

SEMIGROUPS VERSUS EVOLUTION FAMILIES IN THE LOEWNER THEORY

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ABSTRACT. We show that an evolution family of the unit disc is commuting if and only if the associated Herglotz vector field has separated variables. This is the case if and only if the evolution family comes from a semigroup of holomorphic self-maps of the disc.

1. INTRODUCTION

In 1923, Loewner [20] introduced a differential equation to study some extremal problems in the theory of univalent functions, later developed mainly by Kufarev and Pommerenke. Such equation is nowadays known as the radial Loewner equation and it has been used to obtain many fundamental results such as distortion theorems, growth theorems, rotation theorems (see, e.g. [23]). In particular, Loewner's radial equation was a key ingredient in the proof of Bieberbach's conjecture by de Branges in 1985. In the last two decades, many mathematicians have considered and studied a variant of that equation which is called the chordal Loewner differential equation. Such a theory, especially the stochastic version of it, turned out to be useful for solving famous open conjectures. For instance, Lawler, Schramm and Werner solved the Mandelbrot's conjecture about the Hausdorff dimension of the Brownian frontier. For further details, we refer the reader to [21] and references therein.

Recently, the authors and P. Gumenyuk developed a theory which unifies and extends both the radial and the chordal Loewner equations [7], [10]. Indeed, this theory carries out to complex hyperbolic manifolds [8], [2].

Loewner's theory studies the relationships among three notions: Herglotz vector fields, evolution families and Loewner chains. Roughly speaking, a Herglotz vector field $G(z, t)$ is a Carathéodory vector field such that $G(\cdot, t)$ is semicomplete for almost every $t \geq 0$ (see Definition 2.3). An evolution family $(\varphi_{s,t})$ is a family of holomorphic self-maps of the unit disc \mathbb{D} satisfying some algebraic relations in s, t and some regularity hypotheses (see

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Definition 2.5). Finally, a Loewner chain (f_t) is a family of univalent mappings on the unit disc with increasing ranges satisfying some regularity assumptions (see Definition 2.8).

The three objects are related by the following Loewner differential equations:

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t), \quad \frac{\partial f_t(z)}{\partial t} = -f'_t(z)G(z, t), \quad f_s(z) = f_t(\varphi_{s,t}(z)).$$

In [7] it is proved that there is a one-to-one correspondence between evolution families and Herglotz vector fields, while in general Loewner chains are not uniquely associated with Herglotz vector fields [10].

Examples of evolution families are given by semigroups of holomorphic self-maps of the unit disc. Namely, if (Φ_t) is a semigroup (see Subsection 2.2) then setting $\varphi_{s,t} := \Phi_{t-s}$ for $0 \leq s \leq t < +\infty$ we obtain an evolution family [7, Example 3.4]. The associated Herglotz vector field $G(z, t)$ does not depend on t and it is actually the infinitesimal generator of the semigroup. More generally, if $\lambda : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing absolutely continuous function then $(\Phi_{\lambda(t)-\lambda(s)})$ is an evolution family whose Herglotz vector field is *splitting*, in the sense that $G(z, t) = \lambda'(t)\tilde{G}(z)$ with \tilde{G} being the infinitesimal generator of the semigroup. Note that in such cases the evolution family is commuting, namely every element of the family commutes with each other.

The aim of the present paper is to characterize Herglotz vector fields which are splitting (see Definition 2.4). The main result of this paper is the following

Theorem 1.1. *Let $G(z, t)$ be a Herglotz vector field and let $(\varphi_{s,t})$ be its associated evolution family. Then $G(z, t)$ is splitting if and only if $(\varphi_{s,t})$ is commuting.*

Such a result is proved in Theorems 3.4 and 4.3. Moreover, we show in Proposition 3.1 that a Herglotz vector field has an associated Loewner chain of a particular affine form if and only if it is splitting. Also, in Section 3 we describe splitting Herglotz vector fields according to the dynamical properties of related semigroups and we provide their Berkson-Porta like decomposition.

Finally, in Section 5 we introduce the notion of *reversing* evolution family, a natural and weaker notion of commuting, and we show that reversing evolution family are commuting in many cases (see Theorems 5.6 and 5.10).

2. PRELIMINARIES

2.1. Iteration theory. Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc of \mathbb{C} . A holomorphic function $f : \mathbb{D} \rightarrow \mathbb{D}$ such that $f \neq \text{id}$ has at most one fixed point in \mathbb{D} . If f has a fixed point $\tau \in \mathbb{D}$, then f is called *elliptic* and such a point is called the *Denjoy-Wolff point* of f . In case f is not an elliptic automorphism the sequence of iterates $\{f^{on}\}$ converges uniformly on compacta to the constant function $z \mapsto \tau$.

In case f has no fixed points in \mathbb{D} then there exists a unique point $\tau \in \partial\mathbb{D}$, called again the *Denjoy-Wolff point* of f , such that $\{f^{on}\}$ converges uniformly on compacta to the constant function $z \mapsto \tau$. Moreover, $\angle \lim_{z \rightarrow \tau} f(z) = \tau$ and $\angle \lim_{z \rightarrow \tau} f'(z) = \alpha_f$, with

$\alpha_f \in (0, 1]$ (here, as customary, $\angle \lim_{z \rightarrow \tau}$ means angular limit). The function f is said *hyperbolic* if $\alpha_f < 1$ and *parabolic* if $\alpha_f = 1$ (see, e.g. [1]).

If f is parabolic, it is said of *zero parabolic step* if for some, and hence any, $z \in \mathbb{D}$ it follows

$$\lim_{n \rightarrow \infty} \omega(f^{\circ n}(z), f^{\circ(n+1)}(z)) = 0,$$

where ω is the Poincaré distance of \mathbb{D} .

The study of commuting holomorphic self-maps of the unit disc has been a flourishing area of research in the last decades. Some classical papers are [25, 18, 14] and more recently we can mention [9, 15, 16, 19, 29] (see also monographs [17] and [26]). The following result about centralizers of holomorphic self-maps of the disc is true in a more general context without assuming injectivity, but here we only need in the following simple form.

Lemma 2.1. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a univalent function, $f \neq \text{id}$. Let $C(f) := \{g : \mathbb{D} \rightarrow \mathbb{D} : f \circ g = g \circ f\}$ be the centralizer. Then*

- (1) *If f is a hyperbolic automorphism with distinct fixed points $\tau, \tau' \in \partial\mathbb{D}$ then $C(f)$ is abelian and for all $g \in C(f)$ it follows that g is a hyperbolic automorphism with fixed points τ, τ' .*
- (2) *If f is not an automorphism and it is elliptic or hyperbolic then $C(f)$ is abelian.*
- (3) *If f is parabolic of zero hyperbolic step then $C(f)$ is abelian.*

Proof. (1) It is Heins' theorem [18].

(2) It is due to Cowen [14, Corollary 4.2]. However, in the hyperbolic case, one can get a simpler proof with a similar argument to the one we use below in the proof of statement (3) using the uniqueness of the intertwining function for hyperbolic mappings proved in [5].

