Notes on pluripotential theory

Filippo Bracci, Stefano Trapani

DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI ROMA "TOR VERGATA" VIA DELLA RICERCA SCIENTIFICA 1, 00133 ROMA, ITALY.

E-mail address: fbracci@mat.uniroma2.it

E-mail address: trapani@mat.uniroma2.it

Contents

Preface

These are the notes of a PhD course the first named author gave in 2005/06 at Università di Roma "Tor Vergata". The main subject of this course was the study of plurisubharmonic functions and their properties. These are very important tools in complex analysis because plurisubharmonic functions are pretty much related to holomorphic functions but much more flexible to handle and to be constructed. However, these notes contain very few applications of plurisubharmonic functions theory to complex analysis (for instance we included boundary transversality properties of analytic discs as a consequence of Hopf's lemma and a few relations between the pluricomplex Green function and invariant distances).

The present material contains a first part about elementary properties of (pluri-)subharmonic functions (chapters one, two and three), a second part (chapter four) about elementary properties of currents (especially positive currents) and a third part (chapter five) about maximal plurisubharmonic functions and the Monge-Ampère operator. This latter part has been developed in details for smooth plurisubharmonic functions and only sketched for locally bounded ones. Also, in this last chapter there are two sections about the pluricomplex Green function in bounded domains.

The reader—if any—of these notes is assumed to have a basic knowledge of harmonic functions, analysis and geometry.

The PhD course itself and, as consequence these notes, are mainly based on the wonderful books by Klimek [**16**] and Demailly [**9**]. It may happen that some result is stated here in a more general form than in those books and other material has been added from different sources. Some proofs have been completely re-elaborated and might not be contained in the literature in this form but of course we do not claim any original credit on this material.

We thank the participants of the course for their comments and questions which certainly improved these notes. We also thank prof. Sandro Silva for the opportunity of publishing these notes.

CHAPTER 1

Subharmonic functions

1. The sub-mean property and the maximum principle

DEFINITION 1.1. Let $\Omega \subset \mathbb{R}^m$ be a set. A function $u : \Omega \to [-\infty, \infty)$ is *upper semicontinuous* if for all $c \in \mathbb{R}$ the set $\{x \in \Omega : u(x) < c\}$ is open in Ω.

Notice that an upper semicontinuous function is measurable and is not allowed to assume the value $+\infty$ (while it may assume the value $-\infty$). It is easy to show that if $K \subset \mathbb{R}^m$ is a compact set and $u : K \to [-\infty, \infty)$ is upper semicontinuous then it has a maximum on K, but in general it may have no minimum (for instance let $K = [-1, 1]$ and define $u(x) = \log |x|$ for $x \neq 0$ and $u(0) = 0$. Then u is upper semicontinuous on K but has no minimum).

Another useful property that we will use in the sequel is that if u is upper semicontinuous on a compact set K then there exists a decreasing sequence $\{u_j\} \subset C^0(K)$ such that $\lim_{j\to\infty} u_j(x) = u(x)$ for all $x \in K$.

Moreover, if K is a compact set, then $\int_K u$ is well defined (possibly $= -\infty$) for all upper semicontinuous functions u in K, and according to Beppo Levi's theorem on monotone convergence, $\int_K u = \lim_{j\to\infty} \int_K u_j$ with $\{u_j\} \subset C^0(K)$ a sequence decreasing to u on K.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain (not necessarily bounded). Let $u : \Omega \to [-\infty, +\infty)$ be a non-constant upper semicontinuous function. Suppose that for all $a \in \Omega$ there exists $R(a) > 0$ with the following property: for all balls $\mathbb{B}(a, r)$ of center a and radius $0 < r \leq R(a)$ with $\overline{\mathbb{B}(a, r)} \subset \Omega$ it holds*

$$u(a) \leq \frac{1}{\mathrm{v}(\mathbb{B}(\mathrm{a}, \mathrm{r}))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x). \tag{1.1}$$

Then for all $z \in \Omega$

$$u(z) < \sup_{w\in\Omega} u(w).$$

Moreover, if there exists $R_0 \in (0, +\infty]$ such that $R(a) > R_0$ for all $a \in \Omega$ then $u \in L^1_{\mathrm{loc}}(\Omega)$.

PROOF. Let $\alpha := \sup_{z\in\Omega} u(z)$. By hypothesis u does not assume the value $+\infty$ in Ω, thus, if $\alpha = +\infty$ the statement is correct. We can assume then that $\alpha < +\infty$. Let us define

$$\Omega_\alpha := \{z \in \Omega : u(z) \geq \alpha\}.$$

Since u is upper semicontinuous then Ω_α is closed in Ω and by the very definition of α the set Ω_α coincides with the set $\{z \in \Omega : u(z) = \alpha\}$. The theorem will follow if we can prove that

Ω_α is empty. In order to do that, we show that if Ω_α were not empty then it would be open as well, which, by connectedness of Ω, would imply $u \equiv \alpha$ against our hypothesis that u is not constant. Let assume then that there exists $a \in \Omega_\alpha$. Let $\mathbb{B}(a,r) \subset \Omega$ be an open ball relatively compact in Ω, $r \leq R(a)$. We want to show that $\mathbb{B}(a,r) \subset \Omega_\alpha$. If this is not the case then there exists $b \in \mathbb{B}(a,r)$ such that $u(b) < \alpha$ and, since u is upper semicontinuous, there exists an open set $K \subset \mathbb{B}(a,r)$ such that $b \in K$ and $u(x) < \alpha$ for all $x \in K$. Then

$$\begin{aligned}\alpha = u(a) &\leq \frac{1}{\mathrm{v}(\mathbb{B}(\mathrm{a},\mathrm{r}))}\int_{\mathbb{B}(a,r)} u(x)d\lambda(x)\\ &= \frac{1}{\mathrm{v}(\mathbb{B}(\mathrm{a},\mathrm{r}))}\left[\int_{\mathbb{B}(a,r)\setminus K} u(x)d\lambda(x) + \int_K u(x)d\lambda(x)\right]\\ &< \frac{1}{\mathrm{v}(\mathbb{B}(\mathrm{a},\mathrm{r}))}\left[\int_{\mathbb{B}(a,r)\setminus K} u(x)d\lambda(x) + \int_K \alpha d\lambda(x)\right]\\ &\leq \frac{\alpha}{\mathrm{v}(\mathbb{B}(\mathrm{a},\mathrm{r}))}\left[\int_{\mathbb{B}(a,r)\setminus K} d\lambda(x) + \int_K d\lambda(x)\right] = \alpha,\end{aligned}$$

that is, $\alpha < \alpha$, a contradiction. Then $\mathbb{B}(a,r) \subset \Omega_\alpha$ and this latter set is open.

Now assume that $R(a) \geq R_0$ for all $a \in \Omega$. Since u is upper semicontinuous, on each compact subset $K \subset\subset \Omega$ it has a maximum. Moreover, if $u(a) > -\infty$ and $\mathbb{B}(a,\rho)$ with $R(a) \geq \rho > 0$ is relatively compact in Ω, then by (1.1) it follows that $u \in L^1(\mathbb{B}(a,r))$ for all $r \leq \rho$. Therefore the set $W = \{x \in \Omega : \exists U \ni x, u \in L^1_{\mathrm{loc}}(U)\}$ of points where u is locally integrable, is a non-empty open subset of Ω. To show that $u \in L^1_{\mathrm{loc}}(\Omega)$ it is enough to prove that W is closed in Ω. Let $x_0 \in \partial W \cap \Omega$. The condition that $R(a) \geq R_0$ for all $a \in \Omega$ guarantees that there exists a point $a \in W$ with $u(a) > -\infty$ and a number $r > 0$, $r \leq R(a)$, such that $U = \mathbb{B}(a,r)$ is relatively compact in Ω and $x_0 \in \mathbb{B}(a,r)$. Let $c = \max_{z\in\overline{U}} u(z)$. Then $u - c \leq 0$ in U. Therefore for all compact subsets $K \subset \mathbb{B}(a,r)$

$$-\infty < \mathrm{v}(\mathbb{B}(\mathrm{a},\mathrm{r}))(u(a)-c) \leq \int_{\mathbb{B}(a,r)} [u(x)-c]d\lambda(x) \leq \int_K [u(x)-c]d\lambda(x) \leq 0.$$

Hence $u \in L^1(K)$ for all $K \subset\subset \mathbb{B}(a,r)$ and in particular $x_0 \in W$ showing that W is closed in Ω and $u \in L^1_{\mathrm{loc}}(\Omega)$. □

REMARK 1.3. The condition on the existence of $R_0 > 0$ which uniformly bounds $R(a)$ from below for each $a \in \Omega$ is actually not necessary for the conclusion that $u \in L^1_{\mathrm{loc}}(\Omega)$. However this will be a consequence of the equivalence between (3) and (4) in Theorem 2.2.

REMARK 1.4. The previous proof shows that Theorem 1.2 holds if one substitutes the balls $\mathbb{B}(a,r)$ in (1.1) with any other basis of open sets.

REMARK 1.5. Let $\Omega \subset \mathbb{R}^m$ be a connected domain and let $u : \Omega \to [-\infty,\infty)$ be an upper semicontinuous function, $u \not\equiv -\infty$. Define the function $\tilde{u} : \overline{\Omega} \to [-\infty,\infty]$ as follows: $\tilde{u}(x) := u(x)$ for $x \in \Omega$ and $\tilde{u}(y) := \limsup_{\Omega\ni x\to y} u(x)$ for $y \in \partial\Omega$. If $\tilde{u}(y) < +\infty$ for all $y \in \partial\Omega$

then $\tilde{u}$ is upper semicontinuous. In this case we say that $\tilde{u}$ is an upper semicontinuous extension of u to $\overline{\Omega}$. Notice that $\tilde{u}$ is the minimal upper semicontinuous extension of u to $\overline{\Omega}$, namely, if v is another semicontinuous extension of u then $\tilde{u} \leq v$.

COROLLARY 1.6. *Let $\Omega \subset \mathbb{R}^m$ be a connected bounded domain. Let $u : \Omega \to [-\infty, +\infty)$ be a non-constant upper semicontinuous function which satisfies the sub-mean property* (1.1). *For $y \in \partial\Omega$ define $u(y) := \limsup_{\Omega \ni x \to y} u(x)$. Then for all $x \in \Omega$*

$$u(x) < \sup_{y \in \partial\Omega} u(y).$$

In particular, if u extends upper semicontinously on $\overline{\Omega}$ (namely, if $u(y) < \infty$ for all $y \in \partial\Omega$) then $u(z) < \max_{y \in \partial\Omega} u(y)$ for all $z \in \Omega$.

2. Definition and first properties

Let $\mathsf{harm}(\Omega)$ be the space of harmonic functions on a domain $\Omega \subset \mathbb{R}^m$.

DEFINITION 2.1. Let $\Omega \subset \mathbb{R}^m$ be a connected domain. A function $u : \Omega \to [-\infty, \infty)$ is called a *subharmonic* function, $u \in \mathsf{subh}(\Omega)$ if

(1) $u \not\equiv -\infty$.
(2) u is upper semicontinuous.
(3) For all open set $G \subset\subset \Omega$ and all $v \in \mathsf{harm}(G) \cap C^0(\overline{G})$ such that $u(y) \leq v(y)$ for all $y \in \partial G$ it follows that $u(x) \leq v(x)$ for all $x \in G$.

By the very definition, $\mathsf{subh}(\Omega)$ is a cone in the space of all real functions on Ω.

