Holomorphic vector fields transverse to polydiscs

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Abstract

In this paper we study holomorphic vector fields transverse to the boundary of a polydisc in $\mathbb{C}^n$, $n \geq 3$. We prove that, under a suitable hypothesis of transversality with the boundary of the polydisc, the foliation is the pull-back of a linear hyperbolic foliation via a locally injective holomorphic map. This is the $n \geq 3$ version for one-dimensional foliations of a previous result proved for $n = 2$ in [3] and for codimension one foliations in [11].

1 Introduction and main result

One of the main results in the classical theory of codimension one real foliations is a theorem of A. Haefliger [8] which implies that an analytic codimension-one foliation admits no null (homotopic) transversals. In the course of the proof one is led to consider real vector fields in a neighborhood of the closed disc $\overline{D^2} \subset \mathbb{R}^2$ which are transverse to the boundary $\partial D^2 \simeq S^1$. The use of Poincaré-Bendixson Theorem shows the existence of some unilateral hyperbolicity, for some closed orbit $\gamma \subset D^2$, what is not compatible with the analytical behaviour. Unfortunately, there is no feature like the classical Poincaré-Bendixson Theorem in the case of holomorphic vector fields. To overcame this difficult is one of the basic motivations for the present work. Moreover, we have, in the complex setting, natural domains to be considered which are not regular (at the boundary), as polydiscs for instance $\Delta^n \subset \mathbb{C}^n$. Therefore, the study of the consequences of transversality should be somehow extended to such domains.

The case of round domains is studied by T. Ito in [9] where it is proved that if a holomorphic vector field Z in a neighborhood of the closed ball $\overline{B^n}(R) = \{z \in \mathbb{C}^n;\ |z| \leq R\}$, is transverse to the sphere $S^{2n-1}(R) = \partial\overline{B^n}(R) = \{z \in \mathbb{C}^n;\ |z| = R\}$, then such vector field exhibits only one singularity $o \in B^n(R)$, which is accumulated by each orbit of Z in $\overline{B^n(R)}$. Moreover, the germ of Z at o is simple and in the Poincaré domain ([2]). In this paper we are interested in the case of a holomorphic vector field transverse to the boundary of a polydisc in dimension $n \geq 3$. Let us introduce the notions we use.

Let X be a holomorphic vector field in a neighborhood W of the origin $0 \in \mathbb{C}^n$, $n \geq 2$ and denote by $\operatorname{sing}(X)$ its singular set. For $x \in W$ we denote by $L_x^{\mathbb{R}}$ the leaf of the real 2-dimensional foliation defined by the nonsingular orbits of X in $W \setminus \operatorname{sing}(X)$.

Definition 1. Let $M \subset W$ be a smooth real submanifold. We say that X is *transverse* to M if for any $x \in M$ it follows that $X(x) \neq 0$ and $T_xM + T_xL_x^{\mathbb{R}} = T_x(\mathbb{R}^{2n})$.

This definition is natural in case M is smooth and $\operatorname{codim}_{\mathbb{R}} M \leq 2$. However, if M is singular one has to replace this concept suitably. For instance, if $\Delta^2(1) := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$ is the unit bidisc in $\mathbb{C}^2$ then $M = \partial\Delta^2(1)$ is composed by 3 smooth components:

$$\partial\Delta^2(1) = (S^1 \times \Delta) \cup (\Delta \times S^1) \cup (S^1 \times S^1),$$

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where $\Delta = \Delta^1(1) = \{z \in \mathbb{C} : |z| < 1\}$ and $S^1 = \partial\Delta$. In [3] the authors define that X *is transverse to the boundary* $\partial\Delta^2(1)$ provided X is transverse to each smooth component of the boundary. Using such a definition they prove

Theorem ([3]) *Let X be a holomorphic vector field defined in a neighborhood of the closure of $\Delta^2(1) \subset \mathbb{C}^2$. If X is transverse to $\partial\Delta^2(1)$ then there exist*

1. *a locally injective holomorphic map Φ which sends a neighborhood of $\overline{\Delta^2(1)}$ to a neighborhood of the origin $0 \in \mathbb{C}^2$ and*

2. *a linear hyperbolic foliation $\mathcal{L}_\lambda$ on $\mathbb{C}^2$ defined by $xdy + \lambda ydx = 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$*

such that the singular holomorphic foliation $\mathcal{F}(X)$ defined by X is the pull-back $\mathcal{F}(X) = \Phi^*(\mathcal{L}_\lambda)$. Moreover, the map Φ is injective as a map between spaces of leaves.

In this paper we extend this result to one-dimensional foliations in any dimension. When trying to do this, several differences with the two dimensional case appear. One difficult, is the fact that in dimension $n = 2$, a one-dimensional foliation is always of codimension one, and this is no longer true if $n \geq 3$. Usually, holonomy and extension techniques are well-developed for codimension one holomorphic foliations, so we have to develop some material on that either. Another difficult arises when trying to introduce the notion of transversality with the boundary of a polydisc in dimension $n \geq 3$, since there are smooth components of this boundary which have codimension higher than 2.

In what follows we use the following notations: Given real numbers $R > 0$ and $p \geq 2$ we define $\Delta^n(R) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < R, j = 1, \ldots, n\}$ the open polydisc of radius $R > 0$; $\Delta^n[R] := \overline{\Delta^n(R)}$ the closed polydisc of radius R; $B_p^n(R) := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum |z_j|^{2p} < R^{2p}\}$; $B_p^n[R] := \overline{B_p^n(R)}$ and $S_p^{2n-1}(R) := \partial B_p^n(R)$. The following proposition is proved in [12]:

Proposition 1. *Given a holomorphic vector field X in a neighborhood of $\Delta^2[1] \subset \mathbb{C}^2$ the following conditions are equivalent:*

1. *X is transverse to the boundary $\partial\Delta^2(1)$.*

2. *X is transverse to the boundary $\partial B_p^2(1) = S_p^3(1)$ (which is smooth) for all p large enough.*

The proposition suggests the following definition of transversality that we will adopt:

Definition 2. A holomorphic vector field X defined in a neighborhood of the closed polydisc $\Delta^n[1]$ is *transverse to the boundary* $\partial\Delta^n(1)$ if:

1. There exist $\epsilon > 0$ and $p_0 \in (2, +\infty)$ such that if $p \geq p_0$ and $|R - 1| < \epsilon$ then X is transverse to $S_p^{2n-1}(R) = \partial B_p^n[R]$

2. X is transverse $M \subset \mathbb{C}^n$ and to the boundary ∂M for any manifold M such that, up to reordering the affine coordinates, is of the form of $M = S^1 \times B_p^{n-1}(R)$ and, therefore, $\partial M = S^1 \times S_p^{2n-3}(R)$.

Example 1. Let $X = \sum\limits_{j=1}^{n} \lambda_j z_j \frac{\partial}{\partial z_j}$ be a linear vector field on $\mathbb{C}^n$, $n \geq 3$. We assume that X is in the *Poincaré domain*, i.e., the origin $0 \in \mathbb{R}^2$ does not belong to the convex hull of the set $\{\lambda_1, ..., \lambda_n\}$ in $\mathbb{R}^2$. Also we assume that X is *hyperbolic*, i.e., $\lambda_i/\lambda_j \in \mathbb{C} \setminus \mathbb{R}$ for all $i \neq j$. Then X is transverse to the boundary of the unit polydisc $\Delta^n(1) \subset \mathbb{C}^n$.

Our main result is the following:

Theorem 1. *Let X be a holomorphic vector field defined in a neighborhood of the polydisc $\Delta^n[1] \subset \mathbb{C}^n$, $n \geq 2$ and transverse to the boundary $\partial\Delta^n(1)$. Then there exist a holomorphic locally injective map $\Phi\colon W \to \mathbb{C}^n$ from a neighborhood W of the polydisc and a linear vector field Z in $\mathbb{C}^n$ such that in W the singular holomorphic foliation $\mathcal{F}(X)$ defined by X is the pull-back by Φ of the foliation $\mathcal{F}(Z)$ defined by Z. The map Φ is a diffeomorphism in a neighborhood of the origin and is injective as a map between leaf spaces.*

Remark 1. Transversality with $\partial\Delta^n(1)$ is a much stronger condition than transversality with spheres $S^{2n-1}(r) = \partial B_1^n[r]$. Indeed, the holomorphic vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (x+z)\frac{\partial}{\partial z}$ is transverse to *all* spheres $S^5(r), r > 0$ in $\mathbb{C}^3$ but is not transverse to $\partial\Delta^3[1]$.

Sketch of the proof

A brief sketch of the proof of Theorem 1 is as follows. First we use Definition 2 and apply [9] to conclude that X has a unique singularity $\theta \in \Delta^n(1)$ and $DX(\theta)$ is in the Poincaré domain. By the classical Poincaré-Dulac theorem X is holomorphically conjugate in a neighborhood of θ to its linear part or a Poincaré-Dulac resonant normal form. Using the transversality of $\mathcal{F}(X)$ with the components $\Sigma \subset \partial\Delta^n(1)$ diffeomorphic to $S^1 \times B_p^{n-1}[1]$ and with their boundaries, we obtain that (in a neighborhood of) such Σ the foliation $\mathcal{F}(X)$ induces a transversely holomorphic $\mathcal{L}(\Sigma)$ with a single periodic orbit whose holonomy map is an attractive diffeomorphism globally linearizable or globally holomorphically conjugate to a normal form of Poincaré-Dulac resonant type. This suspension flow $\mathcal{L}(\Sigma)$ can therefore be transversely holomorphically conjugated to standard models what allows the construction of suitable systems of closed meromorphic one-forms $\{\eta_k^\Sigma\}_{k=2}^n$ in neighborhoods of (each) Σ, which describe $\mathcal{F}(X)$ in these neighborhoods.