(3) Let $\sigma : \mathbb{D} \rightarrow \mathbb{C}$ be univalent and such that $\sigma \circ f = \sigma + 1$. Such a map σ exists and it is unique in the sense that if $\tilde{\sigma} : \mathbb{D} \rightarrow \mathbb{C}$ is another univalent map such that $\tilde{\sigma} \circ f = \tilde{\sigma} + 1$ then there exists λ such that $\tilde{\sigma} = \sigma + \lambda$ ([12, Theorem 3.1]). Let $g \in C(f)$ and write $\tilde{\sigma} := \sigma \circ g$. It follows that

$$\tilde{\sigma} \circ f = \sigma \circ f \circ g = \sigma \circ g + 1 = \tilde{\sigma} + 1,$$

hence $\sigma \circ g = \sigma + \lambda_g$ for some $\lambda_g \in \mathbb{C}$. Now, if $g, g' \in C(f)$ then

$$\sigma \circ g \circ g' = \sigma + \lambda_g + \lambda_{g'} = \sigma + \lambda_{g'} + \lambda_g = \sigma \circ g' \circ g.$$

Being σ univalent, it follows that $g \circ g' = g' \circ g$, as wanted. \square

Finally, we recall that a point $p \in \partial\mathbb{D}$ is said to be a *boundary repelling fixed point* for a holomorphic map $f : \mathbb{D} \rightarrow \mathbb{D}$ if $\lim_{r \rightarrow 1^+} f(rp) = p$ and $\lim_{r \rightarrow 1^+} f'(rp) = C$ with $C \in (1, +\infty)$.

2.2. Semigroups. A semigroup (Φ_t) of holomorphic self-maps of \mathbb{D} is a continuous homomorphism between the additive semigroup $(\mathbb{R}^+, +)$ of positive real numbers and the semigroup $(\text{Hol}(\mathbb{D}, \mathbb{D}), \circ)$ of holomorphic self-maps of \mathbb{D} with respect to the composition, endowed with the topology of uniform convergence on compacta.

By Berkson-Porta's theorem [4], if (Φ_t) is a semigroup in $\text{Hol}(\mathbb{D}, \mathbb{D})$ then $t \mapsto \Phi_t(z)$ is analytic and there exists a unique holomorphic vector field $F : \mathbb{D} \rightarrow \mathbb{C}$ such that $\frac{\partial \Phi_t(z)}{\partial t} = F(\Phi_t(z))$. Such a vector field F is semicomplete and it is called the *infinitesimal generator* of (Φ_t) . Conversely, any semicomplete holomorphic vector field in \mathbb{D} generates a semigroup in $\text{Hol}(\mathbb{D}, \mathbb{D})$.

Let $F \neq 0$ be an infinitesimal generator with associated semigroup (Φ_t) . Then there exists a unique $\tau \in \overline{\mathbb{D}}$ and a unique $p : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic with $\text{Re } p(z) \geq 0$ such that $F(z) = (z - \tau)(\bar{\tau}z - 1)p(z)$. Such a formula is the well renowned *Berkson-Porta formula*.

The point τ in the Berkson-Porta formula turns out to be the common Denjoy-Wolff point of Φ_t for all $t \geq 0$. Moreover, if $\tau \in \partial\mathbb{D}$ it follows $\angle \lim_{z \rightarrow \tau} \Phi'_t(z) = e^{\beta t}$ for some $\beta \leq 0$, where $\beta = 0$ if and only if Φ_t is parabolic for some—hence any— $t > 0$.

A *boundary repelling fixed point* for a semigroup (Φ_t) is a point $p \in \partial\mathbb{D}$ which is a boundary repelling fixed point for one, and hence any, Φ_t , $t > 0$ [13]. Moreover, if $p \in \partial\mathbb{D}$ is a boundary repelling fixed point for (Φ_t) , then there exists $\beta > 0$ such that $\lim_{r \rightarrow 1^+} \Phi_t(rp) = e^{t\beta}$ (see [11]).

The proof of the following proposition is in [4] and [28] (see also the recent book [17]).

Proposition 2.2. *Let (Φ_t) be a non-trivial semigroup in \mathbb{D} with infinitesimal generator G . Then there exists a unique univalent function $h : \mathbb{D} \rightarrow \mathbb{C}$, called the Königs function of (Φ_t) such that*

- (1) *If (Φ_t) has Denjoy-Wolff point $\tau \in \mathbb{D}$ then $h(\tau) = 0$, $h'(\tau) = 1$ and $h(\Phi_t(z)) = e^{G(\tau)t}h(z)$ for all $t \geq 0$. Moreover, h is the unique holomorphic function from \mathbb{D} into \mathbb{C} such that*
 - (i) $h'(z) \neq 0$, for every $z \in \mathbb{D}$,
 - (ii) $h(\tau) = 0$ and $h'(\tau) = 1$,
 - (iii) $h'(z)G(z) = G'(\tau)h(z)$, for every $z \in \mathbb{D}$.
- (2) *If (Φ_t) has Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ then $h(0) = 0$ and $h(\Phi_t(z)) = h(z) + t$ for all $t \geq 0$. Moreover, h is the unique holomorphic function from \mathbb{D} into \mathbb{C} such that:*
 - (i) $h(0) = 0$,
 - (ii) $h'(z)G(z) = 1$, for every $z \in \mathbb{D}$.

2.3. Loewner theory. The three main objects of the theory are Herglotz vector fields, evolution families and Loewner chains. We give here the actual general definitions from [7] and [10] which include the classical radial and chordal cases.

Definition 2.3. Let $d \in [1, +\infty]$. A *Herglotz vector field of order d* on the unit disc \mathbb{D} is a function $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ with the following properties:

- H1. For all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto G(z, t)$ is measurable;
 H2. For all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto G(z, t)$ is holomorphic;
 H3. For any compact set $K \subset \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t)| \leq k_{K,T}(t)$$

for all $z \in K$ and for almost every $t \in [0, T]$.

- H4. For almost every $t \in [0, +\infty)$ it follows $G(\cdot, t)$ is an infinitesimal generator.

In [7, Theorem 4.8] it is proved that any Herglotz vector field $G(z, t)$ has a decomposition by means of a Berkson-Porta like formula, namely, $G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t)$, where $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ is a measurable function and $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ has the property that for all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto p(z, t) \in \mathbb{C}$ belongs to $L^d_{loc}([0, +\infty), \mathbb{C})$; for all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto p(z, t) \in \mathbb{C}$ is holomorphic; for all $z \in \mathbb{D}$ and for all $t \in [0, +\infty)$, we have $\operatorname{Re} p(z, t) \geq 0$. The data $(\tau(t), p(z, t))$ are called the *Berkson-Porta data* of $G(z, t)$ and they are essentially unique, in the sense that $p(z, t)$ is unique up to a zero measure set in t and $\tau(t)$ is unique if $p(z, t) \not\equiv 0$.

Definition 2.4. A Herglotz vector field $G(z, t)$ of order $d \in [1, +\infty]$ is said to be *splitting* if there exists an infinitesimal generator \tilde{G} and a function $g \in L^d_{loc}([0, +\infty), \mathbb{C})$ such that $G(z, t) = g(t)\tilde{G}(z)$ for all $z \in \mathbb{D}$ and almost every $t \in [0, +\infty)$.

Now we recall the definition of evolution family.

Definition 2.5. A family $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ of holomorphic self-maps of the unit disc is an *evolution family of order d* with $d \in [1, +\infty]$ if

- EF1. $\varphi_{s,s} = id_{\mathbb{D}}$,
 EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$,
 EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi$$

for all $0 \leq s \leq u \leq t \leq T$.

The elements of evolution families are univalent [7, Corollary 6.3]. In [7, Theorem 1.1, Theorem 6.6] it is proved that there is a one-to-one correspondence between evolution families and Herglotz vector fields:

Theorem 2.6. *For any evolution family $(\varphi_{s,t})$ of order $d \geq 1$ in the unit disc there exists a unique (up to changing on zero measure set in t) Herglotz vector field $G(z, t)$ of order d such that for all $z \in \mathbb{D}$*

$$(2.1) \quad \frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad a.e. \ t \in [0, +\infty).$$

Conversely, for any Herglotz vector field $G(z, t)$ of order $d \geq 1$ in the unit disc there exists a unique evolution family $(\varphi_{s,t})$ of order d such that (2.1) is satisfied.

