NORMAL FORMS AND LINEARIZATION OF HOLOMORPHIC DILATION TYPE SEMIGROUPS IN SEVERAL VARIABLES

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ABSTRACT. In this paper we study commuting families of holomorphic mappings in $\mathbb{C}^n$ which form abelian semigroups with respect to their real parameter. Linearization models for holomorphic mappings are used in the spirit of Schröder's classical functional equation.

The one-dimensional linearization models for holomorphic mappings and semigroups, based on Schröder's and Abel's functional equation have been studied by many mathematicians for more than a century.

These models are powerful tools in investigations of asymptotic behavior of semigroups, geometric properties of holomorphic mappings and their applications to Markov's stochastic branching processes.

It turns out that solvability as well as constructions of the solution of Schröder's or Abel's functional equations properly, depend on the location of the so-called Denjoy–Wolff point of the given mappings or semigroups. In particular, recently many efforts were directed to the study of semigroups with a boundary Denjoy–Wolff point [4, 14, 2, 12]. The existence and non-existence of common fixed points for semigroups (and, more generally, for families of commuting holomorphic mappings) has been studied in [11] (see also, [13]).

Multidimensional cases are more delicate even when the Denjoy–Wolff point is inside the underlying domain. It appears that the existence of the solution (the so-called Kœnigs' function) of a multidimensional Schröder's equation depends also on the resonant properties of the linear part of a given mapping (or generator), and its relation to homogeneous polynomials of higher degrees.

In parallel, the study of commuting mappings (or semigroups) is of interest to many mathematicians and goes back to the classical theory of linear operators, differential equations and evolution problems.

In this paper we consider, in particular, the rigidity property of two commuting semigroups. Namely, the question we study is whether those semigroups coincide whenever the linear parts of their generators at their common null point are the same.

Let D be a domain in $\mathbb{C}^n$. We denote the set of holomorphic mappings on D which take values in a set $\Omega \subset \mathbb{C}^m$ by $\mathrm{Hol}(D, \Omega)$. For each $f \in \mathrm{Hol}(D, \mathbb{C}^m)$, the Frechét derivative of f at a point $z \in D$ (which is understood as a linear operator acting from $\mathbb{C}^n$ to $\mathbb{C}^m$ or $n \times m$-matrix) will be denoted by df_z.

For brevity, we write $\mathrm{Hol}(D)$ for $\mathrm{Hol}(D, D)$. The set $\mathrm{Hol}(D)$ is a semigroup with respect to composition operation.

Definition 1. *A family $\mathcal{S} = \{\varphi_t\}_{t \geq 0} \subset \mathrm{Hol}(D)$ of holomorphic self-mappings of D is called a one-parameter continuous semigroup if the following conditions are satisfied:*

(i) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all s, $t \geq 0$;
(ii) $\lim\limits_{t \to 0^+} \varphi_t(z) = z$ for all $z \in D$.

It is more or less known that condition (ii) (the right continuity of a semigroup at zero) actually implies its continuity (right and left) on all of $\mathbb{R}^+ = [0, \infty)$. Moreover, in this case the semigroup is differentiable on $\mathbb{R}^+$ with respect to the parameter $t \geq 0$ (see [4, 14, 2, 12]). Thus, for each $z \in D$ there exists the limit

$$(1) \qquad \lim_{t \to 0^+} \frac{\varphi_t(z) - z}{t} = f(z),$$

which belongs to $\mathrm{Hol}(D, \mathbb{C}^n)$. The mapping $f \in \mathrm{Hol}(D, \mathbb{C}^n)$ defined by (1) is called the *(infinitesimal) generator* of $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$.

Furthermore, the semigroup $\mathcal{S}$ can be defined as a (unique) solution of the Cauchy problem:

$$(2) \qquad \begin{cases} \dfrac{\partial \varphi_t(z)}{\partial t} = f(\varphi_t(z)), & t \geq 0, \\[2mm] \varphi_0(z) = z, & z \in D. \end{cases}$$

The reader may refer to the book [13] for a recent description of the semigroup theory.

Definition 2. *We say that a semigroup $\{\varphi_t\}_{t \geq 0}$ is linearizable if there is a biholomorphic mapping $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ and a linear semigroup $\{\psi_t\}_{t \geq 0}$ such that $\{\varphi_t\}_{t \geq 0}$ conjugates with $\{\psi_t\}_{t \geq 0}$ by h, namely, $h \circ \varphi_t = \psi_t \circ h$ for all $t \geq 0$.*

Linearization methods for semigroups on the open unit disk in $\mathbb{C}\,(=\mathbb{C}^1)$ have been studied by many mathematicians (see, for example, [16, 15, 8]). At the same time, little is known about multi-dimensional cases. For example, in [9] and [7] the problem has been studied for some special class of the so-called one-dimensional type semigroups.

