PLURIPOTENTIAL THEORY, SEMIGROUPS AND BOUNDARY BEHAVIOR OF INFINITESIMAL GENERATORS IN STRONGLY CONVEX DOMAINS

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Abstract. We characterize infinitesimal generators of semigroups of holomorphic self-maps of strongly convex domains using the pluricomplex Green function and the pluricomplex Poisson kernel. Moreover, we study boundary regular fixed points of semigroups. Among other things, we characterize boundary regular fixed points both in terms of the boundary behavior of infinitesimal generators and in terms of pluripotential theory.

Contents

Introduction

A (continuous) semigroup (Φ_t) of holomorphic functions in a domain $D \subset \mathbb{C}^n$ is a continuous homomorphism from the additive semigroup of non-negative real numbers into

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the composition semigroup of all holomorphic self-maps of D endowed with the compact-open topology. Namely, the map $[0, +\infty) \ni t \mapsto (\Phi_t) \in \mathrm{Hol}(D, D)$ satisfies the following conditions:

(1) Φ_0 is the identity map id_D in D,
(2) $\Phi_{t+s} = \Phi_t \circ \Phi_s$, for all $t, s \geq 0$,
(3) Φ_t tends to id_D as t tends to 0 uniformly on compacta of D.

It is well known after the basic work of Berkson and Porta [5] in the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ that the dependence of every semigroup (Φ_t) of holomorphic self-maps of a domain $D \subset \mathbb{C}^n$ on the variable t is analytic and to each continuous semigroup (Φ_t) there corresponds a holomorphic vector field $F : D \to \mathbb{C}^n$ such that $\frac{\partial \Phi_t}{\partial t} = F \circ \Phi_t$. This vector field F is called the *infinitesimal generator* of the semigroup (Φ_t). Conversely, if a holomorphic vector field $F : D \to \mathbb{C}^n$ is *semicomplete*, namely for all $z \in D$ its local flow $\gamma^z(t)$ such that $\gamma^z(0) = z$ is well defined for all $t \geq 0$, then F is the infinitesimal generator of a semigroup of holomorphic self-maps of D. We refer to [1, Section 2.5.3] and [28] for more details. Be aware that in the literature there is not a standard sign convention for the Cauchy problem generating F, namely, sometimes the problem $\frac{\partial \Phi_t}{\partial t} = -F \circ \Phi_t$ is considered and thus all formulas regarding F have reverse inequalities with respect to our formulas. For instance, regarding the bibliography of the present paper, such a convention is adopted in [3], [4], [17], [18], [19], [27], [28] and [30].

It is clear that the analytical properties of an infinitesimal generator are strictly related to the dynamical and geometrical properties of its semigroup. For instance, any zero of F in D corresponds to a common fixed point for (Φ_t).

Therefore one of the main questions in the theory of semigroups of holomorphic functions is that of characterizing (in the most useful way) those holomorphic vector fields which are infinitesimal generators. For $D = \mathbb{D}$, the unit disc of $\mathbb{C}$, there is a very nice representation formula, due to Berkson and Porta [5] (see also [1] and [30]). Namely:

Theorem 0.1 (Berkson-Porta)**.** *A holomorphic function $G : \mathbb{D} \to \mathbb{C}$ is the infinitesimal generator of a semigroup (Φ_t) in $\mathbb{D}$ if and only if there exists a point $b \in \overline{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \to \mathbb{C}$ with $\mathsf{Re}\, p \geq 0$ such that*

$$G(z) = (z - b)(\overline{b}z - 1)p(z), \qquad z \in \mathbb{D}.$$

If the semigroup is not an elliptic group (that is, some/all iterates Φ_t for $t > 0$ are elliptic automorphisms), the point b given in Berkson-Porta's formula is exactly the *Denjoy-Wolff point* of the semigroup (Φ_t). Namely, $\lim_{t \to \infty} \Phi_t(z) = b$ for all $z \in \mathbb{D}$ (see also Section two). Other alternative descriptions of infinitesimal generators in $\mathbb{D}$ can be found in [30, Section 3.6].

In several variables there are various characterizations of infinitesimal generators (see [28] for a good account). All these characterizations reflect the basic fact that holomorphic self-maps of a domain are contractions for the Kobayashi metric of such a domain. In fact, Abate [2] proved that if D is a strongly convex domain with smooth boundary and with Kobayashi metric k_D, then a holomorphic vector field $F: D \to \mathbb{C}^n$ is an infinitesimal generator if and only if $d(k_D \circ F)(z) \cdot F(z) \leq 0$ for all $z \in D$. Unfortunately, even for the case $D = \mathbb{B}^n$, the unit ball of $\mathbb{C}^n$, such a formula is rather complicated and does not give any information on the dynamical properties of the associated semigroup. Later on, still in this optic, C. de Fabritiis gave a better description of a class of infinitesimal generators called "of one-dimensional type" (see [15]). Some rather precise characterizations of infinitesimal generators in the unit ball of infinite dimensional Hilbert spaces are given by D. Aharonov, M. Elin, S. Reich, and D. Shoikhet in [3], [19] and [27].

Part of the present paper is devoted to find characterizations of infinitesimal generators in bounded strongly convex domains with smooth boundary (here and in the rest of the paper "smooth" means at least of class C^3) by means of the pluricomplex Green function G_D of Klimek [21], Lempert [24] and Demailly [16] and the pluricomplex Poisson kernel $u_{D,p}$ introduced by Patrizio and the first named author in [12] (see Section one for definitions and preliminaries about pluripotential theory in strongly convex domains). In particular, we prove (see Theorems 3.5 and 3.11):

Theorem 0.2. *Let $F: D \to \mathbb{C}^n$ be a holomorphic vector field. The following are equivalent:*

(1) *The map F is an infinitesimal generator of a semigroup of holomorphic self-maps of D.*
(2) *For all $z, w \in D$, $z \neq w$, it holds $d(k_D)|_{(z,w)} \cdot (F(z), F(w)) \leq 0$.*
(3) *For all $z, w \in D$, $z \neq w$, it holds $d(G_D)_{(z,w)} \cdot (F(z), F(w)) \leq 0$.*
(4) *For all $z, w \in D$ and for all $r > 0$ such that $z - rF(z), w - rF(w) \in D$ it holds $k_D(z - rF(z), w - rF(w)) \geq k_D(z, w)$.*

Moreover, if F is C^1-regular at a point $p \in \partial D$, then F is an infinitesimal generator whose associated semigroup has Denjoy-Wolff point at $p \in \partial D$ if and only if $d(u_{D,p})_z \cdot F(z) \leq 0$ for all $z \in D$.

In case $D = \mathbb{B}^n$ is the unit ball of $\mathbb{C}^n$ (or more generally for the unit ball of complex Hilbert spaces), equivalence between (1) and (4) (and also with an explicit expression of (3), see Remark 3.7) was proven with different methods by Reich and Shoikhet [27, Theorem 2.1]. The last statement can be seen as a Berkson-Porta like formula at the boundary. Moreover, this last formula is just a particular case of a general one for the existence of *boundary regular fixed points*. We recall that a point $p \in \partial D$ is a boundary regular fixed point—BRFP for short—for a semigroup (Φ_t) if it is a fixed point for non-tangential limits for all Φ_t's and if the *boundary dilatation coefficients* at p of the Φ_t's are all finite (roughly speaking, the boundary dilatation coefficient of a self-map f of D at p is a measure of the velocity f approaches p when moving to p; see Section two for details and precise definitions).

The second part of this paper is devoted to characterize BRFPs of semigroups in terms of the pluricomplex Poisson kernel $u_{D,p}$ and the local behavior of the infinitesimal generator. In this direction, we cite the following result from [19] (see also [17], [4])

Theorem 0.3 (Elin-Shoikhet)**.** *Let $F : \mathbb{B}^n \to \mathbb{C}^n$ be the infinitesimal generator of a semigroup (Φ_t) in $\mathbb{B}^n$ and $p \in \partial\mathbb{B}^n$. Assume that $\lim_{(0,1)\ni r\to 1} F(rp) = 0$. The following are equivalent:*

(1) $\liminf_{(0,1)\ni r\to 1} \mathsf{Re}\,\langle F(rp), p\rangle/(r-1) < +\infty$.
(2) $\lim_{(0,1)\ni r\to 1}\langle F(rp), p\rangle/(r-1) = \beta$ *exists finitely.*
(3) *The point p is a BRFP for the semigroup (Φ_t).*

Moreover, if one of the three conditions holds, then $\beta \in \mathbb{R}$ and the boundary dilatation coefficient of Φ_t at p is $e^{t\beta}$.

The hypothesis in Theorem 0.3 that the infinitesimal generator F has radial limit 0 at p, being essential in the proof of their result, is however not necessary for a point p to be a BRFP (see Example 4.2). Moreover, and surprisingly enough, Theorem 0.3 would be false without such an hypothesis (see Example 4.3 where it is constructed an infinitesimal generator for which (1) holds at some $p \in \partial\mathbb{B}^2$ but p is not a BRFP for the associated semigroup). In fact, it turns out that a point $p \in \partial\mathbb{B}^n$ is a BRFP for the semigroup if and only if a condition similar to (1) holds not just for the radial direction but for all the directions. To be more precise and in order to state the result for general strongly convex domains, we need to use the so called *Lempert projection devices*. For the time being, we can say that a Lempert projection device $(\varphi, \widetilde{\rho}_\varphi)$ is given by a particular holomorphic map $\varphi : \mathbb{D} \to D$ (called *complex geodesic*) which extends smoothly on $\partial\mathbb{D}$ and a holomorphic map $\widetilde{\rho}_\varphi : D \to \mathbb{D}$ such that $\widetilde{\rho}_\varphi \circ \varphi = \mathsf{id}_{\mathbb{D}}$ (actually a Lempert projection device is a triple of maps, we refer the reader to Section one for details). For the unit ball $\mathbb{B}^n$ a Lempert projection device $(\varphi, \widetilde{\rho}_\varphi)$ is nothing but a (suitable) parametrization $\varphi : \mathbb{D} \to \mathbb{B}^n$ of the intersection of $\mathbb{B}^n$ with an affine complex line and $\widetilde{\rho}_\varphi$ is the orthogonal projection on it (see also Section six where the case of $\mathbb{B}^n$ is studied in detail). Our second main result is the following:

Theorem 0.4. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary, let F be the infinitesimal generator of a semigroup (Φ_t) of holomorphic self-maps of D and $p \in \partial D$. The following are equivalent:*

(1) *The semigroup (Φ_t) has a BRFP at p with boundary dilatation coefficients $\alpha_t(p) \leq e^{\beta t}$ for all $t \geq 0$.*
(2) *There exists $\beta \in \mathbb{R}$ such that $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$.*
(3) *There exists $C > 0$ such that for any Lempert's projection device $(\varphi, \widetilde{\rho}_v)$ with $\varphi(1) = p$ it follows*

$$\limsup_{(0,1)\ni r\to 1} \frac{|d(\widetilde{\rho}_\varphi)_{\varphi(r)} \cdot F(\varphi(r))|}{1-r} \leq C.$$

Moreover, if p is a BRFP for (Φ_t) with boundary dilatation coefficients $\alpha_t(p) = e^{-bt}$ then

$$b = \inf_{z \in D} \frac{d(u_{D,p})_z \cdot F(z)}{u_{d,p}(z)},$$

and the non-tangential limit

$$A(\varphi, p) := \angle \lim_{\zeta \to 1} \frac{d(\widetilde{\rho}_\varphi)_{\varphi(\zeta)} \cdot F(\varphi(\zeta))}{\zeta - 1}$$

exists finitely, $A(\varphi, p) \in \mathbb{R}$ and $A(\varphi, p) \leq b$. Also, $b = \sup A(\varphi, p)$, with the supremum taken as φ varies among all Lempert's projection devices $(\varphi, \widetilde{\rho}_v)$ with $\varphi(1) = p$.

This result is contained in Theorem 3.8 and Theorem 4.7. One of the main ingredients in the proof is the remarkable property that the projection of an infinitesimal generator on every complex geodesic is still an infinitesimal generator.

In order to give the proof of the previous results, in the first section we revise pluripotential theory in strongly convex domains and in the second section we study iteration using the pluricomplex Green function and the pluricomplex Poisson kernel. We should say that, even if part of the results in section two are already known, our present formulation seems to be new and, as we will prove later, quite effective. Section three is devoted to the interactions between pluripotential theory and semigroups. In section four we discuss a couple of examples on the boundary behavior of semigroups and complete the proof of our characterization of BRFPs in terms of the boundary behavior of the infinitesimal generator. As a consequence, in Corollary 4.8 we discuss *stationary points* of semigroups (namely those BRFPs for which the boundary dilatation coefficient is less than or equal to 1). In section five we consider the non-linear resolvent of Reich and Shoikhet, proving that every BRFP of the non-linear resolvent is a BRFP for the semigroup (see Proposition 5.2). Finally, in section six we translate our results into the ball $\mathbb{B}^n$ where some more explicit formulations, using automorphisms, are possible. In this case, we also discuss the boundary behavior of the infinitesimal generator at a BRFP under some boundness conditions (see Corollary 6.2).

