International Journal of Mathematics
Vol. 15, No. 5 (2004) 443–466

RESIDUES FOR SINGULAR PAIRS AND DYNAMICS OF BIHOLOMORPHIC MAPS OF SINGULAR SURFACES

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Received 7 November 2003

Dedicated to Professor César Camacho on his 60th birthday

We prove the existence of a parabolic curve for a germ of biholomorphic map tangent to the identity at an isolated singular point of a surface under some conditions. For this purpose, we present a Camacho–Sad type index theorem for fixed curves of biholomorphic maps of singular surfaces and develop a local intersection theory of curves in singular surfaces from an analytic approach by means of Grothendieck residues.

Keywords: Chern classes; residues; holomorphic self-maps; discrete dynamics; parabolic curves; singular surfaces.

Mathematics Subject Classification 2000: Primary 32H50, 32S65, 37F99; Secondary 32C20, 32A10

0. Introduction

Let X be a two-dimensional complex manifold and $Y \subset X$ a (possibly singular) compact complex curve. Also let Σ be a finite set of points of Y containing the singularities of Y and set $Y_0 = Y \setminus \Sigma$. If there is some *holomorphic action* of the tangent bundle of Y_0 on the normal bundle of Y_0 in X, then, by the Bott vanishing theorem (see [4, 16]), the first Chern form for an appropriate connection of the normal bundle vanishes on Y_0. As a consequence, the first Chern class of the line bundle associated to the divisor Y localizes at Σ and gives rise to a *residue* at each point in Σ. This way we have a residue (index) formula saying that the self-intersection number of Y is given as the sum of these residues.

Applications of this principle are given in the paper [15] (see also [12]) by the second named author (where the action comes from Y being a leaf of a foliation on X) and in the joint work [6] of the first named author with Tovena (where the action comes from Y being the fixed points set of a *non-degenerate* holomorphic map of X into itself). These works were motivated by the earlier paper by Camacho and Sad [8], in the first, and the one by Abate [2], in the second. In these papers, Y was assumed to be non-singular and the index theorems were used to solve problems about dynamics of germs of vector fields or maps. After these, some more generalizations (both for foliations and maps) were proposed; here we refer the interested reader to [16, 3, 5] and the bibliography therein.

In the case where X has singularities at some points of Y, there have been no index theorems available. Usually in applications one is lead to first resolve the singularities of X and then apply the above mentioned index theorems. In fact, this is the way Camacho [7] proves the existence of separatrices for a germ of a vector field at an isolated normal singular point of a two-dimensional variety, under the hypothesis that the dual graph of the resolution is a tree, generalizing the result in [8].

In order to generalize the result of [2], one would be tempted to mimic this strategy even for proving the existence of *petals* for germs of biholomorphisms tangent to the identity at an isolated singular point of a two-dimensional variety (with some hypothesis on the type of singularity allowed). But she/he would fail. Indeed, first, the method of desingularization exploited in [7] cannot be applied directly. Since it is possible in general to lift the biholomorphism only after a one point blow-up and not after the blow-up along a submanifold, and even in the case of one point blow-up, it is not always possible to get a desirable lift (see [1]). Secondly (and mainly) because of this intrinsic difference between maps and vector fields: a vector field always has a singularity at a singular point of X whereas a biholomorphism may not (for the definition of *singularity* of a biholomorphism see Sec. 2). Thus, in order to solve the problem of the existence of petals for biholomorphisms tangent to the identity at a singular point, one is forced to remain on singular varieties and thus has to find a new way.

With the aim of giving an answer to such a problem, in this paper we present an index theorem when X is a singular two-dimensional variety nicely embedded in some complex manifold W.

The setting is as follows. Let W be a complex manifold, $P \subset W$ a non-singular hypersurface and X a surface (two-dimensional subvariety) in W such that, if $Y := X \cap P$, then the singular set $\mathrm{Sing}(X)$ of X is a finite set in Y. Assume P intersects $X^r := X \backslash \mathrm{Sing}(X)$ generically transversally. This allows a natural extension of the normal bundle of the regular part of Y in X, namely the restriction of the normal bundle $N_{P,W}$ of P in W to Y. Moreover assume there exists a biholomorphic map f of W into itself so that $f(X) \subset X$ and $f|_P = \mathrm{Id}_P$ (one might substitute the map f by a one-dimensional foliation leaving X and P invariant obtaining a similar result; however we are not going to discuss further of it in here). Suppose f is *tangential* (or *non-degenerate*) on the non-singular part of Y. Roughly speaking f is tangential on Y if Y is "well-fixed" in X (see Sec. 2), and this condition is fulfilled whenever it is so at only one point. Then we obtain an index theorem (see Theorem 2.2) stating that the evaluation of the first Chern class of $N_{P,W}$ on Y equals the sum of the *residues* of f on Y. As often happens, an index theorem is useful only when the residues are explicitly calculated, and we thus perform an explicit calculation of such residues, see Eq. (2.7).

In order to express the index theorem and to apply it properly in our situation, we develop local and global intersection theories of curves in singular surfaces in Appendix. With these, we have a strict generalization of the Abate–Camacho–Sad index theorem (see Theorem 2.6). Also in Sec. 3, we determine the behavior of our residues under one point blow-ups, see formula (3.3). With such tools at hands we can solve the question about the existence of petals for germs of biholomorphisms tangent to the identity at a *t-absolutely isolated point* of X whose resolution graph is a tree (see Theorem 4.1 and Sec. 1 for the terminology). The proof is strongly based on our index theorem, which allows at each step of the desingularization process to select the "good" points to be blown-up, until we find a nice configuration which determines the existence of petals (see Sec. 4 for details).

1. Preliminaries

Let X be a complex analytic variety embedded in a complex manifold W and suppose that X has only one isolated singularity p. The point p is said to be an *absolutely isolated singularity* if it can be resolved by a finite number of blow-ups. That is to say, if there exist

(1) complex manifolds $W_0 = W, W_1, \dots, W_m$,
(2) varieties $X_0 = X, X_1, \dots, X_m$ such that each $X_j \subset W_j$ has only isolated singularities and X_m is non-singular,
(3) proper holomorphic maps $\pi_j : W_j \to W_{j-1}$ such that each $\pi_j : W_j \to W_{j-1}$ is a blow-up with center a singularity of $X_{j-1} \subset W_{j-1}$ and the variety X_j is the strict transform of X_{j-1} under π_j.

Thus $\pi_1 \circ \cdots \circ \pi_m : X_m \to X$ is a *resolution* of p.

If X is a (abstract) variety, an isolated singularity $p \in X$ is said to be an *absolutely isolated singularity* if p can be resolved by finitely many blow-ups of a local embedding of X near p.

In this paper, we consider absolutely isolated singularities satisfying also the following:

(4) at each step of the above process, X_j intersects generically transversally the exceptional divisor.

In the sequel, an absolutely isolated singularity satisfying (4) is simply called a *t-absolutely isolated singularity*.

Example. The variety $\{(x, y, z) \in \mathbb{C}^3 : x^2 - y^2 + z^3 = 0\}$ has a t-absolutely isolated singularity at the origin 0, whereas the Du Val singularity $\{x^2 + y^3 + z^3 = 0\}$ at 0 is an absolutely isolated singularity which does not satisfy (4). More generally the family $\{x^2 - y^2 + z^{2r+1} = 0\}$ with $r \in \mathbb{N}$ has a t-absolutely isolated singularity at the origin of $\mathbb{C}^3$.

Let X be a surface (two-dimensional variety) and $p \in X$ a t-absolutely isolated singularity. Also, let f be a (germ of) biholomorphic self-map of X such that the point p is an isolated fixed point of f and that $df_p = \mathrm{Id}$; such a germ is sometimes called a *biholomorphism tangent to the identity* at p. Let T_pX be the Zariski tangent space of X at p and assume $T_pX = \mathbb{C}^r$ for some r. Then there exists an embedding $j : U \to \mathbb{C}^r$ of a neighborhood U of p in X with $j(p) = 0$.

Lemma 1.1. *There exists a germ of holomorphic self-map $F : \mathbb{C}^r \to \mathbb{C}^r$ at 0 such that $F \circ j = j \circ f$ and $dF_0 = \mathrm{Id}$.*

Proof. Let $V \subset U$ be an open set with $f(V) \subset U$ and $g : j(V) \to j(U)$ the map defined by $j \circ f \circ j^{-1}|_{j(V)}$. Then g has components defined on the locally closed set $j(V)$ and therefore they extend to some neighborhood of $j(V)$. Let F be the germ defined by such extensions. By definition it follows that $F \circ j = j \circ f$. Thus $dF_0 \circ dj_p = dj_p$. Since dj_p is an isomorphism (for $T_pX = \mathbb{C}^r$), we have $dF_0 = \mathrm{Id}$. □

By Lemma 1.1, we may assume that $X \subset \mathbb{C}^r$ is a germ of surface with a t-absolutely isolated singularity at 0, f is a germ of a biholomorphic self-map of $\mathbb{C}^r$ at 0 such that $f(X) \subset X$, 0 is an isolated fixed point of $f|_X$ and $df_0 = \mathrm{Id}$.

Let $W_0 = \mathbb{C}^r$, $X_0 = X$ and $\pi_1 : W_1 \to W_0$ the quadratic blow-up centered at 0. Let $X_1 := \overline{\pi_1^{-1}(X_0 \backslash \{0\})}$ be the strict transform of X_0, $P_1 := \pi^{-1}(0)$, the exceptional divisor and $Y_1 := X_1 \cap P_1$. Note that Y_1 is a (possibly singular) curve.

Let $\mathrm{Sing}(X_1)$ be the set of singular points of X_1 and $X_1^r := X_1 \backslash \mathrm{Sing}(X_1)$ the regular (non-singular) part. Also let $X_1^{nt} := \{\, p \in X_1^r \cap P_1 : T_pX_1^r \subset T_pP_1 \,\}$ be the set of points where X_1^r and P_1 do not intersect transversally. By hypothesis, X_1^{nt} is a discrete set in X_1^r (in fact a finite set, by the following lemma).

