ is a $(2l-1)$-form (that is a 3-form) on $Y \cap U$. Hence

$$(\kappa(\nabla_0, \nabla_1)|_{W_1}, \kappa(\nabla_0, \nabla_1)|_{W_2}, 0)$$

defines an element $\varpi \in A^3(W)$. Note that, since

$$d\kappa(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0) = 0 - 0 = 0,$$

$D\varpi = 0$. On the other hand on $W_1 \cap W_2$ we have the $l$-form (that is a 2-form) $\varphi(M)_{\xi(h_1)\xi(h_2)}^{dh_1 \wedge dh_2}$. We can consider the element $\phi \in A^3(W)$ given by

$$\phi = (0, 0, (2\pi\sqrt{-1})^{-2} \varphi(M)_{\xi(h_1)\xi(h_2)}^{dh_1 \wedge dh_2}).$$

Since the form $\varphi(M)_{\xi(h_1)\xi(h_2)}^{dh_1 \wedge dh_2}$ is holomorphic on $W_1 \cap W_2$ it follows that $D\phi = 0$. The aim is now to show that there exists $\tau \in A^2(W)$ such that

$$\phi - \varpi = D\tau. \quad (3.6)$$

Suppose we proved (3.6). Then $\int_{\partial R_{i,\gamma}} \varpi = \int_{\partial R_{i,\gamma}} \phi$ and the result follows.

Thus we are left to prove (3.6). On $W_1$, since $\xi(h_1) \neq 0$, a basis of $TY$ is given by $\{\xi|_Y, \frac{\partial}{\partial h_2}\}$. Similarly, on $W_2$, $\{\xi|_Y, \frac{\partial}{\partial h_1}\}$ is a basis of $TY$. On $W_i$, $i = 1, 2$, we define the following $\Xi$-connection $\nabla^i$ of type $(1,0)$:

$$\nabla^i_\xi, \delta_j = \Xi(\xi|_Y, \delta_j),$$

$$\nabla^i_\frac{\partial}{\partial h_k}, \delta_j = 0,$$

for $k = i+1 \mod (2)$, $j = r+1, \ldots, m$ (here we recall that $\Xi$ is defined by (2.1) and $\{\delta_{r+1}, \ldots, \delta_m\}$ is the holomorphic frame for $N_{P,M}$ on $\hat{U}_{1,\gamma}$ fixed at the beginning of this section). Recall that for $k \leq 2l = 4$ connections $\hat{\nabla}_j$'s for a vector bundle $E$ over some open set of $Y$, one can define a $2l - k + 1$-form $\varphi(\hat{\nabla}_1, \ldots, \hat{\nabla}_k)$ (see, e.g., [Su3,p.69]), such that

$$\sum_{a=1}^{k} (-1)^{a-1} \varphi(\hat{\nabla}_1, \ldots, \hat{\nabla}_a, \ldots, \hat{\nabla}_k) + (-1)^{k-1} d\varphi(\hat{\nabla}_1, \ldots, \hat{\nabla}_k) = 0. \quad (3.7)$$

Moreover, if all $\hat{\nabla}_1, \ldots, \hat{\nabla}_k$ are $\alpha$-connection with respect to some holomorphic action $\alpha$ on $E$, then $\varphi(\hat{\nabla}_1, \ldots, \hat{\nabla}_k) = 0$ (see, e.g., [Su3, Ch.II,Thm. 9.11]). Going back to our situation, we define

$$\tau = (\varphi(\nabla_0, \nabla_1), \varphi(\nabla_0, \nabla_1, \nabla^1), \varphi(\nabla_0, \nabla_1, \nabla^2), \varphi(\nabla_0, \nabla_1, \nabla^1, \nabla^2)),$$
where we recall that $\nabla_0$ is the original $\Xi$-connection for $N_{Y^0, X^0} = N_{P,M|Y^0}$ on $U_0 \cap Y$ and $\nabla_1$ is the trivial connection for $N_{P,M}$ on $U_{1, \gamma}$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$ (and here we consider its restriction to $Y$). Now we calculate $D\tau$. By (3.7) we get

$$d\varphi(\nabla_0, \nabla_1, \nabla^1) = -\varphi(\nabla_1, \nabla^1) + \varphi(\nabla_0, \nabla^1) - \varphi(\nabla_0, \nabla_1) = -\varphi(\nabla_0, \nabla_1),$$

for $\nabla_1, \nabla^1$ are both $\alpha_1$-connections with respect to the trivial action given by $\alpha_1(\frac{\partial}{\partial z_k}, \delta_j) = 0$, and $\nabla_0, \nabla^1$ are both $\Xi$-connections. Similarly we get

$$d\varphi(\nabla_0, \nabla_1, \nabla^2) = -\varphi(\nabla_0, \nabla_1).$$

Finally by (3.7) and since $\nabla_0, \nabla^1, \nabla^2$ are all $\Xi$-connections

$$\varphi(\nabla_0, \nabla_1, \nabla^2) - \varphi(\nabla_0, \nabla_1, \nabla^1) - d\varphi(\nabla_0, \nabla_1, \nabla^1, \nabla^2) = -\varphi(\nabla_1, \nabla^1, \nabla^2)$$

and

$$d\varphi(\nabla_0, \nabla_1, \nabla^2) = -\varphi(\nabla_1, \nabla^1, \nabla^2).$$