In this paper, we will concentrate on the case when a semigroup has a (unique) interior attractive fixed point, i.e., $\lim_{t\to\infty}\varphi_t(z) = \tau \in D \subset \mathbb{C}^n$ for all $z \in D$. It is well known that this condition is equivalent to that fact that the spectrum $\sigma(A)$ of the linear operator (matrix) A defined by $A := df_\tau$ lies in the open left half-plane (see [1] and [12]) and $d(\varphi_t)_\tau = e^{At}$. Usually, such semigroups are named *of dilation type*. Thus, for the one-dimensional case, it is possible to linearize the semigroup by solving Schröder's functional equation:

$$h\left(\varphi_t(z)\right) = e^{f'(\tau)t}h(z)$$

(see, for example, [16, 14]).

Remark 1. *It should be noted that the latter equation involves the eigenvalue problem for the linear semigroup $\{C_t\}_{t\geq 0}$ of composition operators on the space $\mathrm{Hol}(D,\mathbb{C})$ defined by $C_t : h \mapsto h \circ \varphi_t$.*

It is easy to show that the solvability of a higher dimensional analog of Schröder's functional equation

(3)
$$h\left(\varphi_t(z)\right) = e^{At}h(z), \quad A = df_\tau,$$

is equivalent to a generalized differential equation:

(4)
$$dh_z f(z) = Ah(z).$$

It seems that in general useful criteria (necessary and sufficient conditions) for solvability of (4) are unknown.

Without loss of generality, let us assume that $\tau = 0$.

Proposition 1. *Equation (3), or equivalently, (4) is solvable if and only if there is a polynomial mapping $Q : \mathbb{C}^n \mapsto \mathbb{C}^n$ with $Q(O) = O$ and $dQ_O = \mathrm{id}$, such that the limit*

$$\lim_{t\to\infty} e^{-At}Q(\varphi_t(z)) =: h(z), \quad z \in D,$$

exists.

This proposition is based on the following notation and lemma.

By $\lambda(A)$ we denote the spectrum distortion index of the matrix A, i.e.,

$$\lambda(A) := \frac{\max\limits_{\alpha\in\sigma(A)} |\mathrm{Re}\,\alpha|}{\min\limits_{\alpha\in\sigma(A)} |\mathrm{Re}\,\alpha|}.$$

Lemma 1 (see [6]). *Let $g \in \mathrm{Hol}(D, \mathbb{C}^n)$ admit the expansion: $g(z) = \sum\limits_{\ell \geq m} Q_\ell(z)$, where Q_ℓ is a homogenous polynomial of order ℓ and $m > \lambda(A)$. Then*

$$\lim_{t \to \infty} e^{-At} g(\varphi_t(z)) = O, \quad \text{for all } z \in D.$$

In many cases (and always — in the one dimensional case), a polynomial Q in Proposition 1 can be chosen to be the identity mapping, $Q(z) = z$ for all z. Moreover, in this case $h\left(\varphi_t(z)\right) = e^{At} h(z)$, i.e., the mapping $h(z) = \lim\limits_{t \to \infty} e^{At} \varphi(z)$ forms a conjugation of a given semigroup $\{\varphi_t\}_{t \geq 0}$ with the linear semigroup $\left\{ e^{At} \right\}_{t \geq 0}$.

Definition 3. *Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous one-parameter semigroup of holomorphic self-mappings on a domain $D \subset \mathbb{C}^n$. We say that $\mathcal{S}$ is normally linearizable if the limit*

$$h(z) = \lim_{t \to \infty} e^{-At} \varphi_t(z), \quad z \in D,$$

exists.

A consequence of Lemma 1 is the following assertion.

Proposition 2. *Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a one-parameter semigroup of holomorphic self-mappings on a domain $D \subset \mathbb{C}^n$ generated by $f \in \mathrm{Hol}(D, \mathbb{C}^n)$. If f admits the expansion on the series of homogenous polynomials: $f(z) = Az + \sum\limits_{\ell \geq m} Q_\ell(z)$, where Q_ℓ is a homogenous polynomial of order ℓ and $m > \lambda(A)$, then the semigroup $\mathcal{S}$ is normally linearizable.*

In contrast with the one-dimensional case, for $n > 1$ there are semigroups which are not normally linearizable.