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1. Preliminary results on pluripotential theory in strongly convex domains

For the definition, properties and further results about strongly convex domains, we refer the reader to the nice monograph by Abate [1, Part 2]. Likewise, for an introduction to pluripotential theory with a special emphasis on complex Monge-Ampère operators, we recommend the beautiful book by Klimek [22] (a short introduction is also contained in [9]). Anyhow, for the sake of clearness, we are going to give some basic definitions and define the tools we need later on.

1.1. **The pluricomplex Green function.** Let $D \subset\subset \mathbb{C}^n$ be a domain and $z \in D$. Define

$$\mathcal{K}_{D,z} = \{u \text{ plurisubharmonic in } D : u < 0, u(w) - \log \|z - w\| \leq O(1) \text{ as } w \to z\}.$$

The Klimek [21] *pluricomplex Green function* is defined as

$$G_D(z, w) := \sup_{u \in \mathcal{K}_{D,z}} u(w).$$

Such a function is plurisubharmonic in D, locally bounded in $D \setminus \{z\}$ and has a logarithmic pole at z (see [21] and [22]). If D is hyperconvex (in particular, if D is a convex domain), then Demailly [16] showed that G_D, extended to be 0 on $D \times \partial D$, is continuous as a function $G_D : D \times \overline{D} \to [-\infty, 0)$. Moreover, from the work of Lempert [24] and Demailly [16], it turns out that $G_D(z, w)$ is the unique solution of the following homogeneous Monge-Ampère equation:

$$\begin{cases} u \text{ plurisubharmonic in } D & \\ (\partial\overline{\partial} u)^n = 0 & \text{in } D \setminus \{z\} \\ \lim_{w \to x} u(w) = 0 & \text{for all } x \in \partial D \\ u(w) - \log |w - z| = O(1) & \text{as } w \to z. \end{cases}$$

By the very definition, if $h : D \to D'$ is holomorphic, then for all $z, w \in D$

$$G_{D'}(h(z), h(w)) \leq G_D(z, w). \tag{1.1}$$

In case D is a bounded strongly convex domain with smooth boundary (here and in the rest of the paper "smooth" means at least of class C^3) Lempert [24] proved that $G_D(z, w)$ is smooth and regular for $(z, w) \in \overline{D} \times \overline{D} \setminus \mathsf{Diag}(\overline{D} \times \overline{D})$ and that

$$G_D(z, w) = \log \tanh k_D(z, w), \tag{1.2}$$

where $k_D(z, w)$ is the Kobayashi distance of D (for definition and properties we refer to [1] or to [23]).

For instance, for $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, the pluricomplex Green function coincides with the usual (negative) Green function, while for $D = \mathbb{B}^n$ the unit ball of $\mathbb{C}^n$ we have

$$G_{\mathbb{B}^n}(z, w) = \log \|T_z(w)\|, \tag{1.3}$$

where $T_z : \mathbb{B}^n \to \mathbb{B}^n$ is any automorphism of $\mathbb{B}^n$ with the property that $T_z(z) = 0$.

1.2. **The pluricomplex Poisson kernel.** Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $z_0 \in D$ and let $p \in \partial D$. In the paper [12], Patrizio and the first quoted author introduced a plurisubharmonic function $u_{D,p} : D \to (-\infty, 0)$ which extends smoothly on $\overline{D} \setminus \{p\}$ such that $d(u_{D,p})_z \neq 0$ for all $z \in D$, $u_{D,p}(q) = 0$ for all $q \in \partial D \setminus \{p\}$ and $u_{D,p}$ has a simple pole at p along non-tangential directions. Up to a real positive multiple, we assume here that $u_{D,p}(z_0) = -1$. The function $u_{D,p}$ solves the following homogeneous Monge-Ampère equation:

$$\begin{cases} u \text{ plurisubharmonic in } D & \\ (\partial\overline{\partial}u)^n = 0 & \text{in } D \\ u < 0 & \text{in } D \\ u(w) = 0 & \text{for all } w \in \partial D \setminus \{p\} \\ u(w) \approx \|w - p\|^{-1} & \text{as } w \to p \text{ non-tangentially.} \end{cases}$$

In the papers [12] and [13], the authors prove that $u_{D,p}$ shares many properties with the classical Poisson kernel for the unit disk. In case $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, the function $u_{\mathbb{D},p}$ (normalized so that $u_{\mathbb{D},p}(0) = -1$) is in fact the classical (negative) Poisson kernel. In case $D = \mathbb{B}^n$, the pluricomplex Poisson kernel (normalized so that $u_{\mathbb{B}^n,p}(0) = -1$) is given by

$$u_{\mathbb{B}^n,p}(z) = -\frac{1 - \|z\|^2}{|\langle p - z, p\rangle|^2}.$$

The level sets of $u_{D,p}$ are exactly boundaries of Abate's horospheres. Recall that a *horosphere* $E_D(p, R)$ of center $p \in \partial D$ and radius $R > 0$ (with respect to z_0) is given by

$$E_D(p, R) = \{z \in D : \lim_{w \to p}[k_D(z, w) - k_D(z_0, w)] < \frac{1}{2} \log R\}.$$

Notice that the existence of the limit in the definition of $E_D(p, R)$ is a characteristic of smooth strongly convex domains and follows again from Lempert's theory (see [1, Theorem 2.6.47]). Thanks to our normalization $u_{D,p}(z_0) = -1$, it follows that

$$E_D(p, R) = \{z \in D : u_{D,p}(z) < -1/R\}. \tag{1.4}$$

For the unit disk, these level sets are boundaries of horocycles and, in case $D = \mathbb{B}^n$, these are boundaries of horospheres in $\mathbb{B}^n$ with center p, whose explicit expression is

$$E_{\mathbb{B}^n}(p, R) = \{z \in \mathbb{B}^n : \frac{|1 - \langle z, p\rangle|^2}{1 - \|z\|^2} < R\}.$$

More information about the properties of $u_{D,p}$ (such as smooth dependence on p, extremality, uniqueness, relations with the pluricomplex Green function, usage in representation formulas for pluriharmonic functions) can be found in [13].

1.3. **Lempert's projection devices.** We recall that a complex geodesic $\varphi : \mathbb{D} \to D$ is a holomorphic isometry between $k_{\mathbb{D}}$ (the hyperbolic distance in $\mathbb{D}$) and k_D. By Lempert's work (see [24] and [1]) given two points $z_0 \in D$ and $z \in \overline{D}$, there exists a unique complex geodesic $\varphi : \mathbb{D} \to D$ such that φ extends smoothly past the boundary, $\varphi(0) = z_0$ and $\varphi(t) = z$, with $t \in (0,1)$ if $z \in D$ and $t = 1$ if $z \in \partial D$. Moreover, for any such a complex geodesic there exists a holomorphic retraction $\rho_\varphi : D \to \varphi(\mathbb{D})$, *i.e.* there exists a holomorphic map $\rho_\varphi : D \to D$ such that $\rho_\varphi \circ \rho_\varphi = \rho_\varphi$ and $\rho_\varphi(z) = z$ for any $z \in \varphi(\mathbb{D})$.

Given a complex geodesic, there might exist many holomorphic retractions to such geodesic, but the one constructed by Lempert turns out to be the only one with affine fibers (see [13, Section 3]). We call such a ρ_φ the *Lempert projection* associated to φ.

Furthermore, we let $\widetilde{\rho}_\varphi := \varphi^{-1} \circ \rho_\varphi : D \to \mathbb{D}$ and call it the *left inverse* of φ, for $\widetilde{\rho}_\varphi \circ \varphi = \mathsf{id}_{\mathbb{D}}$. The triple $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ is the so-called *Lempert projection device*.

For $D = \mathbb{B}^n$ the unit ball of $\mathbb{C}^n$ the image of the complex geodesic through the points $z \neq w \in \overline{\mathbb{B}^n}$ is just the *one dimensional slice* $S_{z,w} := \mathbb{B}^n \cap \{z + \zeta(z - w) : \zeta \in \mathbb{C}\}$. The Lempert projection is thus given by the orthogonal projection of $\mathbb{B}^n$ onto $S_{z,w}$.

By Lempert's very definition, if $\varphi : \mathbb{D} \to D$ is a complex geodesic, then

$$G_D(\varphi(\zeta), \varphi(\eta)) = G_{\mathbb{D}}(\zeta, \eta) \tag{1.5}$$

for all $\zeta, \eta \in \mathbb{D}$.

Finally, we mention [12, p. 516] that for any given Lempert projection device $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ in D with $\varphi(1) = p$ there exists $a_\varphi > 0$ such that for all $\zeta \in \mathbb{D}$

$$u_{D,p}(\varphi(\zeta)) = a_\varphi u_{\mathbb{D},1}(\zeta). \tag{1.6}$$

2. Iteration theory by means of pluripotential theory

Both the pluricomplex Green function and the pluricomplex Poisson kernel can be used to describe dynamical properties of holomorphic self-maps of a bounded strongly convex domain with smooth boundary. The aim of this section is exactly to formulate the results we need later on in terms of pluripotential theory.

All the results presented in this section are strongly based on some known results about iteration (mainly due to Abate, see [1]). However, for the aim of completeness, we sometimes provide a sketch of some new direct proofs.

As a matter of notation, for a map $h : D \to D$ we denote by $\mathsf{Fix}(h)$ the set of its fixed points in D, namely

$$\mathsf{Fix}(h) := \{z \in D : h(z) = z\}.$$

To begin with, we can reformulate a Schwarz-type lemma for strongly convex domains as follows:

Theorem 2.1. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary and let $z_0 \in D$. Let $h : D \to D$ be holomorphic. Then $h(z_0) = z_0$ if and only if for all $z \in D$*

$$G_D(z_0, h(z)) \leq G_D(z_0, z). \tag{2.1}$$

Moreover, if equality holds in (2.1) *for some* $z \neq z_0$ *and* $\varphi : \mathbb{D} \to D$ *is the complex geodesic such that* $\varphi(0) = z_0$ *and* $\varphi(t) = z$ *for some* $t \in (0,1)$*, it follows that* $h \circ \varphi : \mathbb{D} \to D$ *is a complex geodesic and* $h : \varphi(\mathbb{D}) \to h(\varphi(\mathbb{D}))$ *is an automorphism.*

Proof. The necessity and sufficiency of (2.1) follows directly from the very definition of G_D and (1.1).

In order to prove the last statement, assume that $G_D(z_0, h(z)) = G_D(z_0, z)$ for some $z \in D$, $z \neq z_0$. Let $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ be the Lempert projective device such that $\varphi(0) = z_0$ and $\varphi(t) = z$ for some $t \in (0,1)$ and let $(\psi, \rho_\psi, \widetilde{\rho}_\psi)$ be the Lempert projective device such that $\psi(0) = z_0$ and $\psi(r) = h(z)$ for some $r \in (0,1)$. Let $\tilde{h}(\zeta) := \widetilde{\rho}_\psi(h(\varphi(\zeta)))$ for $\zeta \in \mathbb{D}$. Notice that $\tilde{h}(0) = 0$. Then, $H(\zeta) := G_{\mathbb{D}}(0, \tilde{h}(\zeta)) - G_{\mathbb{D}}(0, \zeta) \leq 0$. Moreover, the function $\mathbb{D} \ni \zeta \mapsto H(\zeta)$ is harmonic on $\mathbb{D} \setminus \{0, \tilde{h}^{-1}(0)\}$ and bounded from above, thus can be extended in a subharmonic way to all of $\mathbb{D}$. We still call H such an extension. For all $\zeta \in \mathbb{D}$, and by (1.5), it follows that

$$G_{\mathbb{D}}(0, \tilde{h}(t)) = G_D(\psi(0), \psi(\tilde{h}(t))) = G_D(z_0, h(z)) = G_D(z_0, z) = G_{\mathbb{D}}(0, t).$$

By the maximum principle then $H(\zeta) \equiv 0$ and $\tilde{h}(\zeta) = e^{i\theta}\zeta$ for some $\theta \in \mathbb{R}$, proving the statement. □

Definition 2.2. Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $p \in \partial D$, and $h : D \to D$ holomorphic. The *boundary dilatation coefficient* $\alpha_h(p) \in (0, +\infty]$ is defined as

$$\alpha_h(p) = \inf_{q \in \partial D} \{ \sup_{z \in D} \frac{u_{D,p}(z)}{u_{D,q}(h(z))} \}.$$

As we show, this number can be characterized in several ways. Some of them are widely used in the literature (see [1] and [8]). Indeed we have:

Proposition 2.3. *Let* $D \subset\subset \mathbb{C}^n$ *be a strongly convex domain with smooth boundary,* $p \in \partial D$ *and* $h : D \to D$ *holomorphic. Then, the following are equivalent:*

(1) *The boundary dilatation coefficient* $\alpha_h(p) < +\infty$*.*
(2) *There exist a (necessarily unique) point* $q \in \partial D$ *and a number* $\lambda > 0$ *such that*

$$h(E_D(p, R)) \subseteq E_D(q, \lambda R), \ \textit{for all } R > 0. \tag{2.2}$$

(3) *It holds*

$$\frac{1}{2} \log \beta_h(p) := \liminf_{z \to p} [k_D(z, z_0) - k_D(h(z), z_0)] < +\infty. \tag{2.3}$$

Moreover, if one of the statements holds, then

$$\beta_h(p) = \alpha_h(p) = \inf\{\lambda > 0 : \lambda \ \textit{satisfies} (2.2)\}. \tag{2.4}$$

Proof. By the very definition (1) is equivalent to the existence of $q \in \partial D$ such that $u_{D,q}(h(z)) \leq \frac{1}{\alpha_h(p)} u_{D,p}(z)$ for all $z \in D$. By (1.4), (1) and (2) are equivalent and $\alpha_h(p) = \inf\{\lambda > 0 : \lambda \text{ satisfies}(2.2)\}$.

