THE DYNAMICS NEAR QUASI-PARABOLIC FIXED POINTS OF HOLOMORPHIC DIFFEOMORPHISMS IN $\mathbb{C}^2$

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Abstract. Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^2$ fixing O and such that dF_O has eigenvalues 1 and $e^{i\theta}$ with $|e^{i\theta}| = 1$ and $e^{i\theta} \neq 1$. Introducing suitable normal forms for F we define an invariant, $\nu(F) \geq 2$, and a generic condition, that of being *dynamically separating*. In the case F is dynamically separating, we prove that there exist $\nu(F) - 1$ parabolic curves for F at O tangent to the eigenspace of 1.

1. Introduction. Let $\mathsf{End}(\mathbb{C}^2, O)$ denote the group of germs of holomorphic diffeomorphisms at the origin O of $\mathbb{C}^2$ fixing O. One of the main open problems is to understand the dynamics near O of an element $F \in \mathsf{End}(\mathbb{C}^2, O)$ for which the spectrum of the differential dF_O is contained in the unit circle (see Question 2.26 in [9]). The case where O is a parabolic point of F, that is $dF_O = \mathsf{id}$, and O is an isolated fixed point, has been studied by several authors ([7], [17], [10], [1]). To recall their main result we need first a definition:

Definition 1.1. A *parabolic curve* for $F \in \mathsf{End}(\mathbb{C}^2, O)$ at O tangent to (the space spanned by) $v \in \mathbb{C}^2 \setminus \{O\}$ is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^2$ satisfying the following properties:

(1) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$,

(2) φ is continuous on $\partial\Delta$, $\varphi(0) = O$ and $[\varphi(\zeta)] \to [v]$ as $\zeta \to 0$ (where $[\cdot]$ denote the projection of $\mathbb{C}^2 \setminus \{O\}$ to $\mathbb{P}^1$),

(3) $F(\varphi(\Delta)) \subset \varphi(\Delta)$, and $F^n(\varphi(\zeta)) \to O$ as $n \to \infty$ for any $\zeta \in \Delta$.

Then the main result is:

Theorem 1.2. (Écalle, Hakim, Abate) *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ be tangent to the identity and such that O is an isolated fixed point. Let $t(F) \geq 2$ denote the order of vanishing of $F - \mathsf{id}$ at O. Then there exist (at least) $t(F) - 1$ parabolic curves for F at O.*

Actually, Écalle [7] and Hakim [10] proved such a theorem for any dimension, but only for *generic* mappings, while Abate [1] using an ingenious index theorem

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makes the result holds for any map, but just in $\mathbb{C}^2$. The case where there is a curve of fixed points passing through O has also been studied ([11], [5], [2]), and actually one can see Theorem 1.2 as a consequence of results on dynamics near curves of fixed points by means of blow-ups of O in $\mathbb{C}^2$ (see [1], [4]). We also wish to mention that for the semi-attractive case in $\mathbb{C}^n$ (that is one eigenvalue 1 with some multiplicity and the others of modulus strictly less than 1) the existence of parabolic curves is provided by Rivi [13].

Roughly speaking the underlying idea in all previous results is to find "good invariants" attached to F which read dynamical properties of F itself (for instance Hakim's nondegenerate characteristic directions or Abate's indices in [1], and residues in [4]).

In this paper we deal with the case of a map $F \in \mathsf{End}(\mathbb{C}^2, O)$ with $\mathrm{Sp}(dF_O) = \{1, e^{i\theta}\}$ for $\theta \in \mathbb{R}$ and $e^{i\theta} \neq 1$. We call O a quasi-parabolic fixed point for F.

If $e^{i\theta}$ satisfies some Brjuno condition then Pöschel proved that there exists a (germ of) complex curve Γ tangent to the eigenspace of $e^{i\theta}$ which is invariant for F and on which F is conjugated to the rotation $\zeta \mapsto e^{i\theta}\zeta$ (see [12]). However nothing is known about the dynamics in the direction tangent to the eigenspace of 1.

Our starting point is the following trivial observation: the map $F : (z, w) \mapsto (z + z^3, e^{i\theta} w)$ has $\{w = 0\}$ as invariant curve and thus, by the one-dimensional Fatou theory (see, *e.g.*, [6]) there exist two parabolic curves for F at O tangent to the eigenspace of 1, no matter what $e^{i\theta}$ is. However, conjugating F with a map $G \in \mathsf{End}(\mathbb{C}^2, O)$ tangent to id at O, it might be very difficult to check that the new map has an invariant curve tangent to the eigenspace of 1 and two parabolic curves in there.

Motivated by the previous results for germs tangent to the identity, we direct our study in searching invariants for F at a quasi-parabolic point which are related to dynamical properties of F along the direction tangent to the eigenspace of 1.

The main difference between the parabolic and quasi-parabolic case is that in the first, all terms of F are resonant in the sense of Poincaré-Dulac (see, *e.g.*, [3]), while in the second case some are not, and this allows us to dispose of those terms with suitable transformations. More precisely, let $F = (F_1, F_2) \in \mathsf{End}(\mathbb{C}^2, O)$ be given in some system of local coordinates by

$$\begin{cases} F_1(z, w) = z + \sum_{j+k\geq 2} p_{j,k} z^j w^k, \\ F_2(z, w) = e^{i\theta} w + \sum_{j+k\geq 2} q_{j,k} z^j w^k, \end{cases} \tag{1.1}$$

for $p_{j,k}, q_{j,k} \in \mathbb{C}$, $\theta \in \mathbb{R}$ and $e^{i\theta} \neq 1$. A monomial $z^m w^n$ in F_1 is *resonant* if $1 = 1^m e^{i\theta n}$, while a monomial $z^m w^n$ in F_2 is *resonant* if $e^{i\theta} = 1^m e^{i\theta n}$, for $m, n \in \mathbb{N}$, $m+n \geq 2$. A germ F is said to be in Poincaré-Dulac normal form if it is given by (1.1) and $p_{j,k} = q_{j,k} = 0$ for all nonresonant monomials $z^j w^k$. The Poincaré-Dulac Theorem states that it is always possible to formally conjugate F to a (formal) map G in normal form by means of a (formal) transformation tangent to the identity, and actually the method of Poincaré-Dulac is constructive in the sense that given $k \in \mathbb{N}$ it is possible to analytically conjugate F to a (convergent) map G which is in normal form up to order k (that is, nonresonant monomials of degree less than or equal to k are all zero) by means of a (convergent) transformation tangent to the identity.