THEOREM 2.2. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function, $u \not\equiv -\infty$. The following are equivalent:*

(1) *For all open ball $\mathbb{B}(a, r)$ relatively compact in Ω it follows*

$$u(a) \leq \frac{1}{\mu(\partial\mathbb{B}(a,r))} \int_{\partial\mathbb{B}(a,r)} u(\zeta) d\sigma(\zeta). \tag{2.1}$$

(2) *For all open ball $\mathbb{B}(a, r)$ relatively compact in Ω it follows*

$$u(a) \leq \frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x). \tag{2.2}$$

(3) *For all $a \in \Omega$ there exists $R(a) > 0$ such that for all $0 < r < R(a)$ and open balls $\mathbb{B}(a, r)$ relatively compact in Ω it follows*

$$u(a) \leq \frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x). \tag{2.3}$$

(4) $u \in \mathsf{subh}(\Omega)$.

PROOF. (1) implies (2) by integration on r and Fubini's theorem. Obviously (2) implies (3).

Assume (3) holds. Let $G \subset\subset \Omega$ be open and $v \in \mathsf{harm}(G) \cap C^0(\overline{G})$ be such that $u \leq v$ on ∂G. Then $u - v$ has the sub-mean property and, by Theorem 1.6 it satisfies the maximum principle in G. Thus $u - v$ has maximum on ∂G and then $u - v \leq 0$ in G proving that (4) holds.

Finally, assume (4) holds and let $\mathbb{B}(a, r)$ be an open ball relatively compact in Ω. Since $\partial\mathbb{B}(a, r)$ is compact and u is upper semicontinuous on $\partial\mathbb{B}(a, r)$, there exist a decreasing sequence $\{u_j\} \subset C^0(\partial\mathbb{B}(a, r))$ such that $\lim_{j\to\infty} u_j(x) \to u(x)$ for all $x \in \partial\mathbb{B}(a, r)$. Let $U_j \in \mathsf{harm}(\mathbb{B}(a, r)) \cap C^0(\overline{\mathbb{B}(a, r)})$ be such that $U_j = u_j$ on $\partial\mathbb{B}(a, r)$. Since $u \in \mathsf{subh}(\Omega)$ and $u \leq U_j$ on $\partial\mathbb{B}(a, r)$ then $u \leq U_j$ in $\mathbb{B}(a, r)$ for all j. Therefore for all j

$$u(a) \leq U_j(a) = \frac{1}{\mu(\partial\mathbb{B}(a, r))} \int_{\partial\mathbb{B}(a,r)} u_j(\zeta) d\sigma(\zeta).$$

Thus, by Beppo Levi's theorem on monotone convergence

$$u(a) \leq \lim_{j\to\infty} \frac{1}{\mu(\partial\mathbb{B}(a, r))} \int_{\partial\mathbb{B}(a,r)} u_j(\zeta) d\sigma(\zeta) =$$
$$\frac{1}{\mu(\partial\mathbb{B}(a, r))} \int_{\partial\mathbb{B}(a,r)} \lim_{j\to\infty} u_j(\zeta) d\sigma(\zeta) = \frac{1}{\mu(\partial\mathbb{B}(a, r))} \int_{\partial\mathbb{B}(a,r)} u(\zeta) d\sigma(\zeta)$$

and (1) holds. □

REMARK 2.3. By the equivalence between (2) and (3) in Theorem 2.2 it follows that if $u : \Omega \to [-\infty, +\infty)$ is a non-constant upper semicontinuous function which satisfies (1.1) then actually $R(a) = +\infty$ for all $a \in \Omega$ and then $u \in L^1_{\text{loc}}(\Omega)$.

COROLLARY 2.4. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain and let $\{\Omega_k\} \subset \mathbb{R}^m$ be a sequence of connected domains such that $\Omega_k \subseteq \Omega_{k+1}$ and $\bigcup_k \Omega_k = \Omega$. For each k, let $u_k \in \mathsf{subh}(\Omega_k)$ be such that $u_k(x) \geq u_{k+1}(x)$ for all $x \in \Omega_k$ and for all k (that is, $\{u_k\}$ is a decreasing sequence). Let $u(x) := \lim_{k\to\infty} u_k(x)$ for $x \in \Omega$. Then either $u \equiv -\infty$ or $u \in \mathsf{subh}(\Omega)$.*

PROOF. Assume that $u \not\equiv -\infty$. First of all, for $c \in \mathbb{R}$ and $k \in \mathbb{N}$ the set $\{x \in \Omega_k : u(x) \geq c\} = \bigcap_{s\geq k}\{x \in \Omega_s : u_s(x) \geq c\}$ is closed in Ω_k and thus u is upper semicontinuous in Ω_k for all k which implies that u is upper semicontinuous in Ω. Next, according to Theorem 2.2 we just need to prove that u satisfies the sub-mean property. Let $a \in \Omega$ and let $\mathbb{B}(a, r)$ be an open ball relatively compact in Ω. Then, since the u_k's are subharmonic by Beppo Levi's theorem one has

$$u(a) = \lim_{k\to\infty} u_k(a) \leq \lim_{k\to\infty} \frac{1}{\mathrm{v}(\mathbb{B}(a, r))} \int_{\mathbb{B}(a,r)} u_k(x) d\lambda(x) = \frac{1}{\mathrm{v}(\mathbb{B}(a, r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x),$$

as wished. □

Another important consequence of Theorem 2.2 is that subharmonicity is a local property:

PROPOSITION 2.5. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Then a function $u \in \mathsf{subh}(\Omega)$ if and only if for all $x \in \Omega$ there exists an open neighborhood $V_x \subset \Omega$ of x such that $u \in \mathsf{subh}(V_x)$.*

Also

COROLLARY 2.6. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Let $u \in \mathsf{subh}(\Omega)$. Then $u \in L^1_{\text{loc}}(\Omega)$. Moreover, for all $x \in \Omega$*

$$u(x) < \sup_{w\in\Omega} u(w).$$

In particular, if Ω is bounded and u extends upper semi-continuously on $\overline{\Omega}$ then $u(x) < \max_{y\in\partial\Omega} u(y)$ for all $x \in \Omega$.

REMARK 2.7. Corollary 2.6 says that a subharmonic function satisfies the maximum principle, namely, if $u \in \mathsf{subh}(\Omega)$ is not constant then for each $G \subset \Omega$ open it follows $u(z) < \sup_{w\in G} u(w)$. The converse is however not true. For instance consider $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$. Let $u : \Omega \to \mathbb{R}$ be given by $u(x, y) := \sqrt{x + y}$. Then u satisfies the maximum principle (because u is increasing in Ω with respect to the distance from the origin), but it is not subharmonic (for instance we see that $\Delta u < 0$ and by Theorem 4.1 then u cannot be subharmonic).

Also, since harmonic functions are continuous and satisfy the mean-value property, it follows that

PROPOSITION 2.8. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Then $\mathsf{harm}(\Omega) \subset \mathsf{subh}(\Omega)$. Moreover, if $u \in \mathsf{subh}(\Omega)$ and $-u \in \mathsf{subh}(\Omega)$ then $u \in \mathsf{harm}(\Omega)$.*

3. Regularization

Let $\chi \in C^\infty(\mathbb{R}^m)$ be such that $\chi \geq 0$, $\mathsf{supp}(\chi) \subseteq \overline{\mathbb{B}(O, 1)}$, $\chi(x) = \chi(\|x\|)$ and $\int_{\mathbb{R}^m} \chi(x)d\lambda(x) = 1$. Let $\epsilon > 0$ and define

$$\chi_\epsilon(x) := \frac{1}{\epsilon^m}\chi(x/\epsilon).$$

Then $\mathsf{supp}(\chi_\epsilon) \subseteq \overline{\mathbb{B}(O, \epsilon)}$ and $\int_{\mathbb{R}^m} \chi_\epsilon(x)d\lambda(x) = 1$.

For an open connected subset $\Omega \subset \mathbb{R}^m$ let

$$\Omega_\epsilon = \{x \in \Omega : \mathsf{dist}(x, \partial\Omega) > \epsilon\}.$$

From now on, we assume without further comments that ϵ is so small that $\Omega_\epsilon \neq \emptyset$. If $u \in L^1_{\text{loc}}(\Omega)$ then we let for $x \in \Omega_\epsilon$

$$u_\epsilon(x) := u * \chi_\epsilon(x) := \int_{\mathbb{R}^m} u(x - y)\chi_\epsilon(y)d\lambda(y) = \int_{\mathbb{R}^m} u(y)\chi_\epsilon(x - y)d\lambda(y).$$

By Lebesgue's dominated convergence theorem the functions $u_\epsilon \in C^\infty(\Omega_\epsilon)$ and $u_\epsilon \to u$ in the $L^1_{\text{loc}}(\Omega)$-topology as $\epsilon \to 0$ (and thus $u_\epsilon \to u$ pointwise almost everywhere).

THEOREM 3.1. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Let $u \in \mathsf{subh}(\Omega)$. Then $u_\epsilon \in C^\infty(\Omega_\epsilon) \cap \mathsf{subh}(\Omega_\epsilon)$. Moreover, $\{u_\epsilon\}$ is decreasing as $\epsilon \to 0^+$ and for all $x \in \Omega$ it follows $\lim_{\epsilon\to 0} u_\epsilon(x) = u(x)$.*

PROOF. Let $a \in \Omega$ and $\mathbb{B}(a,r)$ an open ball relatively compact in Ω_ϵ. By Fubini's theorem and since u is subharmonic

$$\begin{aligned} \frac{1}{\mathrm{v}(\mathbb{B}(a,r))}\int_{\mathbb{B}(a,r)} u_\epsilon(x)d\lambda(x) &= \int_{\mathbb{R}^m}\chi_\epsilon(y)\left(\frac{1}{\mathrm{v}(\mathbb{B}(a,r))}\int_{\mathbb{B}(a,r)} u(x-y)d\lambda(x)\right)d\lambda(y) \\ &\geq \int_{\mathbb{R}^m}\chi_\epsilon(y)u(a-y)d\lambda(y) = u_\epsilon(a), \end{aligned} \tag{3.1}$$

and thus by Theorem 2.2, $u_\epsilon \in \mathsf{subh}(\Omega_\epsilon)$.

Next we show that u_ϵ is decreasing in ϵ. To this aim, for $a \in \Omega$ and $r > 0$ such that $\mathbb{B}(a,r) \subset\subset \Omega$ we let

$$L(u,a,r) := \frac{1}{\mu(\partial\mathbb{B}(a,r)}\int_{\partial\mathbb{B}(a,r)} u(\zeta)d\sigma(\zeta).$$

We claim that $r \mapsto L(u,a,r)$ is increasing. Indeed, let $r_1 < r_2$ and let $\{u_j\} \subset C^0(\partial\mathbb{B}(a,r_2))$ be a decreasing sequence whose limit is u on $\partial\mathbb{B}(a,r_2)$ (such a sequence exists because u is upper semicontinuous on the compact set $\partial\mathbb{B}(a,r_2)$). Let $U_j \in \mathsf{harm}(\mathbb{B}(a,r_2)) \cap C^0(\overline{\mathbb{B}(a,r_2)})$ be such that $U_j = u_j$ on $\partial\mathbb{B}(a,r_2)$. Since $u \leq u_j$ on $\partial\mathbb{B}(a,r_2)$ then $u \leq U_j$ in $\mathbb{B}(a,r_2)$ for all j. Therefore

$$L(u,a,r_1) \leq L(U_j,a,r_1) = U_j(a) = L(U_j,a,r_2) = L(u_j,a,r_2)$$

for all j. Thus by Beppo Levi's theorem, $L(u,a,r_1) \leq \lim_{j\to\infty} L(u_j,a,r_2) = L(u,a,r_2)$ proving that $r \mapsto L(u,a,r)$ is increasing. Now, a direct computation from the very definition shows that

$$u_\epsilon(x) = \mu(\partial\mathbb{B}(O,1))\int_0^1 \chi(r)r^{m-1}L(u,x,\epsilon r)dr \tag{3.2}$$

and since $r \mapsto L(u,a,r)$ is increasing (and thus decreasing as $r \to 0+$), $\epsilon \mapsto u_\epsilon(x)$ is decreasing for each fixed $x \in \Omega$.