A gluing process based on the rigidity of the transverse dynamics of the flows $\mathcal{L}(\Sigma)$ and followed by application of Hartogs' Extension Theorem for polydiscs gives then a system of closed meromorphic one-forms $\{\eta_j\}_{j=2}^n$ in a which defines $\mathcal{F}(X)$ in a neighborhood of $\Delta^n[1]$. Finally, using the local normal form of X around θ we give global normal forms for the η_j and conclude the linearization of $\mathcal{F}(X)$ as stated.

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Our result is a step towards a compact leaf theorem for holomorphic foliations of dimension one, for its proof states the existence of nontrivial (hyperbolic) holonomy for such foliations which are transverse to the boundary of certain product domains as $S^1 \times \Delta^{n-1}(1)$. Of course, additional hypotheses shall be made on the foliation, like existence of some leaf with subexponential growth or, more generally, of an invariant transverse measure. This is subject of incoming work.

2 Preliminary results

Before going into the foliation framework we need some results on holomorphic diffeomorphisms.

Proposition 2. *Let M be a complex manifold of dimension $m \geq 1$. Let $F : M \to M$ be a holomorphic map having a fixed point $p \in M$ such that dF_p is in the Poincaré domain. If there exist m hypersurfaces which are F-invariant and linearly independent at p then F is holomorphically linearizable in a neighborhood of p.*

Proof. Up to work with F^{-1} we can assume that p is attractive. Moreover, being the result of local nature, we can assume that $M = \mathbb{C}^m$ and $p = 0$ the origin of $\mathbb{C}^m$. We are going to give the proof by induction on m. If $m = 1$ the result is true. Assume it is true for $m - 1$.

First of all we show that dF_0 is diagonalizable. Indeed, if $H_1, \ldots, H_m$ are the F-invariant hypersurfaces, the m (non-singular) curves $\gamma_j := H_1 \cap \ldots \cap \widehat{H_j} \cap \ldots \cap H_m$ (here $\widehat{H_j}$ is omitted) are F-invariant and linearly independent at 0. Since $dF_0(T_0\gamma_j) = T_0\gamma_j$ then dF_0 is diagonalizable. Therefore we can change coordinates in such a way that dF_0 is diagonal. Moreover, up to linear changes of coordinates we can assume that $T_0\gamma_j$ is generated by $E_j = (e_{ji})_{i=1,\ldots,m}$, $e_{ji} = \delta^i_j$. All the changes of coordinates perform in the rest of the proof are tangent to the identity and therefore these properties will survive.

Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of dF_0 labelled so that $1 > |\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| > 0$. We claim that λ_1 is *not* resonant (see [2]). Indeed, if it were $\lambda_1 = \lambda_1^{s_1} \cdots \lambda_m^{s_m}$ with $s_j \in \mathbb{N}$ and $s_1 + \ldots + s_m \geq 2$ then for each $s_j > 0$ we would have $|\lambda_1| < |\lambda_j|$ contrary to our definition of λ_j's. Similarly one can show that λ_j can be resonant only with the λ_k's such that $k < j$ and $|\lambda_k| > |\lambda_j|$. Therefore, by the Poincaré-Dulac Theorem (see, *e.g.*, [2]) we can holomorphically change coordinates to have

$$F(z_1, \ldots, z_m) = (\lambda_1 z_1, \lambda_2 z_2 + f_2(z_1), \ldots, \lambda_m z_m + f_m(z_1, \ldots, z_{m-1})),$$

where f_j are holomorphic polynomials which are the sum of the resonant terms in F_j. The aim is to show that $f_j \equiv 0$ for all j. Assume on the contrary that $f_j \equiv 0$ for $j = 2, \ldots, r-1$ and $f_r \not\equiv 0$, for some $2 \leq r \leq m$. First, we claim that there exists an F-invariant $(r-1)$-dimensional submanifold $Z \subset \mathbb{C}^m$ passing through 0 and such that Z is parameterized by

$$\Gamma : (\zeta_1, \ldots, \zeta_{r-1}) \mapsto (\zeta_1, \ldots, \zeta_{r-1}, \varphi_r(\zeta_1, \ldots, \zeta_{r-1}), \ldots, \varphi_m(\zeta_1, \ldots, \zeta_{r-1})),$$

with $\varphi_j(0) = 0$ and $\frac{\partial \varphi_j}{\partial \zeta_k}(0) = 0$ for $j = r, \ldots, m$ and $k = 1, \ldots, r-1$. Indeed, taking into account our preliminary choice of coordinates, intersecting $m - r + 1$ suitably chosen F-invariant hypersurfaces given by hypothesis, one can find an $(r-1)$-dimensional F-invariant submanifold whose tangent space at 0 is generated by $E_1, \ldots, E_{r-1}$. Then an application of the implicit function theorem gives Z with its parametrization, as wanted. Writing down the equation $F(Z) \subset Z$ in terms of Γ, we obtain the following functional equation

$$F(\Gamma(\zeta_1, \ldots, \zeta_{r-1}) = \Gamma(\zeta_1^0, \ldots, \zeta_{r-1}^0),$$

namely, looking at the first r-terms,

$$\lambda_r \varphi_r(\zeta_1, \ldots, \zeta_{r-1}) + f_r(\zeta_1, \ldots, \zeta_{r-1}) = \varphi_r(\lambda_1 \zeta_1, \ldots, \lambda_{r-1} \zeta_{r-1}), \tag{2.1}$$

for all $\zeta_1, \ldots, \zeta_{r-1} \in \mathbb{C}$ close to 0. Notice that, since $f_r \not\equiv 0$ then $\varphi_r \not\equiv 0$. Now write

$$\varphi_r(\zeta_1, \ldots, \zeta_r) = \sum a_{k_1 \ldots k_{r-1}} \zeta_1^{k_1} \cdots \zeta_{r-1}^{k_{r-1}}$$

and put this expression in (2.1). Equating coefficients with the same $(r-1)$-degrees we find that necessarily $f_r \equiv 0$. Indeed, assume that f_r contains a term like $c\zeta_1^{s_1} \cdots \zeta_{r-1}^{s_{r-1}}$. This is possible only if $\lambda_r = \lambda_1^{s_1} \cdots \lambda_{r-1}^{s_{r-1}}$. Looking at terms of degrees $(s_1, s_2, \ldots, s_{r-1})$ in $(\zeta_1, \ldots, \zeta_{r-1})$ we find

$$\lambda_r a_{s_1 \ldots s_{r-1}} + c = a_{s_1 \ldots s_{r-1}} \lambda_1^{s_1} \cdots \lambda_{r-1}^{s_{r-1}}$$

and hence $c = 0$. □

Lemma 1. *Let $D \subset \mathbb{C}^m$ be a bounded convex domain or a bounded strongly pseudoconvex domain and $F : D \to D$ a holomorphic map. Assume that there exists a point $p \in D$ such that $F(p) = p$ and dF_p is in the Poincaré-domain. Then F is holomorphically conjugate to its Poincaré-Dulac normal form in D, .i.e, there is holomorphic diffeomorphism $\psi \colon D \to \mathbb{C}^n$ such that $F = \psi^{-1} \circ F \circ \psi \colon D \to D$ where $F_0 \colon \mathbb{C}^n \to \mathbb{C}^n$ is a Poincaré-Dulac normal form for F in a neighborhood of $p \in D$.*

Proof. By the Cartan-Carathéodory theorem (see, *e.g.*, [1]) the point p is attractive and its basin of attraction is D then one can extend the local linearizing map to D. $\square$

Finally we collect here, as a lemma, main facts from [9] and [10]:

Lemma 2. *Let X be a holomorphic vector field in a neighborhood of the closed unit ball $B^n[1] \subset \mathbb{C}^n$, $n \geq 2$. Assume that X is transverse to the boundary $S^{2n-1}(1) = \partial B^n[1]$ and singular at the origin 0. Then*

1. *The only singularity of X in $B^n[1]$ is the origin $0 \in \mathbb{C}^n$.*
2. *For all $0 < R < 1$ the vector field X is transverse to the sphere $S^{2n-1}(R) = \partial B^n[R]$.*
3. *The origin is a simple singularity of X and the linear part $DX(0)$ is in the Poincaré domain.*
4. *Each solution L of X which crosses $S^{2n-1}[1]$ tends to the origin, i.e., $0 \in \overline{L}$.*

We will also need the following fact:

Lemma 3. *Let X be a holomorphic vector field with a isolated singularity $p \in U \subset \mathbb{C}^n$, $n \geq 2$, U connected. Assume that the singularity is linearizable and in the Poincaré domain. Let h be a holomorphic function in U such that $dh(X) \equiv 0$ in U (outside the polar set of h). Then h is constant.*

Proof. Assume that $p = 0$ and U contains a small polydisc Δ^n centered at the origin where h has a Taylor expansion. Choose coordinates for which X is linear, *i.e.*, $X = \sum_{j=1}^n \alpha_k z_k \frac{\partial}{\partial z_k}$ and h is given by

$$h = \sum_{j_1,\dots,j_n} h_{j_1,\dots,j_n} z_1^{j_1} \cdots z_n^{j_n},$$

for $j_k \geq 0$ and $k = 1, \dots, n$. The condition $dh(X) \equiv 0$ in Δ^n (off the polar set of h) implies

$$\sum_{j_1,\dots,j_n} h_{j_1,\dots,j_n}(\alpha_1 j_1 + \ldots + \alpha_n j_n) z_1^{j_1} \cdots z_n^{j_n} = 0.$$