Therefore if there exist invariants for F at a quasi-parabolic fixed point they have to be found in normal forms. Unfortunately normal forms are not unique and also they do reflect the character of $e^{i\theta}$, while our leading example does not make differences. Also, normal forms are not stable under blow-ups, which are one of the basic ingredients of parabolic theory. Indeed the only invariant terms are those we call ultra-resonant monomials, that is, for F given by (1.1), of type z^m in F_1 and $z^m w$ in F_2, $m \in \mathbb{N}$. And we say that F is an asymptotic ultra-resonant normal form if $q_{j,0} = 0$ for any j. Note that Poincaré-Dulac normal forms are in fact examples of asymptotic ultra-resonant normal forms but the converse is not true in general, and indeed there are convergent asymptotic ultra-resonant normal forms which have no convergent Poincaré-Dulac normal forms. With a simplified Poincaré-Dulac method we prove that given $F \in \mathsf{End}(\mathbb{C}^2, O)$ with O as quasi-parabolic fixed point, there always exist (possibly formal) asymptotic ultra-resonant normal forms conjugated to F by means of transformations tangent to the identity. Again asymptotic ultra-resonant normal forms are not unique, but we show that the first $j \in \mathbb{N}$ such that $p_{j,0} \neq 0$ is an invariant for (even formal) conjugated ultra-resonant normal forms. Therefore we find the first invariant $\nu(F) \in \mathbb{N} \cap [2, \infty]$ associated to F. Of course this invariant could also have been defined from Poincaré-Dulac normal forms. However, the following result justifies the usage of ultra-resonant normal forms instead of Poincaré-Dulac normal forms:

PROPOSITION 1.3. *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic fixed point of F. Then there exists an invariant nonsingular complex curve Γ for F passing through O and tangent to the eigenspace of* 1 *if and only if F is analytically conjugated to a convergent asymptotic ultra-resonant normal form. Moreover in this case, if $\nu(F) = \infty$ then F pointwise fixes Γ, while if $\nu(F) < \infty$ there exist $\nu(F) - 1$ parabolic curves for F at O contained in Γ.*

For the practical purpose of calculating $\nu(F)$ one does not need to find an asymptotic ultra-resonant normal form. Indeed it is enough to find what we call a ultra-resonant normal form, that is, F given by (1.1) for which the first pure non-zero term in z of F_2 has degree greater than or equal to the first non-zero pure term in z of F_1 (see Section 2).

In the generic case $\nu(F) < \infty$, we can associate to F a second invariant, essentially the sign of $\Theta(F)$. The latter, for F in ultra-resonant normal form given by (1.1), is defined as $\Theta(F) = \nu(F) - j - 1$ where j is the first integer for which $q_{j,1} \neq 0$ and, roughly speaking, measures the "degree of mixing" of the dynamics along the eigenspace associated to 1 and $e^{i\theta}$. Therefore, given any $F \in \mathsf{End}(\mathbb{C}^2, O)$ for which O is quasi-parabolic for F, we say that F is *dynamically separating* if $\nu(F) < \infty$ and $\Theta(\check{F}) \leq 0$ for some ultra-resonant normal form $\check{F}$ of F (see Definition 2.7). Our main result can now be stated as follows:

THEOREM 1.4. *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic point of F. If F is dynamically separating then there exist $\nu(F) - 1$ parabolic curves for F at O tangent to the eigenspace of* 1.

One remarkable consequence of this theorem is that if F is given by (1.1) and $p_{2,0} \neq 0$ then there *always* exists a parabolic curve for F at O tangent to the eigenspace of 1. This is similar to a result in the quasi-hyperbolic case—one eigenvalue 1, the other of modulus < 1—where, under similar hypothesis, the existence of a basin of attraction for F is proved (cf. [8], [14], [15]).

The plan of the paper is the following: In Section 2 we introduce ultra-resonant normal forms, the invariant $\nu(F)$ and dynamically separating maps and give the proof of Proposition 1.3. In Section 3 we prove Theorem 1.4. Finally, in Section 4 we conclude with some remarks and discuss the case $e^{si\theta} = 1$ for some $s \geq 2$, especially relating parabolic curves provided by Theorem 1.4 with the ones given by Hakim's and Abate's theory for F^s.

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2. Ultra-resonant normal forms.

Definition 2.1. Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ be given by (1.1). We call ultra-resonant the monomials of type z^m in F_1 and of type $z^m w$ in F_2, $m \in \mathbb{N}$.

In case there exists $j \in \mathbb{N}$ such that $p_{j,0} \neq 0$ we let

$$\mu(F, z) := \min\{j \in \mathbb{N} : p_{j,0} \neq 0\},$$

and let $\mu(F, z) = +\infty$ if $p_{j,0} = 0$ for all j's. Similarly if there exists $j \in \mathbb{N}$ such that $q_{j,1} \neq 0$, we let

$$\mu(F, w) := \min\{j \in \mathbb{N} : q_{j,1} \neq 0\},$$

setting $\mu(F, w) = +\infty$ if $q_{j,1} = 0$ for all j's.

Finally, if $\mu(F, z) < +\infty$ we let $\Theta(F) := \mu(F, z) - \mu(F, w) - 1$ (with the convention that $\Theta(F) = -\infty$ if $\mu(F, w) = +\infty$).

In general $\mu(F, z)$ and $\mu(F, w)$ are not invariant under change of coordinates. However $\mu(F, z)$ and the sign of $\Theta(F)$ are invariant under a suitable normalization which we are going to describe.

Definition 2.2. We say that a (possibly formal) germ of diffeomorphism $F \in \mathsf{End}(\mathbb{C}^2, O)$ is in *ultra-resonant normal form* if F is given by (1.1) and $q_{j,0} = 0$ for $j = 2, \ldots, \mu(F, z) - 1$. If $q_{j,0} = 0$ for any j we call F an *asymptotic ultra-resonant normal form*.

The first result we prove is the existence of (possibly formal) asymptotic ultra-resonant normal form.

PROPOSITION 2.3. *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic fixed point for F. Then there exists a formal transformation $\check{K} \in \mathsf{End}(\mathbb{C}^2, O)$ tangent to* id *such that $\check{K}^{-1} \circ F \circ \check{K} = \check{F}$, with $\check{F}$ a formal asymptotic ultra-resonant normal form.*

Proof. We may assume F in the form (1.1). Let $q_{s,0} \neq 0$ be the first nonzero coefficient of a pure term in z in F_2. Consider the transformation

$$K_s = \begin{cases} z = Z \\ w = W + aZ^s \end{cases} \tag{2.1}$$

with $a = -q_{s,0}/(e^{i\theta} - 1)$. Then $K_s^{-1} \circ F \circ K_s$ has pure term in Z in the second component of degree $\geq s + 1$. Proceeding this way we can get rid of all pure terms in z in the second component, and $\check{K}$ is given by composition of the K_s's. □

Ultra-resonant normal forms are by no means unique as the following example shows.