Lemma 1.2. *We have $X_1^r \cap \mathrm{Sing}(Y_1) = X_1^{nt}$.*

Proof. Clearly $X_1^r \cap \mathrm{Sing}(Y_1) \subset X_1^{nt}$, for if $X_1 \pitchfork P_1$ at p, Y_1 is non-singular at p. On the other hand suppose that $p \in X_1^r \cap (Y_1 \backslash \mathrm{Sing}(Y_1))$. Then one can choose local coordinates $(z_1, \ldots, z_r)$ on an open set $U \subset W$ such that $p \in U$, $p = (0, \ldots, 0)$, $X_1 \cap U = \{z_2 = \cdots = z_{r-1} = 0\}$ and $Y_1 \cap U = \{z_1 = \cdots = z_{r-1} = 0\}$. In this coordinate system, let $P_1 \cap U = \{h(z_1, \ldots, z_r) = 0\}$ for some holomorphic function h defined on U. The aim is to show that $\frac{\partial h}{\partial z_1}(0, \ldots, 0) \neq 0$ meaning that $X_1 \pitchfork P_1$ at p. Since $Y_1 \cap U = P_1 \cap X_1 \cap U$, $h(z_1, 0 \ldots, 0, z_r) = z_1^t b(z_1, z_r)$ for some $t \geq 1$ and holomorphic function b such that $b(0, z_r) \not\equiv 0$. Moreover $b(0,0) \neq 0$ for otherwise $p \in \mathrm{Sing}(Y_1)$. If $t > 1$, then, $T_qX_1 \subset T_qP_1$ for any $q \in U \cap Y_1$ (and indeed this would hold for all the connected components of Y_1 containing p) against our assumption on the kind of singularity of X_1. Thus $t = 1$ and $\frac{\partial h}{\partial z_1}(0, \ldots, 0) \neq 0$ as wanted. □

We have $\mathrm{Sing}(X_1) \subset \mathrm{Sing}(Y_1)$, for P_1 is generically transversal to X_1^r. Thus $\mathrm{Sing}(Y_1) = \mathrm{Sing}(X_1) \cup X_1^{nt}$, by Lemma 1.2. We set $X_1' := X_1 \backslash (\mathrm{Sing}(X_1) \cup X_1^{nt})$ and $Y_1' := Y_1 \cap X_1' = Y_1 \backslash \mathrm{Sing}(Y_1)$.

It is possible to define a holomorphic self-map $f^1 : W_1 \to W_1$ near P_1 in such a way that $\pi_1 \circ f^1 = f \circ \pi_1$ and $f^1(v) = df_0(v) = v$ for any $v \in P_1$ (see [1]). Thus $f^1(X_1) \subset X_1$ and f^1 pointwise fixes P_1. We call such a map f^1 a *lift* of f.

Let N_{P_1,W_1} be the normal bundle of P_1 in W_1 so that we have the following exact sequence:

$$0 \to TP_1 \to TW_1|_{P_1} \xrightarrow{\rho} N_{P_1,W_1} \to 0\,.$$

Also, let $N_{Y_1',X_1'}$ be the normal bundle of Y_1' in X_1'. Then it follows from the transversality $X_1' \pitchfork P_1$ that $N_{Y_1',X_1'} = N_{P_1,W_1}|_{Y_1'}$. Thus $N_{Y_1',X_1'}$ has a natural extension over Y_1 given by $N_{P_1,W_1}|_{Y_1}$.

Lemma 1.3. *The map f^1 acts as the identity on $N_{Y_1',X_1'}$.*

Proof. By the previous remark, $N_{Y_1',X_1'} = N_{P_1,W_1}|_{Y_1'}$ and therefore we only need to check that f^1 acts as the identity on N_{P_1,W_1}. To see this, let us introduce local coordinates $(x,y) \in \mathbb{C} \times \mathbb{C}^{r-1}$ around $0 \in W_0$ in such a way that $f = (f_1(x,y), f_2(x,y)) \in \mathbb{C} \times \mathbb{C}^{r-1}$ is given by

$$f_1(x,y) = x + A_h(x,y) + \cdots\,,$$

$$f_2(x,y) = y + B_k(x,y) + \cdots\,,$$

where $A_h : \mathbb{C} \times \mathbb{C}^{r-1} \to \mathbb{C}$ is a non-zero homogeneous polynomial of degree $h \geq 2$ and $B_k : \mathbb{C} \times \mathbb{C}^{r-1} \to \mathbb{C}^{r-1}$ is a vector whose coordinates are (not all zero) homogeneous polynomial of degree $k \geq 2$. Let $(u,v) \in \mathbb{C} \times \mathbb{C}^{r-1}$ be local coordinates on W_1 so that $\pi_1(u,v) = (u, uv)$. In such coordinates, $P_1 = \{u = 0\}$. If $f^1(u,v) = (f_1^1(u,v), f_2^1(u,v)) \in \mathbb{C} \times \mathbb{C}^{r-1}$, from $\pi_1 \circ f^1 = f \circ \pi_1$, we obtain

$$f_1^1(u,v) = u + u^h A_h(1,v) + o(u^h)\,,$$

$$f_2^1(u,v) = v + u^{k-1} B_k(1,v) - v u^{h-1} A_h(1,v) + o(u^{\min(k-1,h-1)})\,.$$

Thus the differential df^1 on P_1 is given by

$$df^1{}_{(0,v)} = \begin{pmatrix} 1 & 0 \\ * & \mathrm{Id} \end{pmatrix}\,,$$

where $* = u^{k-2} B_k(1,v) - u^{h-2} v A_h(1,v)$ evaluated at $u = 0$ (it is 0 if, e.g. $k > 2, h > 2$). In these coordinates, $\rho(\frac{\partial}{\partial u})$ is a frame for N_{P_1,W_1} and the action of f^1 on N_{P_1,W_1} is given by

$$\rho\left(\frac{\partial}{\partial u}\right) \mapsto \rho\left(df^1\left(\frac{\partial}{\partial u}\right)\right) = \rho\left(\frac{\partial}{\partial u}\right),$$

as wanted. $\square$

2. Residues for Singular Pairs

Let W_1, P_1, X_1 and Y_1 be as in the previous section. As noted there, the normal bundle $N_{Y_1',X_1'}$ has an extension N_{P_1,W_1} to P_1. We set $N_{Y_1} = N_{P_1,W_1}|_{Y_1}$. In this section we describe a method to localize the first Chern class of the bundle N_{Y_1} and to define residues at either the "singularities" of f^1 on Y_1' or those of Y_1.

The general setting is as follows. Let W be a complex manifold of dimension r, $P \subset W$ a non-singular hypersurface and X a surface with isolated singularities in W. Suppose P intersects X generically transversely. Let X^r be the non-singular part of X and X^{nt} the set of non-transversal points in $P \cap X^r$, as before. Let Y be a curve in $X \cap P$ (note that Y may not be the entire $X \cap P$). We set $X' := X^r \setminus X^{nt}$ and $Y' := Y \cap X'$ (note that Y' is a non-singular curve). We assume that Y is globally irreducible, for otherwise one can work on each irreducible component separately. Suppose $f : W \to W$ is a holomorphic map such that $f|_P = \mathrm{Id}_P$, $f(X) \subset X$ and that f acts as the identity on the normal bundle $N_{P,W}$ of P in W. In this situation, we consider the first Chern class $c^1(N_Y)$ of the line bundle $N_Y = N_{P,W}|_Y$ and will see that it is localized at the singularities of Y and of $f|_X$ on Y'.

First, in the above situation, we may construct a one-dimensional distribution (with singularities) in $TX|_{Y'}$ as follows. Let V be a neighborhood of a point $p \in Y'$ in W. Shrinking V if necessary, we endow V with local coordinates $(z_1, \ldots, z_r)$ in such a way that $P \cap V = \{z_1 = 0\}$, $X' \cap V = \{z_3 = \cdots = z_r = 0\}$ and hence $Y' \cap V = \{z_1 = z_3 = \cdots = z_r = 0\}$. Let $g := f|_{X'}$. By hypothesis $g(X') \subset X'$ and $g|_{Y'} = \mathrm{Id}$. Moreover the transversality $P \pitchfork X'$ implies that $N_{Y',X'} = N_{P,W}|_{Y'}$, and thus g acts as the identity on $N_{Y',X'}$. In the local coordinates (z_1, z_2) on X', setting $g = (g_1, g_2)$, we may write

$$g_1(z_1, z_2) = z_1 + z_1^{\nu} a_1(z_1, z_2)\,, \qquad g_2(z_1, z_2) = z_2 + z_1^{\mu} a_2(z_1, z_2)\,, \tag{2.1}$$

with $a_1(0, z_2) \not\equiv 0,\ a_2(0, z_2) \not\equiv 0$. Using the coherence of the sheaf of ideals of Y one can show that $\nu_g := \min(\nu, \mu) \geq 1$ is constant on each connected component of Y', and, since Y is assumed to be globally irreducible, it is actually constant on Y'. We call ν_g the *order of g on Y* (see [2] or [6]). Let us define the (local) holomorphic vector field

$$X_g := z_1^{\nu-\nu_g} a_1 \frac{\partial}{\partial z_1} + z_1^{\mu-\nu_g} a_2 \frac{\partial}{\partial z_2} = \frac{z_1 \circ g - z_1}{z_1^{\nu_g}} \frac{\partial}{\partial z_1} + \frac{z_2 \circ g - z_2}{z_1^{\nu_g}} \frac{\partial}{\partial z_2}\,. \tag{2.2}$$

To see what happens under a coordinate change, let $(\hat{z}_1, \ldots, \hat{z}_r)$ be another local coordinate system as above. Then, for $j = 1, 2$, we have

$$\hat{z}_j \circ g - \hat{z}_j = \sum_{k=1}^{2} \frac{\partial \hat{z}_j}{\partial z_k} (z_k \circ g - z_k) + R_{2\nu_g}\,, \tag{2.3}$$

where $R_{2\nu_g}$ denotes a term divisible by $z_1^{2\nu_g}$. Now, there exists a nowhere vanishing holomorphic function C such that $\hat{z}_1 = C(z_1, z_2) z_1$. Therefore by (2.3), if $\hat{X}_g$ denotes the vector field defined as (2.2) in the $\hat{z}$ coordinate system, we have

$$\hat{X}_g = \sum_{j=1}^{2} \frac{\hat{z}_j \circ g - \hat{z}_j}{\hat{z}_1^{\nu_g}} \frac{\partial}{\partial \hat{z}_j} = \frac{1}{C^{\nu_g}} \sum_{j=1}^{2} \frac{z_j \circ g - z_j}{z_1^{\nu_g}} \frac{\partial}{\partial z_j} + R_{\nu_g} = C^{-\nu_g} X_g + R_{\nu_g} \,. \quad (2.4)$$

Thus if we let $v_g := X_g|_{Y'}$, it is determined uniquely up to multiplications by non-vanishing holomorphic functions and the set

$$\mathrm{Sing}(g) := \{\, p \in Y' : v_g(p) = 0 \,\}$$

is well-defined. Also the v_g's define a one-dimensional distribution (without singularities) Ξ_g in $TX|_{Y' \setminus \mathrm{Sing}(g)}$. We say that g is *tangential* (or *non-degenerate*, as in [2, 6]) on Y' if $\Xi_g = TY'$ on $Y' \setminus \mathrm{Sing}(g)$.