Thus $h_{j_1,\dots,j_n}(\alpha_1 j_1 + \ldots + \alpha_n j_n) = 0$ for all $j_1, \dots, j_n$. If $\alpha_1 j_1 + \ldots + \alpha_n j_n = 0$ for some $j_1, \dots, j_n \neq 0$ then 0 is not in the Poincaré domain for then 0 belongs to the convex hull of $(\alpha_1, \dots, \alpha_n)$ (indeed it would be $0 = \sum_{k=1}^n \alpha_k (j_k/J)$ with $J = \sum_{k=1}^n j_k > 0$), against our hypothesis. Therefore $h_{j_1,\dots,j_n} = 0$ for all $(j_1, \dots, j_n) \neq (0, \dots, 0)$, and thus h is constant. $\square$

3 Vector fields transverse to certain product domains

In this section we study the transversely holomorphic flows induced by $\mathcal{F}(X)$ on the components of $\partial\Delta^n(1)$ diffeomorphic to $S^1 \times B_p^n[1]$. We will prove these flows correspond to suspensions of holomorphic diffeomorphisms $F\colon V \supset \Delta^{n-1}[1] \to F(V) \supset \Delta^{n-1}[1]$ which are either linearizable diagonal or conjugate to a Poincaré-Dulac normal form. In particular, F always has a single fixed point θ, which is attractive, and dF_θ is in the Poincaré domain. The first result is a generalization of the main construction in [3].

Lemma 4. *Let X be a holomorphic vector field in a neighborhood W of the product $S^1 \times B_p^n[R] \subset \mathbb{C} \times \mathbb{C}^n$ and assume that*

1. *X is transverse to $S^1 \times B_p^n(R)$,*

2. X is transverse to $S^1 \times S_p^{2n-1}(R)$.

Then X induces a transversely holomorphic flow $\mathcal{L}$ in the manifold with boundary $S^1 \times B_p^n[R]$ which is transverse to the boundary and has a single periodic orbit $\gamma \subset S^1 \times B_p^n(R)$ with the following properties:

1. γ meets transversely every fiber $\{z\} \times B_p^n[R]$,

2. the holonomy map associated to γ is defined in all $B_p^n(R)$ and is an attractor.

Proof. Denote by $\mathcal{L}$ the transversely holomorphic flow defined by X in $S^1 \times B_p^n[R]$. By transversality hypothesis this is a non-singular flow. Orient $\mathcal{L}$ in such a way that it points inward $S^1 \times B_p^n[R]$ from the boundary. By the transversality hypothesis it follows that $\mathcal{L}$ is transverse to all the fibers $\{z\} \times B_p^n(R)$, $z \in S^1$. Fix a point $z_0 \in S^1$ and let $(z_0, y) \in \{z_0\} \times B_p^n[R]$. From our orientation of $\mathcal{L}$ and by transversality hypothesis, the orbit of $\mathcal{L}$ starting from (z_0, y) necessarily meets (transversely) all fibers $\{z\} \times B_p^n(R)$ and comes back to the starting fiber at some point $(z_0, h(z_0, y)) \in \{z_0\} \times B_p^n(R)$. Thus we have a well defined first return map

$$\begin{gathered} H \colon S^1 \times B_p^n[R] \to S^1 \times B_p^n(R), \\ (z_0, y) \mapsto (z_0, h(z_0, y)). \end{gathered}$$

This map is clearly holomorphic in y. Moreover, since X is holomorphic, it is also analytic in $z \in S^1$. Now, fix $z_0 \in S^1$ and consider $H_{z_0} : B_p^n[R] \to B_p^n(R)$. By the Brouwer fixed point theorem there exists at least one fixed point $y(z_0) \in B_p^n(R)$. Moreover, by iteration theory (see, *e.g.*, [1]) this fixed point is unique and $H_{z_0}^{\circ k}(y) \to y(z_0)$ as $k \to \infty$ for all $y \in B_p^n[R]$. Thus $y(z_0)$ gives rise to a (analytic) periodic orbit γ of $\mathcal{L}$ in $S^1 \times B_p^n(R)$. This periodic orbit γ is unique and intersects each fiber $\{z\} \times B_p^n[R]$ in $y(z)$, the unique fixed point of H_z. Finally, it is clear by construction that H_z is an attractor for all $z \in S^1$. □

Lemma 5. *Let X be a holomorphic vector field defined in a neighborhood U of the closed set $B_p^n[R]$, $p \geq 2$ and transverse to the boundary $S_p^{2n-1}(R)$. Then X has a unique singularity $\theta \in B_p^n(R)$ in $B_p^n[R]$ which is in the Poincaré domain. Moreover, any leaf L of the foliation $\mathcal{F}(X)$ defined by X which crosses $\partial B_p^n(R)$ tends to θ,* i.e., *$\theta \in \overline{L}$.*

Proof. This is essentially proved in [9] for one can adapt his proof to the case of any strongly convex domain. However, for the reader's convenience, we give here a sketch of the proof. First, the main result of [4] implies that under our hypothesis there exists a biholomorphism ψ from a neighborhood of $B_p^n[R]$ onto a neighborhood of the closed unit ball $B_1^n[1]$ and a holomorphic vector field v in this latter set such that $v = \psi_*(X)$ and v is transverse to the spheres $S_1^{2n-1}(r)$ for all $r \in (0, 1]$. By Lemma 2 and since $X = \psi_*^{-1}(v)$ then the result follows. □

From now on, we assume X to be a holomorphic vector field in a neighborhood W of the polydisc $\Delta^n[1] \subset \mathbb{C}^n$, transverse to $\partial\Delta^n(1)$ in the sense of Definition 2. By Lemma 5 the vector field X admits a unique singularity $\theta \in \Delta^n$ in $\overline{\Delta^n}$ which is in the Poincaré domain. By a Moebius map we can assume that $\theta = 0$ the origin of $\mathbb{C}^n$. We recall that a *local separatrix* of X through 0 is an irreducible (germ of) analytic invariant curve $\Gamma(0)$ passing through the origin. Since $\Gamma(0) \setminus \{0\}$ is connected (of punctured disc type) it follows that for a sufficiently small neighborhood U of the origin we have $\Gamma(0) \setminus \{0\} = L \cap U$ for some leaf L of $\mathcal{F}(X)$. Conversely, by Remmert-Stein extension theorem [7], if a leaf L of $\mathcal{F}(X)$ is such that for some neighborhood U of the origin we have $\overline{L} \setminus L = \{0\}$ (i.e., L is closed off the origin in a neighborhood of the origin and accumulates at the origin) then $L \cap U = \Gamma(0) \setminus \{0\}$ for some local separatrix $\Gamma(0) \subset U$ of $\mathcal{F}(X)$ through the origin. Motivated by these facts we will call a *global separatrix* of $\mathcal{F}(X)$ in W an invariant irreducible analytic curve $\Gamma \subset W$ such that $0 \in \Gamma$, equivalently, $0 \in \Gamma$ and $\Gamma \setminus \{0\} = L$ is a leaf L of $\mathcal{F}(X)$, equivalently, $\Gamma = L \cup \{0\}$ for a leaf L of $\mathcal{F}(X)$ which is closed outside 0 and accumulates the origin.

Lemma 6. *The foliation $\mathcal{F}(X)$ has exactly n local separatrices through the origin.*

Proof. For sake of clearness we give the proof in case $n = 3$. Consider a component $S^1 \times \overline{\Delta^2} \subset \overline{\Delta^3}$ with boundary $\partial(S^1 \times \overline{\Delta^2} = S^1 \times \partial\Delta^2$. By hypothesis X is transverse to the boundary $S^1 \times \partial\Delta^2$ and therefore by definition X is transverse to $S^1 \times \Delta^2$, to $S^1 \times B^2_p(1)$ and to $S^1 \times \partial B^2_p(1)$ for all $p >> 1$. By Lemma 4 the restriction of the foliation $\mathcal{F}(X)$ to $S^1 \times B^2_p[1]$ is a transverse holomorphic flow with a unique periodic orbit $\gamma_1(1)$ whose holonomy map is an attractor defined in $B^2_p[1]$.

Now we are going to slightly perturb the product $S^1 \times \Delta^2[1]$ as $S^1(s) \times \Delta^2[1]$ (here, as usual, $S^1(s) = \partial\Delta^2_1(s))$ for $|s - 1| < \epsilon$. We claim that if $\epsilon > 0$ is small enough than X is transverse to $S^1(s) \times \Delta^2(1)$, to $S^1(s) \times B^2_p(1)$ and to $S^1(s) \times S^3_p(1)$ for all $p \geq 1$. This is clearly true for any p fixed and our definition of transversality implies that we can in fact find such an $\epsilon > 0$ which works for all p. Indeed, assuming by contradiction that for any $m \in \mathbb{N}$ there exist p_m and z_m such that X is not transverse to, say, $S^1(s) \times B^2_p(1)$ at z_m, then passing to the limit we find a point $z \in S^1 \times \Delta^2[1]$ such that X is not transverse to some components of $S^1 \times \Delta^2[1]$ at z.

Fix such a small $\epsilon > 0$. As before, the flow induced by X on $S^1(s) \times B^2_p[1]$ has a unique periodic orbit $\gamma_s(1)$ whose holonomy map is an attractor defined in the ball $B^2_p[1]$.