We have to show that $u_\epsilon \to u$ pointwise as $\epsilon \to 0$. From (3.2), since $u(x) \leq L(u,x,\epsilon r)$ for all $\epsilon > 0$, it follows that $u(x) \leq u_\epsilon(x)$ for all $x \in \Omega_\epsilon$. Let first assume that $u(x) \neq -\infty$ and let $C > 0$. Since u is upper semicontinuous there exists $\epsilon_1 > 0$ such that $u(y) < u(x) + C$ for all $y \in \mathbb{B}(x,\epsilon_1)$. For $\epsilon < \epsilon_1$, since χ_ϵ is supported in $\mathbb{B}(O,\epsilon)$, we have

$$\begin{aligned} u_\epsilon(x) &= \int_{\mathbb{R}^m} u(x-y)\chi_\epsilon(y)d\lambda(y) = \int_{\mathbb{B}(O,\epsilon)} u(x-y)\chi_\epsilon(y)d\lambda(y) \\ &\leq (u(x)+C)\int_{\mathbb{B}(O,\epsilon)}\chi_\epsilon(y)d\lambda(y) = u(x) + C \end{aligned} \tag{3.3}$$

and thus $u(x) \leq u_\epsilon(x) \leq u(x) + C$. Therefore $\lim_{\epsilon\to 0} u_\epsilon(x) = u(x)$. Assume now that $u(x) = -\infty$. Then, since $\limsup_{w\to x} u(x) \leq u(x) = -\infty$, for all $C > 0$ there exists $\epsilon_1 > 0$ such that $u(y) \leq -C$ for all $y \in \mathbb{B}(x,\epsilon)$ and $\epsilon < \epsilon_1$. Arguing as in (3.3) we find that $u_\epsilon(x) \leq -C$ for $\epsilon < \epsilon_1$ and therefore $u_\epsilon(x) \to -\infty$ as $\epsilon \to 0$. □

COROLLARY 3.2. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. If $u, v \in \mathsf{subh}(\Omega)$ and $u = v$ almost everywhere then $u \equiv v$.*

PROOF. Since $u = v$ almost everywhere, then $u_\epsilon \equiv v_\epsilon$. Thus, by Theorem 3.1, $u(x) = \lim_{\epsilon\to 0} u_\epsilon(x) = \lim_{\epsilon\to 0} v_\epsilon(x) = v(x)$ for all $x \in \Omega$. □

REMARK 3.3. Let $\Omega \subset \mathbb{R}^m$. Let $u \in \mathsf{subh}(\Omega)$. Let $\mathbb{B}(x, r)$ be an open ball relatively compact in Ω. Consider the function

$$A(x, u, r) := \frac{1}{\mathsf{v}(\mathbb{B}(\mathsf{x}, \mathsf{r}))} \int_{\mathbb{B}(x,r)} u d\lambda.$$

Then $A(x, u, r)$ is increasing in $r > 0$. Indeed, in the proof of Theorem 3.1 we proved that $L(x, u, r)$ is increasing in $r > 0$ and

$$A(x, u, r) = m \int_0^1 t^m L(x, u, tr) dt.$$

PROPOSITION 3.4. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Let $\{u_j\} \subset \mathsf{subh}(\Omega)$ be a sequence of subharmonic functions which are uniformly bounded from above on compacta of Ω. Let*

$$S(\{u_j\}) := \{x \in \Omega : \exists U \textit{ open neighborhood of } x, \exists C_U > 0 : \sup_j \int_U |u_j| d\lambda \leq C_U\}.$$

Then, either $S(\{u_j\}) = \emptyset$ or $S(\{u_j\}) = \Omega$.

PROOF. The set $S(\{u_j\})$ is clearly open. Since Ω is connected, it is enough to show that it is also closed in Ω. Assume that $S(\{u_j\}) \neq \emptyset$ and let $y \in \overline{S(\{u_j\})} \cap \Omega$. There exist $x \in S(\{u_j\})$ and $r > 0$ such that $\mathbb{B}(x, r)$ is relatively compact in Ω and $y \in \mathbb{B}(x, r)$. In order to show that $y \in S(\{u_j\})$ (proving that Ω is closed) it is enough to show that there exists $M > 0$ such that

$$\sup_j \int_{\mathbb{B}(x,r)} |u_j| d\lambda \leq M. \tag{3.4}$$

Since $\{u_j\}$ are uniformly bounded on compacta, there exists $C \geq 0$ such that $u_j(z) - C \leq 0$ for all $j \in \mathbb{N}$ and $z \in \mathbb{B}(x, r)$. Thus, we can assume that $u_j \leq 0$ on $\mathbb{B}(x, r)$. By hypothesis, $x \in S(\{u_j\})$. Therefore, there exist $0 < r' \leq r$ and $M' > 0$ such that

$$\sup_j \int_{\mathbb{B}(x,r')} |u_j| d\lambda = \sup_j \left(- \int_{\mathbb{B}(x,r')} u_j d\lambda \right) \leq M'.$$

According to Remark 3.3, for all $j \in \mathbb{N}$,

$$\begin{aligned} \frac{1}{\mathsf{v}(\mathbb{B}(x, r))} \int_{\mathbb{B}(x,r)} |u_j| d\lambda &= -A(x, u_j, r) \leq -A(x, u_j, r') \\ &= \frac{1}{\mathsf{v}(\mathbb{B}(x, r'))} \int_{\mathbb{B}(x,r')} |u_j| d\lambda \leq \frac{M'}{\mathsf{v}(\mathbb{B}(x, r'))}, \end{aligned}$$

and we are done. □

4. Subharmonic functions and distributions

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. If $u \in \mathsf{subh}(\Omega)$ then $\Delta u \geq 0$ in the sense of distribution. Conversely, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\Delta u \geq 0$ in the sense of distributions, then there exists $v \in \mathsf{subh}(\Omega)$ such that $v = u$ almost everywhere.*

PROOF. First of all, we assume $u \in C^2(\Omega)$. Suppose $\Delta u \geq 0$. Let $G \subset\subset \Omega$ and $h \in \mathsf{harm}(G) \cap C^0(\overline{G})$ be such that $u \leq h$ on ∂G. Fix $\epsilon > 0$. Let $R := \max\{\|z\|^2 : z \in G\}$, $0 < \delta < \epsilon/R$ and

$$v(z) := u(z) - \epsilon + \delta\|z\|^2.$$

Notice that $v < u$ in $\overline{G}$ and therefore $v < h$ on ∂G. Let $w(z) := v(z) - h(z)$. Then $w(z) < 0$ on ∂G. We claim that $w \leq 0$ in G. Let $a \in \overline{G}$ be such that $w(a) = \max_{z \in \overline{G}} w(z)$. Assume by contradiction that $w(a) > 0$. Since $a \in G$, then there exists $s > 0$ such that $(-s, s) \ni t \mapsto w(a + te_j)$ has a maximum in $t = 0$. Thus

$$\frac{\partial^2}{\partial x_j^2} w(a) = \frac{d^2}{dt^2} w(a + te_j)|_{t=0} \leq 0.$$

Therefore $\Delta w(a) \leq 0$. But

$$\Delta w(a) = \Delta u(a) + \delta \Delta \|z\|^2|_{z=a} - \Delta h(a) = \Delta u(a) + \delta > 0,$$

contradiction. Thus $v \leq h$ in G. Hence, $u \leq h + \epsilon - \delta\|z\|^2 < h + 2\epsilon$ on G. By the arbitrariness of ϵ we obtain $u \leq h$ in G and thus $u \in \mathsf{subh}(\Omega)$. Now, if $u \in C^2(\Omega) \cap \mathsf{subh}(\Omega)$ and $\Delta u(a) < 0$ then there exists an open ball $\mathbb{B}(a, r) \subset \Omega$ such that $\Delta u(x) \leq 0$ for all $x \in \mathbb{B}(a, r)$. Therefore $\Delta(-u) \geq 0$ in $\mathbb{B}(a, r)$ which, by the previous part, implies that $-u \in \mathsf{subh}(\mathbb{B}(a, r))$. Therefore $u, -u \in \mathsf{subh}(\mathbb{B}(a, r))$ and then $u \in \mathsf{harm}(\mathbb{B}(a, r))$. But then $\Delta u(a) = 0$, contradicting $\Delta u(a) < 0$. Thus the theorem holds for C^2 functions.

Now assume $u \in \mathsf{subh}(\Omega)$ (with no regularity assumptions). Let u_ϵ be the regularization sequence given by Theorem 3.1. Then $\Delta u_\epsilon \geq 0$ for all $\epsilon > 0$. If $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, then by the Lebesgue theorem and integration by parts

$$\int_\Omega u \Delta\varphi = \lim_{\epsilon \to 0} \int_\Omega u_\epsilon \Delta\varphi = \lim_{\epsilon \to 0} \int_\Omega (\Delta u_\epsilon)\varphi \geq 0,$$

and thus $\Delta u \geq 0$ in the sense of distributions.

Conversely, let $u \in L^1_{\mathrm{loc}}$ be such that $\Delta u \geq 0$ in the sense of distributions. Let $u_\epsilon := u * \chi_\epsilon$. Recall that $u_\epsilon \to u$ in $L^1_{\mathrm{loc}}(\Omega)$—and thus $u_\epsilon \to u$ almost everywhere. For small ϵ, test function $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$ and by Fubini's theorem we have

$$\int_\Omega u_\epsilon(x)\Delta\varphi(x) d\lambda(x) = \int_{\mathbb{R}^m} \chi_\epsilon(y) \left(\int_\Omega u(x - y)\Delta\varphi(x) d\lambda(x) \right) d\lambda(y) \geq 0,$$

therefore $\Delta u_\epsilon \geq 0$ (this is true in the sense of distributions and, since $u_\epsilon \in C^2(\Omega_\epsilon)$, integrating by parts it is true for all $x \in \Omega_\epsilon$). Hence $u_\epsilon \in \mathsf{subh}(\Omega_\epsilon)$. If we show that $\{u_\epsilon\}$ is decreasing in ϵ, then by Corollary 2.4 the limit is subharmonic (and, as we already noticed, it coincides with u out of a zero-measure set). In order to see that $\{u_\epsilon\}$ is decreasing in ϵ, let $\epsilon_1 < \epsilon_2$. Then $u_{\epsilon_2} = \lim_{\delta\to 0} u_{\epsilon_2} * \chi_\delta$. By Fubini's theorem $u_{\epsilon_2} * \chi_\delta = (u * \chi_{\epsilon_2}) * \chi_\delta = (u * \chi_\delta) * \chi_{\epsilon_2}$. The function $u * \chi_\delta$ is subharmonic and by Theorem 3.1 the regularizing sequence approximating it is decreasing in ϵ, namely $(u * \chi_\delta) * \chi_{\epsilon_2} \geq (u * \chi_\delta) * \chi_{\epsilon_1}$. Thus

$$u_{\epsilon_2} = \lim_{\delta\to 0} u_{\epsilon_2} * \chi_\delta \geq \lim_{\delta\to 0} u_{\epsilon_1} * \chi_\delta = u_{\epsilon_1} \tag{4.1}$$

as needed. □

5. Construction of subharmonic functions

PROPOSITION 5.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Let $V \subset \Omega$ be an open subset. Let $u \in \mathsf{subh}(\Omega)$ and let $v \in \mathsf{subh}(V)$ be such that $\limsup_{z\to y} v(z) \leq u(y)$ for all $y \in \partial V \cap \Omega$. Then the function*

$$w := \begin{cases} \max\{u, v\} \text{ in } V \\ u \text{ in } \Omega \setminus V \end{cases}$$

is subharmonic in Ω. In particular, if $V = \Omega$, namely if $u, v \in \mathsf{subh}(\Omega)$, then $\max\{u, v\} \in \mathsf{subh}(\Omega)$.