Moreover for each $t > 0$ fixed

$$(2.2) \quad \frac{\partial \varphi_{s,t}(z)}{\partial s} = -\varphi'_{s,t}(z)G(z, s)$$

for almost every $s \in (0, t)$ and all $z \in \mathbb{D}$.

Definition 2.7. An evolution family $(\varphi_{s,t})$ is called *commuting* if $\varphi_{m,n} \circ \varphi_{s,t} = \varphi_{s,t} \circ \varphi_{m,n}$ for all $0 \leq s \leq t < +\infty$ and $0 \leq m \leq n < +\infty$

Finally we recall the definition of Loewner chains

Definition 2.8. A family $(f_t)_{0 \leq t < +\infty}$ of holomorphic maps of the unit disc is a *Loewner chain of order d* , with $d \in [1, +\infty]$, if

LC1. $f_t : \mathbb{D} \rightarrow \mathbb{C}$ is univalent for all $t \geq 0$,

LC2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ for all $0 \leq s < t < +\infty$,

LC3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi) d\xi$$

for all $z \in K$ and all $0 \leq s \leq t \leq T$.

In [10, Theorem 1.3, Theorem 4.1] it is proved

Theorem 2.9. (1) For any Loewner chain (f_t) of order $d \in [1, +\infty]$, let

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad 0 \leq s \leq t.$$

Then $(\varphi_{s,t})$ is an evolution family of the same order d . Conversely, for any evolution family $(\varphi_{s,t})$ of order $d \in [1, +\infty]$, there exists a Loewner chain (f_t) of order d such that

$$f_t \circ \varphi_{s,t} = f_s, \quad 0 \leq s \leq t.$$

(2) Moreover let $G(z, t)$ be the Herglotz vector field of order $d \in [1, +\infty]$ associated with the evolution family $(\varphi_{s,t})$. Suppose that (f_t) is a family of univalent functions in the unit disc such that

$$(2.3) \quad \frac{\partial f_s(z)}{\partial s} = -G(z, s)f'_s(z) \quad \text{for every } z \in \mathbb{D}, \text{ a.e. } s \in [0, +\infty).$$

Then (f_t) is a Loewner chain of order d associated with the evolution family $(\varphi_{s,t})$.

We remark that, although we never use this fact in the present paper, given any Loewner chain (f_t) there exists a Herglotz vector field such that (2.3) is satisfied [10, Theorem 4.1].

Throughout the paper, whenever not explicitly needed, in the statements we simply write evolution families, Herglotz vector fields and Loewner chains without mentioning the order.

3. SPLITTING HERGLOTZ VECTOR FIELDS

Evolution families associated with splitting Herglotz vector fields are of “semigroups type” as explained here:

Proposition 3.1. *Let $G(z, t) = g(t)\tilde{G}(z)$ be a splitting Herglotz vector field. Let $(\varphi_{s,t})$ be the evolution family associated with $G(z, t)$. Let (Φ_t) be the semigroup associated with \tilde{G} whose Denjoy-Wolff point is $\tau \in \mathbb{D}$ and let h be the Königs function of \tilde{G} . Set*

$$\lambda(t) := \int_0^t g(\xi) d\xi.$$

(1) *If $\tau \in \mathbb{D}$ then*

- $\varphi_{s,t}(z) = h^{-1}(e^{\tilde{G}'(\tau)[\lambda(t)-\lambda(s)]}h(z))$,
- *there exists a Loewner chain associated with $(\varphi_{s,t})$ of the form $f_s(z) = e^{-\tilde{G}'(\tau)\lambda(s)}h(z)$.*

(2) *If $\tau \in \partial\mathbb{D}$ then*

- $\varphi_{s,t}(z) = h^{-1}(h(z) + \lambda(t) - \lambda(s))$,
- *there exists a Loewner chain associated with $(\varphi_{s,t})$ of the form $f_s(z) = h(z) - \lambda(s)$.*

Proof. Assume first that $\tau \in \mathbb{D}$. Set

$$f_s(z) := e^{-\tilde{G}'(\tau)\lambda(s)}h(z).$$

Recall that $\tilde{G}(z) = \tilde{G}'(\tau)\frac{h(z)}{h'(\tau)}$. Then for all $z \in \mathbb{D}$ and almost every $s \in [0, +\infty)$ it follows

$$\begin{aligned} \frac{\partial f_s(z)}{\partial s} &= -\tilde{G}'(\tau)\lambda'(s)e^{-\tilde{G}'(\tau)\lambda(s)}h(z) = -\tilde{G}'(\tau)g(s)e^{-\tilde{G}'(\tau)\lambda(s)}h(z) \\ &= -g(s)e^{-\tilde{G}'(\tau)\lambda(s)}h'(z)\tilde{G}(z) = -G(z, s)f'_s(z). \end{aligned}$$

Hence (f_s) is a family of univalent maps in the unit disc which satisfies

$$\frac{\partial f_s(z)}{\partial s} = -G(z, s)f'_s(z) \quad \text{for all } z \in \mathbb{D}, \text{ a.e. } s \in [0, +\infty).$$

By Theorem 2.9.(2) it follows that (f_s) is a Loewner chain of order d associated with $G(t, z)$. In particular $f_s(\mathbb{D}) \subseteq f_t(\mathbb{D})$ for all $0 \leq s \leq t < +\infty$ and $\varphi_{s,t}(z) = f_t^{-1} \circ f_s(z)$, proving the statement.

Assume now $\tau \in \partial\mathbb{D}$. Let

$$f_s(z) := h(z) - \lambda(s).$$

Recall that in this case $\tilde{G}(z) = \frac{1}{h'(z)}$. Thus, for all $z \in \mathbb{D}$ and almost every $s \in [0, +\infty)$, it follows

$$\frac{\partial f_s(z)}{\partial s} = -\lambda'(s) = -g(s)\tilde{G}(z)h'(z) = -G(z, s)f'_s(z),$$

and as before we can conclude by Theorem 2.9.(2). □

Remark 3.2. Assuming the notations and hypotheses of Proposition 3.1, as a result of the statement, it follows that $e^{\tilde{G}'(\tau)[\lambda(t)-\lambda(s)]}h(\mathbb{D}) \subseteq h(\mathbb{D})$ in case $\tau \in \mathbb{D}$ and that $h(\mathbb{D}) + (\lambda(t) - \lambda(s)) \subseteq h(\mathbb{D})$ in case $\tau \in \partial\mathbb{D}$, for all $0 \leq s \leq t < +\infty$.

Remark 3.3. Note that if a Herglotz vector field $G(z, t)$ has an associated Loewner chain of the form as in Proposition 3.1 then from (2.3) it follows at once that $G(z, t)$ is splitting.

As an immediate consequence of Proposition 3.1 we have part of Theorem 1.1

Theorem 3.4. *Let $(\varphi_{s,t})$ be an evolution family with associated Herglotz vector field $G(z, t)$. If $G(z, t)$ is splitting then $(\varphi_{s,t})$ is commuting.*

Proposition 3.1 has also the following consequence:

Corollary 3.5. *Let $(\varphi_{s,t})$ be an evolution family of order $d \in [1, +\infty]$ in \mathbb{D} . Then the following are equivalent:*

- (1) *the Herglotz vector field $G(z, t)$ associated with $(\varphi_{s,t})$ is of the form $G(z, t) = g(t)\tilde{G}(z)$ (for all $z \in \mathbb{D}$ and almost every $t \in [0, +\infty)$) for some non-negative function $g \in L_{\text{loc}}^d([0, +\infty), \mathbb{R})$ and some infinitesimal generator \tilde{G} .*
- (2) *There exists a semigroup (Φ_t) of holomorphic self-maps of \mathbb{D} and a locally absolutely continuous and non-decreasing function $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\lambda' \in L_{\text{loc}}^d([0, +\infty), \mathbb{R})$, such that $\varphi_{s,t}(z) = \Phi_{\lambda(t)-\lambda(s)}(z)$.*

Proof. Assume (1). Let h be the Königs function of the semigroup (Φ_t) generated by \tilde{G} . In case $\tau \in \mathbb{D}$ then $\Phi_r(z) = h^{-1}(e^{\tilde{G}'(\tau)r}h(z))$ for all $r \geq 0$, while, if $\tau \in \partial\mathbb{D}$ it follows $\Phi_r(z) = h^{-1}(h(z) + r)$ for all $r \geq 0$. Let $\lambda(t) := \int_0^t g(\xi)d\xi$. Since $g(t) \geq 0$ for almost every $t \in [0, +\infty)$ it follows that $\lambda(t) \geq \lambda(s)$ for $0 \leq s \leq t < +\infty$. Hence, by Proposition 3.1 we have $\varphi_{s,t}(z) = \Phi_{\lambda(t)-\lambda(s)}(z)$, hence (2) holds.