Example 2.4. The germs $F(z, w) = (z + z^2, e^{i\theta} w)$ and $G(z, w) = (z + z^2, e^{i\theta} w - e^{i\theta} wz^2/(1 + z + z^2))$ are both in normal forms and conjugated by the the transformation $(z, w) \mapsto (z, w + zw)$. Moreover $\mu(F, z) = \mu(G, z) = 2$, $\Theta(F) = -\infty$ while $\Theta(G) = -1$.

Using ultra-resonant normal forms we can define some invariants associated to F. Before doing that, we need the following basic lemma.

LEMMA 2.5. *Let $F, G \in \mathsf{End}(\mathbb{C}^2, O)$ be (possibly formal) germs of diffeomorphisms in ultra-resonant normal form. If F is conjugated to G then $\mu(F, z) = \mu(G, z)$. Moreover if $\mu(F, z) = \mu(G, z) < \infty$ then $\Theta(F) \leq 0$ if and only if $\Theta(G) \leq 0$, while if $\mu(F, z) = \mu(G, z) = \infty$ then $\mu(F, w) = \mu(G, w)$.*

Proof. Let F be given by (1.1), and let

$$G(z, w) = (z + \sum_{j+k \geq 2} \tilde{p}_{j,k} z^j w^k, e^{i\theta} w + \sum_{j+k \geq 2} \tilde{q}_{j,k} z^j w^k).$$

If T is the transformation which conjugates F to G, then its differential at the origin must be a diagonal matrix, which we can assume to be the identity. Thus let $T : (z, w) \mapsto (z + \varphi_1(z, w), w + \varphi_2(z, w))$ be the transformation conjugating F to G.

We introduce the following notation: we denote by H_m any term which has order greater than or equal to m. Also, for $m, n \in \mathbb{N}$, $m \leq n$, we write $B_{m,n}$ for indicating terms of order greater than or equal to m but less than or equal to n; we also set $B_{m,n} = 0$ for $m > n$. Moreover we let S_k denote any term of order strictly smaller than k. We also set $a := \mu(F, z)$, $b = \mu(F, w)$ and $\tilde{a} = \mu(G, z)$, $\tilde{b} = \mu(G, w)$. In case $a = \infty$ we agree that terms of type $p_{a,0}z^a$ and symbols like $O(z^a)$ should be understood as zeros (similarly if $\tilde{a} = \infty$). With this convention we can deal with all cases at the same time. Since $F = (F_1, F_2)$ and $G = (G_1, G_2)$ are both in normal form, we can write

$$F(z,w) = \begin{cases} F_1(z,w) = z + p_{a,0}z^a + wB_{1,a-1} + H_{a+1}, \\ F_2(z,w) = e^{i\theta}w + q_{b,1}z^b w + w^2 S_b + O(z^a, z^{b+1}w, w^2 H_b), \end{cases} \tag{2.2}$$

and

$$G(z,w) = \begin{cases} G_1(z,w) = z + \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + wB_{1,\tilde{a}-1} + H_{\tilde{a}+1}, \\ G_2(z,w) = e^{i\theta}w + \tilde{q}_{\tilde{b},1}z^{\tilde{b}} w + w^2 S_{\tilde{b}} + O(z^{\tilde{a}}, z^{\tilde{b}+1}w, w^2 H_{\tilde{b}}). \end{cases} \tag{2.3}$$

Let $c_h \geq 2$ be the order of vanishing of $\varphi_h(z, 0)$ at 0, $h = 1, 2$. Since $F \circ T = T \circ G$, using (2.2) and (2.3) and equating components we obtain

$$\begin{aligned} &\varphi_1(z,w) + p_{a,0}z^a + \varphi_2(z,w)B_{1,a-1} + H_{a+1} + O(w) \\ &\qquad = \varphi_1(G(z,w)) + \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + H_{\tilde{a}+1}, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &e^{i\theta}\varphi_2(z,w) + q_{b,1}(z + \varphi_1(z,w))^b(w + \varphi_2(z,w)) + [2w\varphi_2(z,w) + \varphi_2(z,w)^2]S_b \\ &\qquad + O(z^a, z^{b+1+c_2}, z^{b+1}w) + O(w^2) = \varphi_2(G(z,w)) + \tilde{q}_{\tilde{b},1}z^{\tilde{b}}w + O(z^{\tilde{a}}, z^{\tilde{b}+1}w). \end{aligned} \tag{2.5}$$

Write $\varphi_h(z,w) = \sum_{j+k\geq 2} \varphi_h^{j,k} z^j w^k$, for $\varphi_h^{j,k} \in \mathbb{C}$ and $h = 1, 2$. Then

$$q_{b,1}(z + \varphi_1(z,w))^b(w + \varphi_2(z,w)) = q_{b,1}z^b w + O(w^2, z^{b+1}w, z^{b+c_2}), \tag{2.6}$$

$$\varphi_2(G(z,w)) - e^{i\theta}\varphi_2(z,w) = (1 - e^{i\theta})\varphi_2^{c_2,0}z^{c_2} + O(z^{\tilde{a}}, z^{c_2+1}, w), \tag{2.7}$$

and putting (2.6), (2.7) into (2.5) we get that

$$c_2 \geq \min\{a, \tilde{a}\}, \tag{2.8}$$

where we understood $c_2 = \infty$ (that is $\varphi_2^{j,0} = 0$ for any j) in case $a = \tilde{a} = \infty$. In particular equation (2.4) reads now as

$$\varphi_1(G(z,w)) - \varphi_1(z,w) = p_{a,0}z^a - \tilde{p}_{\tilde{a},0}z^{\tilde{a}} + O(w, z^{a+1}, z^{\tilde{a}+1}). \tag{2.9}$$

We examine the left-hand side of (2.9). Using (2.3) we have

$$\begin{aligned}\varphi_1(G(z,w)) &= \sum_{j+k\geq 2} \varphi_1^{j,k}[z + O(z^{\tilde a}, w)]^j[e^{i\theta}w + O(z^{\tilde a}, wz, w^2)]^k \\ &= \varphi_1(z,w) + O(w, z^{\tilde a+1}).\end{aligned} \tag{2.10}$$

Therefore from (2.9) and (2.10) we get $a = \tilde a$, that is $\mu(F,z) = \mu(G,z)$.