Let $\Gamma_1 := \bigcup_{|s-1|<\epsilon} \gamma_s(1)$ be the union of the periodic orbits. Then $\Gamma_1 \subset A_1 \times B^2_p(1) \subset \mathbb{C} \times \mathbb{C}^2$ where $A_1 := \{\zeta \in \mathbb{C} : |\zeta - 1| < \epsilon\}$. Notice that Γ_1 is a "piece" of an attractive leaf of $\mathcal{F}(X)$ in $A_1 \times \Delta^2(1)$.

Analogously using the components $\Delta \times S^1 \times \Delta$ and $\Delta^2[1] \times S^1$ we obtain corresponding "pieces" of attractive closed leaves $\Gamma_2 \subset \Delta \times A_2 \times \Delta$ and $\Gamma_3 \subset \Delta^2 \times A_3$. By Lemma 5 the leaves $\Gamma_j(0)$'s tend to 0. Therefore X has at least three separatrices at 0. On the other hand, by the Maximum Principle, any separatrix of X must intersect some slice as above and therefore, X has exactly three separatrices through 0. Moreover, this shows that any local separatrix of X is contained in a unique global separatrix of X. $\square$

Now we fix some notation. We denote by $\Lambda(X, 0)$ the union of germs of invariant analytic hypersurfaces through the origin and write $\Lambda(X, 0) = \bigcup_{j=1}^{r} \Lambda_j(X, 0)$ its decomposition in irreducible components. Then we have $1 \leq r \leq n$. Denote $\Lambda(X)$ the union of globally defined invariant hypersurfaces passing through the origin and write $\Lambda(X) = \bigcup_{j=1}^{r_1} \Lambda_j(X)$ its decomposition in irreducible components. We have $r \leq r_1$ and we can assume that $\Lambda_j(X, 0) \subset \Lambda_j(X)$ for all $j = 1, ..., r$. Later on, we will see that any local invariant hypersurface corresponds to a unique global invariant hypersurface and thus $r = r_1$. Also let us denote by $\text{Sep}(X, 0) = \bigcup_{j=1}^{n} \Gamma_j(0)$ the union of local separatrices of X through the origin and by $\text{Sep}(X)$ the set of global separatrices passing through the origin and write $\text{Sep}(X) = \bigcup_{j=1}^{s_1} \Gamma_j$ in irreducible components. From the final part of the proof of Lemma 6 we have that $s = n$ and we can assume that $\Gamma_j(0)$ is the germ of Γ_j at the origin, i.e., $\Gamma_j(0) \subset \Gamma_j$, $j = 1, ..., n$.

For each $j \in \{1, ..., n\}$ we denote by Σ^{2n-1}_j the component of the boundary of $\Delta^n(1)$ which is diffeomorphic to $S^1 \times \Delta^{n-1}[1] \subset \mathbb{C} \times \mathbb{C}^n$ and defined by $|z_j| = 1$. We can consider the induced "holonomy map" $F_j \colon \Delta^{n-1}[1] \to \Delta^{n-1}[1]$, obtained as the holonomy maps of the induced flows on approximations $S^1 \times B^n_p[R]$ of Σ^{2n-1}_j. This map is well-defined up to analytic conjugation in $\text{Aut}(\Delta^{n-1}[1])$ and we can assume that F_j has a single fixed point at the origin, which is an attractor. In particular, according to Lemma 1 we have two possibilities for F_j:

1. F_j is analytically linearizable in a neighborhood of Σ^{2n-1}_j.

2. F_j is analytically conjugate to its Poincaré-Dulac normal form in a neighborhood of Σ_j^{2n-1}.

4 Semi-local constructions of differential forms

In this section we construct systems of closed meromorphic one-forms, in neighborhoods W_j of the components Σ_j^{2n-1}, which define the foliation $\mathcal{F}(X)$ and are closely related to the transverse dynamics in Σ_j^{2n-1}. We fix a component Σ_j^{2n-1} and a sufficiently small neighborhood of Σ_j^{2n-1} in $S^1 \times \mathbb{C}^n$ where we shall work. Given the holonomy map $F_j \colon \Delta^{n-1}[1] \to \Delta^{n-1}[1]$ we assume first that F_j is linearizable. Let $\gamma_j(1)$ be the periodic orbit in Σ_j^{2n-1} and fix a point $z_j^0 \in S^1$. Since the holonomy map F_j is linearizable diagonal one can find holomorphic coordinates $(u_1, ..., u_{n-1})$ in $\{z_1^0\} \times \Delta^{n-1}[1]$ such that the holonomy map is linear of the form $F_j(u_1, ..., u_{n_1}) = (\lambda_1 u_1, ..., \lambda_{n_1} u_n)$ for some $\lambda_1, ..., \lambda_{n-1} \in \mathbb{C}$ with $0 \neq |\lambda_k| < 1$ for all $k = 1, ..., n-1$. Define a system of one-forms $\eta_2^j, ..., \eta_n^j$ in the transverse section $\{z_1^0\} \times \Delta^1[1]$ as follows:

$$\eta_k^j(u_1, ..., u_{n-1}) := \frac{du_k}{u_k}\,;\ k = 2, ..., n.$$

Since $F_j^*(\eta_j^k) = \eta_j^k$, $j = 2, ..., n$, these forms admit holonomy extensions to a neighborhood of Σ_j^{2n-1} which we still denote by η_j^k, and satisfy (by the holonomy extension) $\eta_j^k(X) \equiv 0$, $k = 2, ..., n$. Each form η_k^j is closed and transversely meromorphic with simple poles in Σ_j^{2n-1}. Moreover for each $z_j \in S^1$ the intersections $(\eta_k^j)_\infty \cap (\{z_j\} \times \Delta^{n-1}[1])$, $k = 2, ..., n$, are the invariant hypersurfaces of the corresponding holonomy map.

Arguing as in the proof of Lemma 6 we can indeed construct the system of one-forms $\{\eta_k^j\}_{k=2}^{k=n}$ in a product $W_j = A_j \times V_j^{n-1}$, where $A_j = \{z_j \in \mathbb{C} : |z_j - 1| < \epsilon\}$ and $V_j^{n-1} = \{(z_1, ..., \hat{z}_j, ..., z_n) \in \mathbb{C}^{n-1} : |z_\ell - 1| < \epsilon, \forall \ell = 1, ..., n,\ \ell \neq j\}$ for some small $\epsilon > 0$.

Assume now that F_j is not linearizable. For simplicity of notation we suppose $n = 3$. Let us first make some general considerations. Take a holomorphic vector field Z in a neighborhood of the origin $0 \in \mathbb{C}^2$, with an isolated singularity at the origin, in the Poincaré domain, but not linearizable in a neighborhood of the origin. Then the Poincaré-Dulac theorem implies that we can find local holomorphic coordinates (x, y) in a neighborhood of the origin such that $Z(x, y) = (nx + cy^n)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, for some $n \in \mathbb{N} \setminus \{1\}$ and some $c \in \mathbb{C} \setminus \{0\}$. We can assume that $c = 1$. Straightforward forward integration then shows that the flow Z_t of Z is given in a neighborhood of the origin by $Z_t(x, y) = ((x + ty^n)e^{nt}, ye^t)$, $t \in \mathbb{C}$. If we put $F = Z_a$ for some time $a \in \mathbb{C}$ and $\lambda = e^{an}, \mu = e^a, c = a\lambda$ then we have

$$F(x, y) = (\lambda x + cy^n, \mu y)$$

where $\lambda = \mu^n$.

In particular F preserves the vector field Z, i.e., $F_*(Z) = Z$. Thus it must preserve its dual one-forms. That is the idea we want to use. We look for a pair of independent one-forms $\eta_k, k = 1, 2$ such that $\eta_k(Z) \equiv 1$ and also η_k is closed and meromorphic. We can choose $\eta_2 = \frac{dy}{y}$ which clear is closed and dual to Z in the sense that $\eta_2(Z) \equiv 1$. The one-form η_1 must satisfy $\eta_1(Z) \equiv 1$ and be closed. In order to find η_1 we consider the one-form $\Omega = ydx - (nx + y^n)dx$ which satisfies $\Omega(Z) \equiv 0$ and observe that $\omega := \frac{1}{y^{n+1}}\Omega = \frac{dx}{y^n} - nx\frac{dy}{y^{n+1}} - \frac{dy}{y}$ which is clearly closed, meromorphic and tangent to Z. Thus we can take $\eta_1 = \omega + \eta_2$ satisfies our requirements. We have $\eta_1 = \frac{dx}{y^n} - nx\frac{dy}{y^{n+1}}$ which is closed, dual to Z and independent with η_2 (outside $y = 0$ where both one-forms have poles).