Note that, if g is given by (2.1), then g is tangential at p if and only if $\nu > \nu_g$ and $\mu = \nu_g$. Thus we recover the definition of *non-degenerate map* along a curve of fixed points given in [2, 6]. From the coherence of the sheaf of ideals of Y' it follows that g is tangential at one point if and only if it is tangential everywhere on the connected component of Y' containing such a point (see [6]).

From the point of view of dynamics the non-tangential situation is trivial. First we recall that a *parabolic curve* (sometimes called a *petal*) for a biholomorphism $f : W \to W$ at a point $p \in W$ is a holomorphic map $\varphi : \Delta \to W$, where $\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, such that $\varphi \in C^0(\bar{\Delta})$, $p \in \varphi(\partial\Delta)$, $f(\varphi(\Delta)) \subset \varphi(\Delta)$ and for any $\zeta \in \Delta$ it holds $\lim_{k\to\infty} f^{\circ k}(\varphi(\zeta)) = p$. Then we have (see [5]):

Proposition 2.1. *Let M be a two-dimensional complex manifold, $C \subset M$ a non-singular (possibly non-compact) curve and $f : M \to M$ a holomorphic map ($\neq \mathrm{Id}_M$). Suppose $f|_C = \mathrm{Id}_C$, f acts as the identity on the normal bundle $N_{C,M}$ and f is non-tangential on C. Then for every but a discrete set of points in C there exist parabolic curves for f.*

Thus for our aim we assume that g is tangential on Y'. In this case we may write g locally as $g = (g_1, g_2)$ with

$$g_1(z_1, z_2) = z_1 + z_1^{\nu_g+1} b_1(z_1, z_2)\,, \qquad g_2(z_1, z_2) = z_2 + z_1^{\nu_g} b_2(z_1, z_2)\,, \qquad (2.5)$$

where $b_1 = z_1^{\nu-\nu_g-1} a_1$ and $b_2 = a_2$ (cf. (2.1)), and the vector field X_g is given by

$$X_g = z_1 b_1 \frac{\partial}{\partial z_1} + b_2 \frac{\partial}{\partial z_2} \,.$$

Let $Y_0 := Y' \setminus \mathrm{Sing}(g)$. We may define a holomorphic action (see, e.g. [16, Chap. II, 9])

$$\alpha : \Xi_g \times N_{Y_0,X'} \to N_{Y_0,X'}$$

as follows. Let ρ also denote the canonical projection $TX'|_{Y_0} \to N_{Y_0,X'}$. An element $v \in \Xi_g$ is given by $v = \xi X_g|_{Y_0}$ for some C^∞ function ξ; an element $w \in N_{Y_0,X'}$ is given by $w = \rho(\tilde{w}|_{Y_0})$ for some $\tilde{w} \in TX'$. Then we define

$$\alpha(\xi X_g|_{Y_0}, \rho(\tilde{w}|_{Y_0})) := \rho([\xi X_g, \tilde{w}]|_{Y_0})\,. \qquad (2.6)$$

One easily sees that α is independent of the extension $\tilde{w}$ chosen to define it. Also, whereas X_g depends on the local coordinates chosen, α does not. This is clear if $\nu_g > 1$ because of (2.4) and the very definition of α. To see that this is always the case we need a refinement of (2.4) in the tangential case. In such a case, using the notation as in (2.4) and taking into account that $(z_1 \circ g - z_1)$ is divisible by $z_1^{\nu_g+1}$ and $\hat{z}_1 = C(z_1, z_2)z_1$, we have

$$\begin{aligned}\hat{z}_1 \circ g - \hat{z}_1 &= \sum_{j=1}^{2} \frac{\partial \hat{z}_1}{\partial z_j}(z_j \circ g - z_j) + \sum_{j,k=1}^{2} \frac{\partial^2 \hat{z}_1}{\partial z_j \partial z_k}(z_j \circ g - z_j)(z_k \circ g - z_k) + R_{3\nu_g} \\ &= \sum_{j=1}^{2} \frac{\partial \hat{z}_1}{\partial z_j}(z_j \circ g - z_j) + R_{2\nu_g+1}\,.\end{aligned}$$

From this equation and from (2.3), we obtain

$$\hat{X}_g = \sum_{j=1}^{2} \frac{\hat{z}_j \circ g - \hat{z}_j}{\hat{z}_1^{\nu_g}} \frac{\partial}{\partial \hat{z}_j} = C^{-\nu_g} X_g + T_1 + R_2\,,$$

where T_1 is a vector of the form $\kappa(z_1, z_2)v$ with $\kappa(0, z_2) \equiv 0$ and $\rho(v|_{Y_0}) = 0$ and, as usual, R_2 is a vector whose coefficients are divisible by z_1^2. Using this expression in the very definition of α it is easy to see that it is well-defined.

From the holomorphic action α we can define an α-connection for $N_{Y_0,X'}$ on Y_0 (see [16, Chap. II, 9]), that is a connection ∇_0 such that $(\nabla_0)_v(w) = \alpha(v, w)$ for $v \in TY_0 = \Xi_g$ and $w \in N_{Y_0,X'}$. If ∇_0 is an α-connection for $N_{Y_0,X'}$ on Y_0 we have the "Bott vanishing" $c^1(\nabla_0) = 0$, where c^1 is the first Chern polynomial [4, 16]. Now using the Čech-de Rham cohomology we can localize $c^1(N_{P,W}|_Y)$ as follows.

Let U_0 be a tubular neighborhood of Y_0 in P. If ∇_0 is an α-connection for $N_{Y_0,X'}$ on Y_0, we may endow $N_{P,W}$ on U_0 with the connection given by the pull-back of ∇_0 by the retraction, which we also denote by ∇_0. Let U_1 be a neighborhood of $\Sigma := \mathrm{Sing}(Y) \cup \mathrm{Sing}(g)$ in P such that U_1 is the union of disjoint open sets $U_{1,\gamma}$ each of them containing exactly one point — say p_γ — of Σ and that $N_{P,W}$ is trivial on each $U_{1,\gamma}$. Let $\tilde{R}_{1,\gamma} \subset U_{1,\gamma}$ be a small real $2(r-1)$-dimensional closed disk containing p_γ in its interior and such that $\partial\tilde{R}_{1,\gamma}$ intersects transversally Y_0. Let $R_{1,\gamma} := \tilde{R}_{1,\gamma} \cap Y$ and $L_\gamma := \partial R_{1,\gamma}$, which is the link of the singularity p_γ of Y. Let ∇_1 be a connection for $N_{P,W}$ on U_1. Since U_1 is a trivializing set for $N_{P,W}$, we may choose ∇_1 to be trivial on each $U_{1,\gamma}$ with respect to some frame. We have the Bott difference form $c^1(\nabla_0, \nabla_1)$ of the two connections, which is a 1-form on $U_0 \cap U_1$ with $dc^1(\nabla_0, \nabla_1) = c^1(\nabla_1) - c^1(\nabla_0)$. Let $H_D^*(\mathcal{U})$ denote the Čech-de Rham cohomology associated to the covering $\mathcal{U} = \{U_0, U_1\}$ (see, e.g. [16, Chap. II, 3, Chap. IV, 2], [17]). Then $c^1(N_{P,W})$ in $H_D^2(\mathcal{U})$ is represented by the cocycle

$$(c^1(\nabla_0), c^1(\nabla_1), c^1(\nabla_0, \nabla_1)) = (0, 0, c^1(\nabla_0, \nabla_1))\,,$$

which defines a "localization" of $c^1(N_{P,W})$ in the relative Čech-de Rham cohomology $H_D^2(\mathcal{U}, U_0)$. The localization defines a "residue" for each γ and, if Y is compact, the sum of the residues is equal to $\int_Y c^1(N_{P,W})$ (cf. [16, Chap. IV, Theorem 2.4]). Since the last integral may be written as $(X \cap P) \cdot Y$ (see Appendix below), we have the following "residue theorem".

Theorem 2.2. *Let W be a complex manifold, $P \subset W$ a non-singular hypersurface and X a surface with isolated singularities in W. Suppose P intersects X generically transversely. Let Y be a curve in $X \cap P$. Suppose there exists a holomorphic map $f : W \to W$ such that $f|_P = \mathrm{Id}_P$, $f(X) \subset X$ and $f|_X$ is tangential on the non-singular part of Y. Let $\Sigma := \mathrm{Sing}(Y) \cup \mathrm{Sing}(f|_X)$. Then:*

(1) *For each point p_γ in Σ, we have a residue $\mathrm{Res}(f, Y, X \cap P; p_\gamma) \in \mathbb{C}$, which is determined only by the local behavior of f near p_γ and is given by*
$$\mathrm{Res}(f, Y, X \cap P; p_\gamma) = -\int_{L_\gamma} c^1(\nabla_0, \nabla_1)\,.$$

(2) *If Y is compact, Σ is a finite set and we have*
$$\sum_\gamma \mathrm{Res}(f, Y, X \cap P; p_\gamma) = (X \cap P) \cdot Y\,.$$

Remark 2.3. In the above, $(X \cap P) \cdot Y$ denotes the (global) intersection number of the curves $X \cap P$ (which is Cartier) and Y (which may not be Cartier) in X and, as noted above, is equal to $\int_Y c^1(N_{P,W})$ (see Appendix A.4).