If (3) holds then (2) follows from Abate's version of the *Julia lemma* for strongly convex domains (see [1, Theorem 2.4.16]); also by the same token, $\beta_h(p) \geq \inf\{\lambda > 0 : \lambda \text{ satisfies}(2.2)\}$.

Finally, if (2) holds, let $\varphi : \mathbb{D} \to D$ be the complex geodesic such that $\varphi(0) = z_0$ and $\varphi(1) = p$ and let $\widetilde{\rho}_\varphi : D \to \mathbb{D}$ be its left-inverse. Let $\tilde{h} := \widetilde{\rho}_\varphi \circ h \circ \varphi : \mathbb{D} \to \mathbb{D}$. Since φ is an isometry between the Poincaré distance of $\mathbb{D}$ and the Kobayashi distance of D and $k_{\mathbb{D}}(\widetilde{\rho}_\varphi(z), \widetilde{\rho}_\varphi(w)) \leq k_D(z,w)$ for all $z, w \in D$, then it is easy to check that for all $R > 0$ it holds $\tilde{h}(E_{\mathbb{D}}(1,R)) \subseteq E_{\mathbb{D}}(1, \lambda R)$. Therefore the classical Julia-Wolff-Carathéodory theorem implies that $\beta_{\tilde{h}}(1) < \infty$ and actually $\beta_{\tilde{h}}(1) \leq \lambda$. Now,

$$\begin{aligned}\frac{1}{2}\log\beta_h(p) &= \liminf_{w\to p}[k_D(w,\varphi(0)) - k_D(h(w),\varphi(0))] \\ &\leq \liminf_{\zeta\to 1}[k_D(\varphi(\zeta),\varphi(0)) - k_D(h(\varphi(\zeta)),\varphi(0))] \\ &\leq \liminf_{\zeta\to 1}[k_{\mathbb{D}}(\zeta,0) - k_{\mathbb{D}}(\widetilde{\rho}_\varphi(h(\varphi(\zeta))),0)] = \frac{1}{2}\log\beta_{\tilde{h}}(1),\end{aligned} \tag{2.5}$$

which proves that $\beta_h(p) < +\infty$ and actually $\beta_h(p) \leq \inf\{\lambda > 0 : \lambda \text{ satisfies (2.2)}\}$, ending the proof of the proposition. □

It is worth mentioning that by our very definition $\alpha_h(p)$ does not depend on z_0, while *a priori* the liminf in (2.3) does. However, the independence of such liminf from z_0 can be also shown directly, see [8, Lemma 6.1].

We have the following version of Julia's lemma for strongly convex domains:

Theorem 2.4. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $p \in \partial D$ and $h : D \to D$ holomorphic. If the boundary dilatation coefficient $\alpha_h(p) < +\infty$, then there exists a unique point $q \in \partial D$ such that h has non-tangential limit q at p and, for all $z \in D$,*

$$u_{D,q}(h(z)) \leq \frac{1}{\alpha_h(p)} u_{D,p}(z). \tag{2.6}$$

Moreover, if equality holds in (2.6) *for some $z \in D$ and $\varphi : \mathbb{D} \to D$ is the complex geodesic such that $\varphi(1) = p$ and $\varphi(0) = z$, it follows that $h \circ \varphi : \mathbb{D} \to D$ is a complex geodesic and $h : \varphi(\mathbb{D}) \to h(\varphi(\mathbb{D}))$ is an automorphism.*

Proof. By the very definition, if $\alpha_h(p) < +\infty$ then there exists at least one $q \in \partial D$ and a constant $C > 0$ such that

$$u_{D,q}(h(z)) \leq C u_{D,p}(z), \tag{2.7}$$

for all $z \in D$. Since $u_{D,p}$ has a simple pole as $z \to p$ along non-tangential directions, the above inequality (2.7) implies that h has non-tangential limit q at p. In particular, this implies that there exists *at most one* $q \in \partial D$ such that $\sup_{z\in D} \frac{u_{D,p}(z)}{u_{D,q}(h(z))} < +\infty$. Therefore, (2.6) holds.

In order to prove the last statement, assume that $u_{D,q}(h(z)) = \frac{1}{\alpha_h(p)} u_{D,p}(z)$ for some $z \in D$. Let $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ be the Lempert projective device such that $\varphi(1) = p$ and $\varphi(0) = z$ and let $(\psi, \rho_\psi, \widetilde{\rho}_\psi)$ be the Lempert projective device such that $\psi(1) = q$ and $\psi(0) = h(z)$. Write $\tilde{h}(\zeta) := \widetilde{\rho}_\psi(h(\varphi(\zeta)))$ for $\zeta \in \mathbb{D}$. By (1.6) and (2.6) it follows that $H(\zeta) := u_{\mathbb{D},1}(\zeta) - \lambda u_{\mathbb{D},1}(\tilde{h}(\zeta)) \leq 0$ for $\zeta \in \mathbb{D}$ and $\lambda := \alpha_h(p) a_\psi / a_\varphi$. The function H is harmonic in $\mathbb{D}$ and, by construction,

$$H(0) = u_{\mathbb{D},1}(0) - \lambda u_{\mathbb{D},1}(\tilde{h}(0)) = \frac{1}{a_\varphi} u_{D,p}(z) - \frac{\lambda}{a_\psi} u_{D,q}(h(z)) = 0.$$

Thus the maximum principle implies that $H(\zeta) \equiv 0$, which in turns implies that $\lambda = 1$ and $\tilde{h}$ is the identity on $\mathbb{D}$ and the statement follows. $\square$

Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary, let $z_0 \in D$ and let $p \in \partial D$. Following Abate ([1]) we denote by $K(p, R)$ the *K-region* with vertex p and radius $R > 1$ defined as

$$K(p, R) = \{z \in D : \lim_{w\to p} [k_D(z, w) - k_D(z_0, w)] + k_D(z, z_0) < \log R\}.$$

If $Q : D \to \mathbb{C}^n$ is a function, we write $\mathsf{K}-\lim_{z\to p} Q(z) = L$ if for any sequence $\{z_k\} \subset D$ which tends to p and belongs eventually to a K-region $K(p, R)$ for some $R > 1$, it follows $\lim_{k\to\infty} Q(z_k) = L$. Notice that if Q has $\mathsf{K}-$limit L at p then in particular it has non-tangential limit L at p.

Remark 2.5. If $\alpha_h(p) < +\infty$ and $q \in \partial D$ is the point given by Theorem 2.4 then actually h has $\mathsf{K}-$limit q at p. This follows from Abate's version of the classical Julia-Wolff-Carathéodory theorem, but also from (2.6), since actually $u_{D,p}(z) \to -\infty$ when $z \to p$ inside a K-region (see [13, section 5]).

The reason of the importance of boundary dilatation coefficients in iteration theory is that, while they give a global picture of the dynamics of a self-map of D, they can be easily computed as radial limits along any complex geodesic. We are going to state this fact in a particular case which we need later. Before that we give the following

Definition 2.6. Let $D \subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $h : D \to D$ be holomorphic. We say that a point $p \in \partial D$ is a *boundary regular fixed point*, BRFP for short, if h has non-tangential limit p at p and the boundary dilatation coefficient $\alpha_h(p) < +\infty$. A BRFP with boundary dilatation coefficient ≤ 1 is also called a *stationary point*. Likewise, those boundary regular fixed points with $\alpha_h(p) > 1$ are usually called *boundary repelling fixed points*.

Now we can state the following version of Julia-Wolff-Carathéodory theorem, due essentially to Abate:

Theorem 2.7. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $h : D \to D$ be holomorphic and let $p \in \partial D$. Then p is a BRFP for h if and only if for some—and hence any—Lempert projective device $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ such that $\varphi(1) = p$ it follows*

$$\liminf_{(0,1)\ni r\to 1} \frac{|1 - \widetilde{\rho}_\varphi(h(\varphi(r)))|}{1 - r} < +\infty. \tag{2.8}$$

Moreover, if p is a BRFP for h then

$$\lim_{r\to 1} \frac{1 - \widetilde{\rho}_\varphi(h(\gamma(r)))}{1 - \widetilde{\rho}_\varphi(\gamma(r))} = \alpha_h(p) \tag{2.9}$$

for any curve $\gamma : [0, 1) \to D$ such that $\lim_{r\to 1} \gamma(r) = p$, the curve in $\mathbb{D}$ given by $r \mapsto \widetilde{\rho}_\varphi(\gamma(r))$ converges non-tangentially to 1 and $\lim_{r\to 1} k_D(\gamma(r), \rho_\varphi(\gamma(r))) = 0$. In particular the map $\widetilde{\rho}_\varphi \circ h \circ \varphi : \mathbb{D} \to \mathbb{D}$ has BRFP at 1 with boundary dilatation coefficient $\alpha_h(p)$.

Proof. If p is a BRFP for h then the result follows from [1, Theorem 2.7.14].

Conversely, assume (2.8) holds. Then

$$\liminf_{\zeta\to 1} \frac{1 - |\widetilde{\rho}_\varphi(h(\varphi(\zeta)))|}{1 - |\zeta|} < \liminf_{(0,1)\ni r\to 1} \frac{|1 - \widetilde{\rho}_\varphi(h(\varphi(r)))|}{1 - r} < +\infty.$$

Thus the classical Julia-Wolff-Carathéodory theorem (see, *e.g.*, [1]) implies that 1 is a BRFP for $\zeta \mapsto \widetilde{\rho}_\varphi(h(\varphi(\zeta)))$ with boundary dilatation coefficient $a < +\infty$. Now, by (2.3), taking into account that $k_{\mathbb{D}}(\widetilde{\rho}_\varphi(z), \widetilde{\rho}_\varphi(w)) \leq k_D(z, w)$ and arguing as in (2.5) we find that $\frac{1}{2} \log \alpha_h(p) \leq \frac{1}{2} \log a$, namely $\alpha_h(p) < +\infty$. Theorem 2.4 implies that h has non-tangential limit q at p for some $q \in \partial D$. In order to end the proof we need to show that $q = p$. To this aim, we first notice that $\lim_{r\to 1} \widetilde{\rho}_\varphi(h(\varphi(r))) = 1$ forces $h(\varphi(r))$ to tend to p as $r \to 1$ because $\widetilde{\rho}_\varphi(\overline{D} \setminus \{\varphi(\overline{\mathbb{D}})\}) \subset \mathbb{D}$ by [25, Proposition 1 p. 345]. But $\varphi(\mathbb{D})$ is transverse to ∂D by Hopf's lemma and therefore $\varphi(r) \to p$ non-tangentially. This implies that $\angle \lim_{z\to p} h(z) = p$ and we are done. □

In case a holomorphic self-map of D has no fixed points in D, there always exists a particular stationary point (see [1, Theorem 2.4.23]):

Theorem 2.8 (Abate)**.** *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $h : D \to D$ be holomorphic. If $\mathsf{Fix}(h) = \emptyset$ then there exists a unique point $p \in \partial D$, called the Denjoy-Wolff point of h, such that p is a stationary point for h and the sequence of iterates $\{h^{\circ m}\}$ converges uniformly on compacta to the constant map $D \ni z \mapsto p$.*

Stationary points are quite special, as the following proposition shows:

Proposition 2.9. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $h : D \to D$ be holomorphic. Assume that $p \in \partial D$ is a stationary point.*

(1) *If* $\mathsf{Fix}(h) \neq \emptyset$ *then there exists a complex geodesic* $\varphi : \mathbb{D} \to D$ *such that* $\varphi(1) = p$ *and* $\varphi(D) \subseteq \mathsf{Fix}(h)$*. Moreover, for all* $\theta \in \mathbb{R}$*, the point* $\varphi(e^{i\theta}) \in \partial D$ *is a stationary point for* h *and* $\alpha_h(\varphi(e^{i\theta})) = 1$*.*
(2) *If* $\mathsf{Fix}(h) = \emptyset$ *then* p *is the Denjoy-Wolff point of* h *and* h *has no other stationary point in* ∂D*.*

Proof. (1) Assume $z \in \mathsf{Fix}(h)$. Let $\varphi : \mathbb{D} \to D$ be the complex geodesic such that $\varphi(0) = z$ and $\varphi(1) = p$. Consider the holomorphic self-map of the unit disc $\psi(\zeta) := \widetilde{\rho}_\varphi \circ h \circ \varphi(\zeta)$. Then $\psi(0) = 0$ and by Theorem 2.7, ψ has a stationary point at 1. But then by the Herzig theorem [20] (see also the classical Wolff Lemma in [1]) it follows that $\psi(\zeta) \equiv \zeta$. Thus for any $\zeta, \xi \in \mathbb{D}$

$$\begin{aligned} k_D(\varphi(\zeta), \varphi(\xi)) &\geq k_D(h(\varphi(\zeta)), h(\varphi(\xi))) \geq k_D(\rho_\varphi(h(\varphi(\zeta))), \rho_\varphi(h(\varphi(\xi)))) \\ &= k_D(\varphi(\psi(\zeta)), \varphi(\psi(\xi))) = k_D(\varphi(\zeta), \varphi(\xi)) = k_{\mathbb{D}}(\zeta, \xi), \end{aligned}$$

forcing equality at all the steps. In particular $h \circ \varphi : \mathbb{D} \to D$ is a complex geodesic such that $h(\varphi(0)) = z$ and $h(\varphi(1)) = p$. By the uniqueness of complex geodesics passing through two given points of $\overline{D}$ it follows that $h \circ \varphi = \varphi$. Hence $\varphi(\mathbb{D}) \subset \mathsf{Fix}(h)$.