Example 1. *Let $\{\varphi_t\}_{t \geq 0}$ be a semigroup in $\mathbb{C}^2$ defined by*

$$\varphi_t(z_1, z_2) = \begin{pmatrix} z_1 \exp\left(-(1+i)t\right) \\ \left[a z_1^2 i \left(e^{-it} - 1 \right) + z_2 \right] e^{-(2+i)t} \end{pmatrix}.$$

It is easy to see that

$$\lim_{t \to \infty} e^{-At} \varphi_t(z) = \lim_{t \to \infty} \begin{pmatrix} z_1 \\ a z_1^2 i(\exp(-it) - 1) + z_2 \end{pmatrix}$$

does not exist. Thus, this semigroup is not normally linearizable.
Just differentiating φ_t at $t = 0^+$ we find the semigroup generator:

$$f(z_1, z_2) = \begin{pmatrix} -(1+i)z_1 \\ -(2+i)z_2 + a z_1^2 \end{pmatrix}.$$

For this generator we have $\lambda(A) = m = 2$, i.e., f does not satisfy the conditions of Proposition 2.

Proposition 3. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\{\varphi_t\}$ be a continuous dilation semigroup which is normally linearizable. If for some $t_0 > 0$ the semigroup element φ_{t_0} is a linear map, then all the elements φ_t, $t \geq 0$, are linear.*

Proof. Denote $h(z) := \lim\limits_{t \to \infty} e^{-At} \varphi_t(z)$. Then for all $s > 0$ obviously

$$h(\varphi_s(z)) := e^{As} \lim_{t \to \infty} e^{-A(t+s)} \varphi_t(\varphi_s(z)) = e^{As} h(z),$$

i.e., h is a linearizing conjugation for $\{\varphi_t\}_{t \geq 0}$. Since $\varphi_{t_0} = e^{At_0}$, we have $\varphi_{t_0 n} = e^{At_0 n}$ and

$$h(z) := \lim_{n \to \infty} e^{-At_0 n} \varphi_{t_0 n}(z) = z,$$

so h is the identity mapping. Therefore, $\varphi_s(z) = h^{-1}\left(e^{As} h(z)\right) = e^{As} z$ for all $s \geq 0$. $\qquad \square$

Example 1 above shows that this fact is not generally true. Indeed, for each $t_\ell = 2\pi\ell$, $\ell \in \mathbb{Z}$, the semigroup element φ_{t_ℓ} is a linear mapping. Yet all other elements φ_t, $t \neq 2\pi\ell$, are not linear.

An additional problem is that that with exception of the one-dimensional case, linearizing conjugations may not be unique.

Definition 4. *Let $\mathcal{F} = \{\varphi_s\}_{s \in \mathcal{A}}$ be a family of holomorphic self-mappings of D. We say that $\mathcal{F}$ is uniquely linearizable if there is a unique mapping h biholomorphic in D and normalized by $h(O) = O$, $dh_O = \mathrm{id}$, such that*

$$h \circ \varphi_s = B_s \circ h, \quad s \in \mathcal{A},$$

where $\{B_s\}_{s \in \mathcal{A}}$ is an appropriate family of linear operators on $\mathbb{C}^n$.

Remark 2. *Actually, it follows by the chain rule that $B_s = d(\varphi_s)_O$.*

Remark 3. *A family $\mathcal{F}$ may consist of a single mapping $F \in \mathrm{Hol}(D)$ as well as a discrete or continuous semigroup of holomorphic self-mappings on D.*

Our next example shows that even linear diagonal mappings may not be uniquely linearizable.

Example 2. *Consider a linear mapping $\psi = (\psi_1, \psi_2)$ with*

$$\psi_1(z_1, z_2) = \frac{z_1}{2}, \qquad \psi_2(z_1, z_2) = \frac{z_2}{4}$$

and a holomorphic normalized mapping defined by

$$h(z_1, z_2) = \begin{pmatrix} z_1 \\ z_1{}^2 + z_2 \end{pmatrix}.$$

Then $h \circ \psi = \psi \circ h$, i.e., h and also the identity mapping id *linearize ψ.*

Actually, the question whether a linear mapping $\psi(z) = Bz$ is uniquely linearizable can be formulated as the following rigidity problem:

When do the conditions

$$Q \circ B = B \circ Q \quad and \quad Q'(O) = O$$

on a holomorphic mapping Q imply that $Q \equiv O$?

Remark 4. *In fact, it can be seen that if a matrix B is diagonalizable and $\sigma(B) = \{\beta_1, \ldots, \beta_n\} \subset \Delta$, then ψ is uniquely linearizable if and only if $\beta_1{}^{k_1} \cdot \beta_2{}^{k_2} \cdot \ldots \cdot \beta_n{}^{k_n} \neq \beta_j$ for all $j = 1, \ldots, n$ and $k \in \mathbb{N}^n$.*

Theorem 1. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type, and let ψ be a holomorphic self-mapping of D commuting with $\mathcal{S}$ such that*

(5) $$\psi \circ \varphi_t = \varphi_t \circ \psi$$

for all $t \geq 0$. If ψ is uniquely linearizable by a biholomorphic mapping $h : D \mapsto \mathbb{C}^n$, then all of the elements of the semigroup $\mathcal{S}$ are linearizable by the same mapping h.