Let $a < \infty$. We assume $\Theta(F) \leq 0$ and want to show that $\Theta(G) \leq 0$ (the other implication will follow reversing the role of F and G). We have already proved that $\tilde a = a$ and now we are assuming $b \geq a-1$. Seeking for a contradiction we suppose that $\tilde b < a-1$. Taking into account (2.6) and (2.8), equation (2.5) becomes

$$\varphi_2(G(z,w)) - e^{i\theta}\varphi_2(z,w) = -\tilde q_{\tilde b,1} z^{\tilde b} w + O(wz^{\tilde b+1}, z^a, w^2). \tag{2.11}$$

We examine the left-hand side of (2.11). Since $\varphi_2^{j,0} = 0$ for $j < c_2$ and $c_2 \geq a$ by (2.8), using (2.3) we have

$$\begin{aligned}\varphi_2(G(z,w)) &= \sum_{j\geq 0} \varphi_2^{j+a,0}[z + O(z^a, w)]^{j+a} \\ &\quad + \sum_{j+k\geq 1} \varphi_2^{j,k+1}[z + O(z^a, w)]^j[e^{i\theta}w + O(wz^{\tilde b}, z^a, w^2)]^{k+1} \\ &= \varphi_2(z, e^{i\theta}w) + O(w^2, z^a, wz^{\tilde b+1}).\end{aligned} \tag{2.12}$$

Put (2.12) into (2.11) and noting that $e^{i\theta}\varphi_2(z,w) - \varphi_2(z, e^{i\theta}w)$ does not contain terms in $z^m w$ for any $m \in \mathbb{N}$, we reach a contradiction. Therefore $\tilde b \geq a-1$ and $\Theta(G) \leq 0$ as wanted.

Finally suppose $a = \tilde a = \infty$. Then by hypothesis and by (2.8) the maps $G(z,w), F(z,w)$ and $\varphi_2(z,w)$ do not contain pure terms in z. Therefore, using (2.6), equation (2.5) becomes

$$\varphi_2(G(z,w)) - e^{i\theta}\varphi_2(z,w) = -\tilde q_{\tilde b,1} z^{\tilde b} w + q_{b,1} z^b w + O(wz^{b+1}, wz^{\tilde b+1}, w^2),$$

where, as usual, we set all the terms containing z^b or $z^{\tilde b}$ equal to zero if $b = \infty$ or $\tilde b = \infty$. From this and from (2.12) it follows that $b = \tilde b$. □

Remark 2.6. If F and G are conjugated and in ultra-resonant normal form (and $\mu(F,z) = \mu(G,z) < \infty$), $\mu(F,w)$ might be different from $\mu(G,w)$, as one can see in the Example 2.4.

Now we are in the position to define our invariants:

Definition 2.7. Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic fixed point for F. Let $\check F$ be a (possibly formal) asymptotic ultra-resonant normal form of

F. We let $\nu(F) := \mu(\check{F}, z)$. In case $\mu(\check{F}, z) < \infty$ we call F dynamically separating if $\Theta(\check{F}) \leq 0$.

Remark 2.8. By Lemma 2.5 the previous definition is well posed. Moreover, if $\nu(F) < \infty$ one can find a (convergent) ultra-resonant normal form conjugated to F after a finite number of transformations of type (2.1) .

Let $F \in \mathsf{End}(\mathbb{C}^2, O)$. The Poincaré-Dulac normal form theorem states that it is always possible to find a resonant formal normal form for F. Namely there exists a formal transformation $T : (z, w) \mapsto (z + \ldots, w + \ldots)$ such that $T^{-1} \circ F \circ T(z, w) = (z + R_1(z, w), e^{i\theta} w + R_2(z, w))$, where R_1, R_2 are series of resonant monomials, that is $R_1(z, w)$ is a combination of terms of type z^m, $z^m w^{sn}$, while $R_2(z, w)$ is a combination of terms of type $z^m w, z^m w^{ns+1}$ for $m, n \in \mathbb{N}$, where $s \in \mathbb{N}$ is such that $e^{is\theta} = 1$ (thus $s = 0$ if $e^{i\theta}$ is not a root of unity).

Due to Lemma 2.5 our (formal) asymptotic ultra-resonant form is equivalent to the Poincaré-Dulac normal form for the purpose of calculating $\mu(F, z)$ and $\Theta(F)$. However, asymptotic ultra-resonant normal forms reflect better the dynamics of F, as claimed in Proposition 1.3. Here is its proof.

Proof of Proposition 1.3. If F has a convergent asymptotic ultra-resonant normal form then F is conjugated to a germ of biholomorphism $G = (G_1, G_2)$ such that $G_2(z, w) = wA(z, w)$ for some holomorphic function $A(z, w)$. In particular $w = 0$ is invariant by G. For the converse, if there exists an invariant curve tangent to the eigenspace of 1 we can choose coordinates in such a way that $\Gamma = \{(z, w) : w = 0\}$ and $F(z, w) = (z + \ldots, e^{i\theta} w + wA(z, w))$ for some holomorphic function $A(z, w)$. In particular F has a (convergent) asymptotic ultra-resonant form. By Lemma 2.5, if F has a convergent asymptotic ultra-resonant normal form G then $\mu(G, z) = \nu(F)$. Thus if $\nu(F) = \infty$ then $G_1(z, w) = z + wA_1(z, w)$ and $\{w = 0\}$ is a curve of fixed points for G. If $\nu(F) < \infty$ then the classical one-dimensional Fatou theory gives the result. □

3. Dynamics. In this section we give the proof of Theorem 1.4. The idea is that starting from an ultra-resonant normal form, if $\Theta(F) \leq 0$, it is possible to blow up O a certain number of times in order to find some simpler expression for F, where one can apply a modified Hakim's argument to produce parabolic curves.

We divide the proof into several steps, which might be of some interest on their own.

Recall that if $F \in \mathsf{End}(\mathbb{C}^2, O)$ and $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ is the blow-up (quadratic transformation) of $\mathbb{C}^2$ at O, then there exists a holomorphic map $\tilde{F}$ defined near the exceptional divisor $D := \pi^{-1}(O)$ such that $\pi \circ \tilde{F} = F \circ \pi$, $\tilde{F}(D) = D$ and the action of $\tilde{F}$ on D is given by $D \ni [v] \mapsto [dF_O(v)] \in D$ (see for instance [1], [17]). We call such a $\tilde{F}$ the blow-up of F.