Assertion (2) follows similarly. Indeed, let $q \in \partial D$ be the Denjoy-Wolff point of h. If $q \neq p$ then consider the complex geodesic $\varphi : \mathbb{D} \to D$ such that $\varphi(-1) = q$ and $\varphi(1) = p$ and let $\psi(\zeta) := \widetilde{\rho}_\varphi \circ h \circ \varphi(\zeta)$. As before Theorem 2.7 implies that ψ has stationary points at -1 and $+1$. Now the classical Wolff Lemma (see, *e.g.*, [1]) implies that $\psi(\zeta) \equiv \zeta$. Then we can proceed exactly as before, obtaining that $h(\varphi(\zeta)) = \varphi(\zeta)$ for all $\zeta \in \mathbb{D}$, contradicting the hypothesis. □

3. Pluripotential theory and semigroups

The aim of this section is to use the pluricomplex Green function and the pluricomplex Poisson kernel to characterize infinitesimal generators of semigroups of holomorphic self-maps of a strongly convex domain and their dynamical properties.

We start recalling the following result (see [4] for $D = \mathbb{B}^n$ and [1, Theorem 2.5.24], [10, Theorem A.1] for the general case)

Theorem 3.1. *Let* $D \subset \mathbb{C}^n$ *be a bounded strongly convex domain with smooth boundary. Let* (Φ_t) *be a one-parameter semigroup of holomorphic self-maps of* D*. Then*

- *either* $\bigcap_{t \geq 0} \mathsf{Fix}(\Phi_t) \neq \emptyset$*,*
- *or* $\mathsf{Fix}(\Phi_t) = \emptyset$ *for all* $t > 0$*, there exists a unique* $\tau \in \partial D$ *such that* τ *is the Denjoy-Wolff point of* Φ_t *for all* $t > 0$ *and there exists* $\beta \leq 0$ *such that* $\alpha_{\Phi_t}(\tau) = e^{\beta t}$*.*

If a semigroup (Φ_t) has no fixed points in D, we call the point $\tau \in \partial D$ given by Theorem 3.1 the *Denjoy-Wolff point* of the semigroup.

Definition 3.2. Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary. Let (Φ_t) be a one-parameter semigroup of holomorphic self-maps of D. A point $p \in \partial D$

is called a *boundary regular fixed point for* (Φ_t), or a BRFP for short, if p is a BRFP for Φ_t for all $t \geq 0$. The family of boundary dilatation coefficients of (Φ_t) will be denoted by $(\alpha_t(p))$. A BRFP for (Φ_t) for which $\alpha_t(p) \leq 1$ for some $t > 0$ is called a *stationary point* of the semigroup.

The boundary dilatation coefficients at BRFP's form a semigroup in $(\mathbb{R}_0^+, \cdot)$:

Proposition 3.3. *Let $D \subset \mathbb{C}^n$ be a bounded strongly convex domain with smooth boundary. Let (Φ_t) be a one-parameter semigroup of holomorphic self-maps of D. If $p \in \partial D$ is a BRFP for (Φ_t) then there exists $\beta \in \mathbb{R}$ such that $\alpha_t(p) = e^{\beta t}$ for all $t \geq 0$.*

Proof. Let $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ be the Lempert projection device associated to a complex geodesic such that $\varphi(1) = p$. Consider the following family of functions $T_t : D \to \mathbb{C}$,

$$T_t(z) := \frac{1 - \widetilde{\rho}_\varphi \circ \Phi_t(z)}{1 - \widetilde{\rho}_\varphi(z)}.$$

By Theorem 2.7 it follows that $\lim_{(0,1)\ni r \to 1} T_t(\gamma(r)) = \alpha_t(p)$ for any curve $\gamma : (0,1) \to D$ such that $\lim_{r\to 1} \gamma(r) = p$, the curve $\widetilde{\rho}_\varphi(\gamma(r))$ converges to 1 non-tangentially and $k_D(\gamma(r), \rho_\varphi(\gamma(r))) \to 0$ as $r \to 1$. By [7, Proposition 3.4], it follows that $[0,1) \ni r \mapsto \Phi_t(\varphi(r))$ satisfies the same three properties which are satisfied by γ. Then for $s, t \geq 0$ we have

$$T_{t+s}(\varphi(r)) = \frac{1 - \widetilde{\rho}_\varphi \circ \Phi_t(\Phi_s(\varphi(r)))}{1 - \widetilde{\rho}_\varphi(\Phi_s(\varphi(r)))} \cdot \frac{1 - \widetilde{\rho}_\varphi \circ \Phi_s(\varphi(r))}{1 - \widetilde{\rho}_\varphi(\varphi(r))} = T_t(\Phi_s(\varphi(r)) \cdot T_s(\varphi(r)),$$

and taking the limit as $r \to 1$ it follows that $\alpha_{t+s}(p) = \alpha_t(p)\alpha_s(p)$. Since $\alpha_t(p)$ is clearly measurable in t, this concludes the proof. □

Later we will see how the number β in Proposition 3.3 can be computed using the infinitesimal generator of the semigroup. Now we use the pluricomplex Green function to characterize vector fields which are infinitesimal generators. For this aim we need a lemma whose simple proof is left to the reader:

Lemma 3.4. *Let $T > 0$ be a positive real number and let $g : [0,T] \to \mathbb{R}$ be a function such that*

(1) *for all $a, b \in [0,T]$ and $\lambda \in [0,1]$ it holds*

$$g(\lambda a + (1-\lambda)b) \leq \max\{g(a), g(b)\};$$

(2) *there exists the (right-)derivative of g at 0 and $g'(0) > 0$.*

Then g is non-decreasing.

Now we can state and prove our characterizations of infinitesimal generators:

Theorem 3.5. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $F : D \to \mathbb{C}^n$ be holomorphic. The following are equivalent:*

(1) *The map F is the infinitesimal generator of a semigroup of holomorphic self-maps of D.*

(2) *For all $z, w \in D$ with $z \neq w$ it follows that*

$$d(k_D)|_{(z,w)} \cdot (F(z), F(w)) \leq 0. \tag{3.1}$$

(3) *For all $z, w \in D$ with $z \neq w$ it follows that*

$$d(G_D)|_{(z,w)} \cdot (F(z), F(w)) \leq 0. \tag{3.2}$$

(4) *For each pair $z, w \in D$, it follows*

$$k_D(z - rF(z), w - rF(w)) \geq k_D(z, w) \tag{3.3}$$

for all $r > 0$ such that $z - rF(z)$ and $w - rF(w)$ belong to D.

Proof. First of all we notice that by (1.2) we have $G_D(z, w) = \log \tanh k_D(z, w)$, and thus a simple computation shows that (2) and (3) are equivalent.

Next, we claim that (1) implies (3). Indeed, if F is an infinitesimal generator in D and (Φ_t) is the corresponding semigroup generated by F then by (1.1), for all $z, w \in D$ with $z \neq w$ it follows that for all $t \geq 0$

$$G_D(\Phi_t(z), \Phi_t(w)) - G_D(z, w) \leq 0$$

and it is equal to zero for $t = 0$. Computing the incremental ratio in t for $t = 0$ we obtain (3.2).

Now, assume (2) holds. For $w \in D$, consider the Cauchy problem

$$\begin{cases} \dfrac{d\Phi}{dt} = F \circ \Phi, \\ \Phi(0) = w \end{cases}$$

and denote by $\Phi_w : [0, \delta_w) \to D$ its maximal solution, for some $\delta_w > 0$. To show that F is an infinitesimal generator, it is enough to prove that for all w it holds $\delta_w = +\infty$.

To this aim, let $z, w \in D$ with $z \neq w$ and let $\delta = \min\{\delta_z, \delta_w\}$. Let $g : [0, \delta) \ni t \mapsto k_D(\Phi_z(t), \Phi_w(t))$. By uniqueness of solutions of the above Cauchy problems, we know that, for all $t \in [0, \delta)$, we have $\Phi_z(t) \neq \Phi_w(t)$. According to Lempert's work [24], [25] (see also [1, Proposition 2.6.40]), the function g is smooth and differentiating with respect to t we obtain by (3.1)

$$\begin{aligned} g'(t) &= d(k_D)|_{(\Phi_z(t), \Phi_w(t))} \cdot (\frac{d\Phi_z(t)}{dt}, \frac{d\Phi_w(t)}{dt}) \\ &= d(k_D)|_{(\Phi_z(t), \Phi_w(t))} \cdot (F(\Phi_z(t)), F(\Phi_w(t))) \leq 0. \end{aligned}$$

Therefore g is non-increasing in t, namely

$$k_D(\Phi_z(t), \Phi_w(t)) \leq k_D(\Phi_z(0), \Phi_w(0)) = k_D(z, w). \tag{3.4}$$

This implies that $\delta_z = \delta_w$ because, if for instance $\delta_z < \delta_w$ then as $t \to \delta_z$ it would follow that $\Phi_z(t) \to \partial D$ while $\Phi_w(t) \to \Phi_w(\delta_z) \in D$, and then $k_D(\Phi_z(t), \Phi_w(t)) \to \infty$ contradicting (3.4).

By the arbitrariness of $z, w \in D$, this means that *for all* $z \in D$ we have $\delta_z = \delta$. Hence, by well known results on PDE's, we have a well defined analytic map $\Phi : D \times [0, \delta) \to D$ which is holomorphic in $z \in D$ and such that $\Phi(0, z) = z$ and $\frac{\partial \Phi}{\partial t} = F \circ \Phi$. Also, $\Phi(t+s, z) = \Phi(t, \Phi(s, z))$ for all $s, t \geq 0$ such that $s + t < \delta$ and $z \in D$. This implies that $\delta = \infty$. Indeed, if $\delta < +\infty$, let $2\delta > t > \delta$ and let $s > 0$ be such that $t - s < \delta$, $s < \delta$. Define $\Phi_z(t) := \Phi(t - s, \Phi(s, z))$. This is well defined and solve the Cauchy problem for z, against the maximality of δ.

Thus we have proved that (1), (2) and (3) are equivalent.

Now, let us prove that (4) implies (2). Let $z, w \in D$, $z \neq w$, and $r > 0$ such that $z - rF(z)$ and $w - rF(w)$ belong to D. By convexity, $z - tF(z)$ and $w - tF(w)$ belong to D for all $t \in [0, r]$. Therefore, the function $g : [0, r] \to \mathbb{R}$ given by

$$g(t) = k_D(z - tF(z), w - tF(w))$$

is well-defined and, again by Lempert's result, since $z \neq w$, it is differentiable at 0. By hypothesis, $g(t) \geq g(0)$ for all $t \geq 0$. Therefore $g'(0) \geq 0$. But

$$g'(0) = (dk_D)|_{(z,w)} \cdot (-F(z), -F(w)) = -(dk_D)|_{(z,w)} \cdot (F(z), F(w)).$$

Thus, $(dk_D)|_{(z,w)} \cdot (F(z), F(w)) \leq 0$, and (2) holds.

In order to finish the proof we show that (2) implies (4). To proceed we consider the following two possible cases:

I) $(dk_D)|_{(z,w)} \cdot (F(z), F(w)) < 0$.

II) $(dk_D)|_{(z,w)} \cdot (F(z), F(w)) = 0$.