In particular, suppose that P is a projective space, as in the blow-up situation. Then, denoting by H a hyperplane in P and by L_H the associated line bundle, we have $N_{P,W} = -L_H$. Thus we may write
$$(X \cap P) \cdot Y = \int_Y c^1(-L_H) = -H \cdot Y = -\mathrm{deg} Y\,.$$

Now we wish to find an explicit expression for $\mathrm{Res}(f, Y, X \cap P; p_\gamma)$. For this purpose, let us recall briefly how the form $c^1(\nabla_0, \nabla_1)$ is defined. Consider the vector bundle $E := N_{P,W} \times \mathbb{R}$ over $(U_1 \cap U_0) \times \mathbb{R}$ and let $\bar{\nabla}$ be the connection for E given by $\bar{\nabla} := (1-t)\nabla_0 + t\nabla_1$. Then $c^1(\nabla_0, \nabla_1) := \beta_*(c^1(\bar{\nabla}))$, where β_* is the integration along the fibers of the projection $\beta : (U_1 \cap U_0) \times [0,1] \to U_1 \cap U_0$.

We may assume that there exists an open set $V_{1,\gamma}$ in W with local coordinates $(z_1, \dots, z_r)$ such that $V_{1,\gamma} \cap P = U_{1,\gamma} = \{z_1 = 0\}$. We may take $\rho(\frac{\partial}{\partial z_1})$ as a frame for $N_{P,W}$ and assume that ∇_1 is $\rho(\frac{\partial}{\partial z_1})$-trivial, i.e. the connection form θ_1 of ∇_1 with respect to $\rho(\frac{\partial}{\partial z_1})$ is zero; $\theta_1 = 0$. Therefore, if $\bar{\theta}$ is the connection form of $\bar{\nabla}$, it follows $\bar{\theta} = (1-t)\theta_0$, where θ_0 is the connection form of ∇_0. Then we compute
$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi}\beta_* d\bar{\theta} = \frac{1}{2\pi\sqrt{-1}}\theta_0\,.$$
By the parameterization theorem we may find a holomorphic function h on $V_{1,\gamma}$ such that the restriction of $dz_1 \wedge dh$ to X' does not vanish on a neighborhood of $Y_0 \cap U_{1,\gamma}$. Thus (z_1, h) are local coordinates on X' and h is a local coordinate on Y',

near each point of $Y_0 \cap U_{1,\gamma}$. Once the pair (z_1, h) is chosen as above, the form $\theta_0|_{Y_0}$ can be calculated from the very definition of the action α. The restriction $g = f|_{X'}$ is given by (2.5) with z_2 replaced by h. Recall that $\rho(\frac{\partial}{\partial z_1})$ is the frame chosen for $N_{P,W}$ on $U_{1,\gamma}$ and θ_0 is defined by $\nabla_0(\rho(\frac{\partial}{\partial z_1})) = \theta_0 \otimes \rho(\frac{\partial}{\partial z_1})$. From (2.6), setting $\xi := 1/b_2(z_1, h)$, we compute

$$(\nabla_0)_{\frac{\partial}{\partial h}} \rho\left(\frac{\partial}{\partial z_1}\right) := \alpha\left(\xi X_g|_{Y_0}, \rho\left(\frac{\partial}{\partial z_1}\right)\right) = \rho\left(\left[\frac{b_1(z_1,h)}{b_2(z_1,h)} z_1 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial h}, \frac{\partial}{\partial z_1}\right]|_{Y_0}\right)$$
$$= -\frac{b_1(0,h)}{b_2(0,h)} \rho(\frac{\partial}{\partial z_1}) .$$

Therefore, noting that $b_1(z_1, h) = \frac{(z_1 \circ f - z_1)|_X}{z_1^{\nu_g+1}}$ and $b_2(z_1, h) = \frac{(h \circ f - h)|_X}{z_1^{\nu_g}}$, we get an expression for θ_0 on $Y_0 \cap U_{1,\gamma}$:

$$\theta_0 = -\frac{(z_1 \circ f - z_1)|_X}{z_1 (h \circ f - h)|_X} dh .$$

Hence we have

$$\mathrm{Res}(f, Y, X \cap P; p_\gamma) = \frac{1}{2\pi\sqrt{-1}} \int_{L_\gamma} \frac{(z_1 \circ f - z_1)|_X}{z_1 (h \circ f - h)|_X} dh . \tag{2.7}$$

Note that the residue as defined above is "additive" with respect to decompositions of Y. Namely, let $Y = \cup Y_\lambda$ be a local decomposition of Y at p_γ, with the Y_λ's curves with no common irreducible components, then

$$\mathrm{Res}(f, Y, X \cap P; p_\gamma) = \sum_\lambda \mathrm{Res}(f, Y_\lambda, X \cap P; p_\gamma) , \tag{2.8}$$

since the link L_γ is a disjoint union of those of the Y_λ's.

Lemma 2.4. *If* $df_{p_\gamma} \neq \mathrm{Id}$ *then* $\mathrm{Res}(f, Y, X \cap P; p_\gamma) = 0$.

Proof. By (2.7), it follows that if $h \circ f - h = c z_1 + O(z_1^2)$ for some function c of $(z_2, \ldots, z_r)$ with $c(p_\gamma) \neq 0$, then $\mathrm{Res}(f, Y, X \cap P; p_\gamma) = 0$. Since $df_{p_\gamma}|_{T_{p_\gamma} P} = \mathrm{Id}$ and $z_1 \circ f - z_1$ is divisible by (at least) z_1^2 (for f is tangential on Y) we have

$$df_{p_\gamma} = \begin{pmatrix} 1 & 0 \\ a & \mathrm{Id} \end{pmatrix} ,$$

for some $r-1$ vector a. Assume that $a_2 \neq 0$. If $h \circ f - h = O(z_1^2)$ then consider the function $\tilde{h}(z) = h(z) + \psi(z_2)$ with ψ holomorphic in a neighborhood of p_γ, depending only on z_2 and $\psi'(p_\gamma) \neq 0$. Then $\tilde{h} \circ f - \tilde{h} = c z_1 + O(z_1^2)$ with $c(p_\gamma) \neq 0$ and for a suitable choice of ψ we can still assume that $d\tilde{h} \wedge dz_1|_{X'} \neq 0$ near Y^0. Calculating the residue as in (2.7) using $\tilde{h}$ instead of h we get the result. □

Remark 2.5. Suppose that p_γ is a non-singular point of X. If z_1 is a defining equation of Y in X, i.e. if $Y = X \cap P$ near p_γ, then the residue in (2.7) coincides with the index $\mathrm{Ind}(f, Y; p_\gamma)$ defined in [2] in case Y is non-singular at p_γ, and with the one defined in [6, Definition 6] in case Y is singular at p_γ (see also [5]). More generally if p_γ is a non-singular point of X, the residue coincides with $\mathrm{Ind}(f, Y, X \cap P; p_\gamma)$ in [6, Definition 5]. It would be interesting but seems to be difficult to define an "absolute" residue on Y when p_γ is a singular point of X. We can do it in case the singularity is absolutely isolated as we now show.

Assume we are in the hypotheses of Theorem 2.2 and assume we can write $X \cap P = Y \cup (\bigcup_{j=1}^s Y_j)$ as union of finitely many components. Let p_γ be an absolutely isolated singularity of X. Then by Appendix A.6, we can well define the local intersection numbers $(Y \cdot Y_j)_{p_\gamma}$ and, if Y is compact, the global intersection numbers $Y \cdot Y$ and $Y \cdot Y_j$, for $j = 1, \ldots, s$. Thus in such a situation we define:

$$\mathrm{Res}(f, Y; p_\gamma) := \mathrm{Res}(f, Y, X \cap P; p_\gamma) - \sum_{j=1}^{s} (Y \cdot Y_j)_{p_\gamma} . \tag{2.9}$$

Since

$$(X \cap P) \cdot Y = Y \cdot Y + \sum_{j=1}^{s} Y \cdot Y_j ,$$

if Y is compact, from Theorem 2.2, we have

Theorem 2.6. *In the situation of Theorem* 2.2, *suppose also that X has only absolutely isolated singularities on Y. Then*:

(1) *For each point p_γ in Σ, we have a residue* $\mathrm{Res}(f, Y; p_\gamma) \in \mathbb{C}$, *which is determined only by the local behavior of f near p_γ and is given by* (2.9).
(2) *If Y is compact*, Σ *is a finite set and we have*

$$\sum_{\gamma} \mathrm{Res}(f, Y; p_\gamma) = Y \cdot Y .$$

Explicit formulas for $\mathrm{Res}(f, Y; p_\gamma)$ are obtained using formulas (2.7) and (A.1). We leave the actual calculation to the interested reader. If X is non-singular, the above theorem reduces to [6, Theorem 2], see also [5, Theorem 6.2].

3. Behavior of Residues under Blow-Ups

In this section we examine how the residues introduced in the previous section behave under blow-ups. We use the same notation as in Sec. 2. Let $p \in \Sigma$ be such that $\mathrm{Res}(f, Y, X \cap P; p) \neq 0$.