Proof. Let B denote a linear operator on $\mathbb{C}^n$ defined by $B = d\psi_O$. Also we denote $A = df_O$, where f is the infinitesimal generator of the semigroup $\mathcal{S}$. First, by differentiating (5) at O we obtain $(d\psi_O) \circ e^{At} = e^{At} \circ d\psi_O$, i.e., B commutes with the linear semigroup $\{e^{At}\}_{t \geq 0}$ (in fact, B commutes with A).

By our assumption, $h \circ \psi = B \circ h$. Therefore, for all $t \geq 0$ we have

$$h \circ \psi \circ \varphi_t = B \circ h \circ \varphi_t.$$

On the other hand, $h \circ \psi \circ \varphi_t = h \circ \varphi_t \circ \psi$ by (5). Thus,

$$e^{-At} \circ h \circ \varphi_t \circ \psi = e^{-At} \circ B \circ h \circ \varphi_t = B \circ e^{-At} \circ h \circ \varphi_t.$$

Denoting $h_1 := e^{-At} \circ h \circ \varphi_t$ one rewrites the latter equality in the form

$$h_1 \circ \psi = B \circ h_1.$$

Since $h_1(O) = O$, $d(h_1)_O =$ id and ψ is uniquely linearizable by h, we conclude that $h_1 = e^{-At} \circ h \circ \varphi_t = h$, or

$$h \circ \varphi_t = e^{At} \circ h.$$

The proof is complete.   □

Corollary 1. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type. If there exists $t_0 > 0$ such that φ_{t_0} is uniquely linearizable by a biholomorphic mapping $h : D \mapsto \mathbb{C}^n$, then all the elements of $\mathcal{S}$ are linearizable by the same mapping h which is a unique solution of the differential equation (4)*

$$dh_z f(z) = Ah(z),$$

normalized by the conditions $h(O) = O,\ dh_O = \mathsf{id}$.

Corollary 2. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\mathcal{S}_1 = \{\varphi_t\}_{t \geq 0}$ and $\mathcal{S}_2 = \{\psi_t\}_{t \geq 0}$ be two continuous semigroups on D generated by mappings f_1 and f_2, respectively. Suppose that $d(f_1)_O = d(f_2)_O = A$ with $\mathsf{Re}\,\sigma(A) < 0$ and that there exists $s_0 > 0$ such that*

(i) ψ_{s_0} is uniquely linearizable and
(ii) ψ_{s_0} commutes with the semigroup $\mathcal{S}_1$ such that $\psi_{s_0} \circ \varphi_t = \varphi_t \circ \psi_{s_0}$ for all $t \geq 0$.

Then the semigroups coincide.

Proof. By our assumption, there is a unique biholomorphic mapping h normalized by $h(O) = O,\ dh_O = \mathsf{id}$, such that

$$h \circ \psi_{s_0} = e^{As_0} \circ h.$$

Then Theorem 1 (or Corollary 1) implies that $h \circ \psi_s = e^{As} \circ h$ for all $s \geq 0$. Since the mapping h is biholomorphic, we have:

$$\psi_s = h^{-1} \circ \left(e^{As} \circ h\right).$$

The commutativity of the mapping ψ_{s_0} and the semigroup $\mathcal{S}_1$ implies by the same Theorem 1 that all of the elements of $\mathcal{S}_1$ are linearizable by the mapping h, that is, $h \circ \varphi_t = e^{At} \circ h$ for all $t \geq 0$. Thus

$$\varphi_t = h^{-1} \circ \left(e^{At} \circ h\right).$$

$\square$

Remark 5. *If the semigroups $\mathcal{S}_1 = \{\varphi_t\}_{t \geq 0}$ and $\mathcal{S}_2 = \{\psi_t\}_{t \geq 0}$ commute in the sense: $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all $t, s \geq 0$, then the conclusion that they coincide holds under a formally weaker than condition (i) requirement that differential equation (4) has a unique solution normalized by $h(O) = O, dh_O = \mathsf{id}$.*

Corollary 3. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\mathcal{S}_1 = \{\varphi_t\}_{t \geq 0}$ and $\mathcal{S}_2 = \{\psi_t\}_{t \geq 0}$ be two commuting semigroups on D generated by mappings f_1 and f_2, respectively. Suppose that $d(f_1)_O = d(f_2)_O = A$ with $\mathsf{Re}\,\sigma(A) < 0$. If $\lambda(A) < 2$ then the semigroups coincide.*

The use of the Poincaré–Dulac theorem (see, for example, [3]) is another approach to solve a linearization problem.