We resume our current situation, where $F = F_j$ is not linearizable. Then, it must be of the form $F(x, y) = (\lambda x + ..., \mu y + ...)$ in a local chart (x, y). Since it is not linearizable we must have some resonance in the eigenvalues λ, μ. Let us examine such possible resonances. Since F is attractive we must have $0 < |\lambda| < 1$, $0 < |\mu| < 1$. The only possibility is $\lambda = \lambda^{s_1}\mu^{s_2}$ with $s_1 \geq 0, s_2 \geq 0$ and $s_1 + s_2 \geq 2$. If $s_1 = 1$ then $\mu^{s_2} = 1$ what is not possible. If $s_1 > 1$ then $\mu^{s_2} = \lambda^{1-s_1}$ what is not possible either. Thus the only possibility is, up to reordering the coordinates, $s_1 = 0$ and $\lambda = \mu^{s_2}$. Therefore, we have a normal form like $F(x, y) = (\lambda x + cy^s, \mu y)$ for some $c \in \mathbb{C} \setminus \{0\}$ and $\lambda = \mu^s$ for some $s \in \mathbb{N}, s \geq 2$. By the above considerations we can, arguing as in the linearizable case, construct a system of two one-forms η_1^j, η_2^j in a product $W_j = A_j \times V_j^2$, where $A_j = \{z_j \in \mathbb{C} : |z_j - 1| < \epsilon\}$ and $V_j^2 = \{(z_1, ..., \hat{z}_j, ..., z_3) \in \mathbb{C}^2 : |z_\ell - 1| < \epsilon, \forall \ell = 1, ..., 3,\ \ell \neq j\}$ for some small $\epsilon > 0$. For the general case we obtain therefore:

Lemma 7. *Given any $j \in \{1, ..., n\}$ we can construct a system closed meromorphic of one-forms $\{\eta_k^j\}_{k=2}^{k=n}$ in a product $W_j = A_j \times V_j^{n-1}$, where $A_j = \{z_j \in \mathbb{C} : |z_j - 1| < \epsilon\}$ and $V_j^{n-1} = \{(z_1, ..., \hat{z}_j, ..., z_n) \in \mathbb{C}^{n-1} : |z_\ell - 1| < \epsilon, \forall \ell = 1, ..., n,\ \ell \neq j\}$ for some small $\epsilon > 0$ with the following properties:*

1. *The one-forms $\{\eta_k^j\}_{k=2}^{k=n}$ are meromorphic, closed and the system has rank $n-1$ outside the union of the polar sets, which is of complex codimension one.*
2. *The system $\{\eta_k^j\}_{k=2}^{k=n}$ is integrable and defines the foliation $\mathcal{F}(X)$ in W_j.*
3. *The holonomy map F_j associated to the component Σ_j^{2n-1} is linearizable if and only if the one-forms have simple poles.*

5 Global construction of differential forms

This section consists of a globalization of the semi-local construction performed in Section 4. This is done by gluing together the systems $\{\eta_k^j\}_{k=2}^n$ obtained for each $j = 1, ..., n$. We divide the argumentation in two cases.

5.1 The linearizable case

Let us assume that each component Σ_j^{2n-1} has a linearizable holonomy map F_j for all $j = 1, ..., n$. For the sake of simplicity once again we will assume that $n = 3$. From the above Lemma 7 we can construct a pair of closed meromorphic one-forms $\{\eta_2^1, \eta_3^1\}$ with simple poles in a product $W_1 = A_1 \times V_1^2$, where $A_1 = \{z_1 \in \mathbb{C} : |z_1 - 1| < \epsilon\}$ and $V_1^2 = \{(z_2, z_3) \in \mathbb{C}^2 : |z_2 - 1| < \epsilon, |z_3 - 1| < \epsilon\}$ for some small $\epsilon > 0$. Similarly, for the slices $\Delta^1[1] \times S^1 \times \Delta^1[1]$ and $\Delta^2[1] \times S^1$ we construct pairs of closed meromorphic one-forms $\{\eta_2^2, \eta_3^2\}$ and $\{\eta_2^3, \eta_3^3\}$ with simple poles in suitable neighborhoods W_2, W_3 of $\Delta^1[1] \times A_2 \times \Delta^1[1]$ and $\Delta^2[1] \times A_3$. Moreover $\eta_k^j(X) \equiv 0$ on W_j for $j = 2, 3$ and $k = 2, 3$. We can assume that $W_j \cap W_k$ is connected for $j, k = 1, 2, 3$.

Lemma 8. *There exist constants $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ and*

$$\eta_2^2 = a\eta_2^1 + b\eta_3^1 \quad \text{and} \quad \eta_3^2 = c\eta_2^1 + d\eta_3^1$$

on $W_1 \cap W_2$.

Proof. Since $\{\eta_2^1, \eta_3^1\}$ and $\{\eta_2^2, \eta_3^2\}$ define the same foliation $\mathcal{F}(X)$ in $W_1 \cap W_2$, then $\eta_2^2 = a\eta_2^1 + b\eta_3^1$ and $\eta_3^2 = c\eta_2^1 + d\eta_3^1$ for some meromorphic functions a, b, c, d. We have to show that a, b, c, d are in fact constant for a suitable choice. The main remark is that the functions a, b, c and d can be chosen as *holomorphic*. All one has to do is to reorder the one-forms η_k^1 and η_k^2 where $k = 2, 3$ in such a way that the polar sets $(\eta_k^1)_\infty$ and $(\eta_k^2)_\infty$ coincide in the common domains of the one-forms. We have to show that a, b, c, d are in fact constant.

Taking the exterior derivative of $\eta_2^2 = a\eta_2^1 + b\eta_3^1$ we find

$$0 = da \wedge \eta_2^1 + db \wedge \eta_3^1$$

and then

$$0 = da \wedge \eta_2^1 \wedge \eta_3^1.$$

This implies that the function a is constant along the orbits of X in $W_1 \cap W_2$. Therefore a is holomorphic and constant along the orbits of X. Now we use the fact that η_k^j is F_j-invariant by construction, i.e., $F_j^*(\eta_k^j) = \eta_k^j$ so that from the equation $\eta_2^2 = a\eta_2^1 + b\eta_3^1$ we obtain that$0 = (F_1^*(a) - a)\eta_2^1 + (F_1^*(b) - b)\eta_3^1$ so that, since the one-forms η_k^j are independent off the polar sets, we obtain $a \circ F_1 = a$ and $b \circ F_1 = b$ also from the identity principle. Now the dynamics of the diffeomorphism F_1 implies that the only holomorphic functions constant along its orbits are the constants, thus a and b are constant.

Remark 2. Notice that the set $W_1 \cap W_2$ is an open neighborhood of the intersection $(S^1 \times \Delta^2[1]) \cap (\Delta^1[1] \times S^1 \times \Delta^1[1]) = S^1 \times S^1 \times \Delta^1[1]$. By hypothesis X is transverse to $S^1 \times S^1 \times \Delta^1(1)$ in $\mathbb{C}^3$. Thus $S^1 \times S^1 \times \Delta^1(1)$ is a real transverse section to $\mathcal{F}(X)$ representing the leaf space of $\mathcal{F}(X)$ in a neighborhood of $S^1 \times S^1 \times \Delta^1(1)$. It follows that any holomorphic first integral for $\mathcal{F}(X)$ $W_1 \cap W_2$ is defined on $S^1 \times S^1 \times \Delta^1(1)$ and thus it is constant.

Similarly we prove that c and d are constant. □

Thus we can extend the pair $\{\eta_2^1, \eta_3^1\}$ to $W_1 \cup W_2$ by means

$$\begin{pmatrix} \eta_2^1 \\ \eta_3^1 \end{pmatrix} |_{W_2} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} \eta_2^2 \\ \eta_3^2 \end{pmatrix}.$$

A similar argumentation allows to extend $\{\eta_2^1, \eta_3^1\}$ to W_3. Therefore $\{\eta_2^1, \eta_3^1\}$ can be defined in a neighborhood of $\partial\Delta^3[1]$. Hence, by Hartogs' Extension Theorem (see, *e.g.*, [13]) we have the extension of the one-forms η_2, η_3 to a neighborhood of $\Delta^3[1]$. We state the general n-dimensional conclusion as follows:

Lemma 9. *There exist $n-1$ closed meromorphic one-forms $\{\eta_2, ..., \eta_n\}$, with simple poles, defined in a neighborhood W of $\Delta^n[1]$ such that*

1. $\eta_j(X) \equiv 0,\ j = 2, ..., n.$
2. *The system $\{\eta_k\}_{k=2}^{k=n}$ has rank $n-1$ off the polar sets and defines the foliation $\mathcal{F}(X)$.*
3. $(\eta_2)_\infty \cap ... \cap (\eta_n)_\infty = (\Gamma_1 \cup ... \cup \Gamma_n)$ *and*
4. $(\eta_2)_\infty \cup ... \cup (\eta_n)_\infty = \bigcup_{j=1}^n \Lambda_j(X) = \Lambda(X).$

5.2 The nonlinearizable case

Now we assume that some component Σ_j^{2n-1} has a nonlinearizable holonomy map F_j for some $j = 1, ..., n$. Again we assume that $n = 3$. As in the preceding section, an application of Lemma 7 gives the construction of a pair of closed meromorphic one-forms $\{\eta_2^1, \eta_3^1\}$ in a product $W_1 = A_1 \times V_1^2$, and pairs of closed meromorphic one-forms $\{\eta_2^2, \eta_3^2\}$ and $\{\eta_2^3, \eta_3^3\}$ in suitable neighborhoods W_2, W_3 of $\Delta^1[1] \times A_2 \times \Delta^1[1]$ and $\Delta^2[1] \times A_3$. Moreover $\eta_k^j(X) \equiv 0$ on W_j for $j = 2, 3$ and $k = 2, 3$. We can assume that $W_j \cap W_k$ is connected for $j, k = 1, 2, 3$.

Lemma 10. *There exist constants $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ and*

$$\eta_2^2 = a\eta_2^1 + b\eta_3^1 \quad \textit{and} \quad \eta_3^2 = c\eta_2^1 + d\eta_3^1$$

on $W_1 \cap W_2$.