Let $\pi : \tilde{W} \to W$ be the blowing-up at p. Let $D := \pi^{-1}(p)$. Thus $\pi : \tilde{W} \backslash D \to W \backslash \{p\}$ is biholomorphic. Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{P}$ be the strict transforms of X, Y and P, respectively. By construction, $\tilde{Y}$ is a curve in $\tilde{X} \cap \tilde{P}$ satisfying the conditions as Y in the previous section. In view of Lemma 2.4, we have $df_p = \mathrm{Id}$. Thus there exists a holomorphic map $\tilde{f} : \tilde{W} \to \tilde{W}$ such that $\tilde{f}|_D = \mathrm{Id}$ and $\pi \circ \tilde{f} = f \circ \pi$. Note that $\tilde{f}|_{\tilde{X}'}$ is tangential on $\tilde{Y}'$. First suppose that Y is irreducible at p. Thus $\tilde{Y} \cap D := \{q\}$. We wish to calculate $\mathrm{Res}(\tilde{f}, \tilde{Y}, \tilde{X} \cap \tilde{P}; q)$. To do that, let us introduce coordinates

$(z_1, \ldots, z_r)$ in a neighborhood V of p in W so that $P \cap V = \{z_1 = 0\}$, the restriction of $dz_1 \wedge dz_r$ to X' does not vanish near $Y \cap V$ and the hyperplane $z_r = 0$ is general with respect to Y (see Appendix A.2). Then z_r is a local coordinate around each point in Y'. Let $(w_1, \ldots, w_{r-1}, u)$ be local coordinates on $\tilde{W}$ in such a way that $\pi(w_1, \ldots, w_{r-1}, u) = (uw_1, \ldots, uw_{r-1}, u)$. Thus $\tilde{P} = \{w_1 = 0\}$ and $D = \{u = 0\}$. Moreover the restriction of $dw_1 \wedge du$ to $\tilde{X}' \backslash \{q\}$ does not vanish near $\tilde{Y}$ for, since the restriction $\pi|_{\tilde{X}'}$ of π is biholomorphic outside D,

$$0 \neq (\pi|_{\tilde{X}'})^*(dz_1 \wedge dz_r) = d(uw_1) \wedge du = udw_1 \wedge du\,.$$

Thus by (2.7), we have

$$\mathrm{Res}(\tilde{f}, \tilde{Y}, \tilde{X} \cap \tilde{P}; q) = \frac{1}{2\pi\sqrt{-1}} \int_{\tilde{L}} \frac{(w_1 \circ \tilde{f} - w_1)|_{\tilde{X}}}{w_1(u \circ \tilde{f} - u)|_{\tilde{X}}} du\,,$$

where $\tilde{L}$ is the link of the singularity q of $\tilde{Y}$. Note that $\tilde{L} = \pi^* L$, with L the link of the singularity p of Y. Since $w_1 = u^{-1}(z_1 \circ \pi)$ and $\tilde{f} = \pi^{-1} \circ f \circ \pi$ outside D we have (omitting to write $|_{\tilde{X}}$ in the integrands)

$$\begin{aligned}
&\int_{\tilde{L}} \frac{w_1 \circ \tilde{f} - w_1}{w_1(u \circ \tilde{f} - u)} du \\
&\quad = \int_{\pi^* L} \frac{z_1 \circ f \circ \pi}{z_1 \circ \pi} \cdot \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du + \int_{\pi^* L} \pi^* \left(\frac{z_1 \circ f - z_1}{z_1(z_r \circ f - z_r)} dz_r \right) \\
&\quad = \int_{\pi^* L} \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du + 2\pi\sqrt{-1} \cdot \mathrm{Res}(f, Y, X \cap P; p)\,. \qquad (3.1)
\end{aligned}$$

Now $\tilde{f}|_{\tilde{X}} = \mathrm{Id} + w_1 H$ for some holomorphic map $H = (H_1, H_2)$ near $\tilde{Y}$. Thus, indicating by $\langle .,. \rangle$ the scalar product between vectors, we have

$$\frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} = \frac{w_1 \langle \partial u^{-1}, H \rangle + o(w_1)}{u^{-1} w_1 \langle \partial u, H \rangle + o(w_1)} = -\frac{1 + o(1)}{u + o(1)}\,.$$

Therefore

$$\int_{\pi^* L} \frac{u^{-1} \circ \tilde{f} - u^{-1}}{u^{-1}(u \circ \tilde{f} - u)} du = -\int_{\pi^* L} \frac{du}{u} = -\int_L \frac{dz_r}{z_r} = -2\pi\sqrt{-1} \cdot m(Y, p)\,, \qquad (3.2)$$

where $m(Y, p) \geq 1$ is the multiplicity of Y at p (see Appendix A.2). From (3.1) and (3.2), we obtain

$$\mathrm{Res}(\tilde{f}, \tilde{Y}, \tilde{X} \cap \tilde{P}; q) = \mathrm{Res}(f, Y, X \cap P; p) - m(Y, p)\,. \qquad (3.3)$$

Remark 3.1. In case $p \in X^r$, we have $(z_1 \circ \pi)|_{\tilde{X}} = u^m v$, where v is a defining function of $\tilde{Y}$ in $\tilde{X}$ and $m = m(Y, p)$. Thus $w_1|_{\tilde{X}} = u^{m-1} v$. Therefore, arguing as before,

$$\begin{aligned}
\mathrm{Res}(\tilde{f}, \tilde{Y}, \tilde{X} \cap \tilde{P}; q) &= \frac{1}{2\pi\sqrt{-1}} \left(\int_{\tilde{L}} \frac{v \circ \tilde{f} - v}{v(u \circ \tilde{f} - u)} du + (m(Y, p) - 1) \int_{\tilde{L}} \frac{du}{u} \right) \\
&= \mathrm{Ind}(\tilde{f}, \tilde{Y}; q) + m(Y, p)(m(Y, p) - 1)\,,
\end{aligned}$$

where $\mathrm{Ind}(\tilde{f}, \tilde{Y}; q)$ is the index defined in [6]. Therefore in such a case, from Remark 2.5 and (3.3), we recover [6, (5)], that is

$$\mathrm{Ind}(\tilde{f}, \tilde{Y}; q) = \mathrm{Ind}(f, Y; p) - m(Y, p)^2 .$$

Now suppose Y may not be irreducible at p and let $Y = \cup_{\lambda=1}^{N} Y_\lambda$ be the irreducible decomposition at p. Then the link L is a disjoint union $L = \cup_{\lambda=1}^{N} L_\lambda$, with L_λ the link of Y_λ, and we have the identity (2.8). Let $m(Y_\lambda, p)$ be the multiplicity of Y_λ at p. We set $\tilde{Y} \cap D = \{q_1, \ldots, q_s\}$ (note that $N \geq s$). Then from the relation obtained by replacing Y (respectively $\tilde{Y}$) with Y_λ (respectively $\tilde{Y}_\lambda$) in (3.3) and the identity (2.8), we have

$$\sum_{j=1}^{s} \mathrm{Res}(\tilde{f}, \tilde{Y}, \tilde{X} \cap \tilde{P}; q_j) = \mathrm{Res}(f, Y, X \cap P; p) - \sum_{\lambda=1}^{N} m(Y_\lambda, p) . \tag{3.4}$$

4. Existence of Parabolic Curves for t-absolutely Isolated Singularities whose Resolution Graph is a Tree

The aim of this section is to prove the following result:

Theorem 4.1. *Let X be a surface with an irreducible t-absolutely isolated singularity $p \in X$ whose resolution graph is a tree. Let $f : X \to X$ be a holomorphic map such that p is an isolated fixed point of f and $df_p = \mathrm{Id}$. Then there exists at least one parabolic curve for f at p.*

By the consideration of Sec. 1, we may assume that $X \subset \mathbb{C}^r = W_0$. Blow up the point p. We find an r-dimensional complex manifold W_1, a holomorphic map $f_1 : W_1 \to W_1$ and an $(r-1)$-dimensional projective space $P_1 \subset W_1$ such that $f_1|_{P_1} = \mathrm{Id}_{P_1}$ and f_1 acts as the identity on N_{P_1, W_1}. Moreover there is a surface $X_1 \subset W_1$, the strict transform of X, so that $f_1(X_1) \subset X_1$, $\mathrm{Sing}(X_1) \cup X_1^{nt}$ is a finite set contained in $X_1 \cap P_1$ and $X_1' := X_1 \backslash (\mathrm{Sing}(X_1) \cup X_1^{nt})$ intersects P_1 transversally.

If $f_1|_{X_1 \cap P_1}$ is non-tangential on the non-singular part of some irreducible component of $X_1 \cap P_1$, there exist infinitely many parabolic curves for f_1 contained in X_1 by Proposition 2.1. Those parabolic curves project down to parabolic curves in X for f at p. Thus assume f_1 is tangential on the non-singular part of each irreducible component of $X_1 \cap P_1$. Let us write $X_1 \cap P_1 = \cup_{k_1=1}^{s_1} Y_{1,k_1}$ with each Y_{1,k_1} globally irreducible (note that each Y_{1,k_1} is also locally irreducible for the hypothesis on the dual graph of the resolution of X at p).

We first recall the following result from [2, 5]:

Proposition 4.2. *Assume f is a germ of holomorphic self map of $\mathbb{C}^2$ at 0. Let E be the set of fixed points of f at 0. Suppose that E is a non-singular curve passing through 0 and f is tangential on E. If $\mathrm{Res}(f, E; 0) \notin \mathbb{Q}^+ \cup \{0\}$ then there exists at least one parabolic curve for f at 0.*

Therefore if there exists a point q belonging to the non-singular part of exactly one component Y_{1,k_1} such that $\mathrm{Res}(f_1, Y_{1,k_1}, X_1 \cap P_1; q) \notin \mathbb{Q}^+ \cup \{0\}$, then by Remark 2.5 and Proposition 4.2, there exists a parabolic curve for f_1 at q which projects down to a parabolic curve for f at p.

Thus we may assume that all points where the residue is not rational positive nor zero are contained in $\mathrm{Sing}(X_1) \cup X_1^{nt}$. Now the idea is to blow-up one by one the points for which the residue is not positive rational nor zero (note that, by Theorem 2.2, there always exists such a point and that, by Lemma 2.4, we can well define the lift of f_1 at that point).

We show that, after a finite number of blow-ups, we find either a component of an irreducible curve where the lift of f_1 is non-tangential or a point where we can apply Proposition 4.2. The proof will be by contradiction.

As a matter of notation, let us denote by X_j the strict transform of X at the jth blow-up and by P_i the exceptional divisor born at the ith blow-up and all its strict transforms at the subsequent blow-ups. Also, let us denote by $\cup_{k_j=1}^{s_j} Y_{j,k_j}$ the union in irreducible components of the intersection between X_j and P_j at the jth blow-up. We also denote by the same symbol Y_{j,k_j} the strict transform of the curve Y_{j,k_j} after blow-ups. Note that, for $j > i$, $X_j \cap P_i$ (the intersection of X_j and the strict transform of P_i) is equal to $\cup_{k_i=1}^{s_i} Y_{i,k_i}$ (see Lemma A.6 below). We also denote by f_j the lift of f at the jth blow-up.