PROOF. Clearly $w : \Omega \to [-\infty, \infty)$. We want to prove that w is upper semicontinuous in Ω. By the very definition, w is upper semicontinuous in $\Omega \setminus \overline{V}$. Let $x \in \overline{V} \cap \Omega$. If $x \in V$ then

$$\limsup_{\Omega\ni z\to x} w(z) \leq \max\{u(x), v(x)\} = w(x),$$

while, if $x \in \partial V \cap \Omega$ then

$$\limsup_{\Omega\ni z\to x} w(z) \leq u(x) = w(x),$$

because $\limsup_{z\to y} v(z) \leq u(y)$ for all $y \in \partial V \cap \Omega$. Thus w is upper semicontinuous in Ω.

Now, let $a \in \Omega$. If $w(a) = u(a)$ then

$$\frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} w(x) d\lambda(x) \geq \frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x) \geq u(a) = w(a).$$

If $w(a) = v(a) > u(a)$ (and then necessarily $a \in V$) then we can find $R(a) > 0$ such that $\overline{\mathbb{B}(a,r)} \subset V$ for all $0 < r \leq R(a)$. Thus

$$\frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} w(x) d\lambda(x) \geq \frac{1}{\mathrm{v}(\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} v(x) d\lambda(x) \geq v(a) = v(a),$$

and by Theorem 2.2, $w \in \mathsf{subh}(\Omega)$. □

A simple argument shows that

PROPOSITION 5.2. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Let $\{u_j\} \subset \mathsf{subh}(\Omega)$ be a sequence converging uniformly on compacta. Then the limit $u \in \mathsf{subh}(\Omega)$.*

PROPOSITION 5.3. *Let $\Omega \subset \mathbb{R}^m$ be a connected domain. Let $\{u_\alpha\}_{\alpha\in J}$ be a family of subharmonic functions on Ω. Let $u(x) = \sup_{\alpha\in J} u_\alpha(x)$. Assume that u is locally bounded from above. Let u^* be the upper semicontinuous regularization of u, namely*

$$u^*(x) := \max\{u(x), \limsup_{\Omega\ni w\to x} u(w)\}.$$

Then $u^* \in \mathsf{subh}(\Omega)$ and $u = u^*$ almost everywhere in Ω. Also, $u^* = \lim_{\epsilon\to 0} u * \chi_\epsilon$.

PROOF. By the very definition $u^* : \Omega \to [-\infty, \infty)$ and, since $\limsup_{\Omega\ni w\to x} u^*(w) = u^*(x)$ it is upper semicontinuous. Notice that

$$u^* = \inf\{v \text{ upper semicontinuous } : v \geq u\}.$$

In particular if v is upper semicontinuous and $v \geq u$ then $v \geq u^*$. Let $G \subset\subset \Omega$ be an open set and let $v \in \mathsf{harm}(G) \cap C^0(\partial G)$ be such that $v \geq u^*$ on ∂G. Thus $v \geq u^* \geq u \geq u_\alpha$ on ∂G for all $\alpha \in J$. Since $u_\alpha \in \mathsf{subh}(\Omega)$, $v \geq u_\alpha$ on G for all $\alpha \in J$. Thus $v \geq u$ on G. But v is (upper semi-)continuous and therefore $v \geq u^*$ in G, proving that $u^* \in \mathsf{subh}(\Omega)$.

Now consider the convolutions $u_\epsilon := u * \chi_\epsilon \in C^\infty(\Omega_\epsilon)$. We know that $u_\epsilon \to u$ almost everywhere in Ω. Let $\mathbb{B}(a, r)$ be an open ball relatively compact in Ω. Since for all $\alpha \in J$

$$u_\alpha(a) \leq \frac{1}{\mathrm{v}((\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u_\alpha(x) d\lambda(x) \leq \frac{1}{\mathrm{v}((\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x)$$

then it follows that $u(a) \leq \frac{1}{\mathrm{v}((\mathbb{B}(a,r))} \int_{\mathbb{B}(a,r)} u(x) d\lambda(x)$. Arguing as in (3.1) we find then that the u_ϵ's have the sub-mean property and then $u_\epsilon \in \mathsf{subh}(\Omega)$. Moreover, by Theorem 3.1, $u_\alpha \leq u_\alpha * \chi_\epsilon$ and then $u_\alpha \leq u_\alpha * \chi_\epsilon \leq u * \chi_\epsilon = u_\epsilon$ for all α, showing that $u \leq u_\epsilon$ for all ϵ and thus $u^* \leq u_\epsilon$ (since u_ϵ are C^∞). Arguing as in (4.1) we see that u_ϵ is decreasing in ϵ. Thus by Corollary 2.4 the limit $v := \lim_{\epsilon\to 0} u_\epsilon$ is subharmonic in Ω. Since $v = u$ almost everywhere in Ω, and $u \leq u^* \leq v$ then $u = u^*$ almost everywhere in Ω and $u^* = v$ by Corollary 3.2. □

DEFINITION 5.4. Let $\Omega \subset \mathbb{R}^m$ be a domain. A subset $E \subset \Omega$ is a *polar set* if for each $x \in E$ there exists an open set $V_x \subset \Omega$ with $x \in V_x$ and $v \in \mathsf{subh}(V_x)$ such that $E \cap V_x \subseteq \{v = -\infty\}$.

Since subharmonic functions are L^1_{loc}, then every polar set $E \subset \Omega$ has zero Lebesgue measure and its complementary $\Omega \setminus E$ is dense in Ω.

COROLLARY 5.5. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Let $v \in \mathsf{subh}(\Omega)$ and let E be a closed polar set. Let $u \in \mathsf{subh}(\Omega \setminus E)$ (respectively $u \in \mathsf{harm}(\Omega \setminus E)$) and assume that u is bounded from above. Let*

$$U(x) = \begin{cases} u(x) & \text{if } x \in \Omega \setminus E \\ \limsup_{\Omega\setminus E\ni y\to x} u(y) & \text{if } x \in E \end{cases} \tag{5.1}$$

Then U is subharmonic (respectively harmonic) in Ω.

PROOF. Assume $u \in \mathsf{subh}(\Omega \setminus E)$. Since u is bounded from above, then U is upper semicontinuous on Ω. Let $x \in E$ and let $V_x \subset \Omega$ be an open neighborhood of x such that there exists $v \in \mathsf{subh}(V_x)$ with $E \cap V_x \subset \{v = -\infty\}$. We are going to show that $U \in \mathsf{subh}(V_x)$ and then, by arbitrariness of $x \in E$ and by Proposition 2.5 it will follow that $U \in \mathsf{subh}(\Omega)$.

Let define $U_\epsilon := u + \epsilon v$ (here we consider $U_\epsilon(x) = -\infty$ if $v(x) = -\infty$) in V_x. Then $U_\epsilon \in \mathsf{subh}(\Omega)$. Let $\tilde{U} = \sup_\epsilon U_\epsilon$ on $V_x \setminus E$. According to Proposition 5.3 the upper semicontinuous regularization $\tilde{U}^*$ of $\tilde{U}$ is subharmonic in V_x. By construction $U = \tilde{U}^*$ on V_x and thus $U \in \mathsf{subh}(V_x)$.

If $u \in \mathsf{harm}(\Omega \setminus E)$, let V denotes the function defined as in (5.1) for $-u$. Then $U, V \in \mathsf{subh}(\Omega)$ and $U + V = 0$ in $\Omega \setminus E$ which is a set of full Lebesgue measure. By Corollary 3.2 it follows that $U + V \equiv 0$ and thus $U, -U \in \mathsf{subh}(\Omega)$ which implies that $U \in \mathsf{harm}(\Omega)$. □

THEOREM 5.6. *Let $\Omega \subset \mathbb{R}^m$ be an open set. Assume that one of the following conditions is satisfied:*

(1) $u, v \in \mathsf{harm}(\Omega)$ *with* $v > 0$ *and* $\phi : \mathbb{R} \to \mathbb{R}$ *is a convex function;*
(2) $u \in \mathsf{subh}(\Omega)$, $v \in \mathsf{harm}(\Omega)$ *with* $v > 0$ *and* $\phi : \mathbb{R} \to \mathbb{R}$ *is an increasing convex function;*
(3) $u, -v \in \mathsf{subh}(\Omega)$, *with* $u \geq 0$, $v > 0$ *and* $\phi : \mathbb{R} \to \mathbb{R}^+$ *is a positive convex function with* $\phi(0) = 0$*;*

then $v\phi(u/v) \in \mathsf{subh}(\Omega)$.

PROOF. We only give a proof of (2), the others being similar. First, being ϕ increasing and convex, for $x \in \mathbb{R}$, the tangent line to $\phi(x)$ is given by $y = ax + b$ with $a \geq 0$ and $b \in \mathbb{R}$. Let

$$\mathcal{F}_{a,b}(x) := \{ax + b : a \geq 0, b \in \mathbb{R}, at + b \leq \phi(t) \ \forall t \in \mathbb{R}\}.$$

By convexity $\phi(x) = \sup_{a \geq 0, b} \mathcal{F}_{a,b}(x)$. Now, $v(a\frac{u}{v} + b) = au + bv \in \mathsf{subh}(\Omega)$ for $a \geq 0$ and $b \in \mathbb{R}$. Thus the upper semicontinuous regularization of $v\phi(u/v)(x) = \sup_{a \geq 0, b} \mathcal{F}_{a,b}(au(x) + bv(x))$ is subharmonic by Proposition 5.3. To have the result we only need to show that $v\phi(u/v)$ is upper semicontinuous. But ϕ is increasing and u/v is upper semicontinuous, thus

$$\limsup_{x \to x_0} \phi(u(x)/v(x)) \leq \phi(u(x_0)/v(x_0)).$$

□

COROLLARY 5.7. *Let $\Omega \subset \mathbb{R}^m$ be an open set. If $u \in \mathsf{subh}(\Omega)$ then $e^u \in \mathsf{subh}(\Omega)$.*

PROOF. By Theorem 5.6 with $\phi(x) = e^x$ and $v \equiv 1$. □

6. Boundary behavior: the Hopf lemma

THEOREM 6.1 (Hopf's lemma). *Let $\Omega \subset \mathbb{R}^m$ be a domain. Let $p \in \partial\Omega$ and suppose that $\partial\Omega$ has the inner ball property at p (for instance, if $\partial\Omega$ is C^2 at p). Let U be an open neighborhood* *of p and let $u \in \mathsf{subh}(\Omega \cap U)$ be such that $\lim_{\Omega \ni x \to p} u(x) = u(p)$ and $u < u(p)$ in $U \cap \Omega$. Let $\nu \in \mathbb{R}^m$ be a non-zero vector which does not belong to $T_p\partial\Omega$ and pointing outward. Then*

$$\limsup_{h \to 0} \frac{u(p - h\nu) - u(p)}{h} < 0.$$

PROOF. Since Ω has the inner ball property at p, there exists a ball $\mathbb{B} \subset \Omega \cap U$ such that $\partial\mathbb{B}$ is tangent to $\partial\Omega$ at p. We can assume with no loss of generality that $\mathbb{B} = B(O, 1)$ and that $u(p) = 0$ and $u < 0$ in $\overline{\mathbb{B}} \setminus \{p\}$. Let $v(x) := e^{-\alpha\|x\|^2} - e^{-\alpha}$ for $\alpha > 2m$. A direct computation shows that $\Delta v = e^{-\alpha\|x\|^2}(4\|x\|^2\alpha^2 - 2m\alpha)$ and $v = 0$ on $\partial\mathbb{B}$. Thus, since $\alpha > 2m$, v is subharmonic in $\{z : \|z\| > 1/2\}$. Moreover,

$$\frac{\partial v}{\partial \nu}|_{x=p} = \mathsf{grad}[v(p)] \cdot \nu = -2\alpha e^{-\alpha} p \cdot \nu < 0,$$

since ν points outward, that is $p \cdot \nu > 0$. Fix $\epsilon > 0$. Let M be such that $u \leq -M$ on $\|x\| = 1/2$. Now $v > 0$ in $\mathbb{B}$, but, since $e^{-\alpha/4} - e^{-\alpha} \to 0$ as $\alpha \to \infty$, there exists $\alpha >> 1$ such that $v(x) < M/2\epsilon$ for $\|x\| = 1/2$. Let $V = \mathbb{B} \setminus \{x \in \mathbb{R}^m : \|x\|^2 \leq 1/2\}$. Then $u + \epsilon v \in \mathsf{subh}(V)$. Moreover, by construction $u + \epsilon v \leq 0$ on ∂V. By the maximum principle, $u + \epsilon v \leq 0$ in V and $p \in \partial\mathbb{B}$ is a maximum since $u(p) + \epsilon v(p) = 0$. Let $t_0 > 0$ be such that $p - t\nu \in V$ for $0 < t \leq t_0$. Then

$$\limsup_{t \to 0^+} \frac{(u + \epsilon v)(p - t\nu)}{t} \leq 0.$$

Now let $\{t_k\}$, $t_k > 0$, be any sequence converging to 0. Then for all k

$$0 \geq \frac{(u + \epsilon v)(p - t_k\nu)}{t_k} = \frac{u(p - t_k\nu)}{t_k} + \epsilon\frac{v(p - t_k\nu)}{t_k}$$

and since $\lim_{k \to \infty 0} \frac{v(p - t_k\nu)}{t_k} = -\frac{\partial v}{\partial \nu}(p) > 0$ we obtain

$$\limsup_{k \to \infty} \frac{u(p - t_k\nu)}{t_k} \leq \epsilon\frac{\partial v}{\partial \nu}(p) < 0,$$

proving the statement. □

Hopf's lemma can be used in analysis to prove uniqueness for the solution of von Neumann-type problems. As a matter of example, we give the following:

PROPOSITION 6.2. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain with C^2 boundary. If $u \in \mathsf{harm}(\Omega) \cap C^2(\overline{\Omega})$ is such that $\frac{\partial u}{\partial \mathsf{n}_p}(p) = 0$ for all $p \in \partial\Omega$ (here n_p is the outer normal vector) then $u =$* const.

PROOF. Assume that u is not constant. By the maximum principle it has a strict maximum at some $p \in \partial\Omega$. Thus Hopf's lemma implies that $\frac{\partial u}{\partial \mathsf{n}_p}(p) > 0$, against the hypothesis. □

CHAPTER 2

Pluriharmonic functions

1. Interlude on holomorphic functions

Consider the complex space $\mathbb{C}^n$ as a $2n$-dimensional real space $\mathbb{R}^{2n}$. The multiplication by i in $\mathbb{C}^n$ determines a complex structure J on $\mathbb{R}^{2n}$, called the standard complex structure of $\mathbb{R}^{2n}$. More explicitly, if $v \in \mathbb{C}^n$ and we denote by $v^{\mathbb{R}}$ its image in $\mathbb{R}^{2n}$, then $J(v^{\mathbb{R}}) := (iv)^{\mathbb{R}}$. Let $\Omega \subset \mathbb{C}^n$ be an open set. Then $T\Omega = \Omega \times \mathbb{C}^n$ and one can consider the real structure $\Omega \times \mathbb{R}^{2n}$ with the standard complex structure on each fiber which, being independent of $z \in \Omega$, we still denote by J. If we consider $(T\Omega)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \approx \Omega \times \mathbb{C}^{2n}$ the operator J determines an operator $J^{\mathbb{C}}$ on $(T\Omega)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which has the property that $(J^{\mathbb{C}})^2 = -I$ and thus one has the decomposition $(T\Omega)^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}\Omega \oplus T^{0,1}\Omega$ in terms of the eigenspaces of $J^{\mathbb{C}}$. Namely, $J^{\mathbb{C}}X = iX$ for all $X \in T^{1,0}\Omega$ and $J^{\mathbb{C}}X = -iX$ for all $X \in T^{0,1}\Omega$. Accordingly, one can decompose the cotangent space $(T^*\Omega)^{\mathbb{R}} \otimes \mathbb{C} = (T^*\Omega)^{1,0} \oplus (T^*\Omega)^{0,1}$. In general, given an $\mathbb{R}$-linear map $L : (T\Omega)^{\mathbb{R}} \to T\mathbb{C}$, then L is $\mathbb{C}$-linear (namely there exists a $\mathbb{C}$-linear map $l : T\Omega \to T\mathbb{C}$ such that $l^{\mathbb{R}} = L$) if and only if $L \circ J = iL$, that is, L commutes with the complex structures on Ω and $\mathbb{C}$ respectively, while L is $\mathbb{C}$-antilinear if $L \circ J = -iL$. Notice that L is $\mathbb{C}$-linear (*respectively* $\mathbb{C}$-antilinear) if and only if $L^{\mathbb{C}} \in (T^{1,0}\Omega)^*$ (*respectively* $L^{\mathbb{C}} \in (T^{0,1}\Omega)^*$).

Now, let $u \in C^1(\Omega, \mathbb{R})$. Considering $\mathbb{R} \subset \mathbb{C}$ we can think of $u : \Omega \to \mathbb{C}$ as a function such that $u = \overline{u}$. Thus $du : (T\Omega)^{\mathbb{R}} = \Omega \times \mathbb{R}^{2n} \longrightarrow T\mathbb{R} \subset T\mathbb{C} = \mathbb{C} \times \mathbb{C}$ is an $\mathbb{R}$-linear morphism.

LEMMA 1.1. *Let $\Omega \subset \mathbb{R}^{2m}$ be an open set and let $u \in C^2(\Omega, \mathbb{R})$.*

(1) *The $\mathbb{C}$-linear part of du is given by $\partial u := \frac{1}{2}(du - idu \circ J)$.*
(2) *The $\mathbb{C}$-antilinear part of du is given by $\overline{\partial} u := \frac{1}{2}(du + idu \circ J)$.*

The decomposition $du = \partial u + \overline{\partial} u$ is the unique decomposition in $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts.

PROOF. Clearly $du = \partial u + \overline{\partial} u$. Since

$$(du - idu \circ J) \circ J = (du \circ J + idu) = i(du - idu \circ J)$$

then ∂u is $\mathbb{C}$-linear. Similarly one can prove that $\overline{\partial} u$ is $\mathbb{C}$-antilinear. Finally, if $du = A + B$ is another decomposition in $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts, then $\partial u - A = \overline{\partial} u - B$ and thus

$$i(\partial u - A) = (\partial u - A) \circ J = (\overline{\partial} u - B) \circ J = -i(\overline{\partial} u - B) = -i(\partial u - A)$$

forcing $\partial u = A$ and $\overline{\partial} u = B$. □

In local coordinates $z_j = x_j + iy_j$ in $\mathbb{C}^n$, we define $dz_j = dx_j + idy_j$ and $d\overline{z}_j = dx_j - idy_j$. Also, we let $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ and $\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. A direct computation shows that

$$\partial u = \sum_{j=1}^{n} \frac{\partial u}{\partial z_j} dz_j, \quad \overline{\partial} u = \sum_{j=1}^{n} \frac{\partial u}{\partial \overline{z}_j} d\overline{z}_j.$$

Let us define

$$d^c := i(\overline{\partial} - \partial).$$

LEMMA 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in C^1(\Omega, \mathbb{R})$. Then $d^c u = -du \circ J$.*

PROOF. We have

$$d^c u = i\overline{\partial} u - i\partial u = -\overline{\partial} u \circ J - \partial u \circ J = -du \circ J.$$

□

The classical Cauchy-Riemann equations can be read in terms of d, d^c as follows:

THEOREM 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $f = u+iv \in C^1(\Omega, \mathbb{C})$ is holomorphic in Ω if and only if $d^c u = dv$ in Ω.*

PROOF. Let fix coordinates $\{x_j, y_j\}$ in Ω. The function f is holomorphic in Ω if and only if the Cauchy-Riemann equations

$$\begin{cases} \frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j} \\ \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j} \end{cases}$$

are satisfied. Now, writing $\frac{\partial u}{\partial x} := (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$ (and similarly for $\frac{\partial u}{\partial y}$) we have

$$du \circ J = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \cdot \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = (\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}).$$

Thus Cauchy-Riemann equations become $-du \circ J = dv$. By Lemma 1.2 Cauchy-Riemann equations are then equivalent to $d^c u = dv$, proving the statement. □

2. Pluriharmonic functions

DEFINITION 2.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u \in C^2(\Omega, \mathbb{R})$ is *pluriharmonic*, $u \in \mathsf{Ph}(\Omega)$, if for all $p \in \Omega$ and $v \in \mathbb{C}^n$ the function $\mathbb{C} \ni \zeta \mapsto u(p + \zeta v)$ is harmonic for $|\zeta| << 1$.

For a C^2-real function u we define the complex Hessian (or the Levi form) as the following $(0,2)$-tensor:

$$\mathcal{L}(u) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} dz_j \otimes d\overline{z}_k \tag{2.1}$$

Notice that, since $u = \overline{u}$ then the matrix $(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k})$ is Hermitian.

We have the following characterization of pluriharmonic functions in terms of Levi form:

PROPOSITION 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u \in \mathbb{C}^2(\Omega, \mathbb{R})$ is pluriharmonic if and only if $\mathcal{L}(u) \equiv 0$. Namely, $u \in \mathsf{Ph}(\Omega)$ if and only if $\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(x) = 0$ for all $j, k = 1, \ldots, n$ and $x \in \Omega$.*

PROOF. Recall that in $\mathbb{C}$ with ζ-coordinate, $\Delta = 4\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \overline{\zeta}}$. Therefore

$$\frac{1}{4}\Delta u(p + \zeta v)|_{\zeta=0} = \frac{\partial^2 u(p+\zeta v)}{\partial \zeta \partial \overline{\zeta}}|_{\zeta=0} = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(p) v_j \overline{v}_k = \mathcal{L}(u)(v; v). \tag{2.2}$$

Thus $\Delta u(p + \zeta v)|_{\zeta=0} = 0$ for all $p \in \Omega$ and $v \in \mathbb{C}^n$ if and only if $\mathcal{L}(u) = 0$. □

COROLLARY 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then $\mathsf{Ph}(\Omega) \subset \mathsf{harm}(\Omega)$. If $n > 1$ the inclusion is proper.*

PROOF. By Proposition 2.2 it follows that if $u \in \mathsf{Ph}(\Omega)$ then $\mathcal{L}(u) = 0$ which implies $\Delta u = 0$ and then $u \in \mathsf{harm}(\Omega)$. For $n > 1$ let $z_0 \notin \Omega$ and consider the function $u(z) = \|z - z_0\|^{-2(n-1)}$. Then $\Delta u = 0$ but $\zeta \mapsto u(p + \zeta e_1)$ is not harmonic. □

COROLLARY 2.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\Omega' \subset \mathbb{C}^m$ be another domain and let $f : \Omega' \to \Omega$ be a holomorphic map. If $u \in \mathsf{Ph}(\Omega)$ then $u \circ f \in \mathsf{Ph}(\Omega')$.*

PROOF. In view of Theorem 2.2 it is enough to prove that $\mathcal{L}(u \circ f) = 0$. By the chain rule, since $df = \partial f$

$$\mathcal{L}(u \circ f) = \mathcal{L}(u) \circ (df \otimes df) = 0,$$

because $\mathcal{L}(u) = 0$. □

Summarizing these last two corollary, we can say that a harmonic function is pluriharmonic if and only if it is harmonic under holomorphic changes of coordinates.