Conversely, assuming (2), let \tilde{G} be the infinitesimal generator associated with (Φ_t) . Then, on the one side

$$\frac{\partial\varphi_{s,t}(z)}{\partial t} = \frac{\partial\Phi_{\lambda(t)-\lambda(s)}(z)}{\partial t} = \lambda'(t)\tilde{G}(\Phi_{\lambda(t)-\lambda(s)}(z))$$

and, on the other side by (2.1),

$$\frac{\partial\varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) = G(\Phi_{\lambda(t)-\lambda(s)}(z), t).$$

Hence $G(\Phi_{\lambda(t)-\lambda(s)}(z), t) = \lambda'(t)\tilde{G}(\Phi_{\lambda(t)-\lambda(s)}(z))$ for all $z \in \mathbb{D}$ and almost every $t \in [0, +\infty)$. Setting $s = t$ for those points s where λ is differentiable we obtain (1). \square

Now we are going to see how the function $g(t)$ in the decomposition of a splitting Herglotz vector field depends on the dynamical properties of the associated evolution family.

Proposition 3.6. *Let $G(z, t) = g(t)\tilde{G}(z)$ be a splitting Herglotz vector field. Let $(\varphi_{s,t})$ be the evolution family associated with $G(z, t)$. Let (Φ_t) be the semigroup associated with \tilde{G} . Suppose that either (Φ_t) is hyperbolic or there exists a boundary repelling fixed point for (Φ_t) . Then $g(t) \in \mathbb{R}$ for almost every $t \in [0, +\infty)$. Moreover, either $g(t) \geq 0$ for almost every $t \in [0, +\infty)$ or (Φ_t) is a group of hyperbolic automorphisms and there exist $\tau, \sigma \in \partial\mathbb{D}$, $\tau \neq \sigma$ such that $\tilde{G}(z) = \lambda(z - \tau)(z - \sigma)$ for some $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda(\sigma + \tau) = |\lambda||1 + \tau\sigma| = 0$.*

Proof. Let $\tau \in \partial\mathbb{D}$ be either the Denjoy-Wolff point of the semigroup in case (Φ_t) is hyperbolic, or the boundary repelling fixed point. Then, by the very definition $\lim_{\mathbb{R} \ni r \rightarrow 1^-} \Phi_t(r\tau) = \tau$ and $\lim_{\mathbb{R} \ni r \rightarrow 1^-} \Phi'_t(r\tau) = e^{\beta t}$ for some $\beta \in \mathbb{R} \setminus \{0\}$. Hence by [11, Theorem 1] it follows that

$$\lim_{\mathbb{R} \ni r \rightarrow 1^-} \tilde{G}(r\tau) = 0, \quad \lim_{\mathbb{R} \ni r \rightarrow 1^-} \tilde{G}'(r\tau) = \beta.$$

Since $G(z, t) = g(t)\tilde{G}(z)$, we have for almost every $t \in [0, +\infty)$

$$\lim_{\mathbb{R} \ni r \rightarrow 1^-} G(r\tau, t) = 0, \quad \lim_{\mathbb{R} \ni r \rightarrow 1^-} G'(r\tau, t) = g(t)\beta.$$

Since $G(z, t)$ is an infinitesimal generator for almost every $t \in [0, +\infty)$, another application of [11, Theorem 1], gives

$$g(t)\beta \in \mathbb{R},$$

for almost every $t \in [0, +\infty)$. Since β is a non-zero real number, it follows that $g(t) \in \mathbb{R}$ for a.e. $t \in [0, +\infty)$.

Now, assume there exists $t_0 \in [0, +\infty)$ such that $G(z, t_0)$ is an infinitesimal generator and $g(t_0) < 0$. Then $G(z, t_0) = -|g(t_0)|\tilde{G}(z)$ is an infinitesimal generator. Since infinitesimal generators form a real cone, $-\tilde{G}(z)$ is an infinitesimal generator as well. Hence $\tilde{G}(z)$ is an infinitesimal generator of a group of automorphisms of \mathbb{D} , having τ as a fixed point. Now, semigroups of elliptic and parabolic automorphisms have only one (common) fixed point (see, e.g. [1, Corollary 1.4.20]) which must be in \mathbb{D} in the elliptic case and $\Phi'_t(\tau) = 1$ in the parabolic case. Hence (Φ_t) is a group of hyperbolic automorphisms. This implies that $\tilde{G}(z)$ has the required form by [6, Theorem 2.3]. \square

Example 3.7. Let $\tau \in \partial\mathbb{D}$. Let $\tilde{G}(z) := (z - \tau)(\bar{\tau}z - 1)C_\tau(z)$ with $C_\tau(z) = (\tau + z)/(\tau - z)$ the Cayley transform with pole τ . Then $\tilde{G}(z)$ is an infinitesimal generator of a group of hyperbolic automorphisms with Denjoy-Wolff point τ (see [6, Example 2.7]). Let $G(z, t) = (-1)^{[t]}\tilde{G}(z)$, where $[t]$ denotes the integer part of t . Then $G(z, t)$ is a splitting Herglotz vector field and for each fixed $t \geq 0$, $G(z, t)$ generates a group of hyperbolic automorphisms of \mathbb{D} .

Example 3.8. Let $G(z, t) := (1 + it)(z - 1)^2$. Then $G(z, t)$ is a splitting Herglotz vector field, with $\tilde{G}(z) = (z - 1)^2$ and $g(t) = 1 + it$. Notice that $g(t)\tilde{G}(z)$ generates a semigroup of parabolic type with no boundary repelling fixed points for each fixed $t \geq 0$.

Example 3.9. Let $G(z, t) := -(t(1+i) + 1)z(2+z)$. Then $G(z, t)$ is a splitting Herglotz vector field, with $\tilde{G}(z) = -z(2+z)$ and $g(t) = t(1+i) + 1$. Notice that $g(t)\tilde{G}(z)$ generates a semigroup of elliptic type with no boundary repelling fixed points for each fixed $t \geq 0$.

Theorem 3.10. *Let $G(z, t) = g(t)\tilde{G}(z)$ be a splitting Herglotz vector field of order d such that \tilde{G} is not a generator of a group of hyperbolic automorphisms of \mathbb{D} . Let $\tilde{G}(z) = (z - \tau)(\bar{\tau}z - 1)p(z)$ be the Berkson-Porta decomposition of \tilde{G} . Then the Berkson-Porta data $(p(z, t), \tau(t))$ of $G(z, t)$ is*

$$\tau(t) = \tau, \quad p(z, t) = g(t)p(z).$$

Proof. At those points where $g(t) = 0$ the result is true. Then we can assume that $g(t) \neq 0$ for almost every t .