As said, we blow up a point $q \in X_j$ if

$$\mathrm{Res}(f_j, Y_{i,k_i}, X_j \cap P_i; q) \notin \mathbb{Q}^+ \cup \{0\}\,,$$

for some $i \leq j$ and $k_i \in \{1, \ldots, s_i\}$. We remark once more that by Lemma 2.4, it follows that $(df_j)_q = \mathrm{Id}$ and therefore we can well define the lift f_{j+1}. Note also that by (3.3), if $\tilde{q}$ is the point of intersection between the strict transform of Y_{i,k_i} and the exceptional divisor P_{j+1},

$$\mathrm{Res}(f_{j+1}, Y_{i,k_i}, X_{j+1} \cap P_i; \tilde{q}) \notin \mathbb{Q}^+ \cup \{0\}\,.$$

Therefore if we do not find either a component of an irreducible curve where the blow-up of f_j is non-tangential or a point where we can apply Proposition 4.2, we keep on blowing up.

Notice that if $q \in Y_1 \cap Y_2$ for some $Y_1, Y_2 \in \{Y_{i,k_i}\}$, $i = 1, \ldots, j$, $k_i = 1, \ldots, s_i$, but $q \notin \mathrm{Sing}(X_j)$ then necessarily $(df_j)_q = \mathrm{Id}$. One way to see this is to consider the two residues $r_1 := \mathrm{Res}(f_j, Y_1; q)$ and $r_{12} := \mathrm{Res}(f_j, Y_1, Y_1 \cup Y_2; q)$, which coincide with the ones defined in [6] for X is non-singular at q, see Remark 2.5. Since $r_{12} = r_1 + (Y_1 \cdot Y_2)_q$, r_1 and r_{12} cannot be both equal to zero. However, if $(df_j)_q \neq \mathrm{Id}$, then by Lemma 2.4 it would follow $r_1 = r_{12} = 0$. Thus $(df_j)_q = \mathrm{Id}$ and we can freely blow-up at q.

Then we can assume that after a certain number — say m — of blow-ups, the strict transform X_m contains a family of irreducible curves $\{Y_{j,k_j}\}$, $j = 1, \ldots, r$, $k_j = 1, \ldots, s_j$ such that, outside $\mathrm{Sing}(X_m)$, the family has only normal crossing intersections, i.e. if $q \notin \mathrm{Sing}(X_m)$ and $q \in Y_{i,k_i}$ then there exists at most one curve

Y_{j,k_j} such that Y_{i,k_i} intersects transversally Y_{j,k_j} at q. Moreover if q is such a point of transversal intersection between two components we can define (as in (2.9)) the intrinsic residue $\mathrm{Res}(f_m, Y_{j,k_j}; q)$ by subtracting from $\mathrm{Res}(f_m, Y_{j,k_j}, X_m \cap P_j; q)$ the "excess" given by the intersection number at q between $(X_m \cap P_j)\backslash Y_{j,k_j}$ and Y_{i,k_i}. Note that if $\mathrm{Res}(f_m, Y_{j,k_j}, X_m \cap P_j; q) \notin \mathbb{Q}^+ \cup \{0\}$ then this is so for $\mathrm{Res}(f_m, Y_{j,k_j}; q)$. Also, by Remarks 2.5 and 3.1, it follows that $\mathrm{Res}(f_m, Y_{j,k_j}; q)$ is the index defined in [2, 5].

In particular, blowing up some more if necessary, we can exploit the *reduction theorem of singularities for* f_m (see [2, Theorem 2.3], [5, Theorem 3.3, Lemma 7.2]) which, in our settings, reads as follows. If $q \notin \mathrm{Sing}(X_m)$ and $q \in Y_{i,k_i} \cap Y_{j,k_j}$ then either

$$\mathrm{Res}(f_m, Y_{i,k_i}; q) \cdot \mathrm{Res}(f_m, Y_{j,k_j}; q) = 1\,, \qquad \text{or} \qquad (*_1)$$

$$\mathrm{Res}(f_m, Y_{i,k_i}; q) = 0\,, \qquad \mathrm{Res}(f_m, Y_{j,k_j}; q) \neq 0\,. \qquad (*_2)$$

Summing up, we have a surface X_m, a holomorphic map $f_m : X_m \to X_m$ fixing a family of irreducible curves $\{Y_{j,k_j}\}$ and being tangential on the non-singular part of such curves. Outside $\mathrm{Sing}(X_m)$, the family $\{Y_{j,k_j}\}$ has only normal crossing intersections. Also, by the hypothesis on the dual graph of the resolution, each two curves intersect each other in at most one point (three or more curves might intersect at one point only if such a point is in $\mathrm{Sing}(X_m)$). The residues of f_m at points not belonging to the normal crossing intersections of the family $\{Y_{j,k_j}\}$ are all positive rational or zero, while the intrinsic residues at crossings are of type $(*_1)$ or $(*_2)$. Note that the residue at each point in $\mathrm{Sing}(X_m)$ is also positive rational or zero so that, even if we blow up the point, we may not be able to lift f_m.

If $\mathrm{Sing}(X_m) = \emptyset$, one might argue as in the proof of [7, Proposition 3.3] to find a contradiction. However in general we have points in $\mathrm{Sing}(X_m)$ which might not be resolved (for the map cannot be lifted there) and we have to argue differently.

We define a new family $\{Z_i\}$ from the family $\{Y_{j,k_j}\}$ as follows. We say that $Y_{j_1,k_{j_1}}, \ldots, Y_{j_t,k_{j_t}}$ form a *chain* if $j_1 = \cdots = j_t$ and there exist $\{q_1, \ldots, q_{t-1}\} \subset \mathrm{Sing}(X_m)$ such that $Y_{j_1,k_{j_1}} \cap Y_{j_2,k_{j_2}} = \{q_1\}$, …, $Y_{j_{t-1},k_{j_{t-1}}} \cap Y_{j_t,k_{j_t}} = \{q_{t-1}\}$. We give an equivalence relation on $\{Y_{j,k_j}\}$ saying that $Y_{i,k_i} \sim Y_{j,k_j}$ if there exists a chain joining Y_{i,k_i} and Y_{j,k_j} (also, by definition, each curve is equivalent to itself). In particular if two curves are equivalent then they were born at the same blow-up. We define the Z_j's to be the union of equivalent curves. Thus, for instance, Z_1 is the union of $Y_{1,1}$ and all the curves Y_{1,k_1} equivalent to $Y_{1,1}$. Note that by hypotheses on the dual graph of the resolution, for any $i \neq j$, Z_j intersects Z_i in at most one point (but three or more Z_j's might intersect at one point of $\mathrm{Sing}(X_m)$).

Now we want to select a "good" subfamily of $\{Z_j\}$. We say that Z_j is *younger than* Z_i if there exists $q \in \mathrm{Sing}(X_m)$ such that $Z_i \cap Z_j = \{q\}$ and if $Y_{j,k_j} \subset Z_j$, $Y_{i,k_i} \subset Z_i$ it follows that $j > i$. Note that given Z_i, Z_j, either $Z_i \cap Z_j \notin \mathrm{Sing}(X_m)$ or Z_i is younger than Z_j or Z_j is younger than Z_i. In other words we say that Z_j is younger than Z_i if they intersect at one singular point of X_m and Z_j is made of curves generated by a blow-up happened after the blow-up which generated the curves contained in Z_i.

Remark 4.3. By definition, if the element Z_j is the union of curves generated at the ith blow-up and $q \in \mathrm{Sing}(X_m) \cap Z_j$, then Z_j is the youngest element at q if and only if $Z_j = X_m \cap P_i$ at q.

Lemma 4.4. *There exists a subfamily $\mathcal{B} = \{Z_{j_a}\}$ of $\{Z_j\}$ such that*

(1) $\bigcup Z_{j_a}$ *is connected*,
(2) Z_{j_a} *is not younger than* Z_{j_b} *for any* j_a, j_b,
(3) *if* $Z_i \notin \mathcal{B}$ *intersects* $Z_j \in \mathcal{B}$ *at* q, *then* $q \in \mathrm{Sing}(X_m)$ *and* Z_j *is younger than* Z_i.

Proof. We prove by induction on the number N of singularities of X_m contained in at least two different Z_j's. If $N = 0$, then we are done with $\mathcal{B} = \{Z_j\}$. Suppose $N > 0$ and let q be a singularity of X_m contained in two (or more) different Z_j's. Let Z_{j_0} be the youngest element passing through q. Then discard all the other Z_k's containing q. Since the dual graph of the resolution of the singularity is a tree, eliminating those Z_k's divides the dual graph of $\{Z_j\}$ into (at least) two connected components. Take the one containing Z_{j_0}, where we have at most $N-1$ singularities of X_m contained in (at least) two different Z_j's. □

Note that two elements of $\mathcal{B}$ either do not intersect or intersect transversally at one point; thus the dual graph of $\mathcal{B}$ is a tree. Also, by Corollary A.8, the intersection matrix $(Z_{j_a} \cdot Z_{j_b})$ is negative definite.

Now, for any element Z_{j_a} of the family $\mathcal{B}$, we define the intrinsic residue at a point $q \in Z_{j_a}$ as follows. If $q \notin \mathrm{Sing}(X_m)$ then $\mathrm{Res}(f_m, Z_{j_a}; q) = \mathrm{Res}(f_m, Y_{j,k_j}; q)$, provided $q \in Y_{j,k_j} \subset Z_{j_a}$ (by construction there is only one curve in Z_{j_a} passing through q in such a case). If $q \in \mathrm{Sing}(X_m)$ then

$$\mathrm{Res}(f_m, Z_{j_a}; q) = \sum_{k_j \,|\, q \in Y_{j,k_j} \subset Z_{j_a}} \mathrm{Res}(f_m, Y_{j,k_j}, X_m \cap P_j; q)\,,$$

which is rational positive or 0. By Remark 4.3 and Theorem 2.6, for any $Z_{j_a} \in \mathcal{B}$ we have then

$$\sum_{q \in Z_{j_a}} \mathrm{Res}(f_m, Z_{j_a}; q) = Z_{j_a} \cdot Z_{j_a}\,. \tag{4.1}$$

Now we can argue as in the [7, Proof of Proposition 3.3] to reach the contradiction. We recall such an argument here for the reader convenience.