THEOREM 2.5. *Let $\Omega \subset \mathbb{C}^n$ be a domain.*

(1) *If f is holomorphic in Ω then $\mathsf{Re}\, f, \mathsf{Im}\, f \in \mathsf{Ph}(\Omega)$.*
(2) *Suppose $H^1(\Omega, \mathbb{R}) = 0$. If $u \in \mathsf{Ph}(\Omega)$ then there exists $v \in \mathsf{Ph}(\Omega)$ such that $u + iv$ is holomorphic in Ω.*

PROOF. (1) If $f : \Omega \to \mathbb{C}$ is holomorphic then for all $p \in \Omega$ the function $\mathbb{C} \ni \zeta \mapsto f(p+\zeta v)$ is holomorphic for $|\zeta| << 1$. Thus its real and imaginary parts are harmonic and then $\mathsf{Re}\, f, \mathsf{Im}\, f$ are pluriharmonic.
(2) Let $\omega := d^c u$. Then $d\omega = 0$ and since $H^1(\Omega, \mathbb{R}) = 0$, Poincaré lemma implies that ω is exact. Thus, there exists $v \in C^1(\Omega)$ such that $dv = \omega$. Hence $d^c u = dv$ and the function $u + iv$ is holomorphic in view of Theorem 1.3. □

REMARK 2.6. The previous theorem says that locally every pluriharmonic function is the real part of a holomorphic function.

CHAPTER 3

Plurisubharmonic functions

1. Definition and first properties

DEFINITION 1.1. Let $\Omega \subset \mathbb{C}^n$ be a connected domain. A function $u : \Omega \to [-\infty, \infty)$ is *plurisubharmonic*, $u \in \mathsf{Psh}(\Omega)$, if

(1) $u \not\equiv -\infty$.
(2) u is upper semicontinuous.
(3) For all $p \in \Omega$ and $v \in \mathbb{C}^n$ the function $\mathbb{C} \ni \zeta \mapsto u(p + \zeta v)$ is either subharmonic or $\equiv -\infty$ for $|\zeta| << 1$.

PROPOSITION 1.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in C^2(\Omega)$. Then $u \in \mathsf{Psh}(\Omega)$ if and only if for all $v \in \mathbb{C}^n$ it follows*

$$\mathcal{L}(u)(v; v) \geq 0. \tag{1.1}$$

PROOF. By Theorem 4.1, $u \in \mathsf{Psh}(\Omega)$ if and only if $\Delta u(p + \zeta v)|_{\zeta=0} \geq 0$ for all $v \in \mathbb{C}^n$. By (II.2.2) it follows that $\Delta u(p + \zeta v)|_{\zeta=0} \geq 0$ if and only if $\mathcal{L}(u)(v; v) \geq 0$. □

COROLLARY 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then*

(1) $\mathsf{Ph}(\Omega) \subset \mathsf{Psh}(\Omega)$.
(2) $\mathsf{Psh}(\Omega) \cap C^2(\Omega) \subset \mathsf{subh}(\Omega)$.

PROOF. (1) By Proposition II.2.2, if $u \in \mathsf{Ph}(\Omega)$ it follows that $\mathcal{L}(u) = 0$, thus by Proposition 1.2 we have $u \in \mathsf{Psh}(\Omega)$.

(2) Let $u \in C^2(\Omega)$. If $u \in \mathsf{Psh}(\Omega)$ then by Proposition 1.2 it follows that $\mathcal{L}(u)(v; v) \geq 0$ for all $v \in \mathbb{C}^n$, namely the matrix $(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k})$ is positive semi-definite. In particular its trace is ≥ 0. Since a direct computation shows that

$$\Delta u = 4\mathsf{tr}(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}) \geq 0$$

it follows that $u \in \mathsf{subh}(\Omega)$ in view of Proposition I.4.1. □

DEFINITION 1.4. Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in C^2(\Omega)$. We say that u is *strictly plurisubharmonic* if for all $v \in \mathbb{C}^n \setminus \{O\}$ it follows $\mathcal{L}(u)(v; v) > 0$.

REMARK 1.5. One could define strictly plurisubharmonic functions without the requirement of C^2-regularity. Namely, one can say that a function $u \in \mathsf{Psh}(\Omega)$ is *strictly plurisubharmonic in the weak sense* if for each $p \in \Omega$ there exists $c > 0$ such that $z \mapsto u(z) - c\|z\|^2$

is plurisubharmonic near p. Note that $u \in C^2(\Omega)$ is strictly plurisubharmonic in the weak sense if and only if it is strictly plurisubharmonic in the sense of Definition 1.4. This follows easily from the fact that $(z, v) \mapsto \mathcal{L}(u)(v; v)$ is continuous and thus it has a minimum for $(z, v) \in \overline{\mathbb{B}(p, r)} \times \partial\mathbb{B}(O, 1)$, where $\mathbb{B}(p, r) \subset\subset \Omega$ is any open ball.

LEMMA 1.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $u \in \mathsf{Psh}(\Omega)$ then for all $a \in \Omega$ and $b \in \mathbb{C}^n$ such that $\{a + \zeta b : |\zeta| \leq 1\} \subset \Omega$ it holds*

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta} b) d\theta. \tag{1.2}$$

Conversely, if $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous, $u \not\equiv -\infty$ and (1.2) *holds, then $u \in \mathsf{Psh}(\Omega)$.*

In particular a plurisubharmonic functions has the sub-mean property with respect to polydiscs.

PROOF. If $u \in \mathsf{Psh}(\Omega)$ then $\zeta \mapsto u(a + \zeta b)$ is subharmonic and then (1.2) follows from Theorem 2.2. Conversely, again by Theorem 2.2, if (1.2) holds then $\zeta \mapsto u(a + \zeta b)$ is either $\equiv -\infty$ or it is subharmonic and thus $u \in \mathsf{Psh}(\Omega)$.

Finally, assume $u \in \mathsf{Psh}(\Omega)$. Let $P(a, r) \subset\subset \Omega$ be a polydisc with multiradius $r = (r_1, \ldots, r_n)$ be relatively compact in Ω. Let $\rho_j \in (0, r_j)$ for $j = 1, \ldots, n$. By (1.2) we have

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} & u(a_1 + \rho_1 e^{i\theta_1}, \ldots, a_n + \rho_n e^{i\theta_n})) d\theta_1 \ldots d\theta_n \\ & \geq \frac{1}{(2\pi)^{n-1}} \int_0^{2\pi} \cdots \int_0^{2\pi} u(a_1, a_2 + \rho_2 e^{i\theta_2}, \ldots, a_n + \rho_n e^{i\theta_n})) d\theta_2 \ldots d\theta_n \\ & \geq u(a). \end{aligned}$$

Now we multiply both sides of the previous inequality by $\rho_1 \cdots \rho_n$ and integrate for $\rho_j \in (0, r_j)$ obtaining

$$\frac{1}{\mathsf{v}(P(a, r)} \int_{P(a,r)} u(z) d\lambda(z) \geq u(a).$$

Thus u has the sub-mean property with respect to polydiscs. □

REMARK 1.7. Again, it should be remarked that being plurisubharmonic is a local property (which follows directly from the fact that being subharmonic is a local property).

PROPOSITION 1.8. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then $\mathsf{Psh}(\Omega) \subset \mathsf{subh}(\Omega)$, and for $n > 1$ the inclusion is proper.*

PROOF. Let $B(a, r) \subset\subset \Omega$ be an open ball. Then (1.2) holds for all $\|b\| = r$. Consider $\pi : \partial\mathbb{B}(a, r) \to \mathbb{CP}^{n-1}$ the Hopf fibration with fiber S^1 given by $\pi(z) = [z]$. For any real $2n - 1$ form ω on $\partial B(a, r)$ with upper semicontinuous coefficients it is possible to define a real $2n - 2$ form $\pi_*(\omega)$ on $\mathbb{CP}^{n-1}$ obtained by integration along the fibers (see [**4**, p.61-63] for the continuous case, the semi-continuous is analogous). In our case we set $\omega = u d\sigma$ and then in local coordinates $(\theta, b) \in (0, 2\pi) \times \mathbb{R}^{2n-2}$ which trivialize the Hopf fibration and for which $d\sigma = d\theta \wedge d\sigma'(b)$ with $d\sigma'(b)$ a $2n-2$ form, it follows that $\pi_*(ud\sigma) := \int_0^{2\pi} u(a+e^{i\theta}b)d\theta\wedge d\sigma'(b)$. Thus, by the projection formula [**4**, Proposition 6.15 p. 63] and by (1.2) it follows that

$$\int_{\partial B(a,r)} ud\sigma = \int_{\mathbb{CP}^{n-1}} \pi_*(ud\sigma) \geq u(a)2\pi \int_{\mathbb{CP}^{n-1}} d\sigma'.$$

Repeating the argument with $u \equiv 1$ it follows that

$$\mu(\partial B(a,r)) = \int_{\partial B(a,r)} d\sigma = \int_{\mathbb{CP}^{n-1}} \pi_*(d\sigma) = 2\pi \int_{\mathbb{CP}^{n-1}} d\sigma'.$$

Putting these two inequalities together we obtain

$$\mu(\partial\mathbb{B}(a,r))u(a) \leq \int_{\partial\mathbb{B}(a,r)} u(\zeta)d\sigma(\zeta),$$

by Theorem 2.2 it follows that $u \in \mathsf{subh}(\Omega)$.

To see that for $n > 1$ the inclusion is proper, we exhibit an example. Let $u(x_1, x_2) = 4(x_1^2 - x_2^2)$. Then $\Delta u = 0$ and hence $u \in \mathsf{harm}(\mathbb{C}^2) \subset \mathsf{subh}(\mathbb{C}^2)$. Now, $u(x_1, x_2) = z_1^2 - z_2^2 + \overline{z}_1^2 - \overline{z}_2^2 - 2z_1\overline{z}_1 + 2z_2\overline{z}_2$ and a direct computation shows that $\mathcal{L}(u)(v; v) = -2|v_1|^2 + 2|v_2|^2$ proving that $u \notin \mathsf{Psh}(\Omega)$. □

REMARK 1.9. Proposition 1.8 can be proved straightly using the regularization sequence to be constructed in Theorem 2.1 and the fact that smooth plurisubharmonic functions are subharmonic in view of Corollary 1.3. Of course, proceeding this way, the proof of Theorem 2.1 is more complicated (for this way of arguing see[**16**]).

In view of Proposition 1.8, plurisubharmonic functions enjoy all properties of subharmonic functions such as being L^1_{loc}, the maximum principle and Hopf's lemma.

LEMMA 1.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\{\Omega_k\} \subset \mathbb{C}^n$ be a sequence of connected domains such that $\Omega_k \subseteq \Omega_{k+1}$ and $\bigcup_k \Omega_k = \Omega$. For each k, let $u_k \in \mathsf{Psh}(\Omega_k)$ be such that $u_k(x) \geq u_{k+1}(x)$ for all $x \in \Omega_k$ and for all k (that is, $\{u_k\}$ is a decreasing sequence). Let $u(x) = \lim_{j\to\infty} u_j(x)$. Then either $u \equiv -\infty$ or $u \in \mathsf{Psh}(\Omega)$.*

PROOF. Assume that $u \not\equiv -\infty$. According to Corollary I.2.4, the limit $u \in \mathsf{subh}(\Omega)$. Then the result follows by Lemma 1.6, since for all (suitably chosen) a, b and $j >> 1$

$$u(a) \leq u_j(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u_j(a + e^{i\theta}b)d\theta$$

and the latter integral converges to $\frac{1}{2\pi}\int_0^{2\pi} u(a+e^{i\theta}b)d\theta$ by Beppo Levi's monotone convergence theorem. □

COROLLARY 1.11. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then $u \in \mathsf{Ph}(\Omega)$ if and only if $u, -u \in \mathsf{Psh}(\Omega)$.*

PROOF. If $u \in \mathsf{Ph}(\Omega)$ then by Proposition II.2.2 then $\mathcal{L}(u) = 0$ and then $\mathcal{L}(u), \mathcal{L}(-u) = 0$ which implies $u, -u \in \mathsf{Psh}(\Omega)$ by Proposition 1.2. Conversely, let $u, -u \in \mathsf{Psh}(\Omega)$. By Proposition 1.8 and Proposition I.2.8 it follows that $u \in \mathsf{harm}(\Omega)$. In particular $u \in C^\infty(\Omega)$. By Proposition 1.2 and since both u and $-u$ are plurisubharmonic, then $\mathcal{L}(u)(v; v) \geq 0$ and $\mathcal{L}(-u)(v; v) \geq 0$ implying that $\mathcal{L}(u) = 0$ and, by Proposition II.2.2, $u \in \mathsf{Ph}(\Omega)$. □