For simplicity, we assume in the sequel that A is a diagonal matrix, $A = \mathrm{diag}(\alpha_1 \ldots, \alpha_n)$ with $\mathrm{Re}\,\alpha_n \leq \ldots \leq \mathrm{Re}\,\alpha_1 < 0$.

Let $k := (k_1, \ldots, k_n) \in \mathbb{N}^n$ be such that $|k| := \sum k_j \geq 2$.

Definition 5. *We say that A is resonant (or the n-tuple $(\alpha_1, \ldots \alpha_n)$ of the eigenvalues of A is resonant) if for some $\ell = 1, \ldots, n$*

$$(\alpha, k) := \sum_{j=1}^n k_j \alpha_j = \alpha_\ell.$$

Such a relation is called a resonance. *The number $|k|$ is called the* order *of the resonance.*

If $\alpha_\ell = (\alpha, k)$, we call any map $G : \mathbb{C}^n \mapsto \mathbb{C}^n$ *resonant monomial* if it has the form $G(z) = (g_1(z), \ldots, g_n(z))$ with $g_j \equiv 0$ for $j \neq \ell$ and $g_\ell(z) = az^k$.

Lemma 2. *If $\mathrm{Re}\,\alpha_n \leq \ldots \leq \mathrm{Re}\,\alpha_1 < 0$ then there is at most a finite number of resonances for α. Moreover, if $\alpha_j = (k, \alpha)$ then $k_j = \ldots = k_n = 0$.*

Proof. Both statements follow from the simple observation that if $\alpha_j = (k, \alpha)$, then $\mathrm{Re}\,\alpha_j = (k, \mathrm{Re}\,\alpha)$, and by the ordering of α_j.   □

For simplicity of notation, let

$$M_j := \begin{cases} 0, & \text{if there is no } k \text{ with } \alpha_j = (k, \alpha), \\ \max\{|k| : \alpha_j = (\alpha, k)\} & \text{otherwise.} \end{cases}$$

and $M(\alpha) := \max\{M_j : j = 1, \ldots, n\}$.

A vector polynomial map $R : \mathbb{C}^n \mapsto \mathbb{C}^n$, $R(O) = O$, is *triangular* if by switching coordinates $R(z) = (R_1(z), \ldots, R_n(z))$ assumes the form

$$R_j(z) = a_j z_j + r_j(z_1, \ldots, z_{j-1}), \quad j = 1, \ldots, n$$

where r_j is a polynomial.

Theorem 2. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\{\varphi_t\}_{t \geq 0}$ be a continuous dilation type semigroup generated by $f \in \mathrm{Hol}(D, \mathbb{C}^n)$ with $df_O = A$. Then there exists an injective holomorphic map $h : D \mapsto \mathbb{C}^n$ (independent of t) such that $h(O) = O$, $dh_O = \mathrm{id}$ and*

$$h \circ \varphi_t = P_t \circ h,$$

where $P_t(z) = e^{At}z + R_t(z)$ is a triangular polynomial group of automorphisms of $\mathbb{C}^n$ whose degree is less than or equal to $M(\alpha)$, and $R_t(z)$ containing only resonant monomials. In particular, if there are no resonances then $\{\varphi_t\}_{t \geq 0}$ is linearizable.

Proof. Let $\varphi_t(z) = e^{At}z + \sum_{|m|\geq 2} P_{m,t}(z)$ be the homogeneous expansion at O (which is defined on a small ball containing O and contained in D). It follows from the theory of semigroups of holomorphic maps that each $P_{m,t}(z)$ is real analytic in t.

By our assumption, A is diagonal and the convex hull in $\mathbb{C}$ of its eigenvalues does not contain 0. Therefore by the classical Poincaré–Dulac theorem, there exist an open neighborhood U of O and a holomorphic map $h : U \mapsto \mathbb{C}^n$ normalized by $h(O) = O$ and $dh_O = \mathrm{id}$ such that $dh_z(f(z)) = \widehat{f}(h(z))$, where $\widehat{f}(z) = Az + T(z)$ with T being a polynomial vector field containing only resonant monomials.