Case I). Fix $r > 0$ such that $z - rF(z)$ and $w - rF(w)$ belong to D. Then we have that $z - tF(z)$ and $w - tF(w)$ belong to D for all $t \in [0, r]$. Therefore, the function $g : [0, r] \to \mathbb{R}$ given by

$$g(t) = k_D(z - tF(z), w - tF(w))$$

is well-defined and, since $z \neq w$, it is differentiable at 0 with derivative given by $g'(0) = (dk_D)|_{(z,w)} \cdot (-F(z), -F(w)) > 0$. Moreover, by [28, Proposition 3.8], given $z_1, z_2, w_1, w_2 \in D$ and $\lambda \in [0, 1]$, we have that

$$k_D(\lambda z_1 + (1 - \lambda) z_2, \lambda w_1 + (1 - \lambda) w_2) \leq \max\{k_D(z_1, w_1), k_D(z_2, w_2)\}.$$

In particular, if $a, b \in [0, r]$ and $\lambda \in [0, 1]$, we have that

$$\begin{aligned} g(\lambda a + (1 - \lambda) b) &= k_D(z - (\lambda a + (1 - \lambda) b) F(z), w - (\lambda a + (1 - \lambda) b) F(w)) \\ &= k_D(\lambda(z - aF(z)) + (1 - \lambda)(z - bF(z)), \lambda(w - aF(w)) + (1 - \lambda)(w - bF(w))) \\ &\leq \max\{k_D(z - aF(z), w - aF(w)), k_D(z - bF(z), w - bF(w))\} \\ &= \max\{g(a), g(b)\}. \end{aligned}$$

Therefore, g satisfies the hypothesis of Lemma 3.4 and thus it is non-decreasing. Namely,

$$k_D(z-tF(z), w-tF(w)) \geq k_D(z,w)$$

for all $t \in [0,r]$.

Case II). Let $G : D \to \mathbb{C}^n$ holomorphic be an infinitesimal generator in D such that $(dk_D)|_{(z,w)} \cdot (G(z), G(w)) < 0$. Such a map can be constructed as follow. Up to translations we can assume that $z = O$ the origin in $\mathbb{C}^n$. Let $a < 0$. By convexity, the family of functions $\Phi_t : z \mapsto e^{at}z$ is a semigroup of holomorphic self-maps of D. The associated infinitesimal generator is $G(z) = az$. Therefore

$$(dk_D)|_{(O,w)} \cdot (G(O), G(w)) = a(dk_D)|_{(O,w)} \cdot (O, w).$$

Now, the vector (O, w) points outward with respect to the boundary of the Kobayashi ball of center O and radius $k_D(O, w)$ because Kobayashi balls of convex domains are convex (see, *e.g.*, [1, Proposition 2.3.46]). Since Kobayashi balls are level sets of k_D, this implies that $(dk_D)|_{(O,w)} \cdot (O, w) \neq 0$. Hence $d(k_D)|_{(O,w)} \cdot (G(O), G(w)) \neq 0$ and, by the already proved equivalence between (1) and (2), actually $(dk_D)|_{(z,w)} \cdot (G(z), G(w)) < 0$.

Now fix $\epsilon > 0$ and consider the vector field $H := F + \epsilon G$. This is an infinitesimal generator of a semigroup of holomorphic self-maps in D (because $F+\epsilon G$ satisfies (3.1) and by the equivalence between (1) and (2)). Now, by construction, $(dk_D)|_{(z,w)} \cdot (H(z), H(w)) < 0$ and, for what we proved in Case I), $k_D(z - rH(z), w - rH(w)) \geq k_D(z, w)$ for all $r > 0$ such that $z - rH(z), w - rH(w) \in D$. Now, letting ϵ tends to 0 we end the proof. □

As a corollary we have the following characterization of groups of biholomorphisms of D:

Corollary 3.6. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $F : D \to \mathbb{C}^n$ be holomorphic. The following are equivalent:*

(1) *The map F is the infinitesimal generator of a group of holomorphic self-maps of D.*
(2) *For all $z, w \in D$ with $z \neq w$ it follows that*

$$d(k_D)|_{(z,w)} \cdot (F(z), F(w)) = 0.$$

(3) *For all $z, w \in D$ with $z \neq w$ it follows that*

$$d(G_D)|_{(z,w)} \cdot (F(z), F(w)) = 0.$$

Proof. Apply Theorem 3.5 to F and $-F$. □

Remark 3.7. In case $D = \mathbb{B}^n$ the unit ball of $\mathbb{C}^n$, using (1.3), equation (3.2) assumes a simple expression given by

$$\frac{\mathsf{Re}\,\langle z, F(z)\rangle}{1-\|z\|^2} + \frac{\mathsf{Re}\,\langle w, F(w)\rangle}{1-\|w\|^2} \leq \mathsf{Re}\,\frac{\langle F(z), w\rangle + \langle z, F(w)\rangle}{1-\langle z, w\rangle}. \tag{3.5}$$

In fact, in case $D = \mathbb{B}^n$, Theorem 3.5 with (3.5) replacing (3.2), was proven with different methods by Reich and Shoikhet [27, Theorem 2.1].

For boundary regular fixed points, we have the following result:

Theorem 3.8. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $F : D \to \mathbb{C}^n$ be a holomorphic infinitesimal generator of a semigroup (Φ_t), $\beta \in \mathbb{R}$ and $p \in \partial D$. The following are equivalent:*

1. *The semigroup (Φ_t) has a BRFP at p with boundary dilatation coefficients $\alpha_t(p) \leq e^{\beta t}$ for all $t \geq 0$.*
2. *$d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$.*

Moreover, if p is a BRFP for (Φ_t) then the boundary dilatation coefficient of Φ_t is $\alpha_t(p) = e^{-tb}$ with $b = \inf_{z\in D} d(u_{D,p})_z \cdot F(z)/u_{D,p}(z)$.

Proof. Suppose (1) holds. Then $u_{D,p}(\Phi_t(z)) - e^{-t\beta} u_{D,p}(z) \leq 0$ for all $t \geq 0$ and $z \in D$. In particular,

$$\begin{aligned} 0 &\geq \lim_{t\to 0^+} \frac{u_{D,p}(\Phi_t(z)) - e^{-t\beta}u_{D,p}(z)}{t} \\ &= \frac{\partial}{\partial t}[u_{D,p}(\Phi_t(z)) - e^{-t\beta}u_{D,p}(z)]|_{t=0} = d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z), \end{aligned}$$

and (2) follows.

Conversely, assume (2) holds. Fix $z \in D$ and let $g(t) := u_{D,p}(\Phi_t(z)) - e^{-t\beta}u_{D,p}(z)$. We have to show that $g(t) \leq 0$ for all $t \geq 0$. Deriving g, we obtain

$$\begin{aligned} g'(t) &= d(u_{D,p})_{\Phi_t(z)} \cdot \frac{\partial \Phi_t}{\partial t}(z) + \beta e^{-\beta t} u_{D,p}(z) \\ &= d(u_{D,p})_{\Phi_t(z)} \cdot F(\Phi_t(z)) + \beta e^{-\beta t} u_{D,p}(z) \\ &= d(u_{D,p})_{\Phi_t(z)} \cdot F(\Phi_t(z)) + \beta u_{D,p}(\Phi_t(z)) - \beta g(t). \end{aligned}$$

Therefore, using hypothesis (2) we have that for all $t \geq 0$

$$g'(t) + \beta g(t) \leq 0. \tag{3.6}$$

Now let $h(t) := -(g'(t)+\beta g(t)) \geq 0$. Solving the differential equation $g'(t)+\beta g(t)+h(t) = 0$ with initial value $g(0) = 0$, we obtain

$$g(t) = -e^{-\beta t} \int_0^t e^{\beta s} h(s) ds \leq 0,$$

and thus (1) follows.

Finally, the last statement comes directly from Proposition 3.3 and the equivalence between (1) and (2). $\square$

Remark 3.9. If $\beta \leq 0$ in Theorem 3.8, it follows that for all $z \in D$ the function $[0, +\infty) \ni t \mapsto u_{D,p}(\Phi_t(z)) - e^{-t\beta} u_{D,p}(z)$ is non-increasing. Indeed, for $s > t$, we have

$$\begin{aligned} u_{D,p}(\Phi_s(z)) - e^{-s\beta} u_{D,p}(z) &= u_{D,p}(\Phi_{s-t}(\varphi_t(z))) - e^{-s\beta} u_{D,p}(z) \\ &\leq e^{-(s-t)\beta}[u_{D,p}(\Phi_t(z)) - e^{-t\beta} u_{D,p}(z)] \\ &\leq u_{D,p}(\Phi_t(z)) - e^{-t\beta} u_{D,p}(z). \end{aligned}$$

We end up this section with a Berkson-Porta like characterization of infinitesimal generators.

Definition 3.10. Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $F : D \to \mathbb{C}^n$ holomorphic and $p \in \partial D$. We say that $F \in C^1_E(p)$ if for any horosphere $E_D(p, R)$ there exists a $(n \times n)$-matrix A such that

$$\lim_{E_D(p,R) \ni z \to p} dF_z = A.$$

Theorem 3.11. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $p \in \partial D$, $F : D \to \mathbb{C}^n$ holomorphic and assume that $F \in C^1_E(p)$. Then F is the infinitesimal generator of a semigroup of holomorphic self-maps of D with a stationary point at p if and only if*

$$d(u_{D,p})_z \cdot F(z) \leq 0 \tag{3.7}$$

for all $z \in D$.

Proof. One direction follows directly from Theorem 3.8.

Conversely, assume that (3.7) holds. Fix $w_0 \in D$ and let $\gamma : [0, \delta) \to D$ be the maximal solution of the Cauchy problem

$$\begin{cases} \frac{d\gamma}{dt} = F \circ \gamma \\ \gamma(0) = w_0. \end{cases}$$

It is enough to prove $\delta = +\infty$. Assume by contradiction that $\delta < +\infty$. Let $g(t) := u_{D,p}(\gamma(t))$ for $t \in [0, \delta)$. Deriving g, we obtain by (3.7)

$$g'(t) = d(u_{D,p})_{\gamma(t)}(\gamma'(t)) = d(u_{D,p})_{\gamma(t)}(F(\gamma(t))) \leq 0.$$

Thus for all $t \in [0, \delta)$ it follows $u_{D,p}(\gamma(t)) \leq u_{D,p}(\gamma(0)) = u_{D,p}(w_0)$. This means that if $w_0 \in E_D(p, R)$ then $\gamma(t)$ belongs to $E_D(p, R)$ for all $t \in [0, \delta)$. In particular, since $\overline{E_D(p, R)} \cap \partial D = \{p\}$, it means that $\lim_{t \to \delta} \gamma(t) = p$.

Since $F \in C^1_E(p)$ and $\partial E_D(p, R)$ is Lipschitz (it is actually $C^{1,1}$ at p and smooth elsewhere, see [13, Section 4]) by (a very simple form of) Whitney extension theorem there exists a function $\tilde{F} : \mathbb{C}^n \to \mathbb{C}^n$ of class C^1 such that $\tilde{F}|_{E_D(p,R)} = F$. If $\tilde{F}(p) = 0$ then the Cauchy problem

$$\begin{cases} \frac{d\eta}{dt} = \tilde{F} \circ \eta \\ \eta(\delta) = p \end{cases}$$

has the unique solution $\eta(t) \equiv p$. In particular, γ cannot reach p in a finite time, which gives us the searched contradiction to $\delta < +\infty$.

To conclude the proof we are left to show that necessarily $\tilde{F}(p) = 0$. But this follows at once from the fact that for any $z \in E_D(p, R)$ the solution of the Cauchy problem

$$\begin{cases} \frac{d\gamma^z}{dt} = \tilde{F} \circ \gamma^z \\ \gamma^z(0) = z \end{cases}$$

is such that $\gamma^z(t) \in D$ for $t \in [0, \delta^z)$ for a suitable $\delta^z \in (0, +\infty]$ and, arguing as for γ, $\lim_{t \to \delta^z} \gamma^z(t) = p$. $\square$

Remark 3.12. If $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, then Theorem 3.11 holds without any regularity assumption on F at $p \in \partial\mathbb{D}$. Indeed, a direct computation shows that (3.7) reduces exactly to the Berkson-Porta formula [5].

If $D = \mathbb{B}^n$ the unit ball of $\mathbb{C}^n$, a direct computation shows that (3.7) corresponds to

$$\frac{\mathsf{Re}\,\langle F(z), z\rangle}{1 - \|z^2\|} \leq \mathsf{Re}\,\frac{\langle F(z), p\rangle}{1 - \langle z, p\rangle}.$$

In fact, for $D = \mathbb{B}^n$ and with the additional hypothesis that F extends holomorphically through $\partial\mathbb{B}^n$, Theorem 3.11 follows from [3, Theorem 3.1].

4. Boundary behavior of infinitesimal generators

In all this section, D denotes a bounded strongly convex domain in $\mathbb{C}^n$ with smooth boundary.

Before proving the main result of this section, we examine two significative examples. We will use a lemma whose proof can be derived from the proof of [11, Theorem 1.4].