2. Regularization of plurisubharmonic functions

THEOREM 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega)$. Let $u_\epsilon := \chi_\epsilon * u$. Then $u_\epsilon \in \mathsf{Psh}(\Omega_\epsilon) \cap C^\infty(\Omega_\epsilon)$. Moreover $\{u_\epsilon\}$ is decreasing in ϵ and converges pointwise to u.*

PROOF. Since plurisubharmonic functions are subharmonic by Proposition 1.8 and in view of Theorem I.3.1 we have only to prove that $u_\epsilon \in \mathsf{Psh}(\Omega)$. By Fubini's theorem, if $\{a + \zeta b : |\zeta| \leq 1\} \subset \Omega_\epsilon$, we have

$$
\begin{aligned}
u_\epsilon(a) = \int_{\mathbb{C}^n} u(a-y)\chi_\epsilon(y)d\lambda(y) &\leq \int_{\mathbb{C}^n} \frac{1}{2\pi}\int_0^{2\pi} u(a+e^{i\theta}b-y)\chi_\epsilon(y)d\theta d\lambda(y) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}^n} u(a+e^{i\theta}b-y)\chi_\epsilon(y)d\lambda(y)d\theta = \frac{1}{2\pi}\int_0^{2\pi} u_\epsilon(a+e^{i\theta}b)d\theta,
\end{aligned}
$$

and thus by Lemma 1.6, $u_\epsilon \in \mathsf{Psh}(\Omega_\epsilon)$. □

COROLLARY 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega)$. Then there exists $\{v_\epsilon\} \subset C^\infty(\Omega_\epsilon)$ strictly plurisubharmonic in Ω_ϵ such that $\{v_\epsilon\}$ is decreasing in ϵ and converges pointwise to u.*

PROOF. Let $\{u_\epsilon\}$ be given by Theorem 2.1. Let $v_\epsilon(z) := u_\epsilon(z) + \epsilon\|z\|^2$. Then for $\mathbb{C}^n \ni b \neq O$,

$$\mathcal{L}(v_\epsilon)(b; b) = \mathcal{L}(u_\epsilon)(b; b) + \epsilon\mathcal{L}(\|z\|^2)(b; b) > 0$$

hence v_ϵ is strongly plurisubharmonic in Ω_ϵ and the remaining properties follow from the properties of the u_ϵ's. □

As a consequence, arguing as in Theorem I.4.1, one can prove the following

PROPOSITION 2.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $u \in \mathsf{Psh}(\Omega)$ then $\mathcal{L}(u)(v; v) \geq 0$ in the sense of distribution for all $v \in \mathbb{C}^n$. Conversely, if $u \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathcal{L}(u)(v; v) \geq 0$ in the sense of distribution for all $v \in \mathbb{C}^n$, then there exists $v \in \mathsf{Psh}(\Omega)$ such that $v = u$ almost everywhere.*

Also, the arguments in the proofs of results in section 5 of Chapter I work with only minor changes for plurisubharmonic functions allowing to construct new plurisubharmonic functions starting from given plurisubharmonic functions. We leave details to the reader.

We end up this section with an application of Hopf's lemma for plurisubharmonic functions to analytic discs attached to pseudoconvex domains:

PROPOSITION 2.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Assume there exists a neighborhood U of Ω and $\rho \in \mathsf{Psh}(U) \cap C^1(U)$ such that $\Omega = \{z \in U : \rho(z) < 0\}$ and $d\rho_x \neq 0$ for all $x \in \partial\Omega$. Let $\varphi : \mathbb{D} \to \Omega$ be a holomorphic disc such that $\varphi \in C^1(\overline{\mathbb{D}})$ and $\varphi(\partial\mathbb{D}) \subset \partial\Omega$. Then $\varphi(\partial\mathbb{D})$ is transverse to $\partial\Omega$ at every point.*

PROOF. By hypothesis $\rho(\varphi(\zeta)) < 0$ for all $\zeta \in \mathbb{D}$ and $\rho(\varphi(\zeta)) = 0$ for all $\zeta \in \partial\mathbb{D}$. Then Hopf's Lemma implies that for all $\zeta \in \partial\mathbb{D}$

$$d\rho_{\varphi(\zeta)}(\varphi'(\zeta)) = \lim_{\mathbb{R}\ni t \to 1^-} \frac{-\rho(\varphi(t\zeta))}{1-t} \neq 0.$$

Therefore $\varphi'(\zeta) \notin T_{\varphi(\zeta)}\partial D$ and hence $\varphi(\partial\mathbb{D})$ is transverse to $\partial\Omega$ at every point. □

3. Plurisubharmonic and subharmonic functions under changes of coordinates

We begin with the following example:

EXAMPLE 3.1. Let $(x, y) \in \mathbb{R}^2$ and let $u(x, y) = x^2 - y^2$. Then $\Delta u(x, y) = 0$ and $u \in \mathsf{harm}(\mathbb{R}^2)$. Consider the following linear change of coordinates: $x = X, y = X - Y$. Then $u(X, Y) = -Y^2 + 2XY$ and thus $\Delta u(X, Y) = -2$ which implies that $u(X, Y)$ is not subharmonic in the new coordinates.

Very roughly, the reason why subharmonic functions do not behave well under changes of coordinates is that in general a change of coordinates is not conformal, thus it does not preserve balls and spheres and the sub-mean property is no longer true.

Contrarily, not degenerate holomorphic mappings are conformal in $\mathbb{C}$ and thus one might expect some better behavior for plurisubharmonic functions. Indeed we have

PROPOSITION 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u : \Omega \to [-\infty, \infty)$. Then $u \in \mathsf{Psh}(\Omega)$ if and only if for all $f : \Omega' \to \Omega$ holomorphic (with Ω' a domain of $\mathbb{C}^m$) it follows that either $u \circ f \in \mathsf{subh}(\Omega')$ or $u \circ f \equiv -\infty$.*

PROOF. First assume that $u \in \mathsf{Psh}(\Omega) \cap C^2(\Omega)$ and let $f : \Omega' \to \Omega$ be holomorphic. Then for all $v \in \mathbb{C}^m$

$$\mathcal{L}(u \circ f)(v; v) = \mathcal{L}(u)(df(v); df(v)) \geq 0,$$

proving that $u \circ f \in \mathsf{Psh}(\Omega')$. If $u \in \mathsf{Psh}(\Omega)$ (no regularity assumptions) let $\{u_\epsilon\}$ be the sequence given by Theorem 2.1. Then $u_\epsilon \circ f \in \mathsf{Psh}(f^{-1}(\Omega_\epsilon))$ and since the sequence $\{u_\epsilon \circ f\}$ is decreasing in ϵ, it follows that the limit (which is $u \circ f$) is either $\equiv -\infty$ or plurisubharmonic (and thus subharmonic) in Ω by Lemma 1.10.

Conversely, if $u \circ f$ is subharmonic or $\equiv -\infty$ for all holomorphic mappings $f : \Omega' \to \Omega$ then it is so also for holomorphic map $\mathbb{C} \ni \zeta \mapsto a + \zeta b$ (for $a \in \Omega$, $b \in \mathbb{C}^n$ and $|\zeta| << 1$) and this is exactly the definition of plurisubharmonic function. □

REMARK 3.3. With some more effort it can be proven that $u \in \mathsf{Psh}(\Omega)$ if and only if for all $\mathbb{C}$-linear isomorphism $T : \mathbb{C}^n \to \mathbb{C}^n$ it follows that $u \circ T \in \mathsf{subh}(T^{-1}(\Omega))$ (see, *e.g.*, [**16**, Theorem 2.9.12 p. 68].

As an application we have the following result:

THEOREM 3.4. *Let* $u \in \mathsf{subh}(\mathbb{R}^2)$. *If there exists* $M < \infty$ *such that* $u(x) \leq M$ *for all* $x \in \mathbb{R}^2$ *then* u *is constant.*

PROOF. We consider $u : \mathbb{C} \to [-\infty, \infty)$ with complex variable z. If u is not constant we can assume that $u(0) < u(1)$. Let $v(z) := u(1/z)$. By Proposition 3.2 the function $v \in \mathsf{subh}(\mathbb{C} \setminus \{0\})$ and $v(z) \leq M$ for all $z \in \mathbb{C} \setminus \{0\}$. By construction

$$\limsup_{|z|\to\infty} v(z) = \limsup_{|z|\to\infty} u(1/z) = \limsup_{|w|\to 0} u(w) \leq u(0) < u(1) = v(1).$$

Thus there exists $R > 0$ such that

$$\sup_{z\in\mathbb{C}\setminus\{0\}} v(z) = \sup_{z\in B(0,R)\setminus\{0\}} v(z). \tag{3.1}$$

Let define $v(0) := \limsup_{z\to 0} v(z)$. Since v is bounded, then Corollary I.5.5 implies that $v \in \mathsf{subh}(\mathbb{R}^2)$. By (3.1) it follows that there exists $z \in \overline{B(0,R)}$ such that $v(z) = \sup_{w\in\mathbb{R}^2} v(w)$, but this contradicts the maximum principle in Corollary I.2.6. □

As a corollary:

COROLLARY 3.5. *Let* $u \in \mathsf{Psh}(\mathbb{C}^n)$. *If there exists* $M < \infty$ *such that* $u(z) \leq M$ *for all* $z \in \mathbb{C}^n$ *then* u *is constant.*

PROOF. Apply Theorem 3.4 to all complex lines passing through O. □

Notice that the previous result would be false for subharmonic functions (which are not plurisubharmonic):

EXAMPLE 3.6. In $\mathbb{R}^3$ let $u(x) = -1/\|x\|$. Then $\Delta u(x) = 0$ for $x \neq O$ and $u(0) = -\infty$, therefore $u \in \mathsf{subh}(\mathbb{R}^3)$. However $u(x) \leq 0$ for all $x \in \mathbb{R}^3$ and it is not constant.

REMARK 3.7. Proposition 3.2 allows to define the sheaf of plurisubharmonic functions on complex manifolds. In other words, if M is a complex manifold and $U \subset M$ is an open set, then $u : U \to [-\infty, \infty)$ is plurisubharmonic if for all $x \in U$ there exists a local chart (V, φ) such that $x \in V$ and $u \circ \varphi^{-1} \in \mathsf{Psh}(\varphi(V))$. Proposition 3.2 guarantees that such a definition does not depend on the (holomorphic) local chart chosen to define it.

The maximum principle (as well as the previous results on plurisubharmonic functions) extends easily to plurisubharmonic functions on complex manifolds. For instance, this implies that $\mathsf{Psh}(\mathbb{CP}^1) = \mathbb{R}$. With this, we have a simple alternative proof of Theorem 3.4 as follows: if $u \in \mathsf{subh}(\mathbb{C})$ is bounded from above, then its extension to $\mathbb{CP}^1$ given by defining $u(\infty) = \limsup_{|z|\to\infty} u(z)$ is subharmonic on $\mathbb{CP}^1$ (by the analogous of Corollary I.5.5 for complex manifolds) thus it is constant.

CHAPTER 4

Currents

1. Distributions

Let $\Omega \subset \mathbb{R}^m$ be a domain. We write $f \in C_0^k(\Omega)$ if $f : \Omega \to \mathbb{C}$ is such that $f \in C^k(\Omega)$ and $\mathsf{supp}(f) \subset\subset \Omega$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we denote by $|\alpha| = \sum_{j=1}^m \alpha_j$ and by $D^\alpha f = \frac{\partial f^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$. We let $C_0(\Omega) := C_0^0(\Omega)$.