If $\tau \in \mathbb{D}$ then $\tilde{G}(\tau) = 0$ and hence $G(\tau, t) = g(t)\tilde{G}(\tau) = 0$ for almost every $t \in [0, +\infty)$. Therefore for almost every $t \in [0, +\infty)$ it follows that

$$G(z, t) = (z - \tau)(\bar{\tau}z - 1)p(z, t).$$

By the “essential” uniqueness of the Berkson-Porta data it follows that $\tau(t) = \tau$ and $p(z, t) = g(t)p(z)$ for almost every $t \in [0, +\infty)$.

Assume $\tau \in \partial\mathbb{D}$. Let (Φ_t) be the semigroup generated by \tilde{G} and let $\lim_{\mathbb{R}\ni r \rightarrow 1^-} \Phi'_t(r\tau) = e^{\beta t}$. There are two cases.

If (Φ_t) is hyperbolic then $\beta < 0$, and by hypothesis and by Proposition 3.6 it follows that $g(t) \geq 0$ for almost every $t \in [0, +\infty)$ (and actually $g(t) > 0$ for almost every $t \geq 0$ because we are assuming $g(t) \neq 0$ almost everywhere). If (Φ_t) is parabolic then $\beta = 0$. In both cases

$$\lim_{\mathbb{R}\ni r \rightarrow 1^-} G(r\tau, t) = 0, \quad \lim_{\mathbb{R}\ni r \rightarrow 1^-} G'(r\tau, t) = g(t)\beta \leq 0,$$

for almost every t . By [11, Theorem 1] this implies that τ is the Denjoy-Wolff point of the semigroup generated by $g(t)\tilde{G}(z)$ for almost every t . Hence by the Berkson-Porta formula

$$G(z, t) = (z - \tau)(\bar{\tau}z - 1)p(z, t), \quad \text{a.e. } t \geq 0,$$

and again by the uniqueness of the Berkson-Porta data it follows $\tau(t) = \tau$ and $p(z, t) = g(t)p(z)$ for almost every $t \in [0, +\infty)$. \square

4. COMMUTING EVOLUTION FAMILIES

The aim of the present section is to prove that the Herglotz vector field of a commuting evolution family is splitting. To this aim we need some preliminary results, interesting by themselves.

Recall that if $X, Y : \mathbb{D} \rightarrow \mathbb{C}$ are holomorphic vector fields then $[X, Y] := YX - XY$.

It is well known that $[X, Y] \equiv 0$ if and only if their flows are commuting. Moreover, since $T\mathbb{D}$ is generated by $\frac{\partial}{\partial z}$ then $[X, Y] \equiv 0$ is equivalent to $X(z) = \lambda Y(z)$ for some $\lambda \in \mathbb{C}$. Thus the following holds.

Lemma 4.1. *Let $G(z, t)$ be a Herglotz vector field. For almost every $t \geq 0$ fixed, let (ϕ_r^t) be the semigroup generated by $G(\cdot, t)$. Then the following are equivalent:*

- (1) $G(z, t)$ is splitting,
- (2) $[G(\cdot, t), G(\cdot, s)] \equiv 0$ for almost every $s, t \in [0, +\infty)$,
- (3) $\phi_r^t \circ \phi_r^s = \phi_r^s \circ \phi_r^t$ for all $r \geq 0$ and almost all $t, s \geq 0$.

The next proposition is a technical result about the differentiability of evolution families (cfr. [7, Theorem 6.4]).

Proposition 4.2. *Let $G(z, t)$ be a Herglotz vector field of order $d \in [1, +\infty]$ and let $(\varphi_{s,t})$ be its associated evolution family. Then there exists a zero measure set $M \in [0, +\infty)$ such that for all $t \in [0, +\infty) \setminus M$ it holds*

$$\lim_{h \rightarrow 0^+} \frac{\varphi_{t,t+h}(z) - z}{h} = G(z, t),$$

uniformly on compacta of \mathbb{D} .

Proof. Let $f_h^t(z) := \frac{\varphi_{t,t+h}(z) - z}{h}$ for $0 < h < 1$. We first show that $\{f_h^t\}$ is a normal family for all $t \in [0, +\infty) \setminus N_0$ for some set N_0 of zero measure.

To this aim, let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of compacta of \mathbb{D} such that $K_n \subset \overset{\circ}{K}_{n+1}$ and $\cup_n K_n = \mathbb{D}$. Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of positive real number such that $T_n < T_{n+1}$ and $\lim_{n \rightarrow \infty} T_n = +\infty$. By the very definition of evolution family (Property EF3) it follows that for each n there exists a non-negative function $k_n := k_{K_n, T_n} \in L^d([0, T_n], \mathbb{R})$ such that

$$|\varphi_{t,t+h}(z) - z| \leq \int_t^{t+h} k_n(\xi) d\xi$$

for all $t, h \geq 0$ with $t + h < T_n$ and all $z \in K_n$. Hence

$$|f_h^t(z)| \leq \frac{1}{h} \int_t^{t+h} k_n(\xi) d\xi,$$

and, since the function on the right hand side tends to $k_n(t)$ for almost every $t \in [0, T_n)$, it follows that $\{f_h^t\}$ is equibounded in K_n for almost every $t \in [0, T_n)$. Since the countable union of zero measure sets has zero measure, it follows that $\{f_h^t\}$ is equibounded on compacta of \mathbb{D} for almost all $t \geq 0$. Montel's theorem implies that $\{f_h^t\}$ is normal.

By [7, Theorem 6.4] there exists a zero measure set $N_1 \subset [0, +\infty)$ such that for all $t \in [0, +\infty) \setminus N_1$ the limit

$$\lim_{h \rightarrow 0^+} \frac{\varphi_{0,t+h}(z) - \varphi_{0,t}(z)}{h} = G(\varphi_{0,t}(z), t)$$

uniformly on compacta of \mathbb{D} . Hence for all $t \in [0, +\infty) \setminus N_1$ it follows

$$\lim_{h \rightarrow 0^+} \frac{\varphi_{t,t+h}(\varphi_{0,t}(z)) - \varphi_{0,t}(z)}{h} = \lim_{h \rightarrow 0^+} \frac{\varphi_{0,t+h}(z) - \varphi_{0,t}(z)}{h} = G(\varphi_{0,t}(z), t).$$

Now, let $t \in [0, +\infty) \setminus (N_0 \cup N_1)$. Let $f^t : \mathbb{D} \rightarrow \mathbb{C} \cup \{\infty\}$ be any limit of $\{f_h^t\}$. By the previous equation it follows that $f^t(\cdot) = G(\cdot, t)$ on the open set $\varphi_{0,t}(\mathbb{D})$. Therefore $f^t(\cdot) = G(\cdot, t)$ on \mathbb{D} and this proves the theorem with $M = N_0 \cup N_1$. \square

Now we are in good shape to prove the remaining part of Theorem 1.1:

Theorem 4.3. *Let $G(z, t)$ be a Herglotz vector field of order $d \in [1, +\infty]$ and let $(\varphi_{s,t})$ be its associated evolution family. If $(\varphi_{s,t})$ is commuting then $G(z, t)$ is splitting.*

Proof. Let $g_h^t(z) := \varphi_{t,t+h}(z)$ for $t \geq 0$ and $0 \leq h \leq 1$. Then $g_h^t \rightarrow \text{id}$ as $h \rightarrow 0$. Moreover, by Proposition 4.2 it follows

$$\lim_{h \rightarrow 0^+} \frac{g_h^t(z) - z}{h} = G(z, t),$$

uniformly on compacta of \mathbb{D} for almost all $t \geq 0$.