Lemma 4.1. *Let $a, b \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$ and*

$$G(z) = a - \langle z, a\rangle z - [Az + \langle z, b\rangle z].$$

Then, G is the infinitesimal generator of a continuous semigroup of holomorphic self-maps of $\mathbb{B}^n$ if and only if

$$|\langle b, u\rangle| \leq \mathsf{Re}\,\langle Au, u\rangle, \tag{4.1}$$

for all $u \in \partial\mathbb{B}^n$. Moreover, if equality holds at every point of $\partial\mathbb{B}^n$, then G is the infinitesimal generator of a continuous group of holomorphic self-maps of $\mathbb{B}^n$.

Example 4.2. Let us consider $F : \mathbb{B}^2 \to \mathbb{C}^2$ given by $F(z_1, z_2) = (0, -z_2/(1 - z_1))$. Let $e_1 = (1, 0)$. By a direct computation one can see that F is an infinitesimal generator of a semigroup (Φ_t) of holomorphic self-maps of $\mathbb{B}^2$ which pointwise fixes the slice $\mathbb{D} \ni \zeta \mapsto (\zeta, 0)$. Clearly $d(u_{\mathbb{B}^2, e_1})_z \circ F(z) \leq 0$ for all $z \in \mathbb{B}^2$. Thus F has a stationary point at e_1.

Also, $\langle F(z), e_1\rangle = 0$ for all $z \in \mathbb{B}^2$ and therefore the radial limit $\lim_{(0,1)\ni r\to 1} \frac{F(re_1)}{1-r} = 0$, as predicted by Theorem 0.3. Now let us consider the following map

$$\eta(z_1, z_2) = \left(\frac{-sz_2 + (1-\beta)z_1 + \beta}{-sz_2 - \beta z_1 + 1 + \beta}, \frac{z_2 + sz_1 - s}{-sz_2 - \beta z_1 + 1 + \beta}\right)$$

where $\mathsf{Re}\,\beta > 0$ and $s = \sqrt{2\mathsf{Re}\,\beta}$. Notice that $\eta : \mathbb{B}^2 \to \mathbb{B}^2$ is a parabolic automorphism such that $\eta(e_1) = e_1$ (see [6, Example 5.1]). Hence $\tilde{F}(z_1, z_2) := d\eta^{-1}_{\eta(z_1,z_2)} \cdot F(\eta(z_1, z_2))$ is the infinitesimal generator of the semigroup $(\tilde{\Phi}_t)$ of holomorphic self-maps of $\mathbb{B}^2$, where $\tilde{\Phi}_t = \eta^{-1}\circ\Phi_t\circ\eta$. Thus $\tilde{\Phi}_t$ pointwise fixes the slice $\eta^{-1}(\zeta, 0)$ for $\zeta \in \mathbb{D}$. A direct computation shows that $F(\eta(r, 0)) = (0, s)$ for $r \in (0, 1)$. Thus, since

$$d\eta_{re_1} = \frac{1}{(-r\beta + 1 + \beta)^2}\begin{pmatrix} 1 & s(r-1) \\ s & \bar{\beta}(r-1) + 1 \end{pmatrix},$$

it follows that $\tilde{F}(r, 0) = (\beta(1-r)+1)(-s^2(r-1), s)$ for $r \in (0, 1)$. In particular the radial limit of $\tilde{F}$ at e_1 is not zero. A direct computation shows that

$$\lim_{(0,1)\ni r\to 1} \frac{\tilde{F}_1(re_1)}{r-1} = -s^2 < 0.$$

Hence the semigroup of $\mathbb{D}$ generated by the "projection" $\tilde{F}_1(\zeta e_1)$ of $\tilde{F}$ to the slice $\mathbb{D} \ni \zeta \mapsto \zeta e_1$ has Denjoy-Wolff point at 1, with boundary dilatation coefficients $(e^{-s^2 t})$. However, by construction, e_1 is a stationary point for the semigroup $(\tilde{\Phi}_t)$, with boundary dilatation coefficients all equal to 1.

Example 4.3. Let $G : \mathbb{B}^2 \to \mathbb{C}^2$ be the map defined by $G(z_1, z_2) = (\frac{-1}{3} iz_1, iz_2)$. The map G is an infinitesimal generator of a (semi)group (Φ_t) of elliptic automorphisms fixing the origin (see Lemma 4.1). Let us consider the automorphism of $\mathbb{B}^2$ given by

$$\psi(z_1, z_2) = \left(\frac{-\sqrt{3}z_1}{2 - z_2}, \frac{1 - 2z_2}{2 - z_2}\right)$$

(see [1, Lemma 2.2.1]). We note that $\psi = \psi^{-1}$. Such an automorphism maps the slice $\{(z_1, z_2) \in \mathbb{B}^2 : z_2 = 0\}$ to the slice $\{(z_1, z_2) \in \mathbb{B}^2 : z_2 = 1/2\}$. Let us define $P(z_1, z_2) := d\psi_{\psi^{-1}(z_1,z_2)} \cdot G(\psi^{-1}(z_1, z_2))$. A direct computation shows that

$$P(z_1, z_2) = \left(\frac{-2i}{3} z_1 z_2, \frac{-i}{3}(z_2 - 2)(2z_2 - 1)\right).$$

Let (Φ^P_t) be the group of automorphisms generated by P. Such a group is obtained by conjugation from the group (Φ_t) and therefore has only one fixed point. Since $P(z_1, 0) = (0, \frac{-2i}{3})$ (and P is holomorphic past the boundary) then (Φ^P_t) cannot have BRFP at any point of the boundary of the slice $\{(z_1, z_2) \in \mathbb{B}^2 : z_2 = 0\}$. However, obviously

$$\lim_{\mathbb{D} \ni \zeta \to 1} \frac{P_1(\zeta, 0)}{\zeta - 1} = 0.$$

Finally, let

$$H(z_1, z_2) = \left(\frac{2i}{3} z_2(z_1 - 1), \frac{2i}{3}(1 + z_2^2 - z_1) \right).$$

The map $H : \mathbb{B}^2 \to \mathbb{C}^2$ is the infinitesimal generator of a group of automorphisms (see again Lemma 4.1) with the property that $H(e_1) = (0, 0)$. Let us define $F = P + H$. Then F is the infinitesimal generator of a group of automorphisms and a direct computation shows that

$$F(z_1, z_2) = \left(-\frac{2i}{3} z_2, -\frac{5i}{3} z_2 - \frac{2i}{3} z_1 \right).$$

Thus the group generated by F has a unique fixed point at $(0, 0)$ and no BRFPs on $\partial\mathbb{B}^2$. However the semigroup generated by $\mathbb{D} \ni \zeta \mapsto F_1(\zeta, 0)$ on $\mathbb{D}$ is the trivial semigroup and

$$\lim_{\mathbb{D} \ni \zeta \to e^{i\theta}} \frac{F_1(\zeta, 0)}{\zeta - 1} = 0$$

for all $\theta \in \mathbb{R}$. Notice that the slice $\{(z_1, z_2) \in \mathbb{B}^2 : z_2 = 0\}$ contains the fixed point of the semigroup.

The previous two examples show that, on the one hand, the requirement that the radial limit exists in Theorem 0.3 is sufficient but not necessary for the existence of BRFP's. Also, even if a BRFP exists, say at $p \in \partial\mathbb{B}^n$, the radial limit of the incremental ratio of the projection of the infinitesimal generator along p might not give information on the boundary dilatation coefficients of the semigroup at p. On the other hand, the sole information on the existence of the limit of the incremental ratio along a given point $p \in \partial\mathbb{B}^n$ does not imply existence of a BRFP at p. Last but not least, an unexpected phenomenon takes place for infinitesimal generators: the behavior of the semigroup generated by the "restriction" of the infinitesimal generator to one complex geodesic—even a complex geodesic containing fixed points of the semigroup—can be completely different from the behavior of the semigroup in the ball (cfr. Theorem 2.7).

Definition 4.4. Let $F : D \to \mathbb{C}^n$ be a holomorphic infinitesimal generator. For a Lempert projection device $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ we will denote by $f_\varphi(\zeta) := d(\widetilde{\rho}_\varphi)_{\varphi(\zeta)} \cdot F(\varphi(\zeta))$ the holomorphic vector field on $\mathbb{D}$.

Proposition 4.5. *Let $F : D \to \mathbb{C}^n$ be a holomorphic infinitesimal generator. Let $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ be a Lempert projection device with $p = \varphi(1) \in \partial D$. Then the vector field $f_\varphi(\zeta)$ is a holomorphic infinitesimal generator in $\mathbb{D}$. Moreover, if there exists $\beta \in \mathbb{R}$ such that $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$, then $d(u_{\mathbb{D},1})_\zeta \cdot f_\varphi(\zeta) + \beta u_{\mathbb{D},1}(\zeta) \leq 0$ for all $\zeta \in \mathbb{D}$.*

Proof. Considering the pluricomplex Green function $G_D : D \times D \to \mathbb{R}$. Its differential $dG_D : TD \times TD \to T\mathbb{R}$ can be decomposed as $dG_D = d_z G_D + d_w G_D$ where, if $(u, v) \in TD \times TD$ we have $dG(u, v) = d_z G_D(u) + d_w G_D(v)$. With this notation, Theorem 3.5 implies that for all $z \neq w$

$$d(G_D)_{(z,w)} \cdot (F(z), F(w)) = d_z(G_D)|_{(z,w)} \cdot F(z) + d_w(G_D)|_{(z,w)} \cdot F(w) \leq 0.$$

Now let $z = \varphi(\eta)$ and $w = \varphi(\zeta)$ for $\eta \neq \zeta \in \mathbb{D}$. We claim that

$$d_w(G_D)|_{(\varphi(\eta),\varphi(\zeta))} \cdot F(\varphi(\zeta)) = d_w(G_D)|_{(\varphi(\eta),\varphi(\zeta))}(d(\rho_\varphi)_{\varphi(\zeta)} \cdot F(\varphi(\zeta))). \tag{4.2}$$

Assume that (4.2) is true. According to (1.5) we also have $G_D(\varphi(\eta), \varphi(\zeta)) = G_{\mathbb{D}}(\eta, \zeta)$ for all $\zeta \in \mathbb{D}$, thus by (4.2)

$$\begin{aligned} d_w(G_{\mathbb{D}})|_{(\eta,\zeta)}(f_\varphi(\zeta)) &= d_w(G_D)|_{(\varphi(\eta),\varphi(\zeta))} \cdot (d\varphi_\zeta(f_\varphi(\zeta))) \\ &= d_w(G_D)|_{(\varphi(\eta),\varphi(\zeta))} \cdot (d(\rho_\varphi)_{\varphi(\zeta)}(F(\varphi(\zeta)))) \\ &= d_w(G_D)|_{(\varphi(\eta),\varphi(\zeta))} \cdot F(\varphi(\zeta)). \end{aligned}$$

A similar equation holds for $d_z(G_D)|_{(z,w)}|_{(\varphi(\eta),\varphi(\zeta))} \cdot F(\varphi(\eta))$, swapping the roles of η and ζ in the previous argument. Thus

$$d(G_{\mathbb{D}})|_{(\eta,\zeta)} \cdot (f_\varphi(\eta), f_\varphi(\zeta)) = d(G_D)|_{(\varphi(\eta),\varphi(\zeta))} \cdot (F(\varphi(\eta)), F(\varphi(\zeta)))) \leq 0,$$

for all $\zeta, \eta \in \mathbb{D}$ with $\zeta \neq \eta$, which implies that f_φ is an infinitesimal generator on $\mathbb{D}$ by Theorem 3.5.

Now we are left to prove claim (4.2). Since ρ_φ is holomorphic, then $d\rho_\varphi = \partial\rho_\varphi$ and the Lempert projection ρ_φ determines a holomorphic splitting of the exact sequence of holomorphic bundles

$$0 \longrightarrow T\varphi(\mathbb{D}) \xrightarrow{\iota} TD|_{\varphi(\mathbb{D})} \longrightarrow N_{\varphi(\mathbb{D}),D} \longrightarrow 0,$$

given by

$$T_{\varphi(\zeta)}D = \iota(d\rho_\varphi(T_{\varphi(\zeta)}D)) \oplus \mathsf{Ker} d(\rho_\varphi)_{\varphi(\zeta)}.$$

Now let $B_D(\varphi(0), R) = \{z \in D : k_D(z, \varphi(0)) < R\}$ be a Kobayashi ball for D and let $\zeta_1 \in \mathbb{D}$ be such that $\varphi(\zeta_1) \in \partial B_D(\varphi(0), R)$. It is known—and can be easily proven using Lempert's special coordinates, see [24], [25]—that

$$T^{\mathbb{C}}_{\varphi(\zeta_1)}\partial B_D(\varphi(0), R) = \mathsf{Ker} d(\rho_\varphi)_{\varphi(\zeta_1)} \tag{4.3}$$

where, as usual, $T^{\mathbb{C}}_{\varphi(\zeta_1)}\partial B_D(\varphi(0), R)$ denotes the complex tangent space of $\partial B_D(\varphi(0), R)$ at $\varphi(\zeta_1)$. By (1.2) it follows that

$$\partial B_D(\varphi(0), R) = \{z \in D : G_D(\varphi(0), z) = r\}$$

for a suitable $r < 0$, and in particular

$$T_{\varphi(\zeta_1)}\partial B_D(\varphi(0), R) = \mathsf{Ker}(d_w G_D)_{(\varphi(0),\varphi(\zeta_1))}. \tag{4.4}$$

Now, consider the Kobayashi ball $B_D(\varphi(\eta), R)$ with $R = R(\varphi(\zeta)) > 0$ such that $\varphi(\zeta) \in \partial B_D(\varphi(\eta), R)$. Equations (4.3) and (4.4) yield

$$\mathsf{Ker} d(\rho_\varphi)_{\varphi(\zeta)} = T^{\mathbb{C}}_{\varphi(\zeta)} \partial B_D(\varphi(\eta), R) \subset T_{\varphi(\zeta)} \partial B_D(\varphi(\eta), R) = \mathsf{Ker}(d_w G_D)_{(\varphi(\eta),\varphi(\zeta))}$$

from which equation (4.2) follows, and the claim is proved.