Proof. We proceed similarly to the proof of Lemma 8. Therefore, because $\{\eta_2^1, \eta_3^1\}$ and $\{\eta_2^2, \eta_3^2\}$ define the same foliation $\mathcal{F}(X)$ in $W_1 \cap W_2$, we have $\eta_2^2 = a\eta_2^1 + b\eta_3^1$ and $\eta_3^2 = c\eta_2^1 + d\eta_3^1$ for some meromorphic functions a, b, c, d and $ad - bc \neq 0$ and we have to show that a, b, c, d are in fact constant. Again this comes from the fact that the functions a, b, c and d can be chosen as *holomorphic* if we reorder the one-forms η_k^1 and η_k^2 where $k = 2, 3$ in such a way that the polar sets $(\eta_k^1)_\infty$ and $(\eta_k^2)_\infty$ coincide in the common domains of the one-forms. Then, as before, the invariance of the η_k^j with respect to the holonomy maps F_j as well as the local dynamics of the F_j imply that the functions are constant indeed. □

Lemma 11. *There exist $n - 1$ closed meromorphic one-forms $\{\eta_2, ..., \eta_n\}$, not all having simple poles, defined in a neighborhood W of $\Delta^n[1]$ such that*

1. $\eta_j(X) \equiv 0$, $j = 2, ..., n$.
2. *The system $\{\eta_k\}_{k=2}^{k=n}$ has rank $n - 1$ off the polar sets and defines the foliation $\mathcal{F}(X)$.*
3. $(\eta_2)_\infty \cap ... \cap (\eta_n)_\infty = (\Gamma_1 \cup ... \cup \Gamma_n)$ *and*
4. $(\eta_2)_\infty \cup ... \cup (\eta_n)_\infty = \bigcup_{j=1}^r \Lambda_j(X) = \Lambda(X)$ *where $1 \leq r \leq n$ and r is the number of global invariant hypersurfaces of $\mathcal{F}(X)$.*

6 Linearization of foliations

In this section we assume that each component Σ_j^{2n-1} has a linearizable holonomy map F_j. Under this hypothesis, we will prove that the foliation $\mathcal{F}(X)$ is linearizable in the sense of Theorem 1. We shall state it in a more general context as follows:

Proposition 3. *Let X be a holomorphic vector field in a neighborhood W of the closed polydisc $\Delta^n[1] \subset \mathbb{C}^n$, $n \geq 3$ and assume that:*

1. $\mathrm{sing}(X) \cap \Delta^n[1] = \{0\}$ *and the origin is a singularity in the Poincaré domain.*
2. *There exists a system $\{\eta_j\}_{j=2}^n$ of closed meromorphic one forms, with simple poles in W, such that the system has rank $n - 1$ (outside the polar sets) and $\eta_j(X) \equiv 0$ for $j = 2, \ldots, n$.*

Then there exists a holomorphic map $F\colon W \to \mathbb{C}^n$ such that $F(0) = 0$ and nonsingular at the origin, and a linear polynomial vector field $Z = \sum_{j=1}^n \lambda_j z_j \frac{\partial}{\partial z_j}$ in $\mathbb{C}^n$, which is in the Poincaré domain, such that the foliation $\mathcal{F}(X)$ is the pull-back $F^*(\mathcal{F}(Z))$. In this sense, $\mathcal{F}(X)$ is globally the pull-back by F of its normal form at the origin.

Our first step is the following lemma:

Lemma 12. *Let X be a holomorphic vector field in a neighborhood W of the closed polydisc $\Delta^n[1] \subset \mathbb{C}^n$, $n \geq 3$. Let $\{\eta_j\}_{j=2}^n$ be a system of closed meromorphic one forms with simple poles in W such that the system has rank $n - 1$ and $\eta_j(X) \equiv 0$ for $j = 2, \ldots, n$. Assume that X has an isolated singularity at the origin 0 which is in the Poincaré domain. Then X is linearizable in* *a neighborhood of the origin and there exist holomorphic functions $\hat{f}_j$, $j = 1,\dots,n$ defined in a neighborhood of $\Delta^n[1]$ and complex numbers $\mu_j \in \mathbb{C}^*$ such that, up to a linear automorphism of the system $\{\eta_j\}_{j=2}^n$, one can write for $j = 2,\dots,n$*

$$\eta_j = \frac{d\hat{f}_1}{\hat{f}_1} - \mu_j \frac{d\hat{f}_j}{\hat{f}_j}.$$

Proof. Assume that we have proved that X is linearizable at 0, thus there exist a neighborhood of 0 and local coordinates $x_1,\dots,x_n$ on U such that $x_j(0) = 0$ and

$$X(x_1,\dots,x_n) = \sum_{k=1}^n \alpha_k x_k \frac{\partial}{\partial x_k}, \tag{6.1}$$

for $\alpha_j \in \mathbb{C}^*$ and the origin is not in the convex hull of $\{\alpha_1,...,\alpha_n\}$ in $\mathbb{R}^2$.

In particular, the foliation $\mathcal{F}(X)$ defined by X has n separatrices at 0, whose union in U is given by

$$\mathrm{Sep}(\mathcal{F}(X),0) = \bigcup_{k=1}^n \{x_1 = \ldots = \hat{x}_k = \ldots = x_n = 0\},$$

where $\hat{x}_k$ means omitted. Moreover there are n local X-invariant hypersurfaces through the origin, whose union is given by

$$\Lambda(\mathcal{F}(X),0) := \bigcup_{k=1}^n \{x_k = 0\}.$$

Therefore the analytic set $(\eta_j)_\infty$ (which is not irreducible in general but it is $\mathcal{F}(X)$-invariant by the condition $\eta_j(X) \equiv 0$) coincides with an some component of the subset $\Lambda(\mathcal{F}(X))$, the saturated of the germ $\Lambda(\mathcal{F}(X),0)$.

In particular $\Lambda(\mathcal{F}(X))$ is analytic of pure codimension one, writes as $\Lambda(\mathcal{F}(X)) = \bigcup_{j=2}^n (\eta_j)_\infty$ and has exactly n irreducible components of codimension one. Thus we can find n reduced holomorphic functions $f_1,\dots,f_n : W \to \mathbb{C}$ such that

$$\Lambda(\mathcal{F}(X)) = \bigcup_{k=1}^n \{f_k = 0\},$$

and each $\{f_k = 0\}$ is irreducible. In particular $(\eta_j)_\infty \subset \bigcup_{k=1}^n \{f_k = 0\}$ and by the *Integration Lemma* ([5], [6]) we can write

$$\eta_j = \sum_{k=1}^n \lambda_k^{(j)} \frac{df_k}{f_k} + dF_j, \tag{6.2}$$

for some $\lambda_k^{(j)} \in \mathbb{C}$ and some holomorphic functions $F_j : W \to \mathbb{C}$, $j = 1,\dots,n$.

We claim that in U

$$\eta_j(x_1,\dots,x_n) = \sum_{k=1}^n \lambda_k^{(j)} \frac{dx_k}{x_k}, \tag{6.3}$$

and

$$\sum_{k=1}^n \alpha_k \lambda_k^{(j)} = 0 \tag{6.4}$$

for $j = 2, \ldots, n$. To prove the claim we argue as follows. Up to reordering we can assume that $\{x_k = 0\} = \{f_k = 0\} \cap U$, for $k = 1, \ldots, n$. Thus in U we can write $f_k(x_1, \ldots, x_n) = x_k g_k(x_1, \ldots, x_n)$ for some never vanishing holomorphic function g_k defined in U. Thus (6.2) implies that on U

$$\eta_j = \sum_{k=1}^{n} \lambda_k^{(j)} \frac{dx_k}{x_k} + \sum_{k=1}^{n} \lambda_k^{(j)} \frac{dg_k}{g_k} + dF_j.$$

Since $\eta_j(X) \equiv 0$, setting

$$\eta_j := \sum_{k=1}^{n} \lambda_k^{(j)} \frac{dg_k}{g_k} + dF_j,$$

we have

$$\sum_{k=1}^{n} \lambda_k^{(j)} \alpha_k + \theta_j(X) \equiv 0. \tag{6.5}$$

The one-form θ_j is closed and holomorphic in U. Therefore $\theta_j(X)(0) = 0$ for $X(0) = 0$. Using this, from (6.5) evaluated at 0 we obtain (6.4). And then, again from (6.5) we get that $\theta_j(X) \equiv 0$ in U. Since θ_j is closed and holomorphic in U, up to shrink U, we can assume that θ_j is also exact in U and set $\theta_j = dG_j$ for some holomorphic function G_j in U. Thus $dG_j(X) = 0$, $j = 2, \ldots, n$. By Lemma 3 the functions G_j are then constant and hence $dG_j \equiv 0$ proving that $\theta_j \equiv 0$ and thus (6.3).

Since by hypothesis the system $\{\eta_j\}$ has rank $n-1$, it follows from (6.3) that the complex vectors $v_j := (\lambda_1^{(j)}, \ldots, \lambda_n^{(j)}) \in \mathbb{C}^n$, $j = 2, \ldots, n$ span an $(n-1)$-dimensional subspace of $\mathbb{C}^n$. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a *linear* invertible transformation which fixes $(\alpha_1, \ldots, \alpha_n)$ and sends v_j to $(1, 0, \ldots, 0, \mu_j, 0 \ldots, 0)$ with $\mu_j = -\alpha_1/\alpha_j$, $j = 2, \ldots, n$.