The semigroup $\{\varphi_t\}_{t\geq 0}$ is (locally around O) conjugated to the semigroup $\{\psi_t\}_{t\geq 0}$, $\psi_t = h \circ \varphi_t \circ h^{-1}$, generated by $\widehat{f} = A + T$. Since T contains only resonant monomials and $\mathrm{Re}\,\alpha_n \leq \ldots \leq \mathrm{Re}\,\alpha_1 < 0$, Lemma 2 implies that $\widehat{f}$ is triangular, i.e., $\{\psi_t\}_{t\geq 0}$ satisfies the following system:

$$\begin{cases} \dot{x_1} = \alpha_1 x_1 \\ \dot{x_2} = \alpha_2 x_2 + r_2(x_1) \\ \ldots \\ \dot{x_n} = \alpha_n x_n + r_n(x_1, x_2, \ldots, x_{n-1}), \end{cases}$$

where the r_j's are polynomials in $x_1, \ldots, x_{j-1}$ containing only resonant monomials. Such a system can be integrated directly by first solving $\dot{x_1} = \alpha_1 x_1$, then substituting such solution into $\dot{x_2} = \alpha_2 x_2 + r_2(x_1)$, and so on. In the end, ψ_t is of the form

$$\psi_t(z) = (e^{t\alpha_1}z_1, e^{t\alpha_2}(z_2 + R_{2,t}(z_1)), \ldots, e^{t\alpha_n}(z_n + R_{n,t}(z_1, z_2, \ldots, z_{n-1}))),$$

with $R_{j,t}$ a polynomial in $z_1, \ldots, z_{j-1}$ of (at most) degree M_j containing with only resonant monomials. Moreover, $R_{j,t}$ depends also polynomially on t. It can be shown by induction. It is true for $j = 1$, so assume it is true for $j - 1$. Then the l-th component of (ψ_t) for $l = 1, \ldots, j - 1$ is of the form $\psi_{t,l}(z) = e^{t\alpha_l}(z_l + R_{l,t}(z_1, z_2, \ldots, z_{l-1}))$ with $R_{l,t}$ a polynomial in $z_1, \ldots, z_{l-1}$ of degree at most M_l and depending polynomially on t. Substituting these into the differential equation $\dot{x_j} = \alpha_j x_j + r_j(x_1, x_2, \ldots, x_{j-1})$, one obtains

$$\dot{x_j} = \alpha_j x_j + r_j(e^{\alpha_1 t}z_1, e^{t\alpha_2}(z_2 + R_{2,t}(z_1)), \ldots$$
$$\ldots, e^{t\alpha_{j-1}}(z_{j-1} + R_{j-1,t}(z_1, z_2, \ldots, z_{j-2}))).$$

Therefore the solution is of the form $e^{\alpha_j t}g(t)$ for some function g such that $g(0) = z_j$ and

$$\dot{g}(t) = e^{-\alpha_j t}r_j(x_1, x_2, \ldots, x_{j-1}).$$

Now, r_j contains only resonant monomials for α_j. Let z^m be such a resonant monomial. Then, taking into account that $m_j = \ldots = m_n = 0$ by Lemma 2, it follows

$$z^m = az_1^{m_1} \cdots z_{j-1}^{m_{j-1}} = e^{(m,\alpha)t}[z_1^{m_1} \cdots (z_{j-1}+R_{j-1,t}(z_1, z_2, \ldots, z_{j-2}))^{m_{j-1}}].$$

Hence

$$\dot{g}(t) = e^{(-\alpha_j+(m,\alpha))t}[z_1^{m_1} \cdots (z_{j-1} + R_{j-1,t}(z_1, z_2, \ldots, z_{j-2}))^{m_{j-1}}],$$

and, being $\alpha_j = (m, \alpha)$, then actually

$$\dot{g}(t) = z_1^{m_1} \cdots (z_{j-1} + R_{j-1,t}(z_1, z_2, \ldots, z_{j-2}))^{m_{j-1}}.$$

Since this holds for all resonant monomials in r_j, this proves that $R_{t,j}(z)$ is a polynomial in both $z_1, \ldots, z_{j-1}$ and t. The degree of $R_{t,j}$ is at most M_j because it contains only resonant monomials for α_j. This proves the induction and the claim about the $R_{j,t}$'s.

This fact implies that $\psi_{-t}(z)$ is well defined for all $t \geq 0$ and $z \in \mathbb{C}^n$. Therefore, $\{\psi_t\}_{t \in \mathbb{R}}$ is a group of polynomial automorphisms of $\mathbb{C}^n$.

Finally, since O is an attracting fixed point by hypothesis, then h can be extended to all D by imposing $h(w) = \psi_{-t}(h(\varphi_t(w)))$ for all $w \in D$. $\square$

Example 3. *For $n = 2$ there is only one possible resonance, namely, $\alpha_2 = m\alpha_1$. Hence, up to conjugation, the dilation semigroups in $\mathbb{C}^2$ are of the form:*

$$\varphi_t(z) = (e^{\alpha_1 t}z_1, e^{\alpha_2 t}(z_2 + atz_1^m))$$

for some $a \in \mathbb{C}$.