In order to prove the last assertion of the proposition, we argue similarly as before. Let $\zeta \in \mathbb{D}$ and let $E_D(\varphi(1), R)$ be the horosphere in D which contains $\varphi(\zeta)$ on its boundary. Then (again using Lempert's special coordinates, see [12, p. 517])

$$T^{\mathbb{C}}_{\varphi(\zeta_1)} \partial E_D(\varphi(1), R) = \mathsf{Ker} d(\rho_\varphi)_{\varphi(\zeta_1)}, \tag{4.5}$$

and

$$T_{\varphi(\zeta_1)} \partial E_D(\varphi(1), R) = \mathsf{Ker} d(u_{D,\varphi(1)})_{\varphi(\zeta_1)}. \tag{4.6}$$

Since $T^{\mathbb{C}}_{\varphi(\zeta)} \partial E_D(\varphi(1), R) \subset T_{\varphi(\zeta)} \partial E_D(\varphi(1), R)$, equation (4.5) yields

$$d(u_{D,\varphi(1)})_{\varphi(\zeta)} \cdot F(\varphi(\zeta)) = d(u_{D,\varphi(1)})_{\varphi(\zeta)} (d(\rho_\varphi)_{\varphi(\zeta)} \cdot F(\varphi(\zeta))),$$

and since $u_{D,\varphi(1)} \circ \varphi(\zeta) = a_\varphi u_{\mathbb{D},1}(\zeta)$ for some $a_\varphi > 0$ by (1.6), we have

$$d(u_{\mathbb{D},1})_\zeta \cdot f_\varphi(\zeta) + \beta u_{\mathbb{D},1}(\zeta) = \frac{1}{a_\varphi} [d(u_{D,p})_{\varphi(\zeta)} \cdot F(\varphi(\zeta)) + \beta u_{D,p}(\varphi(\zeta))]. \tag{4.7}$$

If $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \le 0$ for all $z \in D$ then $d(u_{\mathbb{D},1})_\zeta \cdot f_\varphi(\zeta) + \beta u_{\mathbb{D},1}(\zeta) \le 0$ for all $\zeta \in \mathbb{D}$ as stated. □

Before stating and proving the main result of this section we need a preliminary lemma.

Lemma 4.6. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc in $\mathbb{C}$. Let G be the infinitesimal generator of a semigroup (η_t) of holomorphic self-maps of $\mathbb{D}$. The following are equivalent:*

1. *The point 1 is a boundary regular fixed point for (η_t).*
2. *There exists $C > 0$ such that the radial limit*

$$\limsup_{(0,1)\ni r \to 1} \frac{|G(r)|}{1-r} \le C.$$

Moreover, if 1 is a BRFP for (η_t) with boundary dilatation coefficients $\alpha_t(1) = e^{bt}$ then

$$\angle \lim_{\zeta \to 1} \frac{G(\zeta)}{\zeta - 1} = b.$$

Proof. If (1) holds then the result follows directly from [14, Theorem 1].

Conversely, hypothesis (2) implies that $\lim_{r\to 1} G(r) = 0$. By Berkson-Porta's theorem 0.1, there exists a point $b \in \overline{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \to \mathbb{C}$ with $\mathsf{Re}\, p \ge 0$ such that

$$G(z) = (z - b)(\overline{b} z - 1) p(z), \qquad z \in \mathbb{D}.$$

If $b = 1$, we have that 1 is the Denjoy-Wolff point of the semigroup (η_t) and (1) follows. Otherwise, we have that $\lim_{r\to 1} p(r) = 0$. Then the function $\varphi(z) = \frac{1-p(z)}{1+p(z)}$ is a self-map of the unit disc and $\lim_{r\to 1}\varphi(r) = 1$. By [26, Proposition 4.13], the function φ has angular derivative (possibly infinite) at 1. Thus, p, and so G, has angular derivative at 1. That is, there exists the radial limit

$$\lim_{r\to 1}\frac{G(r)}{r-1}.$$

By (2), such a limit is finite and again by [14, Theorem 1], we obtain that 1 is a boundary regular fixed point of the semigroup. □

Theorem 4.7. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, F the infinitesimal generator of a semigroup (Φ_t) of holomorphic self-maps of D, and $p \in \partial D$. The following are equivalent:*

1. *The point p is a BRFP for (Φ_t).*
2. *There exists $C > 0$ such that for any Lempert's projection device $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ with $\varphi(1) = p$ it follows*

$$\limsup_{(0,1)\ni r\to 1}\frac{|f_\varphi(r)|}{1-r} \leq C.$$

Moreover, if p is a BRFP for (Φ_t) with boundary dilatation coefficients $\alpha_t(p) = e^{\beta t}$ then the non-tangential limit

$$A(\varphi, p) := \angle\lim_{\zeta\to 1}\frac{f_\varphi(\zeta)}{\zeta - 1}$$

exists finite, $A(\varphi, p) \in \mathbb{R}$ and $A(\varphi, p) \leq \beta$. Also, $\beta = \sup A(\varphi, p)$, with the supremum taken as φ varies among all complex geodesics with $\varphi(1) = p$.

Proof. Suppose (1) holds. By Theorem 3.8 there exists $\beta \in \mathbb{R}$ such that $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$. Let $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ be a Lempert's projection device with $\varphi(1) = p$ and let f_φ be the associated vector field. By Proposition 4.5, the map f_φ is an infinitesimal generator and satisfies $d(u_{\mathbb{D},1})_\zeta \cdot f_\varphi(\zeta) + \beta u_{\mathbb{D},1}(\zeta) \leq 0$ for all $\zeta \in \mathbb{D}$. By Theorem 3.8, the semigroup generated by f_φ in $\mathbb{D}$ has a BRFP at 1 with boundary dilatation coefficients $\alpha_t(1) \leq e^{t\beta}$. By Lemma 4.6, it follows that the non-tangential limit $\angle\lim_{\zeta\to 1}\frac{f_\varphi(\zeta)}{\zeta-1}$ exists finite and it is a real number less than or equal to β, and thus (1) and part of the last statement are proved.

Suppose (2) holds. By Theorem 3.8, it is enough to show that there exists $\beta \in \mathbb{R}$ such that for all $z \in D$ it holds $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$. Fix $z \in D$ and let $\varphi : \mathbb{D} \to D$ be the complex geodesic such that $\varphi(0) = z$ and $\varphi(1) = p$. By Proposition 4.5, the vector field f_φ is an infinitesimal generator in $\mathbb{D}$. Hypothesis (2) and Lemma 4.6 imply that 1 is a BRFP for the semigroup generated by f_φ with boundary dilatation coefficients less than or equal to e^{Ct}. Therefore Theorem 3.8 applied to f_φ yields $d(u_{\mathbb{D},1})_z \cdot f_\varphi(\zeta) + Cu_{\mathbb{D},1}(\zeta) \leq 0$ for all $\zeta \in \mathbb{D}$. By (4.7) it follows that $d(u_{D,p})_{\varphi(\zeta)} \cdot F(\varphi(\zeta)) + Cu_{D,p}(\varphi(\zeta)) \leq 0$ for all $\zeta \in \mathbb{D}$ and thus, in particular for $\zeta = 0$, we have $d(u_{D,p})_z \cdot F(z) + Cu_{D,p}(z) \leq 0$ as needed.

Finally, notice that, again by Theorem 3.8, the previous arguments show also that if p is a BRFP with boundary dilatation coefficients $\alpha_t(p) = e^{\beta t}$ then β is the supremum of all $A(\varphi, p)$. □

Corollary 4.8. *Let $F : D \to \mathbb{C}^n$ be the holomorphic infinitesimal generator of a semigroup (Φ_t) with a stationary point $p \in \partial\mathbb{B}^n$. Then for any Lempert's projection device $(\varphi, \rho_\varphi, \widetilde{\rho}_\varphi)$ with $\varphi(1) = p$*

1. $\angle \lim_{\zeta\to 1} f_\varphi(\zeta) = 0$,
2. $\angle \lim_{\zeta\to 1} f_\varphi(\zeta)/(\zeta - 1) = A(\varphi, p)$ *is a finite real number,* $A(\varphi, p) \leq 0$ *and the boundary dilatation coefficients* $\alpha_t(p) = e^{t\beta}$ *are such that* $A(\varphi, p) \leq \beta \leq 0$ *for all* φ.

Moreover,

a) *if $F(z) = 0$ for some $z \in D$ then there exists a complex geodesic $\varphi : \mathbb{D} \to D$ with $\varphi(1) = p$ such that $F(\varphi(\zeta)) = 0$ for all $\zeta \in \mathbb{D}$ and all points of $\varphi(\partial\mathbb{D})$ are stationary points for (Φ_t) with boundary dilatation coefficients $\alpha_t(\varphi(e^{i\theta})) = 1$ for all $\theta \in \mathbb{R}$. Also $A(\varphi, \varphi(e^{i\theta})) = 1$ for all $\theta \in \mathbb{R}$.*
b) *If $F(z) \neq 0$ for all $z \in D$ then p is the Denjoy-Wolff point of (Φ_t).*

Proof. Taking into account that if $z \in D$ then $F(z) = 0$ if and only if $z \in \mathsf{Fix}(\Phi_t)$, the statement is a direct consequence of Theorem 4.7 and Proposition 2.9. □

5. Boundary repelling fixed points and the non-linear resolvent

In [27] Reich and Shoikhet proved the following result.

Theorem 5.1. *Let $D \subset \mathbb{C}^n$ be a bounded convex domain (not necessarily strongly convex). Let $F : D \to \mathbb{C}^n$ be a holomorphic infinitesimal generator of a semigroup (Φ_t) of holomorphic self-maps of D. Then there exists a family $\{G_t\}$ of holomorphic self-maps of D, with $G_0 = \mathsf{id}_D$, depending on the parameter $t \in [0, +\infty)$ such that for all $z \in D$ and $t \in [0, +\infty)$*

$$G_t(z) - z = tF(G_t(z)), \tag{5.1}$$

and for all $z \in D$

$$F(z) = \lim_{t\to 0^+} \frac{G_t(z) - z}{t}. \tag{5.2}$$

Moreover, if $z, w \in D$ are such that $w - z = tF(w)$ then $w = G_t(z)$.

The above family $\{G_t\}$ is called the *non-linear resolvent of F*. This non-linear resolvent reads some dynamical properties of the semigroup. Indeed, by (5.1) (and uniqueness) it follows easily that

$$\mathsf{Fix}(G_t) = \{z \in D : F(z) = 0\} = \mathsf{Fix}(\Phi_t).$$

In [27, Proof of Corollary 1.6], it is also proved that if $D \subset\subset \mathbb{C}^n$ is a strongly convex domain with smooth boundary and (Φ_t) has no fixed points in D, and $\tau \in \partial D$ is the Denjoy-Wolff point of the semigroup, then

$$G_t(E_D(\tau, R)) \subseteq E_D(\tau, R),$$

for all $R > 0$ and all $t > 0$. Then, by Theorem 2.3, we obtain that τ is a stationary point for all G_t. Since F has no zeros in D, then $\mathsf{Fix}(G_t) = \emptyset$ and, by Proposition 2.9, we conclude that τ is the Denjoy-Wolff point of G_t, for all $t > 0$. That is, the functions Φ_t and G_t share the same Denjoy-Wolff point.