Applying this linear transformation to the system $\{\eta_j\}_{j=2}^{j=n}$ we obtain a new system $\{\tilde\eta_j\}_{j=2}^{j=n}$ of the form

$$\tilde\eta_j = \frac{df_1}{f_1} + \eta_j \frac{df_j}{f_j} + d\tilde F_j,$$

for some holomorphic function $\tilde F_j : W \to \mathbb{C}$.

Now define for each $j \geq 2$ the functions $\hat f_j := f_j \exp(\tilde F_j/\mu_j)$. Then

$$\mu_j \frac{d\hat f_j}{\hat f_j} = \mu_j \frac{df_j}{f_j} + \mu_j \frac{d\tilde F_j}{\mu_j} = \mu_j \frac{df_j}{f_j} + d\tilde F_j.$$

Let us set $\hat f_1 = f_1$. Thus

$$\tilde\eta_j = \frac{d\hat f_1}{\hat f_1} + \mu_j \frac{d\hat f_j}{\hat f_j},$$

as wanted.

It remains to prove that X is indeed linearizable at the origin. This is a consequence of the fact that, by hypothesis the one-forms η_j have simple poles and therefore, any local separatrix of X through the origin has linearizable holonomy map. The singularity is therefore linearizable as a consequence of the Poincaré-Dulac theorem. □

The previous lemma implies

Corollary 1. *Let X be a holomorphic vector field defined in a neighborhood W of the polydisc $\Delta^n[1] \subset \mathbb{C}^n$, $n \geq 3$. Assume that X has an isolated singularity at $0 \in \Delta^n(1)$ which is linearizable, hyperbolic and in the Poincaré domain. Let $\{\eta_j\}_{j=2}^n$ and $\{\tilde\eta_j\}_{j=2}^n$ be two systems of closed* *meromorphic one-forms in W, with rank $n-1$ and such that for every $j = 2, \ldots, n$ we have $\eta_j(X) \equiv \tilde{\eta}_j(X) \equiv 0$. Then the two systems differ by a linear transformation of $\mathbb{C}^n$. Namely, there exists an $n \times n$ invertible matrix $T = (T_{jl})$ such that for every $j = 2, \ldots, n$ it follows*

$$\tilde{\eta}_j = \sum_{l=2}^{n} T_{jl} \eta_l.$$

Finally we have the following result:

Lemma 13. *Let $\eta_2, \ldots, \eta_n$ and X be as in Lemma 12. The foliation $\mathcal{F}(X)$ induced by X on $\Delta^n[1]$ is the pull-back of a linear hyperbolic foliation $\mathcal{F}_Z$, $Z = \sum_{j=1}^{n} \alpha_j z_j \frac{\partial}{\partial z_j}$ on $\mathbb{C}^n$ by some holomorphic map $\Phi : W \to \mathbb{C}^n$ defined in a neighborhood W of $\Delta^n[1]$. Such a map Φ is injective as a map between leaves spaces.*

Proof. Apply Lemma 12 in order to find n holomorphic function $f_1, \ldots, f_n : W \to \mathbb{C}$ in a neighborhood W of $\Delta^n[1]$ such that $\eta_j = \frac{df_1}{f_1} + \mu_j \frac{df_j}{f_j}$, for $j = 2, \ldots, n$ and also $f_k = x_k g_k$ in a neighborhood U of 0, with g_k never vanishing and $(x_1, \ldots, x_n)$ are local coordinates in U such that X is given by (6.1). Define $\Phi : W \to \mathbb{C}^n$ by

$$\Phi(p) = (f_1(p), \ldots, f_n(p)).$$

The Jacobian matrix of Φ at 0 is given by

$$\mathrm{Jac}(\Phi)(0) = \left[\frac{\partial f_k}{\partial x_l}(0)\right]_{k,l=1}^{n} = \begin{pmatrix} g_1(0) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & g_n(0) \end{pmatrix}$$

which is non singular. In particular, up to shrink U, we can assume that Φ is invertible on U and then it is a biholomorphism between U and $V = \Phi(U) \subset \mathbb{C}^n$, a open neighborhood of 0. Let $z_1, \ldots, z_n$ be global coordinates in $\mathbb{C}^n$ and let $Z = \sum_{j=1}^{n} \alpha_j z_j \frac{\partial}{\partial z_j}$. Define

$$\tilde{\eta}_j := \frac{dz_1}{z_1} + \mu_j \frac{dz_j}{z_j},$$

$j = 2, \ldots, n$. The one-forms $\tilde{\eta}_j$ are linear logarithmic on $\mathbb{C}^n$ and define a $(n-1)$-rank system $\{\tilde{\eta}_j\}_{j=2}^{n}$ such that $\tilde{\eta}_j(Z) \equiv 0$, $j = 2, \ldots, n$.

By construction $\eta_j = \Phi^*(\tilde{\eta}_j$ and then the foliation $\mathcal{F}(X)$ is the pull back of the linear foliation $\mathcal{F}_Z$ and the map $\Phi : W \to \mathbb{C}^n$ is injective near the origin $0 \in \mathbb{C}^n$ and each leaf of $\mathcal{F}(X)$ tends to the origin, therefore the map Φ is injective as a map between leaf spaces.

□

Proposition 3 is now a straightforward consequence of Lemmas 12 and 13.

7 The nonlinearizable case

In this section we deal with the nonlinearizable case for some holonomy map F_j of a component Σ_j^{2n-1}. Indeed, we study this situation in a more general context and prove the following analogous of Proposition 3:

Proposition 4. *Let X be a holomorphic vector field in a neighborhood W of the closed polydisc $\Delta^n[1] \subset \mathbb{C}^n$, $n \geq 3$ and assume that:*

1. $\operatorname{sing}(X) \cap \Delta^n[1] = \{0\}$ *and the origin is a singularity in the Poincaré domain.*

2. *There exists a system $\{\eta_j\}_{j=2}^n$ of closed meromorphic one forms in W such that the system has rank $n-1$ (outside the polar sets) and $\eta_j(X) \equiv 0$ for $j = 2, \ldots, n$.*

Then there exists a holomorphic map $F\colon W \to \mathbb{C}^n$ such that $F(0) = 0$ and nonsingular at the origin, and a polynomial vector field Z in $\mathbb{C}^n$, which is a Poincaré-Dulac normal form for X is a neighborhood of the origin, such that the foliation $\mathcal{F}(X)$ is the pull-back $F^*(\mathcal{F}(Z))$. In this sense, $\mathcal{F}(X)$ is globally the pull-back by F of its normal form at the origin.

Proof. Proposition 3 in Section 6 above refers to the linearizable case. Thus we will assume that the vector field X has resonances and has a nonlinearizable Poincaré-Dulac normal for Z in a neighborhood of the origin. Again, for the sake of simplicity we assume that $n = 3$. There are therefore three holonomy maps $F_j\colon \Delta^{n-1}[1] \to \Delta^{n-1}[1]$ corresponding to the components Σ_j^{2n-1} of the boundary of $\Delta^n[1]$, with $j = 1, 2, 3$. Some of these maps is nonlinearizable. We will consider the following situation in coordinates $(x, y, z) = (z_1, z_2, z_3) \in \mathbb{C}^3 :$ $F_1(y, z)$ is linearizable, $F_2(x, z)$ is not linearizable, $F_3(x, y)$ is not linearizable. We recall that according to Proposition 2 the holonomy map F_j is linearizable if and only if it admits two invariant hypersurfaces through the origin, and it is nonlinearizable if and only if it admits only one invariant hypersurface through the origin. Each such an invariant hypersurfaces corresponds to an invariant hypersurface through the origin $0 \in \mathbb{C}^3$ for the foliation $\mathcal{F}(X)$ and to an irreducible component of the polar set of the corresponding system of one-forms η_j^k, $k = 1, 2$ and therefore appears as one irreducible component of the polar set of the system of one-forms $\{\eta\}_{k=1}^{k=3}$ defined in W. Therefore, the nonlinearizable case for X corresponds to the case $\mathcal{F}(X)$ has one or two invariant hypersurfaces through the origin and the case that we are considering corresponds to the case we have two such hypersurfaces. Indeed, according to the possibility for F_1, F_2 and F_3 that we are considering we can assume that Poincaré-Dulac normal form Z for X writes as

$$Z(x, y, z) = (nx + y^n)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \nu z\frac{\partial}{\partial z}$$

where $\nu \in \mathbb{C}\backslash\mathbb{R}_-$. Thus we a local analytic conjugacy between X and Z is a neighborhood of the origin. We also have from the hypothesis on F_1, F_2, F_3 that the set of separatrices of $\mathcal{F}(X)$ through the origin $\Lambda(\mathcal{F}(X))$ is analytic of pure codimension one, writes as $\Lambda(\mathcal{F}(X)) = \bigcup_{j=2}^3 (\eta_j)_\infty$ and has exactly 2 irreducible components of codimension one. Thus we can find 2 reduced holomorphic functions $f_1, f_2 : W \to \mathbb{C}$ such that

$$\Lambda(\mathcal{F}(X)) = \bigcup_{k=1}^{2} \{f_k = 0\},$$

and each $\{f_k = 0\}$ is irreducible. In particular $(\eta_j)_\infty \subset \bigcup_{k=1}^2 \{f_k = 0\}$ and by the *Integration Lemma* (see, [5] and [6]) we can write

$$\eta_j = \sum_{k=1}^{2} \lambda_k^{(j)} \frac{df_k}{f_k} + d\bigg(\frac{g_j}{f_1^{r_j-1} f_2^{s_j-1}}\bigg), \tag{7.1}$$

for some $\lambda_k^{(j)} \in \mathbb{C}$ and some holomorphic functions $g_j : W \to \mathbb{C}$, $j = 1, 2$. The numbers $r_j, s_j \in \mathbb{N}$ are the order of $\{f_1 = 0\}$ and the order of $\{f_2 = 0\}$ as pole of η_j respectively.