Thus

$$D\tau = (-\varphi(\nabla_0, \nabla_1), -\varphi(\nabla_0, \nabla_1), -\varphi(\nabla_1, \nabla^1, \nabla^2)).$$

Therefore (3.6) will follow as soon as we show that

$$-\varphi(\nabla_1, \nabla^1, \nabla^2) = (2\pi \sqrt{-1})^{-2} \varphi(\mathcal{M}) \frac{dh_1 \wedge dh_2}{\xi(h_1)\xi(h_2)}.$$  \hspace{1cm} (3.8)

To prove (3.6) we first calculate the connection matrices $\theta_1, \theta^1, \theta^2$ of $\nabla_1, \nabla^1, \nabla^2$ with respect to the frame $\{\delta_{r+1}, \ldots, \delta_m\}$. Clearly $\theta_1 \equiv 0$. As for $\theta^1$ (and similarly for $\theta^2$) we have

$$\Xi(\xi|Y, \delta_j) = \nabla^1_{\xi|Y} \delta_j = \nabla^1_{\xi(h_1)\frac{\partial}{\partial h_1} + \xi(h_2)\frac{\partial}{\partial h_2}} \delta_j = \xi(h_1)\nabla^1_{\frac{\partial}{\partial h_1}} \delta_j.$$  \hspace{1cm} (3.9)

Now, by the very definition of $\Xi$ (see (2.1) and (3.4)) we have

$$\Xi(\xi|Y, \delta_j) = \chi(\xi, \frac{\partial}{\partial z_j}|Y) = \chi(\sum_{k=r+1}^{m} \xi(z_k) \frac{\partial}{\partial z_j} + \sum_{i=1}^{2} \xi(h_i) \frac{\partial}{\partial h_i} | \frac{\partial}{\partial z_j}|Y)$$

$$= - \sum_{k=r+1}^{m} \frac{\partial \xi(z_k)}{\partial z_j} |_Y \delta_k.$$  \hspace{1cm} (3.10)

Thus from (3.9) and (3.10) we find

$$\theta^i = - \frac{dh_i}{\xi(h_i)} \mathcal{M}, \hspace{0.5cm} i = 1, 2.$$
Now let $\nabla$ denote the connection for the bundle $TY \times \mathbb{R}^2$ over $W_1 \cap W_2 \times \mathbb{R}^2$ defined by $\nabla = (1 - \sum_{k=1}^{2} t_k) \nabla_1 + \sum_{j=1}^{2} t_j \nabla^j$. The connection matrix of $\tilde{\nabla}$ is then given by $\tilde{\theta} = -\sum_{i=1}^{2} t_i / \xi(h_i) M dh_i$ and the curvature matrix $\tilde{K}$ is

$$\tilde{K} = -\sum_{i=1}^{2} dt_i \wedge \frac{dh_i}{\xi(h_i)} M + \text{terms not containing } dt_i.$$

Let $\Delta^2$ be the standard 2-simplex in $\mathbb{R}^2$ and denote by $\beta : Y \times \Delta^2 \to Y$ the projection. By the very definition $\varphi(\nabla_1, \nabla^1, \nabla^2)$ is given by $(2\pi \sqrt{-1})^{-2} \beta_* (\varphi(\tilde{K}))$, where $\beta_* : \Omega^*(Y \times \Delta^2) \to \Omega^{*-2}(Y)$ denotes the integration along the fibers. From the expression of $\tilde{K}$ formula (3.8) follows and we are done. □

**Final Remarks**

1. When $\{ q_\gamma \}$ is such that $q_\gamma \in X'$, then one can take $h_1, \ldots, h_l$ to be part of local coordinates in an open (in $M$) neighborhood of $q_\gamma$. Thus in such a case Theorem (3.1) reduced to [LS, Thm.2], which indeed is the classical Camacho-Sad formula for $l = 1, n = 2$.

2. When $l = n - 1$, instead of a foliation $\mathcal{F}$ of $X$ one can consider a holomorphic self-map $f : X \to X$ which pointwise fixes $Y$. Generically (see [ABT]) this allows to define a one-dimensional foliation $\mathcal{F}_f$ of $Y$ and a holomorphic action $\Xi_f$ of $\mathcal{F}_f$ on $N_{Y', X'}$ outside some “singularities” of $f$ on $Y$. Arguing as in section 2 one has a residue theorem for this case as well, which generalizes [BS, Thm. 2.2] where this result was achieved for $l = 1, n = 2$ and under the assumption that $f$ were a holomorphic self-map of all the ambient $M$ pointwise fixing $P$ as well.

**References**


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