Fix an element Z in $\mathcal{B}$. Since the dual graph of $\mathcal{B}$ is a tree, we can give it an order so that Z is the maximal element. We define the level of each element in $\mathcal{B}$ as in [7] and suppose Z is at level N (that is, the highest). In general, an element Z_{j_a} is at level ℓ, if the (only) element that is greater than Z_{j_a} and is intersecting with (is connected to) Z_{j_a} is at level $\ell + 1$. The minimal elements are at level 0 (that is, the lowest). In what follows, we denote by Z_{ℓ,k_ℓ}, $k_\ell = 1, 2, \ldots$, the elements at level ℓ. Since the matrix $(Z_{j_a} \cdot Z_{j_b})$ is negative definite and either Z_{j_a} does not intersect Z_{j_b} or Z_{j_a} intersects transversally Z_{j_b} at one point, by [7, Proposition 2.1], there exists a function $h : \mathcal{B} \to \mathbb{R}^-$ such that

(1) $h(Z_{\ell,k_\ell}) < 0$, for any ℓ, k_ℓ,
(2) $h(Z_{0,k_0}) = Z_{0,k_0} \cdot Z_{0,k_0}$, for any k_0,
(3) if $\ell \geq 1$, $h(Z_{\ell,k_\ell}) = Z_{\ell,k_\ell} \cdot Z_{\ell,k_\ell} - \sum_{k=1}^{t} \frac{1}{h(Z_{\ell-1,k})}$, for any k_ℓ, where $Z_{\ell-1,k}$, $k = 1, \ldots, t$, are all the elements at level $\ell - 1$ intersecting with Z_{ℓ,k_ℓ}.

Recall again that all the residues of f_m outside the points of intersections of two Z_{j_a}'s are rational positive or zero. Let Z_{1,k_1} be an element at level one and let $Z_{0,k}$, $k = 1, \ldots, t$, be all the elements at level 0 intersecting with Z_{1,k_1}. We denote by q_k the points of intersection of Z_{1,k_1} and $Z_{0,k}$. By (4.1) it follows that

$$\mathrm{Res}(f_m, Z_{0,k}; q_k) \leq Z_{0,k} \cdot Z_{0,k} = h(Z_{0,k}).$$

Since the residues at q_k satisfy $(*_1)$ or $(*_2)$, for $k = 1, \ldots, t$,

$$\mathrm{Res}(f_m, Z_{1,k_1}; q_k) \geq \frac{1}{h(Z_{0,k})}\,.$$

Let Z_{2,k_2} be the element at level two intersecting with Z_{1,k_1} and q the point of intersection. By (4.1), it follows

$$\begin{aligned}\mathrm{Res}(f_m, Z_{1,k_1}; q) &\leq Z_{1,k_1} \cdot Z_{1,k_1} - \sum_{k=1}^{t} \mathrm{Res}(f_m, Z_{1,k_1}; q_k)\\ &\leq Z_{1,k_1} \cdot Z_{1,k_1} - \sum_{k=1}^{t} \frac{1}{h(Z_{0,k})} = h(Z_{1,k_1})\,.\end{aligned}$$

Thus

$$\mathrm{Res}(f_m, Z_{2,k_2}; q) \geq \frac{1}{h(Z_{1,k_1})}\,.$$

Proceeding by induction, if $Z_{N-1,k}$, $k = 1, \ldots, s$, are the elements at level $N-1$ (each of them intersects with Z) and if p_k is the point of intersection of Z and $Z_{N-1,k}$,

$$\sum_{k=1}^{s} \mathrm{Res}(f_m, Z; p_k) \geq \sum_{k=1}^{s} \frac{1}{h(Z_{N-1,k})} = Z \cdot Z - h(Z) > Z \cdot Z\,,$$

which contradicts (4.1), proving the theorem.

Appendix A. Intersection Theory on Singular Surfaces

In the sequel, a variety will be a reduced analytic space. A curve or a surface will be a variety of pure dimension one or two, respectively. For a subvariety V and a divisor D in a complex manifold W, we denote by $D \cdot V$ the pull-back $\iota^* D$ of

D by the embedding $\iota : V \hookrightarrow W$. We use the symbol $\cap$ to denote set theoretic intersections.

A.1. *Grothendieck residues relative to a subvariety*

Let U be a neighborhood of 0 in $\mathbb{C}^r$ and V a subvariety of pure dimension n in U which contains 0 as at most an isolated singular point. Also, let $f_1, \ldots, f_n$ be holomorphic functions on U with $\bigcap_{i=1}^n \{p \in U : f_i(p) = 0\} \cap V = \{0\}$. For a holomorphic n-form ω on U, the *Grothendieck residue relative to* V is defined by

$$\mathrm{Res}_0 \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix}_V = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_\Gamma \frac{\omega}{f_1 \cdots f_n},$$

where Γ is an n-cycle in V defined by $\Gamma = \bigcap_{i=1}^n \{p \in U : |f_i(p)| = \varepsilon_i\} \cap V$ with ε_i small positive numbers (cf. [16, Chap. IV, 8], [18]).

A.2. *Multiplicities*

Let V be as above and let $C_0(V)$ denote the tangent cone of V at 0. Recall that $C_0(V)$ is an analytic space whose support is the zero set of all the leading homogeneous polynomials of germs in the ideal of V at 0, and has the same dimension as V (see, e.g. [19]). We say that a collection of hyperplanes $(H_1, \ldots, H_i)$ through 0, $1 \le i \le n$, is general with respect to V if $\dim C_0(V) \cap H_1 \cap \cdots \cap H_i = n - i$.

We define the *multiplicity* of V at 0 by

$$m(V, 0) = \mathrm{Res}_0 \begin{bmatrix} d\ell_1 \wedge \cdots \wedge d\ell_n \\ \ell_1, \ldots, \ell_n \end{bmatrix}_V ,$$

where $\ell_1, \ldots, \ell_n$ denote defining linear functions of n hyperplanes general with respect to V. This definition of multiplicity coincides with the one in [9, p. 79]:

Lemma A.1. *Let $PC_0(V)$ denote the projective cone of V at* 0 (*which is in* $\mathbb{P}^{r-1}$). *Then*

$$m(V, 0) = \deg PC_0(V) .$$

We give a proof in the case necessary for our purpose in A5 below (a similar proof works in general case).

A.3. *Intersections, local theory*

Let X be a surface in a small neighborhood U of 0 in $\mathbb{C}^r$ possibly with an isolated singularity at 0. Let D_1 and D_2 be (effective, for simplicity) Cartier divisors on X. Defining functions for D_1 and D_2 are the restrictions of holomorphic functions f_1 and f_2 on U. Suppose f_1 and f_2 have no common irreducible factors at 0. Then the *intersection number* of D_1 and D_2 at 0 is defined by

$$(D_1 \cdot D_2)_0 = \mathrm{Res}_0 \begin{bmatrix} df_1 \wedge df_2 \\ f_1, f_2 \end{bmatrix}_X .$$

In particular, we may write

$$m(X, 0) = (H_1' \cdot H_2')_0\,,$$

where (H_1, H_2) is a pair of hyperplanes general with respect to X and $H_i' = H_i \cdot X$, $i = 1, 2$ (note that H_i' may have a non-reduced structure).

If D is a Cartier divisor defined by f and if Y is a Cartier curve, by the projection formula, we have

$$(D \cdot Y)_0 = \mathrm{Res}_0 \begin{bmatrix} df \\ f \end{bmatrix}_Y\,,$$

which may be used to define the intersection number of D and Y, even if Y is not Cartier. In particular, for a curve Y (which may not be Cartier) in X,

$$m(Y, 0) = (H' \cdot Y)_0,$$

where H is a hyperplane general with respect to Y and X, and $H' = H \cdot X$. From the above arguments, we have

Lemma A.2. *Let P be a non-singular hypersurface (through 0) in a neighborhood of 0 in $\mathbb{C}^r$. If P intersects transversely $X \backslash \{0\}$ near 0 and set $Y = X \cap P$, then*

$$m(Y, 0) = m(X, 0)\,.$$

A.4. *Intersections, global theory*

Let X be a surface with isolated singularities in a complex manifold W. Let D be a Cartier divisor on X and denote by L_D the associated line bundle over X. The bundle L_D admits a canonical section s and its first Chern class $c^1(L_D)$ is localized at the support $|D|$ of D. We denote the localization by $c^1(L_D, s)$, which is in $H^2(X, X \backslash |D|)$ (cf. [16–18]). If $|D|$ is compact, we have the Alexander homomorphism

$$H^2(X, X \backslash |D|) \to H_2(|D|)\,,$$

which sends $c^1(L_D, s)$ to the class $[D]$ of the Weil divisor associated to D (this can be proved in the framework of the Čech-de Rham cohomology as in [17]).

Let D_1 and D_2 be Cartier divisors on X such that the support of at least one of them, say D_2, is compact (X may not be compact). Then the (global) intersection number of D_1 and D_2 in X is defined by

$$D_1 \cdot D_2 = \int_X c^1(L_{D_1}) \cdot c^1(L_{D_2}, s_2)\,.$$

In the algebraic category this definition coincides with the one in [9]. We may also write

$$D_1 \cdot D_2 = c^1(L_{D_1}) \frown [D_2]\,,$$

which makes sense even if D_2 is not Cartier, but only Weil. If D_1 extends to a divisor on W and if D_1 and D_2 do not have common components, then the Čech-de Rham theory applies (see, e.g. [16, 17]) so that we have

$$D_1 \cdot D_2 = \sum_p (D_1 \cdot D_2)_p \,,$$

where p runs through the intersection points of D_1 and D_2.

The *self-intersection number* of a Cartier divisor D with compact support in X is given by

$$D \cdot D = c^1(L_D) \frown [D] \,.$$

A.5. *Effect of blowing-up*

Let X be a surface with isolated singularities in W, as in the previous section, and p a point of X. Let $\pi : \tilde{W} \to W$ be the blowing-up of W at p, $D = \pi^{-1}(p)$ the exceptional divisor, $\tilde{X}$ the strict transform of X and $\rho : \tilde{X} \to X$ the restriction of π. We set $E = D \cdot \tilde{X}$. Note that the support of E is $\pi^{-1}(p) \cap \tilde{X} = \rho^{-1}(p)$ and as an analytic subspace of $D = \mathbb{P}^{r-1}$, it coincides with the projective cone $PC_p(X)$ of X at p. It is also considered as a Cartier divisor in $\tilde{X}$. In the sequel, we assume that $\tilde{X}$ has only isolated singularities.