Claim 1. *In U we have*

$$\eta_j(x,y,z) = \alpha_j \left[d\left(\frac{x}{y^n}\right) - \frac{dy}{y}\right] + \beta_j \left(\nu\frac{dy}{y} - \frac{dz}{z}\right) \tag{7.2}$$

for some $\alpha_j, \beta_j \in \mathbb{C}$ and $j = 1, 2$.

Proof. We define in the local one-forms $\eta_1^0 = d\left(\frac{x}{y^n}\right) - \frac{dy}{y}$ and $\eta_2^0 = \nu\frac{dy}{y} - \frac{dz}{z}$. Clearly we have $\eta_j^0(X) \equiv 0$ for $j = 1, 2$. Thus we must have $\eta_j = a_j\eta_1^0 + b_j\eta_2^0$ for some meromorphic functions a_j, b_j in U for $j = 1, 2$. Taking the exterior derivatives in these expressions we obtain

$$0 = da_j \wedge \eta_1^0 + db_j \wedge \eta_2^0.$$

Multiplying conveniently we get

$$0 = da_j \wedge \eta_1^0 \wedge \eta_2^0 = db_j \wedge \eta_1^0 \wedge \eta_2^0.$$

Therefore, a_j and b_j are constant along the leaves of $\mathcal{F}(X)$ in U. Nevertheless, the dynamics of such a Poincaré-Dulac normal form admits no meromorphic first integral in a neighborhood of the singularity, except for the constants. Thus, a_j and b_j are constant what proves the claim. □

In U we can write $f_2 = y.h_2$ and $f_3 = z.h_3$ for some never vanishing holomorphic functions h_2 and h_3 defined in U. Thus (7.1) implies that we must have

$$\eta_j = a_j\, d\left(\frac{g_j}{f_2^n}\right) + b_j\,\frac{df_2}{f_2} + c_j\,\frac{df_3}{f_3}.$$

for some constants $a_j, b_j, c_j \in \mathbb{C}$. Using $f_2 = yh_2$ and $f_3 = z_3$ we obtain

$$\eta_j = a_j\, d\left(\frac{g_j}{y^n h_2^n}\right) + b_j\,\frac{dy}{y} + c_j\,\frac{dz}{z} + b_j\frac{dh_2}{h_2} + c_j\frac{dh_3}{h_3}.$$

The one-form $b_j\frac{dh_2}{h_2} + c_j\frac{dh_3}{h_3}$ is closed and holomorphic so that we can write it as $b_j\frac{dh_2}{h_2} + c_j\frac{dh_3}{h_3} = d\psi_j$ for some holomorphic function defined in a suitable neighborhood of the origin, that we can assume to be U. From the claim we then obtain

$$\eta_j = a_j\, d\left(\frac{g_j}{y^n h_2^n}\right) + b_j\,\frac{dy}{y} + c_j\,\frac{dz}{z} + d\psi_j = \alpha_j \left[d\left(\frac{x}{y^n}\right) - \frac{dy}{y}\right] + \beta_j \left(\nu\frac{dy}{y} - \frac{dz}{z}\right).$$

Comparing the residues along $\{z = 0\}$ we obtain $c_j = -\beta_j$ and therefore

$$a_j\, d\left(\frac{g_j}{y^n h_2^n}\right) + b_j\,\frac{dy}{y} + d\psi_j = \alpha_j \left[d\left(\frac{x}{y^n}\right) - \frac{dy}{y}\right] + \beta_j \left(\nu\frac{dy}{y}\right).$$

Comparing now residues along $\{y = 0\}$ we get $b_j = -\alpha_j + \nu\beta_j$ and therefore

$$a_j\, d\left(\frac{g_j}{y^n h_2^n}\right) + d\psi_j = \alpha_j\, d\left(\frac{x}{y^n}\right).$$

Since ψ_j is holomorphic necessarily we have $d\psi_j \equiv 0$ and then

$$a_j\, d\left(\frac{g_j}{y^n h_2^n}\right) = \alpha_j\, d\left(\frac{x}{y^n}\right).$$

Thus we have $\frac{g_j}{y^n h_2^n} = A_j\frac{x}{y^n} + B_j$ for some contants $A_j, B_j \in \mathbb{C}$. Hence $\frac{g_j}{f_2^n} = B_j + A_j\frac{x}{y^n}$ in U. Replacing g_j by $g_j - B_j f_2^n$ we can assume that $B_j = 0$. With this we get $\frac{g_j}{f_2^n} = A_j\frac{x}{y^n}$ in U and therefore g_1/g_2 is constant in U and therefore in W. Moreover, from the above equations we we get that $g_j = A_j x h_2^n$ in U. Thus finally we obtain

$$\eta_j = \tilde{a}_j \left[d\left(\frac{g}{f_2^n}\right) - \frac{df_2}{f_2}\right] + \tilde{b}_j \left[\nu \frac{df_2}{f_2} - \frac{df_3}{f_3}\right] + d\varphi_j.$$

for some constants $\tilde{a}_j, \tilde{b}_j \in \mathbb{C}$ and some holomorphic functions φ_j in W. If we write $\theta = d\left(\frac{g}{f_2^n}\right) - \frac{df_2}{f_2}$ and $\xi = \frac{df_2}{f_2}$ then $\eta_j = \tilde{a}_j\theta + \tilde{b}_j\xi + d\varphi_j$ so that suitable linear combinations of η_1 and η_2 can be written as $\tilde{\eta}_1 = \theta_1 + d\psi_1$ and $\tilde{\eta}_2 = \xi + d\psi_2$ for some holomorphic functions ψ_j in W. Now we define $\hat{f}_2 = f_2 e^{-\psi_1}$ so that we have

$$\frac{d\hat{f}_2}{\hat{f}_2} = \frac{df_2}{f_2} - d\psi_1.$$

We also define $\hat{f}_1 = g e^{-n\psi_1}$. The functions $\hat{f}_1$ and $\hat{f}_2$ are holomorphic in W and we can write

$$\tilde{\eta}_1 = d\left(\frac{\hat{f}_1}{\hat{f}_2^n}\right) - \frac{d\hat{f}_2}{\hat{f}_2}.$$

Similarly, we define $\hat{f}_3 = f_3 e^{-(\psi_2 + \nu\psi_1)}$ so that $\hat{f}_3$ is holomorphic and we can write

$$\tilde{\eta}_2 = \nu \frac{d\hat{f}_2}{\hat{f}_2} - \frac{d\hat{f}_3}{\hat{f}_3}.$$

Now we define $\Phi : W \to \mathbb{C}^3$ by

$$\Phi = (\hat{f}_1, \hat{f}_2, \hat{f}_3).$$

The Jacobian matrix of Φ at the origin $0 \in \mathbb{C}^3$ is clearly non singular. In particular, up to shrink U, we can assume that Φ is invertible on U and then it is a biholomorphism between U and $V = \Phi(U) \subset \mathbb{C}^3$, a open neighborhood of the origin $0 \in \mathbb{C}^3$. Given global affine coordinates $(x, y, z) \in \mathbb{C}^3$ we recall the definition of the vector field Z given by

$$Z(x, y, z) = (nx + y^n)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \nu z \frac{\partial}{\partial z}$$

as well as the one-forms η_j^0 introduced above as $\eta_1^0 = d\left(\frac{x}{y^n}\right) - \frac{dy}{y}$ and $\eta_2^0 = \nu\frac{dy}{y} - \frac{dz}{z}$. Then by construction we have $\tilde{\eta}_j = \Phi^*(\eta_j^0)$ in W so that the foliation $\mathcal{F}(X)$ is the pull-back by Φ of the foliation $\mathcal{F}(Z)$ defined by Z. The map Φ is a diffeomorphism in a neighborhood of the origin and each leaf of $\mathcal{F}(X)$ tends to the origin, therefore the map Φ is injective as a map between leaf spaces. This ends the proof of Proposition 4. □

8 End of the proof of Theorem 1

In order to finish the proof of Theorem 1 we must eliminate the nonlinearizable case dealt with in Section 7 so that we will just have to apply Proposition 3 from Section 6 and conclude. Suppose by contradiction that we are in the nonlinearizable situation of Proposition 4 from Section 7. Thus, $\mathcal{F}(X)$ is the pull-back of a resonant nonlinear Poincaré-Dulac foliation $\mathcal{F}(Z)$ by a holomorphic map $\Phi\colon W \supset \Delta^n[1] \to \mathbb{C}^n$, such that $\Phi(0) = 0$ and Φ is a local diffeomorphism at the origin. According to Lemma 6 the foliation $\mathcal{F}(X)$ has exactly n global separatrices through 0. Therefore, also $\mathcal{F}(Z)$ exhibits exactly n separatrices through the origin. Then an immediate analysis on the Poincaré-Dulac normal form Z in a neighborhood of 0 shows that this singularity is still analytically linearizable, yielding a contradiction. Once we have proved that $\mathcal{F}(X)$ is the pull-back of a linear foliation $\mathcal{F}(Z)$, it is easy to conclude that Z must be hyperbolic, due to the attractive behavior of the holonomy maps F_j associated to the components Σ_j^{2n-1} and therefore to all the (global) separatrices of $\mathcal{F}(Z)$. This shows Theorem 1.

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