LEMMA 3.1. *Suppose F is given by (1.1). If*

(1) $q_{j,0} = 0$ *for* $j < \mu(F, z)$ *and*

(2) $q_{j,1} = 0$ *for* $j < \mu(F, z) - 1$,

then one can perform a finite number of blow-ups and changes of coordinates in such a way that the blow-up map $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ *is given by*

$$
\begin{cases} \tilde{F}_1(z, w) = z - z^{\nu(F)} + O(z^{\nu(F)+1}, z^{\nu(F)}w), \\ \tilde{F}_2(z, w) = e^{i\theta}w - \lambda w z^{\nu(F)-1} + O(wz^{\nu(F)}, z^{\nu(F)-1}w^2, z^{\nu(F)+2}), \end{cases} \tag{3.1}
$$

with $Re(\lambda e^{-i\theta}) < 0$.

Proof. Note that by hypothesis F is an ultra-resonant normal form, thus $\nu(F) = \mu(F, z)$. First of all, we can use transformations of type (2.1), for $s = \nu(F)$, as in the proof of Proposition 2.3, to dispose of $q_{\nu(F),0}$. Note that K_s does not decrease the order of vanishing of $F_1(z, w) - z$ and $F_2(z, w) - e^{i\theta}w$, nor it effects the ultra-resonant monomials of order $\leq \nu(F)$. Now we blow-up the point O in $\mathbb{C}^2$. Recalling that $1/(1+\xi) = \sum_{k\geq 0}(-1)^k\xi^k$ for $|\xi| < 1$, in coordinates $(z = u, w = uv)$ we have that the blow-up map $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ is given by

$$
\begin{aligned}
\tilde{F}_1(u, v) &= u + \sum_{j+k\geq 2} p_{j,k}u^{j+k}v^k = u + \sum_{j+k\geq 2} \tilde{p}_{j,k}u^j v^k, \\
\tilde{F}_2(u, v) &= \left[e^{i\theta}v + \sum_{j+k\geq 2} q_{j,k}u^{j+k-1}v^k\right]\left[1 - \sum_{j+k\geq 2} p_{j,k}u^{j+k-1}v^k \right. \\
&\quad \left. + \left(\sum_{j+k\geq 2} p_{j,k}u^{j+k-1}v^k\right)^2 + \cdots\right] = e^{i\theta}v + \sum_{j+k\geq 2} \tilde{q}_{j,k}u^j v^k.
\end{aligned} \tag{3.2}
$$

Thus, setting $p_{j,k} = 0$ for $j + k < 2$, it follows that $\tilde{p}_{j,k} = p_{j-k,k}$. In particular $\mu(F, z) = \mu(\tilde{F}, u)$ and $p_{\mu(F,z),0} = \tilde{p}_{\mu(\tilde{F},u),0}$. Moreover, if m_1 was the order of vanishing of $F_1(z, w) - z$ (that is $p_{j,k} = 0$ for $j + k < m_1$), then the order of vanishing of $\tilde{F}_1(u, v) - u$ is at least $m_1 + 1$ if $m_1 < \nu(F)$ or it is equal to m_1 if $m_1 = \nu(F)$. Also, the lowest nonzero non ultra-resonant term in $\tilde{F}_1$, i.e., the one of type $w^a z^b$, $a \geq 1$, $b \geq 0$, has degree strictly greater than the lowest one in F_1.

The terms $\tilde{q}_{j,k}$ in the second component of $\tilde{F}$ are more difficult to write explicitly. We use the notations H_m and $B_{m,n}$ introduced in the proof of Lemma 2.5. Denote by m_2 the order of vanishing of $F_2(z, w) - e^{i\theta}w$. Note that, since we assumed that $q_{j,0} = 0$ for $j < \nu(F) + 1$ and by hypothesis (2), then for every $q_{j,k} \neq 0$ with $j + k < \nu(F)$ it follows that $k \geq 2$. Thus, using hypothesis (1) and (2) we have

$$
\begin{aligned}
\tilde{F}_2(u,v) &= [e^{i\theta}v + q_{\nu(F)-1,1}u^{\nu(F)-1}v + v^2B_{m_2-1,\nu(F)-2} + H_{\nu(F)+1}][1 \\
&+ \sum_{k=1}^{\infty}(-1)^k(p_{\nu(F),0}u^{\nu(F)-1} + p_{\nu(F)+1,0}u^{\nu(F)} + v\sum_{j=m_1-1}^{\nu(F)-1}p_{j,1}u^j + v^2B_{m_1-1,\nu(F)-2} \\
&+ H_{\nu(F)+1})^k] = [e^{i\theta}v + q_{\nu(F)-1,1}u^{\nu(F)-1}v + v^2B_{m_2-1,\nu(F)-2}][1 - p_{\nu(F),0}u^{\nu(F)-1} \\
&+ p_{\nu(F),0}^2u^{2\nu(F)-2} - v\sum_{j=m_1-1}^{\nu(F)-1}p_{j,1}u^j - v^2B_{m_1-1,\nu(F)-2} - \sum_{k=2}^{\infty}v^kH_{2(m_1-1)}] + H_{\nu(F)+1} \\
&= e^{i\theta}v + (q_{\nu(F)-1,1} - e^{i\theta}p_{\nu(F),0})u^{\nu(F)-1}v + v^2H_{m_1-1} + v^2H_{m_2-1} + H_{\nu(F)+1}.
\end{aligned}
$$

In particular note that the ultra-resonant terms in $\tilde{F}_2$ are vanishing up to order $\nu(F)-1$. Also $\tilde{q}_{\nu(F)-1,1} = (q_{\nu(F)-1,1} - e^{i\theta}p_{\nu(F),0})$ and then

$$\mathrm{Re}(e^{-i\theta}\tilde{q}_{\nu(F)-1,1}/\tilde{p}_{\nu(F),0}) = \mathrm{Re}(e^{-i\theta}q_{\nu(F)-1,1}/p_{\nu(F),0}) - 1.$$

Finally note that the order of vanishing of $\tilde{F}_2(u,v)-e^{i\theta}v$ is at least $\min\{\nu(F), m_1+1, m_2+1\}$. This time the lowest nonzero non ultra-resonant term in $\tilde{F}_2$ might be of degree strictly smaller than the one in F_2. However, its degree is at least $\min\{\nu(F)+1, m_1+1, m_2+1\}$. In particular the map $\tilde{F}$ has properties (1), (2) in the hypothesis and its lowest nonzero non ultra-resonant term (in both components) has degree at least $\min\{\nu(F)+1, m_1+1, m_2+1\}$. Moreover $\mathrm{Re}(e^{-i\theta}\tilde{q}_{\nu(F)-1,1}/\tilde{p}_{\nu(F),0})$ is one less than $\mathrm{Re}(e^{-i\theta}q_{\nu(F)-1,1}/p_{\nu(F),0})$.