So, if the matrix $A = df_\tau$ is resonant, it may happen that all elements of the semigroup generated by f are not linearizable. In this connection the following question arises naturally. Suppose that one of the elements of the semigroup $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ (say, φ_{t_0}) is linearizable. Find conditions which ensure that all other elements φ_t, $t \neq t_0$, are linearizable too.

To answer this question we need the following notion.

Definition 6. *We say that the matrix $A = \operatorname{diag}(\alpha_1 \ldots \alpha_n)$ has pure real resonance if there are $j = 1, \ldots, n$ and $k \in \mathbb{N}^n$ such that $\operatorname{Re} \alpha_j = \operatorname{Re}(\alpha, k)$ but $\alpha_j \neq (\alpha, k)$.*

In particular, if all eigenvalues α_j have the same argument, then A doesn't have pure real resonances.

Theorem 3. *Let $D \subset \mathbb{C}^n$ be a domain containing O. Let $\{\varphi_t\}_{t \geq 0}$ be a continuous dilation semigroup generated by $f \in \operatorname{Hol}(D, \mathbb{C}^n)$ with $df_O = A$, where A doesn't have pure real resonances. If there exists $t_0 > 0$ such that φ_{t_0} is linearizable by biholomorphic mapping $h : D \mapsto \mathbb{C}^n$, $h(O) = O$. Then the semigroup $\{\varphi_t\}_{t \geq 0}$ is linearizable by h.*

Not that even for the non-resonant case Theorem 3 completes Theorem 2 since it asserts the following fact: if $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ is a linearizing mapping for φ_{t_0}, it also can serve as a linearizing mapping for all φ_t, $t \geq 0$.

Proof. Let us define $\psi_t := h \circ \varphi_t \circ h^{-1}$. Then ψ_t is a semigroup on $h(D)$.

Let $\psi_t(z) = e^{At}z + \sum_m P_{m,t}(z)$ be the homogeneous expansion at O (which is defined on a small ball containing O and contained in $g(D)$), where $m \geq 2$ is the least positive integer such that $P_{m,t} \not\equiv 0$ for all t, z. If the theorem holds then $m = +\infty$ (namely, (ψ_t) is linear). Seeking a contradiction, we assume that $m < +\infty$.

It follows from the theory of semigroups of holomorphic maps that each $P_{m,t}(z)$ is real analytic in t.

Since by hypothesis $\psi_{t_0} = h \circ \varphi_{t_0} \circ h^{-1}$ is linear, then $P_{m,t_0} \equiv 0$.

Now, from $\psi_{t+s} = \psi_t \circ \psi_s$ it follows that

$$(6) \qquad P_{m,t+s}(z) = e^{At}P_{m,s}(z) + P_{m,t}(e^{As}z).$$

Write $P_{m,t}(z) = (\sum_{|k|=m} p_k^1(t)z^k, \ldots, \sum_{|k|=m} p_k^n(t)z^k)$, where, as usual, $z^k = z_1^{k_1} \cdots z_n^{k_n}$. From (6) it follows that for $j = 1, \ldots, n$

$$p_k^j(t+s) = e^{\alpha_j t}p_k^j(s) + p_k^j(t)e^{(\alpha,k)s}.$$

Differentiating such an expression with respect to t and setting $t = 0$, we obtain the following differential equation:

$$(7) \qquad \frac{d}{dt}p_k^j(s) = \alpha_j p_k^j(s) + a_k^j e^{(\alpha,k)s},$$

where we set $a_k^j = \frac{dp_k^j(t)}{dt}|_{t=0}$. There are two cases:

(1) if $\mathrm{Re}\,\alpha_j \neq \mathrm{Re}\,(\alpha, k)$, then imposing the condition $p_k^j(0) = 0$, equation (7) has the solution

$$(8) \qquad p_k^j(t) = a_k^j \frac{e^{(\alpha,k)t} - e^{\alpha_j t}}{(\alpha, h) - \alpha_j}.$$

(2) if $\mathrm{Re}\,\alpha_j = \mathrm{Re}\,(\alpha, k)$, then by our assumption $\alpha_j = (\alpha, k)$. In this case, imposing the condition $p_k^j(0) = 0$, equation (7) has the solution

$$(9) \qquad p_k^j(t) = a_k^j e^{t\alpha_j}t.$$

By (8) and (9) it follows that $p_k^j(t_0) = 0$ if and only if $p_k^j(t) = 0$ for all $t \geq 0$, and hence $P_{m,t_0} \equiv 0$ if and only if $P_{m,t} \equiv 0$ for all $t \geq 0$, reaching a contradiction with our hypothesis. $\square$

Example 1 above shows that if A has pure real resonance, Theorem 3 fails.