Let (ϕ_r^t) be the semigroup associated with $G(\cdot, t)$ for $t \geq 0$ fixed (this is well defined for almost all $t \geq 0$). By the product formula (see [24, Theorem 6.12]) we have

$$\phi_r^t = \lim_{n \rightarrow \infty} (g_{r/n}^t)^{\circ n},$$

where the limit is uniform on compacta of \mathbb{D} . Hence, for almost all $s \neq t$ and for all $r \geq 0$ we have

$$\phi_r^t \circ \phi_r^s = \lim_{m,n \rightarrow \infty} (g_{r/n}^t)^{\circ n} \circ (g_{r/m}^s)^{\circ m} = \lim_{m,n \rightarrow \infty} (g_{r/n}^s)^{\circ n} \circ (g_{r/m}^t)^{\circ m} = \phi_r^s \circ \phi_r^t.$$

Lemma 4.1 implies that $G(z, t)$ is splitting. \square

5. REVERSING EVOLUTION FAMILIES

Definition 5.1. An evolution family $(\varphi_{s,t})$ is called *reversing* if $\varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$, for all $0 \leq s \leq u \leq t < +\infty$.

According to Definition 2.5, an evolution family is reversing if the functions $\varphi_{s,u}$ and $\varphi_{u,t}$ commute for all $0 \leq s \leq u \leq t < +\infty$.

Remark 5.2. Note that if $(\varphi_{s,t})$ is a commuting evolution family then it is also reversing because $\varphi_{u,t} \circ \varphi_{s,u} = \varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$ for all $0 \leq s \leq u \leq t < +\infty$.

Now we study common fixed points of reversing evolution families. First, we show that, although in principle a reversing family is not commuting, one can always find a finite chain of mappings such that each commutes with the previous one, relating any two elements of the family.

Lemma 5.3. *Let $(\varphi_{s,t})$ be a reversing evolution family. Then for any $0 \leq u \leq v < +\infty$ and $0 \leq s \leq t < +\infty$ such that $\varphi_{s,t} \neq \text{id}$ and $\varphi_{u,v} \neq \text{id}$ there exist $1 \leq m \leq 4$ and $\{(s_0, t_0), \dots, (s_m, t_m)\}$ such that $s_0 = u$, $t_0 = v$, $s_m = s$, $t_m = t$, $0 \leq s_j \leq t_j < +\infty$, $\varphi_{s_j, t_j} \neq \text{id}$ for all $j = 0, \dots, m$, and*

$$\varphi_{s_j, t_j} \circ \varphi_{s_{j+1}, t_{j+1}} = \varphi_{s_{j+1}, t_{j+1}} \circ \varphi_{s_j, t_j}$$

for $j = 0, \dots, m - 1$.

Proof. Let $0 \leq u \leq v < +\infty$ and $0 \leq s \leq t < +\infty$ be such that $\varphi_{u,v}$ and $\varphi_{s,t}$ are not the identity and they do not commute (otherwise the result is true with $m = 1$). We can assume that $v \leq t$. First, let $v < t$.

By hypothesis of reversing, $\varphi_{l,r}$ commutes with $\varphi_{r,n}$ for all $0 \leq l \leq r \leq n < +\infty$.

Hence $\varphi_{u,v}$ commutes with $\varphi_{v,n}$ for $n \geq v$. In particular, it commutes with $\varphi_{v,t}$. Suppose that $\varphi_{v,t} \neq \text{id}$. Then $\varphi_{v,t}$ commutes with $\varphi_{t,r}$ for all $r \geq t$. Also, $\varphi_{s,t}$ commutes with $\varphi_{t,r}$ for all $r \geq t$. If $\varphi_{t,r} = \text{id}$ for all $t \leq r$ then by (2.1) it follows that

$$(5.1) \quad 0 = \frac{\partial \varphi_{t,r}(z)}{\partial r} = G(\varphi_{t,r}(z), r) = G(z, r)$$

for almost every $r \geq t$. Hence $G(z, r) \equiv 0$ for almost every $r \geq t$. Therefore, again by (2.1), $\varphi_{s,r} = \text{id}$ for all $r \geq t$, hence $\varphi_{s,t} = \text{id}$, contradicting our hypothesis. Therefore, if $\varphi_{v,t} \neq \text{id}$ the result is proved with $m = 3$.

Assume that $\varphi_{v,t} = \text{id}$. We claim that there exists $v < t' < t$ such that $\varphi_{v,t'} \neq \text{id}$, $\varphi_{t',t} \neq \text{id}$. If this is the case, then $\varphi_{u,v}$ commutes with $\varphi_{v,t'}$ which commutes with $\varphi_{t',t}$ which commutes with $\varphi_{t,r}$ for all $r \geq t$ and such elements—which cannot be all $\equiv \text{id}$ as we saw before—commute with $\varphi_{s,t}$, concluding the result with $m = 4$.

We need to show that we can choose $v < t' < t$ such that $\varphi_{v,t'} \neq \text{id}$, $\varphi_{t',t} \neq \text{id}$. Indeed, arguing by contradiction, let $N := \{r \in [v, t] : \varphi_{v,r} = \text{id}\}$ and $M := [v, t] \setminus N$. Thus $\varphi_{r,t} = \text{id}$ for all $r \in M$. Hence $\frac{\partial \varphi_{r,t}}{\partial r} = 0$ for almost all $r \in M$ and $\frac{\partial \varphi_{v,r}}{\partial r} = 0$ for almost all $r \in N$. The last condition, as in (5.1), implies that $G(z, r) \equiv 0$ for almost every $r \in N$. By (2.2) we have also that for almost every $r \in M$

$$0 = \frac{\partial \varphi_{r,t}}{\partial r} = -\varphi'_{r,t}(z)G(z, r) = -G(z, r),$$

hence $G(z, r) \equiv 0$ for almost every $r \in M$. Since $[v, t] = M \cup N$, we have $G(z, r) \equiv 0$ for almost every $r \in [v, t]$. Therefore, by (2.1), $\varphi_{s,r} = \text{id}$ for all $r \in [v, t]$. Thus $\varphi_{s,t} = \text{id}$ against our hypothesis.

Finally, the case $v = t$ follows easily by noting that $\varphi_{u,t}$ and $\varphi_{s,t}$ commute with $\varphi_{t,r}$ for any $r \geq t$. \square

The previous lemma has several interesting consequences. We start with the following result about hyperbolic automorphisms:

Proposition 5.4. *Let $(\varphi_{s,t})$ be a reversing evolution family such that $\varphi_{u,v}$ is a hyperbolic automorphism of \mathbb{D} for some $0 \leq u < v < +\infty$. Then for all $0 \leq s < t < +\infty$ with $\varphi_{s,t} \neq \text{id}$ it follows that $\varphi_{s,t}$ is a hyperbolic automorphism of \mathbb{D} . Moreover if $G(z, t)$ is the associated Herglotz vector field of $(\varphi_{s,t})$ then $G(z, t)$ is splitting and the family is commuting. In particular, there exist two distinct points $\tau, \tau' \in \partial\mathbb{D}$ such that $\varphi_{s,t}(\tau) = \tau, \varphi_{s,t}(\tau') = \tau'$ for all $0 \leq s \leq t < +\infty$.*

Proof. Let $\varphi_{s,t} \neq \text{id}$. Let $\{\varphi_{s_0,t_0}, \dots, \varphi_{s_m,t_m}\}$ be a chain such that $\varphi_{s_j,t_j} \neq \text{id}$ for all $j = 0, \dots, m$, $\varphi_{s_0,t_0} = \varphi_{u,v}$, $\varphi_{s_m,t_m} = \varphi_{s,t}$ and $\varphi_{s_j,t_j} \circ \varphi_{s_{j+1},t_{j+1}} = \varphi_{s_{j+1},t_{j+1}} \circ \varphi_{s_j,t_j}$ for $j = 0, \dots, m-1$. By Lemma 5.3 such a chain exists. By Lemma 2.1.(1) $\varphi_{s,t}$ commutes with $\varphi_{u,v}$ and it is a hyperbolic automorphism with the same fixed points. By Theorem 4.3 the associated Herglotz vector field is splitting. We note that one can even prove directly the last assertion without applying Theorem 4.3. In fact, moving to the right half plane by means of a conjugation with a Cayley transform, one sees that all the elements of the evolution family are of the form $\lambda(s,t)w$, and hence by (2.1), we see that $G(z,t)$ is splitting. \square