Repeating the previous arguments (conjugation with K_s followed by blow-up) we will eventually find a map in ultra-resonant normal form given by (1.1) with

(i) $q_{j,k} = 0$ for $j+k < \nu(F)$,

(ii) $p_{j,k} = 0$ for $j+k < \nu(F)$,

(iii) $\mathrm{Re}(e^{-i\theta}q_{\nu(F)-1,1}/p_{\nu(F),0}) < 1$.

Note that $\nu(F)$ is the same as for the starting map. Eventually performing some more transformations K_s as in (2.1), with $s = \nu(F), \nu(F)+1, \nu(F)+2$, we can assume $q_{j,0} = 0$ for $j < \nu(F)+3$.

Let $\alpha^{\nu(F)-1} = -p_{\nu(F),0}$ and let T be the transformation given by $Z = \alpha z, W = w$. The map $\hat{F} = T \circ F \circ T^{-1}$ satisfies (i), (ii) and $\nu(\hat{F}) = \nu(F)$. Moreover, denoting with $\hat{}$ the coefficients of $\hat{F}$, we have $\hat{p}_{\nu(F),0} = -1$, $\hat{q}_{j,0} = 0$ for $j < \nu(F)+3$ and $\hat{q}_{\nu(F)-1,1} = -q_{\nu(F)-1,1}/p_{\nu(F),0}$. In particular property (iii) becomes $\mathrm{Re}(e^{-i\theta}\hat{q}_{\nu(F)-1,1}) > -1$.

Now we perform a final blow-up of O. Let $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ be the blow-up and $\tilde{F}$ the blow-up map. In the coordinates (z,w) such that the projection $\pi(z,w) = (z, zw)$, we have that $\tilde{F} = (\tilde{F}_1, \tilde{F}_2)$ is given by (3.1), with $\lambda = -(e^{i\theta} + \hat{q}_{\nu(F)-1,1})$. $\square$

Now we prove that form (3.1) is actually useful.

LEMMA 3.2. *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ be given by (3.1), with $\nu(F) \geq 2$ and $\lambda \in \mathbb{C}$ such that $Re(\lambda e^{-i\theta}) < 0$. Then there exist $\nu(F) - 1$ parabolic curves for F at O tangent to* $[1 : 0]$.

Proof. The proof is a modification of that of Theorem 3.1 of [1]. Let $r = \nu(F) - 1$. Let $D_{\delta,r} := \{\zeta \in \mathbb{C} : |\zeta^r - \delta| < \delta\}$ and let $\mathcal{E}(\delta) := \{u \in \mathrm{Hol}(D_{\delta,r}, \mathbb{C}) : u(\zeta) = \zeta^2 u^0(\zeta), \|u^0\|_\infty < \infty\}$. The set $\mathcal{E}(\delta)$ is a Banach space with norm $\|u\|_{\mathcal{E}(\delta)} = \|u^0\|_\infty$. For $u \in \mathcal{E}(\delta)$ we let $F^u(\zeta) = F_1(\zeta, u(\zeta))$. The classical Fatou theory for mappings of the form $\zeta - \zeta^{r+1} + O(\zeta^{r+2})$ implies that there exists $\delta_0 = \delta_0(\|u^0\|_\infty)$ such that if $0 < \delta < \delta_0$ then F^u maps each component of $D_{\delta,r}$ into itself and moreover

$$|(F^u)^n| = O(\frac{1}{n^{1/r}}). \tag{3.3}$$

Suppose we find $u \in \mathcal{E}(\delta)$ such that $u(F_1(\zeta, u(\zeta)) = F_2(\zeta, u(\zeta))$ for any $\zeta \in D_{\delta,r}$. Thus the map $\varphi^u(\zeta) := (\zeta, u(\zeta))$ restricted to each connected component of $D_{\delta,r}$ is a parabolic curve for F.

For $(z, w) \in \mathbb{C}^2$ let $z_1 := F_1(z, w)$ and $w_1 := F_2(z, w)$. Suppose z, z_1 belong to the same connected component of $D_{\delta,r}$. Let $\mu := \lambda e^{-i\theta}$ and define

$$H(z, w) := w - e^{-i\theta} \frac{z^\mu}{z_1^\mu} w_1.$$

Thus a direct computation shows that

$$\begin{aligned} H(z, w) &= w - z^\mu \frac{w - \mu z^r w + O(wz^{r+1}, w^2 z^r, z^{r+3})}{z^\mu (1 - z^r + O(z^{r+1}, z^r w))^\mu} \\ &= w - [w - \mu z^r w + O(wz^{r+1}, w^2 z^r, z^{r+3})][1 + \mu z^r + O(z^{r+1}, z^r w)] \\ &= O(z^{r+1} w, z^r w^2, z^{r+3}). \end{aligned}$$

Now $F_2(z, w) = w_1 = e^{i\theta} \frac{z_1^\mu}{z^\mu}(w - H(z, w))$ and therefore we are left to solve the following functional equation:

$$u(z_1(\zeta, u(\zeta)) = e^{i\theta} \frac{z_1^\mu}{\zeta^\mu}(u(\zeta) - H(\zeta, u(\zeta)). \tag{3.4}$$

For $\zeta_0 \in D_{\delta,r}$ let $\zeta_n := (F^u)^n(\zeta_0)$. For $u \in \mathcal{E}(\delta)$ let

$$Tu(\zeta_0) := \zeta_0^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)).$$

If u is such that $\|u^0\| < c_0$ and $\delta \leq \delta_0(c_0)$, then $H(\zeta_n, u(\zeta_n))$ is defined for any $\zeta_0 \in D_{\delta,r}$. Moreover one can show exactly as in [1] and [10] that the series converges normally and $Tu \in \mathcal{E}(\delta)$ (essentially because $|e^{in\theta}| = 1$ and thus all the estimates for the parabolic case in [1] go through in this case as well).