Corollary 4. *Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type generated by $f \in \mathrm{Hol}(D, \mathbb{C}^n)$ with $df_O = A = \mathsf{diag}(\alpha_1, \ldots, \alpha_n)$. Suppose that there is $t_0 > 0$ such that φ_{t_0} is a linear mapping. Assume that one of the following conditions holds:*

(i) A doesn't have pure real resonances;
(ii) $e^{(\alpha,k)t_0} \neq e^{\alpha_j t_0}$ for all $j = 1, \ldots, n$ and $k \in \mathbb{N}^n$.
Then all elements of $\mathcal{S}$ are linear mappings.

Proof. If condition (i) holds, the assertion follows immediately by Theorem 3.

Assume that condition (ii) holds. First, we show that φ_{t_0} is uniquely linearizable. Indeed, let $h(z) = z + \ldots$ be a linearizing mapping different from id. This means that $h \circ \varphi_{t_0} = \varphi_{t_0} \circ h$ and for some $j = 1, \ldots, n$, the j-th coordinate of h contains a non-zero monomial $a_k z_1{}^{k_1} \ldots z_n{}^{k_n}$ with $|k| \geq 2$. Therefore,

$$h_j\left(e^{\alpha_1 t_0} z_1, \ldots, e^{\alpha_n t_0}\right) = e^{\alpha_j t_0} h_j(z),$$

and so

$$a_k e^{(\alpha,k)t_0} z^k = a_k e^{\alpha_j t_0} z^k.$$

The contradiction provides that φ_{t_0} is uniquely linearizable by the identity mapping id.

Now, Corollary 1 implies that the all mappings φ_t, $t \geq 0$, are linearizable by the identity mapping. Hence, they are linear. $\square$

Combining Corollary 4 with Proposition 3, we get the following result.

Corollary 5. *Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$ and let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type generated by $f \in \mathrm{Hol}(\mathbb{B}, \mathbb{C}^n)$ with $df_O = A = \mathsf{diag}(\alpha_1, \ldots, \alpha_n)$. Suppose that there is $t_0 > 0$ such that φ_{t_0} is a linear fractional self-mapping of $\mathbb{B}^n$. Assume that one of the following conditions holds:*

(i) A doesn't have pure real resonances;
(ii) $\mathcal{S}$ is normally linearizable;
(iii) $e^{(\alpha,k)t_0} \neq e^{\alpha_j t_0}$ for all $j = 1, \ldots, n$ and $k \in \mathbb{N}^n$.
Then for all $t \geq 0$ the mapping φ_t is a linear fractional self-map of $\mathbb{B}^n$.

Proof. According to [5, Thm. 3.2 and Rmk. 3.4] (and its proof) there exists $h : \mathbb{B}^n \mapsto \mathbb{C}^n$ a linear fractional mapping fixing O such that $h \circ \varphi_{t_0} \circ h^{-1}$ is linear. By Corollary 4 and Proposition 3, it follows that $h \circ \varphi_t \circ h^{-1}$ is linear for all $t \geq 0$. Therefore, φ_t is the composition of linear fractional maps and hence linear fractional for all $t \geq 0$. $\square$

Corollary 6. *Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type generated by $f \in \mathrm{Hol}(\mathbb{B}, \mathbb{C}^n)$, $f(z) = Az + \sum_{\ell \geq m} Q_\ell(z)$, where Q_ℓ is a*

homogenous polynomial of order ℓ and $m > \lambda(A)$. If for some $t_0 > 0$, the semigroup element φ_{t_0} is a linear (respectively, linear fractional) mapping, then all the elements of $\mathcal{S}$ are linear (respectively, linear fractional) mappings.

A direct consequence of our Theorems 2 and 3 and a recent Forelli type extension theorem (see [10, Theorem 6.2]) is the following assertion.

Corollary 7. *Let $\mathcal{S} = \{\varphi_t\}_{t \geq 0}$ be a continuous semigroup of dilation type generated by $f \in \text{Hol}(D, \mathbb{C}^n)$ with $df_O = A = \text{diag}(\alpha_1, \ldots, \alpha_n)$, where all eigenvalues α_j have the same argument. Suppose that a function F defined on D is real analytic at O, and that its restrictions to the integral curves of the vector field f are holomorphic. If at least one of the following conditions holds:*

(i) A is not resonant,
or
(ii) there is t_0 such that φ_{t_0} is linearizable,
then F is holomorphic on D.

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