For boundary regular fixed points, we can prove:

Proposition 5.2. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $F : D \to \mathbb{C}^n$ be a holomorphic infinitesimal generator with associated semigroup (Φ_t), non-linear resolvent $\{G_t\}$, and $p \in \partial D$. Suppose there exists $\beta \in \mathbb{R}$ such that for any $t > 0$ the point p is a BRFP for G_t with boundary dilatation coefficients $\alpha_{G_t}(p) \leq e^{t\beta}$. Then p is a BRFP for (Φ_t) with boundary dilatation coefficients $\alpha_t(p) \leq e^{t\beta}$.*

Proof. By Theorem 3.8 it is enough to prove that $d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$. Fix $z \in D$. Since p is a BRFP for $\{G_t\}$ and $\alpha_{G_t}(p) \leq e^{t\beta}$, by Theorem 2.4, we have

$$g(t) := u_{D,p}(G_t(z)) - e^{-t\beta} u_{D,p}(z) \leq 0$$

for all $t \in [0, \infty)$. Since $g(0) = 0$, using (5.1) and the fact that $G_t(z) \to z$ for $t \to 0^+$ by (5.2), we have

$$\begin{aligned} 0 \geq \lim_{t \to 0^+} \frac{g(t)}{t} &= \lim_{t \to 0^+} \frac{u_{D,p}(G_t(z)) - u_{D,p}(z)}{t} + \beta u_{D,p}(z) \\ &= \lim_{t \to 0^+} \frac{u_{D,p}(tF(G_t(z)) + z) - u_{D,p}(z)}{t} + \beta u_{D,p}(z) \\ &= d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z), \end{aligned}$$

proving the statement. □

Remark 5.3. If $p \in \partial D$ is a BRFP for $\{G_t\}$ with boundary dilatation coefficients $\alpha_{G_t}(p) \leq e^{\beta t}$ for some $\beta \in \mathbb{R}$, then (see Remark 2.5), for any $t \in [0, \infty)$, it follows that $\mathsf{K}\text{-}\lim_{z \to p} G_t(z) = p$. In particular, from (5.1), it follows that for all $t > 0$ and $R > 1$

$$\lim_{G_t(K(p,R)) \ni z \to p} F(z) = 0.$$

However, even in this case, F might not have radial limit 0 at p. In fact, looking at the infinitesimal generator $\tilde{F}$ in Example 4.2, one easily sees that the non-linear resolvent $\{G_t\}$ has a BRFP at e_1 with boundary dilatation coefficients $\alpha_{G_t}(e_1) = 1$ (because by construction $G_t(z) = z$ on a complex geodesic containing e_1 on its boundary). But $\tilde{F}$ does not have radial limit 0 at e_1.

The converse to Proposition 5.2 is false, as the following example shows:

Example 5.4. Let $f(\zeta) = 1 - \zeta^2$. Then f is the infinitesimal generator of a group of hyperbolic automorphisms in $\mathbb{D}$, with Denjoy-Wolff point 1 and boundary repelling fixed point -1. It is easy to check that the non-linear resolvent of f is given by

$$G_t(z) = \frac{1}{2t}(-1 + \exp(\frac{1}{2}\mathrm{Log}(1 + 4t(t + z)))$$

for all $t > 0$ and $z \in \mathbb{D}$. Now, a direct computation shows $G_t(-1) = \frac{|2t-1|-1}{2t} \neq -1$.

6. Boundary behavior in the the unit ball

In this section we translate our results on BRFP's for semigroups of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$, where most expressions have computable forms.

In order to simplify our statements and without loss of generality, we will assume that, up to conjugation, the base point is $e_1 = (1, 0, \ldots, 0) \in \partial\mathbb{B}^n$.

Theorem 6.1. *Let $F : \mathbb{B}^n \to \mathbb{C}^n$ be the infinitesimal generator of a semigroup (Φ_t) of holomorphic self-maps of $\mathbb{B}^n$. The following are equivalent:*

1. *The point $e_1 \in \partial\mathbb{B}^n$ is a BRFP for (Φ_t).*
2. *There exists $C > 0$ such that for all automorphisms $H = (H_1, \ldots, H_n) : \mathbb{B}^n \to \mathbb{B}^n$ such that $H(e_1) = e_1$ it follows*

$$\limsup_{(0,1)\ni r\to 1} \frac{|d(H_1)_{H^{-1}(re_1)}(F(H^{-1}(re_1)))|}{1-r} \leq C. \tag{6.1}$$

Moreover, if e_1 is a BRFP for (Φ_t) with boundary dilatation coefficients $\alpha_t(e_1) = e^{\beta t}$ then the non-tangential limit

$$A(H, e_1) := \angle\lim_{\zeta\to 1} \frac{d(H_1)_{H^{-1}(\zeta e_1)}(F(H^{-1}(\zeta e_1)))}{\zeta - 1}$$

exists finitely, $A(H, e_1) \in \mathbb{R}$ and $A(H, e_1) \leq \beta$. Also, $\beta = \sup A(H, e_1)$, with the supremum taken as H varies among all automorphisms of $\mathbb{B}^n$ with $H(e_1) = e_1$.

Proof. The result follows from Theorem 4.7 as soon as one realizes how Lempert's projection devices in the unit ball are related to automorphisms of $\mathbb{B}^n$. Indeed, thanks to the double transitivity of the group of automorphisms of $\mathbb{B}^n$ on $\partial\mathbb{B}^n$, any complex geodesic $\varphi : \mathbb{D} \to \mathbb{B}^n$ of $\mathbb{B}^n$ passing through e_1 can be written as $\zeta \mapsto H^{-1}(\zeta e_1)$ for some suitable automorphism $H : \mathbb{B}^n \to \mathbb{B}^n$. The associated Lempert projection ρ_φ is thus given by $\rho_\varphi(z) = H^{-1}(\langle H(z), e_1\rangle e_1) = H^{-1}(H_1(z), 0 \ldots, 0)$ and the left inverse is $\widetilde{\rho}_\varphi(z) = H_1(z)$. Therefore

$$f_\varphi(\zeta) = d(\widetilde{\rho}_\varphi)_{\varphi(\zeta)} \cdot F(\varphi(\zeta)) = d(H_1)_{H^{-1}(\zeta e_1)} \cdot F(H^{-1}(\zeta e_1)),$$

from which the statement follows. ☐

In the statement of Theorem 6.1, the sufficient condition for e_1 to be a BRFP can be checked considering only the class of *parabolic* automorphisms H (namely, those for which the boundary dilatation coefficient at e_1 is 1). For the sake of clearness, we examine in detail the case $n = 2$. In such a case we can limit ourselves to (parabolic) automorphisms of $\mathbb{B}^2$ of the form

$$H_{s,\theta}(z) = \frac{(-sz_2 + (1-\beta)z_1 + \beta, e^{i\theta}(z_2 + sz_1 - s))}{-sz_2 - \beta z_1 + 1 + \beta}, \tag{6.2}$$

where $\beta \geq 0$, $s = \sqrt{2\beta}$ and $\theta \in \mathbb{R}$. Notice that $(H_{s,\theta})^{-1}$ also has the same form of $H_{s,\theta}$ and the differential of $H_{s,\theta}$ at e_1 is

$$d(H_{s,\theta})_{e_1} = \begin{pmatrix} 1 & 0 \\ se^{i\theta} & e^{i\theta} \end{pmatrix}.$$

If $\varphi : \mathbb{D} \to \mathbb{B}^2$ is a complex geodesic with $\varphi(1) = e_1$ and we write $\varphi'(1)$ in projective coordinates as $\varphi'(1) = [1 : se^{i\theta}]$, with $s \geq 0$ and $\theta \in \mathbb{R}$, the corresponding $H_{s,\theta}$ in (6.2) is such that $H_{s,\theta}(\mathbb{D} \times \{0\}) = \varphi(\mathbb{D})$ and therefore, by uniqueness of complex geodesics, $\varphi(\zeta) = H_{s,\theta}(\psi(\zeta), 0)$ for some automorphism ψ of $\mathbb{D}$. Thus, in the statement of Theorem 6.1 for $n = 2$, it is enough to check condition (6.1) for H belonging to the class of $H_{s,\theta}$'s.

Theorem 6.1 and the previous observation can be used to obtain the boundary behavior of infinitesimal generators with some bounds on the image. To explain this fact, we prove the following corollary in $\mathbb{B}^2$, which can be easily generalized to $\mathbb{B}^n$ for any $n \geq 2$, and can be considered a Julia-Wolff-Carathéodory type theorem for infinitesimal generators.

Corollary 6.2. *Let $F : \mathbb{B}^2 \to \mathbb{C}^2$ be the infinitesimal generator of a semigroup with a BRFP at e_1. Suppose there exist a horosphere $E_{\mathbb{B}^2}(e_1, R)$ and two distinct points $a_0, a_1 \in \mathbb{C}$ such that $F_1(E_{\mathbb{B}^2}(e_1, R)) \subset \mathbb{C} \setminus \{a_0, a_1\}$. Then*

(1) *F_1 has non tangential limit 0 at e_1, namely, $\angle \lim_{z \to e_1} F_1(z) = 0$.*
(2) *$\angle \lim_{\zeta \to 1}(1-\zeta) F_2(\frac{((1-\beta)\zeta + \beta, e^{i\theta}(s\zeta - s))}{-\beta\zeta + 1 + \beta}) = 0$ for all $\beta \geq 0$, $s = \sqrt{2\beta}$ and $\theta \in \mathbb{R}$.*

Proof. By the very definition of horospheres in $\mathbb{B}^2$, there exists a ball $B \subset E_{\mathbb{B}^2}(e_1, R)$ such that B is tangent to $\mathbb{B}^2$ at e_1. Let $\{z_k\} \subset \mathbb{B}^2$ be any sequence converging to e_1 non-tangentially. Then the sequence $\{z_k\}$ is eventually contained in B. Hence

$$k_{E_{\mathbb{B}^2}(e_1,R)}(z_k, \langle z_k, e_1 \rangle e_1) \leq k_B(z_k, \langle z_k, e_1 \rangle e_1). \tag{6.3}$$

For $k \to \infty$ we have $k_B(z_k, \langle z_k, e_1 \rangle e_1) \to 0$ because $z_k \to e_1$ non-tangentially in B (and non-tangential sequences are special in the sense of Abate [1, Lemma 2.2.24]). Therefore

$$\lim_{k \to \infty} k_{E_{\mathbb{B}^2}(e_1,R)}(z_k, \langle z_k, e_1 \rangle e_1) = 0. \tag{6.4}$$

Now let $g := F_1|_{E_{\mathbb{B}^2}(e_1,R)} : E_{\mathbb{B}^2}(e_1, R) \to \mathcal{L} := \mathbb{C} \setminus \{a_0, a_1\}$. By the monotonicity of Kobayashi distance we have

$$\omega_{\mathcal{L}}(g(z_k), g(\langle z_k, e_1 \rangle e_1)) \leq k_{E_{\mathbb{B}^2}(e_1,R)}(z_k, \langle z_k, e_1 \rangle e_1),$$

and (6.4) forces

$$\lim_{k\to\infty} \omega_{\mathcal{L}}(g(z_k), g(\langle z_k, e_1\rangle e_1)) = 0.$$

Since $\mathcal{L}$ is hyperbolic, this means that if $g(\langle z_k, e_1\rangle e_1)$ tends to some $b \in \mathbb{C}$ then $g(z_k)$ must have the same limit as $k \to \infty$. By (6.1) it follows that $g(\zeta e_1)$ has non-tangential limit 0 at 1. Since $z_k \to e_1$ non-tangentially, the same does $\{\langle z_k, e_1\rangle e_1\}$. Then

$$\lim_{k\to\infty} F_1(z_k) = \lim_{k\to\infty} g(z_k) = \lim_{k\to\infty} g(\langle z_k, e_1\rangle e_1) = 0,$$

proving that F_1 has non-tangential limit 0 at e_1.

As for (2), from Theorem 6.1 with $H = H_{s',\theta'}$ (for $s' \geq 0$, $\theta' \in \mathbb{R}$) as in (6.2), we have

$$\angle \lim_{\zeta\to 1} d((H_{s',\theta'})_1)_{H^{-1}_{s',\theta'}(\zeta e_1)}(F(H^{-1}_{s',\theta'}(\zeta e_1))) = 0.$$

By the very definition of $H_{s',\theta'}$ (and keeping in mind that $(H_{s',\theta'})^{-1} = H_{s,\theta}$ for some $s \geq 0$ and $\theta \in \mathbb{R}$) an easy computation shows that

$$d((H_{s',\theta'})_1)_{H^{-1}_{s',\theta'}(\zeta e_1)}(F(H^{-1}_{s',\theta'}(\zeta e_1))) = C(\zeta)[F_1(H_{s,\theta}(\zeta e_1)) + (1-\zeta)F_2(H_{s,\theta}(\zeta e_1))],$$

where $C(\zeta)$ is a smooth function which tends to some real number $C \neq 0$ for $\zeta \to 1$. Thus, since F_1 has non-tangential limit 0 at e_1 by (1), statement (2) follows. $\square$

Example 6.3. The infinitesimal generator $F(z_1, z_2) = (0, -z_2/(1-z_1))$ in Example 4.2 has the boundary behavior prescribed by Corollary 6.2 at e_1. Notice that F_2 has not (non-tangential) limit 0 at e_1.

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