Now suppose u is a fixed point for T. Then φ^u is a parabolic curve for F. indeed if

$$u(\zeta_0) = Tu(\zeta_0) = \zeta_0^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)),$$

then

$$\begin{aligned} u(\zeta_1) &= \zeta_1^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_{n+1}^{-\mu} H(\zeta_{n+1}, u(\zeta_{n+1})) = e^{i\theta} \zeta_1^\mu \sum_{n=1}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)) \\ &= \frac{\zeta_1^\mu}{\zeta_0^\mu} e^{i\theta} \left(\zeta_0^\mu \sum_{n=0}^{\infty} e^{-in\theta} \zeta_n^{-\mu} H(\zeta_n, u(\zeta_n)) - H(\zeta_0, u(\zeta_0)) \right) \\ &= \frac{\zeta_1^\mu}{\zeta_0^\mu} e^{i\theta} (u(\zeta_0) - H(\zeta_0, u(\zeta_0))), \end{aligned}$$

solving thus (3.4).

It remains to show that T does have a fixed point. For doing this we only need to show that T is a contraction on a suitable closed convex subset of $\mathcal{E}(\delta)$. This can be done arguing exactly as in Theorem 3.1 of [1], for all the estimates holding in there actually hold in this case, and we are done. □

Now we are in a good shape to prove our main theorem.

Proof of Theorem 1.4 Since having parabolic curves is obviously a property invariant under changes of coordinates and by Remark 2.8, we can assume F to be in ultra-resonant normal form. By definition of dynamically separating map, $\Theta(F) \leq 0$ and we can thus apply Lemma 3.1 to F and Lemma 3.2 to its blow-up $\tilde{F}$ in order to produce $\nu(F) - 1$ parabolic curves for $\tilde{F}$ at some point of the exceptional divisor. These parabolic curves blow down to $\nu(F) - 1$ parabolic curves for F tangent to the eigenspace of 1 and we are done. □

4. Final remarks.

1. Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and suppose O is a quasi-parabolic fixed point for F. In case $e^{i\theta s} = 1$ for some $s \geq 2$ one can try to apply Hakim and Abate's theory to produce parabolic curves for F^s. If F is dynamically separating one always obtains $\nu(F) - 1$ parabolic curves for F by Theorem 1.4 and these are obviously parabolic curves for F^s as well. The question is whether these parabolic curves are the ones predicted by Hakim's and Abate's theory for F^s (if such a theory applies). To give an appropriate answer we need some tools from [10] and [1]. For the reader's convenience we quickly recall them here.

Let $G \in \mathsf{End}(\mathbb{C}^2, O)$ be such that $dG_O = \mathsf{id}$. Let $G = \mathsf{id} + \sum_{m\geq 2} G_m$ be the homogeneous expansion of G. Then the *order of* G, which we denote by $t(G)$, is the first m such that $G_m \neq 0$. A vector $v \in \mathbb{C}^2 \setminus \{O\}$ is called a *characteristic direction* for G if $G_{t(G)}(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Moreover if $\lambda \neq 0$ the vector v is called a *nondegenerate characteristic direction* while it is called *degenerate* in case $\lambda = 0$. Hakim's theory [10] predicts the existence of at least $t(G) - 1$ parabolic curves tangent to each nondegenerate characteristic direction.

We have the following relations:

Proposition 4.1. *Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic fixed point for F. Suppose F is given by (1.1) and $e^{i\theta s} = 1$ for some $s \geq 2$. Let $G := F^s$ and assume F is dynamically separating. Then:*

(1) $G \neq \mathsf{id}$ *and* $t(G) \leq \nu(F)$.

(2) $[1 : 0]$ *is a characteristic direction for G. Moreover* $[1 : 0]$ *is a nondegenerate characteristic direction for G if and only if $\nu(F) = t(G)$.*

(3) *The $\nu(F)-1$ parabolic curves tangent to* $[1 : 0]$ *at O given by Theorem 1.4 for G can be found applying Hakim's and Abate's theory to G after a finite number of blow-ups.*

Proof. Since F is dynamically separating then there exist parabolic curves for F by Theorem 1.4 which are obviously parabolic curves for G. Thus $G \neq \mathsf{id}$. It is then clear that $\nu(F) \geq t(G)$. To prove the other statements we notice that everything involved is invariant under conjugation and thus, using transformations as (2.1) we can assume that $q_{j,0} = 0$ for $j \leq \nu(F)$. Therefore for $F = (F_1, F_2)$ we can write

$$F(z, w) = \begin{cases} F_1(z, w) = z + p_{\nu(F),0} z^{\nu(F)} + O(z^{\nu(F)+1}, zw, w^2) \\ F_2(z, w) = e^{i\theta} w + O(z^{\nu(F)-1} w, w^2, z^{\nu(F)+1}). \end{cases}$$

Iterating we find that $F^s = G = (G_1, G_2)$ is given by

$$G(z, w) = \begin{cases} G_1(z, w) = z + s p_{\nu(F),0} z^{\nu(F)} + O(z^{\nu(F)+1}, zw, w^2) \\ G_2(z, w) = w + O(z^{\nu(F)-1} w, w^2, z^{\nu(F)+1}). \end{cases} \tag{4.1}$$

From this it follows that $[1 : 0]$ is a characteristic direction for G. Moreover it is nondegenerate if and only if $t(G) = \nu(F)$ for in that case $G_{t(G)} = (p_{\nu(F),0} z^{\nu(F)} + wQ'(z, w), wQ''(z, w))$ with Q', Q'' suitable homogeneous polynomials of degree $t(G) - 1$.