In case there are no hyperbolic automorphisms in a reversing family we get:

Proposition 5.5. *Let $(\varphi_{s,t})$ be a reversing evolution family. Suppose that $\varphi_{s,t}$ is not a hyperbolic automorphism of \mathbb{D} for all $0 \leq s \leq t < +\infty$. Then there exists $\tau \in \overline{\mathbb{D}}$ such that τ is the Denjoy-Wolff point of $\varphi_{s,t}$ for all $0 \leq s \leq t < +\infty$ with $\varphi_{s,t}(z) \neq z$.*

Proof. Let $\varphi_{s,t}, \varphi_{u,v} \neq \text{id}$. Let $\{\varphi_{s_0,t_0}, \dots, \varphi_{s_m,t_m}\}$ be a chain such that $\varphi_{s_j,t_j} \neq \text{id}$ for all $j = 0, \dots, m$, $\varphi_{s_0,t_0} = \varphi_{u,v}$, $\varphi_{s_m,t_m} = \varphi_{s,t}$ and $\varphi_{s_j,t_j} \circ \varphi_{s_{j+1},t_{j+1}} = \varphi_{s_{j+1},t_{j+1}} \circ \varphi_{s_j,t_j}$ for $j = 0, \dots, m-1$. By Lemma 5.3 such a chain exists with $m \leq 4$. By Behan's theorem [3] it follows that either $\varphi_{s_j,t_j}, \varphi_{s_{j+1},t_{j+1}}$ are hyperbolic automorphisms or they share the same Denjoy-Wolff point. By hypothesis there are no hyperbolic automorphisms and hence the result is proved. \square

Next we show that in many cases a reversing evolution family is commuting:

Theorem 5.6. *Let $(\varphi_{s,t})$ be a reversing evolution family. Suppose that one of the following holds:*

- (1) *there exists $0 \leq u \leq v < +\infty$ such that $\varphi_{u,v}$ is elliptic,*
- (2) *there exists $0 \leq u \leq v < +\infty$ such that $\varphi_{u,v}$ is hyperbolic,*
- (3) *for all $0 \leq u \leq v < +\infty$ such that $\varphi_{u,v} \neq \text{id}$, the maps $\varphi_{u,v}$ are parabolic of zero hyperbolic step.*

Then $(\varphi_{s,t})$ is commuting.

Proof. In case $\varphi_{u,v}$ is a hyperbolic automorphism the result follows from Proposition 5.4. In case all $\varphi_{s,t} \neq \text{id}$ are elliptic automorphisms, by Proposition 5.5, there exists $\tau \in \mathbb{D}$ which is a common fixed point for all the family. Thus, up to conjugation with a fixed automorphism which maps τ to 0, we can assume that $\tau = 0$. Hence $\varphi_{s,t}(z) = \lambda(s,t)z$ for some $|\lambda(s,t)| = 1$ and the family is commuting.

We can assume that $\varphi_{u,v}$ is not a hyperbolic or elliptic automorphism (but note that we are not excluding it can be a parabolic automorphism). Let $\varphi_{s,t} \neq \text{id}$. Let $\{\varphi_{s_0,t_0}, \dots, \varphi_{s_m,t_m}\}$ be a chain of minimal length such that $\varphi_{s_j,t_j} \neq \text{id}$ for all $j = 0, \dots, m$, $\varphi_{s_0,t_0} = \varphi_{u,v}$, $\varphi_{s_m,t_m} = \varphi_{s,t}$ and $\varphi_{s_j,t_j} \circ \varphi_{s_{j+1},t_{j+1}} = \varphi_{s_{j+1},t_{j+1}} \circ \varphi_{s_j,t_j}$ for $j = 0, \dots, m-1$. By Lemma 5.3 such a chain exists with $m \leq 4$. By Lemma 2.1 it must be $m = 1$, hence

$\varphi_{s,t}$ commutes with $\varphi_{u,v}$. Thus $(\varphi_{s,t}) \subset C(\varphi_{u,v})$, the centralizer of $\varphi_{u,v}$. Again by Lemma 2.1 such a centralizer is abelian, and hence the family is indeed commuting. \square

Remark 5.7. Note that if a reversing evolution family $(\varphi_{s,t})$ contains a parabolic element $\varphi_{u,v}$ then for all $0 \leq s \leq t < +\infty$ such that $\varphi_{s,t} \neq \text{id}$ it follows that $\varphi_{s,t}$ is parabolic (but the hyperbolic step can be zero or positive). This follows at once by Lemma 5.3 and [14, Corollary 4.1].

Theorem 5.6 together with Theorem 1.1 implies that the Herglotz vector field of a reversing evolution family satisfying the hypothesis of Theorem 5.6 is splitting. In the elliptic and hyperbolic cases, such a result can be proved directly by looking at the Herglotz vector field. We provide here such a proof, which can be also extended to the parabolic case at the price of assuming some regularity for the vector field.

First, we relate reversing with a property of the Herglotz vector field.

Lemma 5.8. *Let $(\varphi_{s,t})$ be an evolution family associated with the Herglotz vector field $G(z, t)$. Then the following conditions are equivalent*

- (1) $(\varphi_{s,t})$ is reversing.
- (2) $G(\varphi_{s,t}(z), u) = \varphi'_{s,t}(z)G(z, u)$ for every s, t and almost every u such that $0 \leq s \leq u \leq t < +\infty$.

Proof. Assume (1) is satisfied. Fixing s, t and differentiating with respect to u the equation $\varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$, using (2.1) and (2.2) we obtain for almost every u

$$\begin{aligned} 0 &= \frac{\partial \varphi_{s,u}}{\partial u}(\varphi_{u,t}(z)) + \varphi'_{s,u}(\varphi_{u,t}(z)) \frac{\partial \varphi_{u,t}(z)}{\partial u} \\ &= G(\varphi_{s,u}(\varphi_{u,t}(z)), u) - \varphi'_{s,u}(\varphi_{u,t}(z)) \varphi'_{u,t}(z) G(z, u) \\ &= G(\varphi_{s,u} \circ \varphi_{u,t}(z), u) - (\varphi_{s,u} \circ \varphi_{u,t})'(z) G(z, u) \\ &= G(\varphi_{s,t}(z), u) - \varphi'_{s,t}(z) G(z, u). \end{aligned}$$

Thus (2) holds.

Conversely, assume (2) holds. Fix $z \in \mathbb{D}$ and $0 \leq s \leq t < +\infty$. For $s \leq u \leq t$ let $f(u) := \varphi_{s,t}(z) - \varphi_{s,u} \circ \varphi_{u,t}(z)$. Note that f is absolutely continuous by [7, Proposition 3.7] and $f(s) = 0$. Differentiating f with respect to u , by the previous computations, we obtain that $f'(u) = 0$ almost everywhere. Thus $f \equiv 0$ and (1) holds. \square

Proposition 5.9. *Let $(\varphi_{s,t})$ be a reversing evolution family with associated Herglotz vector field $G(z, t)$. Let $0 \leq u \leq v < +\infty$ be such that $\varphi_{u,v} \neq \text{id}$. Assume that $\varphi_{u,v}$ is elliptic or hyperbolic. Then $G(z, t)$ is splitting.*

Proof. If $\varphi_{u,v}$ is a hyperbolic automorphism then the result follows from Proposition 5.4. In case $\varphi_{s,t} \neq \text{id}$ are all elliptic automorphisms then one can argue as in the proof of Theorem 5.6.