Before we proceed further, we give:

Proof of Lemma A.1 in the case $n = 2$. Let H_1 and H_2 be two hyperplanes through p, general with respect to X, and set $H_i' = H_i \cdot X$, $i = 1, 2$. Let $\tilde{H}_i$ be the strict transform of H_i and set $\tilde{H}_i' = \tilde{H}_i \cdot \tilde{X}$. Then, from $\pi^* H_1 = \tilde{H}_1 + D$, we may write $\rho^* H_1' = \tilde{H}_1' + E$. Since $\tilde{H}_1' \cdot \tilde{H}_2' = 0$, we have

$$m(X, p) = (H_1' \cdot H_2')_p = \rho^* H_1' \cdot \tilde{H}_2' = E \cdot \tilde{H}_2' = \deg(PC_p(X)) \,. \qquad \square$$

Let Y be a curve through p in X. Note that the strict transform of Y by ρ is equal to that of Y by π, which is denoted by $\tilde{Y}$.

Lemma A.3. *We have*

$$m(Y, p) = E \cdot \tilde{Y} \,.$$

Proof. Let H be a hyperplane through p, general with respect to Y and X, and set $H' = H \cdot X$. Let $\tilde{H}$ be the strict transform of H and set $\tilde{H}' = \tilde{H} \cdot \tilde{X}$. Then we have $\rho^* H' = \tilde{H}' + E$, as above. Since $\tilde{H}' \cdot \tilde{Y} = 0$, we have

$$m(Y, p) = (H' \cdot Y)_p = \rho^* H' \cdot \tilde{Y} = E \cdot \tilde{Y} \,. \qquad \square$$

Lemma A.4. *If Y is Cartier, the multiplicity $m(Y,p)$ is divisible by $m(X,p)$ and if we set $m(Y,X;p) = m(Y,p)/m(X,p)$, we have*

$$\rho^* Y = \tilde{Y} + m(Y,X;p)E\,.$$

Proof. Since Y is Cartier, we may write $\rho^*Y = \tilde{Y} + kE$ for some integer k. Let H, H' and $\tilde{H}'$ be as in the proof of Lemma A.4. Then, since $\tilde{Y}\cdot\tilde{H}' = 0$, we have

$$m(Y,p) = (Y\cdot H')_p = \rho^*Y\cdot\tilde{H}' = kE\cdot\tilde{H}'\,,$$

where $E\cdot\tilde{H}' = m(X,p)$ by Lemma A.1. □

From Lemmas A.3 and A.4, we have the following

Theorem A.5. *Let $\rho : \tilde{X} \to X$ be the blowing-up at p, as above.*
(1) *Let Y_1 and Y_2 be curves in X near p. Suppose $Y_1 \cap Y_2 = \{p\}$ and Y_1 is Cartier. Then*

$$(Y_1\cdot Y_2)_p = \sum_{q\in\rho^{-1}(p)} (\tilde{Y}_1\cdot\tilde{Y}_2)_q + m(Y_1,X;p)\cdot m(Y_2,p)\,.$$

(2) *Let Y_1 be a compact Cartier curve in X and Y_2 a curve in X (which may be equal to Y_1). Suppose $Y_1\cap Y_2 = \{p\}$. Then*

$$Y_1\cdot Y_2 = \tilde{Y}_1\cdot\tilde{Y}_2 + m(Y_1,X;p)\cdot m(Y_2,p)\,.$$

We give the following expression for the multiplicity:

$$m(X,p) = -E\cdot E\,,$$

which follows from $E\cdot E = c^1(L_D) \frown [E]$ and the fact that $L_D = -$ hyperplane bundle.

We finish this section with

Lemma A.6. *Let P be a non-singular hypersurface in W. Suppose P intersects generically transversely (the non-singular part of) X and set $Y = X\cap P$. Let $\pi : \tilde{W} \to W$ be the blowing-up at p and let $\tilde{P}$, $\tilde{X}$ and $\tilde{Y}$ be the strict transforms of P, X and Y, respectively. Then $\tilde{P}$ intersects generically transversely $\tilde{X}$ and $\tilde{Y} = \tilde{X}\cap\tilde{P}$.*

Proof. From Lemmas A.2 and A.4, we have $\rho^*Y = \tilde{Y} + E$. On the other hand, from $\pi^*P = \tilde{P} + D$, we compute

$$\tilde{P}\cdot\tilde{X} = \pi^*P\cdot\tilde{X} - E = \rho^*Y - E = \tilde{Y}\,.$$ □

A.6. *Intersections of Weil curves*

Let X be a surface in a complex manifold W. In this section, we assume that X has only absolutely isolated singularities. Let Y_1 and Y_2 be two (distinct) curves in X. If at least one of them is Cartier, the previous Secs. A.3 and A.4 give a way to define the local and global intersection numbers of Y_1 and Y_2. If Y_1 and Y_2 are only Weil curves, we proceed as follows. Let $p \in Y_1 \cap Y_2$ and let $\pi : \tilde{W} \to W$ be the blowing-up at p. We use the notation of the Sec. A.5 for strict transforms etc. In view of Theorem A.5, we define

$$(Y_1 \cdot Y_2)_p = \sum_{q \in \rho^{-1}(p)} (\tilde{Y}_1 \cdot \tilde{Y}_2)_q + \frac{m(Y_1, p) \cdot m(Y_2, p)}{m(X, p)}, \tag{A.1}$$

where $(\tilde{Y}_1 \cdot \tilde{Y}_2)_q$ is defined as in the Sec. A.3, if $\tilde{Y}_1$ or $\tilde{Y}_2$ is Cartier at q, or by recursion of the above formula if either is not Cartier at q.

If at least one of Y_1 and Y_2 is compact, define

$$Y_1 \cdot Y_2 = \sum_{p \in Y_1 \cap Y_2} (Y_1 \cdot Y_2)_p \,. \tag{A.2}$$

Note that if either of Y_1 and Y_2 is not Cartier at p then $(Y_1 \cdot Y_2)_p$ is only a rational number, in general, for $m(X, p)$ might not divide $m(Y_1, p) \cdot m(Y_2, p)$.

Also, in view of Lemma A.4, for a compact curve Y in X, we define the inverse image (total transform) by

$$\rho^* Y = \tilde{Y} + \frac{m(Y, p)}{m(X, p)} E \,.$$

Then we can define by recursion the self-intersection number of Y as

$$Y \cdot Y = \rho^* Y \cdot \rho^* Y \,. \tag{A.3}$$

Note that, in the above, we need not to resolve the singularities of X, we only need to take blowing-ups sufficiently many times so that the curve becomes Cartier. Note also that our definitions (A.2) and (A.3) coincide with those of [13] (see also [14]).

We finish this appendix by stating some results about the negative definiteness of the intersection matrix of the resolution of a t-absolutely isolated singularity at each step of the resolution process. The proof of this fact is essentially that of [14, Theorem 1.2]. However we sketch it here for the reader convenience.

Let $p \in X$ be a t-absolutely isolated singularity. As in Sec. 4, let X_j denote the strict transform of X at the jth blow-up. Also, let us denote by P_j the exceptional divisor at the jth blow-up and $Y_j = X_j \cap P_j$. If $k > j$, we still denote by Y_j the strict transform of Y_j at the kth blow-up. We also write $Y_j = \cup_{k_j=1}^{s_j} Y_{j,k_j}$ for the decomposition in (global) irreducible components.

Theorem A.7. *Let $n \in \mathbb{N}$. The matrix $(Y_{i,k_i} \cdot Y_{j,k_j})$ with $i, j = 1, \ldots, n$ and $k_i = 1, \ldots, s_i$, $k_j = 1, \ldots, s_j$, is negative definite.*

Proof. Let $Y_1, \dots, Y_r$ be the irreducible components at the nth blow-up. Blow-up one point and let $\rho : \tilde{X} \to X$ be the blow-up morphism. Denote by $\tilde{Y}_j$ the strict transform of Y_j, $j = 1, \dots, r$. Also, let $Z_1, \dots, Z_s$ be the irreducible curves coming from the last blow-up. Using (A.1) and performing symmetric operations one can see that the intersection matrix of $\{\tilde{Y}_j, Z_i\}$ is equivalent to the matrix

$$\begin{pmatrix} (\rho^* Y_i \cdot \rho^* Y_j)_{i,j=1,\dots,r} & 0 \\ 0 & (Z_i \cdot Z_j)_{i,j=1,\dots,s} \end{pmatrix} = \begin{pmatrix} (Y_i \cdot Y_j) & 0 \\ 0 & (Z_i \cdot Z_j) \end{pmatrix}.$$

Proceeding this way until X has been desingularized, Grauert's theorem (see, e.g. [11]) implies that each block is negative definite and in particular $(Y_i \cdot Y_j) < 0$ as wanted. $\square$

Now fix $N \in \mathbb{N}$ and for $j = 1, \dots, N$ let Z_j be the sum of distinct irreducible curves Y_{j,k_j} in such a way that $Z_i \cap Z_j$ for $i \neq j$ does not contain any curve.

Corollary A.8. *The intersection matrix* $(Z_i \cdot Z_j)_{i,j=1,\dots,N}$ *is negative definite.*

Proof. Let V be the real vector space generated by the Y_{j,k_j}'s; $V = \bigoplus \mathbb{R} Y_{j,k_j}$. Let $\beta : V \times V \to \mathbb{R}$ be the bilinear form defined by bilinearly extending the function β defined on the natural basis of V as $\beta(Y_{j,k_j}, Y_{i,k_i}) = Y_{j,k_j} \cdot Y_{i,k_i}$. By Theorem A.7, the form β is negative definite. Let U be the subspace of V generated by Z_j. Then $\beta|_U$ is negative definite and its matrix in the basis $\{Z_j\}$ is exactly $(Z_i \cdot Z_j)$, which is thus negative definite as wanted. $\square$

Acknowledgment

Part of this research has been carried out while each one of the authors was visiting the home institution of the other. The authors would like to thank sincerely Università di Roma "Tor Vergata" and Hokkaido University for their generous hospitality and support.

The first author is partially supported by Progetto MIUR di Rilevante Interesse Nazionale *Proprietà geometriche delle varietà reali e complesse*. The second author is partially supported by a grant from the Japan Society for the Promotion of Science.

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