To prove part (3), we make some preliminary observations. If $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ is a blow-up at O and $\tilde{F}$ is the blow-up of F, since $\pi \circ \tilde{F}^s = F^s \circ \pi$ and π is a biholomorphism outside the exceptional divisor then $\tilde{G} = \tilde{F}^s$. Notice that while $\nu(F) = \nu(\tilde{F})$, in general $t(G) \leq t(\tilde{G})$ (see Lemma 2.1(ii) and (2.1) in [1]). We may assume that after finitely many blow-ups and changes of coordinates F is given by (3.1) with $\mathrm{Re}(\lambda e^{-i\theta}) < 0$. A simple computation shows that G has order $\nu(F)$ and $G_{\nu(F)}(z,w) = (-sz^{\nu(F)}, -s\lambda e^{i\theta}z^{\nu(F)-1}w)$. Thus $[1:0]$ is a nondegenerate characteristic direction for G, and Hakim's theory produces (at least) $\nu(F)-1$ parabolic curves for G tangent to $[1:0]$. Now we have to show that such curves are the same as the ones given by Lemma 3.2. To see this, notice that G is of the form (3.5) at p. 201 of [1]. The $\nu(F)-1$ parabolic curves for G are then unique in the class of curves of the form $\zeta \mapsto (\zeta, u(\zeta))$ with $u \in \mathcal{E}(\delta)$ as in Lemma 3.2 (see p. 201–203 in [1]). Since the parabolic curves produced in Lemma 3.2 are in such a class then they must be the ones given by Hakim's and Abate's theory, and we are done. □

Example 4.2. The map $F(z,w) = (z+z^5, -w+w^3+z^5)$ is dynamically separating, $\nu(F) = 5$ and thus it has 4 parabolic curves tangent to $[1:0]$ at O by Theorem 1.4. The map $G(z,w) = F^2(z,w) = (z+2z^5+O(z^6), w-2w^3+O(w^4, z^7, w^2z^5))$ has therefore 4 parabolic curves tangent to $[1:0]$ at O. Moreover $t(G) = 3$ and the vector $[1:0]$ is a degenerate characteristic direction for G. However $\tilde{G}$ has order 5 at $[1:0]$ and has $[1:0]$ as a nondegenerate characteristic direction as a simple computation shows. Notice that $[0:1]$ is a nondegenerate characteristic direction for G and Hakim's results give 2 parabolic curves for G tangent to $[0:1]$ at O. These are contained into $\{z=0\}$ and are exchanged into each other by F.

Remark 4.3. Let $F \in \mathsf{End}(\mathbb{C}^2, O)$, and assume O is a quasi-parabolic fixed point for F and $e^{i\theta s} = 1$ for some $s \geq 2$. Suppose F is not dynamically separating. A calculation similar to the one performed in the proof of Proposition 4.1 shows that $[1:0]$ is always a degenerate characteristic direction for F^s, providing $F^s \neq \mathsf{id}$.

2. Let $F \in \mathsf{End}(\mathbb{C}^2, O)$ and assume O is a quasi-parabolic fixed point. In case F is not dynamically separating, there might be no parabolic curves tangent to the eigenspace of 1. A first simple example is when $F^s = \mathsf{id}$. However note that in such a case, if $p_j : \mathbb{C}^2 \to \mathbb{C}$ is the projection on the jth component, setting

$$\sigma(z,w) = \left(\sum_{m=0}^{s-1} p_1 \circ F^m(z,w), \sum_{m=0}^{s-1} e^{-i\theta m} p_2 \circ F^m(z,w) \right)$$

then $\sigma \circ F \circ \sigma^{-1}(z,w) = (z, e^{i\theta}w)$, thus $F_1(z,w) = z$, and in particular $\nu(F) = \infty$.

Less trivial examples of nondynamically separating map without parabolic curves are provided by the following construction. Let $f(u,v) = (f_1(u,v), f_2(u,v)) \in \mathsf{End}(\mathbb{C}^2, O)$ be given by

$$\begin{cases} f_1(u,v) = e^{i\theta}u + (a_{20}u^2 + a_{11}uv + a_{02}v^2) + \cdots \\ f_2(u,v) = e^{i\theta}v + (b_{20}u^2 + b_{11}uv + b_{02}v^2) + \cdots \end{cases} \tag{4.2}$$

with $e^{i\theta}$ satisfying the Bryuno condition

$$|e^{i\theta m} - 1| \geq cm^{-N}, \quad m \in \mathbb{N}$$

for some $c > 0$ and some large N. Note that the set of points on the circle satisfying such a condition has full measure. It is a classical result (see, e.g., [3] and [12]) that such a germ f is linearizable, and in particular there cannot exist parabolic curves for f. Now suppose that $a_{02} = 0$ in (4.2). Blow up the point O in $\mathbb{C}^2$ and consider the blow up map F of f at the point $[0 : 1]$ of the exceptional divisor. If the projection $\pi : \widetilde{\mathbb{C}^2} \to \mathbb{C}^2$ is given by $(u, v) = \pi(z, w) = (zw, w)$ then $F = (F_1, F_2)$ is given by

$$\begin{cases} F_1(z, w) = z + e^{-i\theta} w \frac{(a_{11} - b_{02})z + a_{03}w + \cdots}{1 + e^{-i\theta} w[b_{02} + \cdots]}, \\ F_2(z, w) = e^{i\theta} w + w[b_{02} w + (b_{11} zw + b_{03} w^2 + \cdots]. \end{cases} \tag{4.3}$$

Then $[0 : 1]$ is a quasi-parabolic point for F but there cannot exist parabolic curves tangent to the eigenspace of 1 for otherwise these would be parabolic curves for f at O. Note that even in this case $\nu(F) = \infty$.

We have to say that at the present we do not have any example of a nondynamically separating mapping F with $\nu(F) < \infty$ and without parabolic curves, even if we believe such a map should exist.

We conclude this work by mentioning a simple family of nondynamically separating maps for which nothing is known, but the understanding of which might unlock the general theory. Let $F_a = (F_{1,a}, F_{2,a})$ be given by

$$F_a(z, w) = \begin{cases} F_{1,a}(z, w) = z + z^3 + aw^2 \\ F_{2,a}(z, w) = e^{i\theta} w + zw + z^3, \end{cases} \tag{4.4}$$

with $a \in \mathbb{C}$. If $a = 0$, then $\{z = 0\}$ is invariant by F_0. Moreover, once fixed $w \in \mathbb{C}$, by the classical Leau-Fatou theory there exist two petals $P_1, P_2 \subset \mathbb{C}$ for $z \mapsto F_{1,0}(z, w)$ at $z = 0$. Then the two open sets $D_j = P_j \times \mathbb{C}$, $j = 1, 2$ are invariant by F_0. However we do not know whether there exist parabolic curves contained in D_1 or D_2.

If $a \neq 0$ and $e^{i\theta}$ is not a root of unity we do not even know whether there exists $P \in \mathbb{C}^2$ such that $F_a^n(P) \neq O$ for any n but $F_a^n(P) \to O$ as $n \to \infty$.

Notice that in case $e^{i\theta s} = 1$ for some $s \geq 2$ then Theorem 1.2 provides some parabolic curves for F^s. A direct computation shows that these curves are not tangent to $[1 : 0]$. In fact the known techniques for the parabolic case are not applicable to F^s along the direction $[1 : 0]$, not even after blow-ups.

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