We can thus assume that $\varphi_{u,v}$ is not an automorphism. Let $\tau \in \overline{\mathbb{D}}$ be the Denjoy-Wolff point of $\varphi_{u,v}$. By Proposition 5.5, it follows τ is the Denjoy-Wolff point of $\varphi_{s,t}$ for all $0 \leq s \leq t < +\infty$ such that $\varphi_{s,t} \neq \text{id}$. By [7, Theorem 6.7] it follows then that for almost every $t \geq 0$

$$G(z, t) = (z - \tau)(\bar{\tau}z - 1)p(z, t),$$

with $\text{Re } p(z, t) \geq 0$. In particular, by Berkson-Porta formula, the semigroup generated by $G(\cdot, t)$ has Denjoy-Wolff point τ .

Claim. There exists $t_0 \geq 0$ such that $G(\cdot, t_0)$ is an infinitesimal generator with $G(z, t_0) \not\equiv 0$ and $p(\tau, t_0) \neq 0$ in case $\tau \in \mathbb{D}$, or the angular limit

$$\beta(t_0) := \angle \lim_{z \rightarrow \tau} \frac{G(z, t_0)}{z - \tau}$$

exists and it is different from 0 in case $\tau \in \partial\mathbb{D}$.

In case $\tau \in \mathbb{D}$ this holds for any $t_0 \geq 0$ such that $G(z, t_0) \not\equiv 0$ is an infinitesimal generator with Denjoy-Wolff point τ (namely for almost every $t_0 \geq 0$ such that $G(z, t_0) \not\equiv 0$) for otherwise $p(\tau, t_0) = 0$ which implies actually $p(z, t_0) \equiv 0$ and hence $G(z, t_0) \equiv 0$.

In case $\tau \in \partial\mathbb{D}$, by [11, Theorem 1] for almost every $t \geq 0$ it follows that

$$(5.2) \quad \angle \lim_{z \rightarrow \tau} \frac{G(z, t)}{z - \tau} = \beta(t) \in (-\infty, 0].$$

By hypothesis $\varphi'_{u,v}(\tau) < 1$ (in the sense of angular limits). According to [7, Theorem 7.1], $\varphi'_{u,v}(\tau) = \exp(\lambda(s) - \lambda(t))$ with

$$\lambda(t) = \int_0^t \left(\angle \lim_{z \rightarrow \tau} \frac{2|\tau - z|^2 p(z, \xi)}{1 - |z|^2} \right) d\xi.$$

Hence, since $z \rightarrow \tau$ non-tangentially, if it were $\beta(t) = 0$ for almost every $t \geq 0$ it would follow that $\lambda \equiv 0$ and $\varphi'_{u,v}(\tau) = 1$, a contradiction. Thus the claim is proven.

Now let us fix $t \geq 0$ such that $G(\cdot, t)$ is an infinitesimal generator with Denjoy-Wolff point τ (this happens for almost every $t \geq 0$ such that $G(z, t) \not\equiv 0$). Consider the function

$$A(z) := \frac{G(z, t)}{G(z, t_0)}.$$

Then in case $\tau \in \mathbb{D}$

$$(5.3) \quad \lim_{z \rightarrow \tau} A(z) = \lim_{z \rightarrow \tau} \frac{G(z, t)}{G(z, t_0)} = \frac{p(\tau, t)}{p(\tau, t_0)}.$$

While, in case $\tau \in \partial\mathbb{D}$, by (5.2) and since $\beta(t_0) < 0$,

$$(5.4) \quad \angle \lim_{z \rightarrow \tau} A(z) = \angle \lim_{z \rightarrow \tau} \frac{G(z, t)}{z - \tau} \frac{z - \tau}{G(z, t_0)} = \frac{\beta(t)}{\beta(t_0)}.$$

Now the family is reversing, therefore by Lemma 5.8 we have

$$A(z) = \frac{\varphi'_{u,v}(z)G(z,t)}{\varphi'_{u,v}(z)G(z,t_0)} = \frac{G(\varphi_{u,v}(z),t)}{G(\varphi_{u,v}(z),t_0)} = A(\varphi_{u,v}(z)).$$

By induction then

$$(5.5) \quad A(z) = A(\varphi_{u,v}^{\circ n}(z))$$

for all $z \in \mathbb{D}$. But $\lim_{n \rightarrow \infty} \varphi_{u,v}^{\circ n}(z) = \tau$ being τ the Denjoy-Wolff point of $\varphi_{u,v}$ and, in case $\tau \in \partial\mathbb{D}$, the sequence $\{\varphi_{u,v}^{\circ n}(z)\}$ converges to τ non-tangentially (see for instance [5]). Thus by (5.5) and either (5.3) in case $\tau \in \mathbb{D}$ or (5.4) in case $\tau \in \partial\mathbb{D}$

$$A(z) = \lim_{n \rightarrow \infty} A(\varphi_{u,v}^{\circ n}(z)) = g(t)$$

with $g(t) := \frac{p(\tau,t)}{p(\tau,t_0)}$ in case $\tau \in \mathbb{D}$ and $g(t) = \frac{\beta(t)}{\beta(t_0)}$ in case $\tau \in \partial\mathbb{D}$. Thus $G(z,t) = g(t)G(z,t_0)$ for almost every $t \geq 0$. Hence $G(z,t)$ is splitting. \square

The previous proof can be adapted to the parabolic case in the following way:

Theorem 5.10. *Let $(\varphi_{s,t})$ be a reversing evolution family with common Denjoy-Wolff point $\tau \in \partial\mathbb{D}$. Let $G(z,t)$ be its associated Herglotz vector field. Suppose that $G(\cdot,t)$ has derivatives up to order three at $z = \tau$ for almost every $t \in [0, +\infty)$. Then $G(z,t)$ is splitting.*

Proof. If the evolution family is not trivial, we can find $t \geq 0$ such that $G(\cdot,t) \not\equiv 0$, $G(\cdot,t)$ is an infinitesimal generator and it is differentiable up to the third order at τ . We claim that there exists $\beta(t) \in \{1, 2, 3\}$ such that

$$\lim_{z \rightarrow \tau} \frac{G(z,t)}{(z-\tau)^{\beta(t)}} \neq 0.$$

Indeed, if $\lim_{z \rightarrow \tau} \frac{G(z,t)}{(z-\tau)^3} = 0$ by the Shoikhet version of Burns-Krantz type theorem for semigroups [27] it follows $G(z,t) \equiv 0$.

Let $\beta = \inf \beta(t)$, where t is chosen among those $t \geq 0$ such that $G(z,t) \not\equiv 0$, $G(z,t)$ is an infinitesimal generator and $G(z,t)$ is differentiable up to order three at τ . Let t_0 be such that $\beta(t_0) = \beta$.

As in the proof of Proposition 5.9, for almost every $t \geq 0$ fixed, we can define the function $A(z) := G(z,t)/G(z,t_0)$. Thus

$$\lim_{z \rightarrow \tau} A(z) = \lim_{z \rightarrow \tau} \frac{G(z,t)}{(z-\tau)^\beta} \frac{(z-\tau)^\beta}{G(z,t_0)} = C(t)$$

exists. Now one can argue exactly as in the proof of Proposition 5.9. \square

Question: is there an example of a reversing evolution family which is not commuting?

Such a family, if exists, should be of parabolic type and contains parabolic mappings of positive hyperbolic step, moreover, the associated Herglotz vector field should not be differentiable at the Denjoy-Wolff point.

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