DEFINITION 1.1. Fix $p \leq k$. Given $f \in C_0^k(\Omega)$ and K a compact subset of Ω such that $\mathsf{supp}(f) \subseteq K$ and $\epsilon > 0$, the sets

$$V_p(f, \epsilon, K) = \{g \in C_0^k(\Omega) : \mathsf{supp}(g) \subseteq K, \sup_{x \in \Omega} |D^\alpha (f - g)(x)| < \epsilon, \forall \alpha \in \mathbb{N}^m, |\alpha| \leq p\}$$

form a basis of open neighborhoods of f. We call the C^p-topology on $C_0^k(\Omega)$ the topology defined by $V_p(f, \epsilon, K)$ when f, ϵ, K (with $\mathsf{supp}(f) \subset K$) vary.

Notice that a sequence $\{g_j\} \subset C_0^k(\Omega)$ converges to $f \in C_0^k(\Omega)$ in the C^p-topology provided that

(1) $\cup_j \mathsf{supp}(g_j) \cup \mathsf{supp}(f)$ is relatively compact in Ω and
(2) $D^\alpha g_j$ converges uniformly in Ω to $D^\alpha f$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq p$.

The space $C^k(\Omega)$ endowed with the topology of uniform convergence (on Ω) is a Banach space. We can thus consider the induced topology, denoted by $\tilde{C}_p$, on $C_0^k(\Omega)$. More in details, a basis of open neighborhoods for such a topology is provided by

$$\tilde{V}_p(f, \epsilon) = \{g \in C_0^k(\Omega) : \sup_{x \in \Omega} |D^\alpha (f - g)(x)| < \epsilon, \forall \alpha \in \mathbb{N}^m, |\alpha| \leq p\}$$

when $f \in C_0^k(\Omega)$, $\epsilon > 0$ vary. Thus a sequence $\{g_j\} \subset C_0^k(\Omega)$ converges to $f \in C_0^k(\Omega)$ in the $\tilde{C}^p$-topology if and only if the sequence $\{D^\alpha g_j\}$ converges uniformly in Ω to $D^\alpha f$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq p$.

Notice that the topology C_p is finer then the $\tilde{C}_p$-topology on $C_0^k(\Omega)$, that is, an open set in $\tilde{C}_p$ is open also in C_p because, clearly, $V_p(f, \epsilon, K) \subset \tilde{V}_p(f, \epsilon)$ for all K compact sets which contain the support of f. However the C_p-topology does not coincide with the $\tilde{C}_p$-topology. For instance, consider a sequence $\{g_j\}$ defined as follows: let $\{a_j\} \subset \Omega$ be a sequence with no accumulation points in Ω. For each j let $B(a_j, r_j)$ be an open ball relatively compact in Ω with $r_j < 1$. Let g_j be a function with compact support in $B(a_j, r_j)$ such that $\max |D^\alpha g_j| \leq 1/j$ for

$\alpha \in \mathbb{N}^n$, $|\alpha| \leq p$. Then $g_j \to 0$ uniformly in Ω (and thus in the $\tilde{C}_p$-topology). However, since $\cup \mathsf{supp}(g_j)$ is not relatively compact in Ω, then g_j does not converge to 0 in the C_p-topology (indeed $\{g_j\}$ does not eventually belong to any open neighborhood of the form $V_p(0, \epsilon, K)$).

As a consequence, the identity map $I : (C_0^k(\Omega), C_p) \longrightarrow (C_0^k(\Omega), \tilde{C}_p)$ is continuous but not open. Thus, a continuous linear functional T on $(C_0^k(\Omega), \tilde{C}_p)$ gives rise to a continuous linear functional $T \circ I$ on $(C_0^k(\Omega), C_p)$ (but not all continuous linear functionals on $(C_0^k(\Omega), C_p)$ are of this form).

DEFINITION 1.2. Let $p \leq k$. A *distribution of order* p is a linear functional $T : C_0^k(\Omega) \to \mathbb{C}$ which is continuous with respect to the C^p-topology of $C_0^k(\Omega)$. We denote by $\mathcal{D}is_p^k(\Omega)$ the space of distributions of order p on $C_0^k(\Omega)$. We omit to write the subindex p in case $p = k$, that is, $\mathcal{D}is^k(\Omega) := \mathcal{D}is_k^k(\Omega)$.

Clearly, $\mathcal{D}is_{p-1}^k(\Omega) \subset \mathcal{D}is_p^k(\Omega)$ for all $p \leq k$ and $\mathcal{D}is_p^k(\Omega) \subset \mathcal{D}is_p^{k-1}(\Omega)$ for all $k \geq 1$ and $p \leq k-1$.

The elements of $\mathcal{D}is^0(\Omega)$ are called *Radon measures*. This is justified by the following version of Riesz' representation theorem:

THEOREM 1.3. *Let $\Omega \subset \mathbb{R}^m$ be a domain. To any Radon measure T there corresponds a unique (generalized) complex Borel measure μ_T such that*

$$T(\varphi) = \int_\Omega \varphi d\mu_T,$$

for all $\varphi \in C_0(\Omega)$. Moreover, any positive linear functional T on $C_0(\Omega)$ (namely, $T(\varphi) \geq 0$ for all $\varphi \geq 0$ with $\varphi \in C_0(\Omega)$) is necessarily continuous and μ_T is a real positive measure. Conversely, if μ_T is a real positive measure then $T \geq 0$.

REMARK 1.4. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of Ω and let $\mathsf{Mes}(\mathbb{C})$ be the space of regular finite complex measure. Let $\mathcal{B}_c = \{E \in \mathcal{B} : E \subset\subset \Omega\}$. The *(generalized) complex Borel measure* μ_T as defined in Theorem 1.3 is a function $\mu_T : \mathcal{B}_c \to \mathsf{Mes}(\mathbb{C})$ such that for any $E \in \mathcal{B}_c$ the measure $\mu_T(E)$ (also denoted by $\mu_T|_E$) is a finite regular complex Borel measure, namely, $\mu_T|_E$ is σ-addictive, regular and with finite total variation. Moreover, if $E \subset E'$ then $\mu_T|_E = (\mu_T|_{E'})|_E$. Since Ω is union of compact subsets, by the Radon-Nicodym theorem there exists a positive measure ν_T on Ω (possibly with $\nu_T(\Omega) = \infty$) and a complex function $h : \Omega \to \mathbb{C}$ with $|h| \equiv 1$ such that $\mu_T(E) = \int_E h d\nu_T$ for all Borel sets $E \subset\subset \Omega$. Moreover ν_T equals the total variation of μ_T on each relatively compact Borel set $E \subset \Omega$. Thus ν_T can be defined as the *total variation* $|T|$ of T. If ν_T is a finite measure on Ω, then the corresponding μ_T is a regular finite complex Borel measure. Moreover, if the Radon measure T defines a continuous linear functional on $(C_0(\Omega), \tilde{C}_0)$ (that is on $C_0(\Omega)$ with the induced topology of uniform convergence) then μ_T is a regular finite complex Borel measure on Ω, and conversely, to any regular finite complex Borel measure on Ω there corresponds a unique Radon measure which is continuous on $(C_0(\Omega), \tilde{C}_0)$ (see, *e.g.*, [**18**]).

From now on, we will consider $\mathcal{D}is_p^k(\Omega)$ endowed with the weak* topology. Note that, a sequence $\{T_j\} \subset \mathcal{D}is_p^k(\Omega)$ (weakly*-)converges to $T \in \mathcal{D}is_p^k(\Omega)$ if and only if for all $f \in C_0^k(\Omega)$ it follows that $\lim_{j\to\infty} T_j(f) = T(f)$. In particular, by the Banach-Alaoglu theorem, the open ball in $\mathcal{D}is_p^k(\Omega)$ is relatively compact in the weak*-topology.

We collect here a few useful and known facts about distributions:

LEMMA 1.5. *Let $\Omega \subset \mathbb{R}^m$ be a domain. Then*

(1) *Let $T_j, T \in \mathcal{D}is^0(\Omega)$. Then $T_j \to T$ (in the weak* topology) if and only if $T_j(\varphi) \to T(\varphi)$ for all $\varphi \in C_0^\infty(\Omega)$ and $\sup_j\{|dT_j|(K)\} < \infty$ for all compact subset $K \subset \Omega$ (here dT_j denotes the complex Borel measure given by Riesz' Theorem* 1.3 *and $|dT_j|$ is its total variation). Moreover the condition $\sup_j\{|dT_j|(K)\} < \infty$ is not necessary if $T_j, T \geq 0$.*
(2) *If $T \in \mathcal{D}is^\infty(\Omega)$ and $T \geq 0$, then $T \in \mathcal{D}is^0(\Omega)$.*

At this point, it is worth to mention two results about subharmonic functions when considering their Laplacian as a measure. We state them for $\mathbb{C}$, referring to [**16**, Section 4.1] for generalizations to $\mathbb{R}^m$ and proofs. Let $\Omega \subset \mathbb{C}$ be an open set and let $u \in \mathsf{subh}(\Omega)$. By Theorem I.4.1 and Lemma 1.5.(2), Δu is a positive linear functional on $C_0(\Omega)$. By Theorem 1.3, Δu is thus a Radon measure and there exists a complex Borel measure μ_u such that

$$\Delta u(\varphi) = \int_\Omega \varphi d\mu_u$$

for all $\varphi \in C_0(\Omega)$. For each open subset $U \subset\subset \Omega$ let μ_u^U be the finite complex measure in $\mathbb{C}$ obtained by extending with 0 on $\mathbb{C} \setminus U$ the restriction of μ_u to U. The measure μ_u^U has compact support contained in $\overline{U}$. We define the *potential of* u

$$P_u^U(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-\zeta| d\mu_u^U(\zeta).$$

Since μ_u^U is a finite complex measure compactly supported in $\overline{U}$, it follows that $P_u^U \in L^1_{\text{loc}}(\mathbb{C})$. Moreover the following result (known as the *Riesz decomposition theorem*) holds:

THEOREM 1.6. *Let $\Omega \subset \mathbb{C}$ be a domain. Let $u \in \mathsf{subh}(\Omega)$. If U is an open set relatively compact in Ω, then there exists $h_u \in \mathsf{harm}(U)$ such that*

$$u(z) = P_u^U(z) + h_u(z)$$

for all $z \in U$.

PROOF. The key point is to prove the following equality:

$$\Delta P_u^U = \mu_u^U \quad \text{in } \mathcal{D}is^0(\mathbb{C}). \tag{1.1}$$

Once this is obtained, we have that $\Delta(u - P_u^U) = \Delta u - \Delta P_u^U = \mu_u^U - \Delta P_u^U = 0$. By Theorem I.4.1 there exist $h, g \in \mathsf{subh}(U)$ such that $h = u - P_u^U$ almost everywhere and $g = -(u - P_u^U)$ almost everywhere. In particular $h + g = 0$ almost everywhere and by Corollary I.3.2

then $h = -g$ everywhere and thus $h \in \mathsf{harm}(U)$. By the same token, $h = u - P_u^U$ everywhere as required. We are left to prove (1.1). First, we recall the well known fact

$$\frac{1}{2\pi}\Delta_\zeta \log|z-\zeta| = \delta_z \quad \text{in } \mathcal{D}is^0(\mathbb{C}),$$

where δ_z here denotes the Dirac delta defined by $\delta_z(\varphi) = \varphi(z)$ for all $\varphi \in C_0(\mathbb{C})$. From this and from Fubini's theorem we have for all $\varphi \in C_0^\infty(\mathbb{C})$

$$\begin{aligned}\Delta P_u^U(\varphi) &= \int_{\mathbb{C}} P_u^U(z)\Delta\varphi(z)d\lambda(z) = \int_{\mathbb{C}} \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-\zeta| d\mu_u^U(\zeta)\Delta\varphi(z)d\lambda(z)\\ &= \int_{\mathbb{C}} \frac{1}{2\pi}\int_{\mathbb{C}} \log|z-\zeta|\Delta\varphi(z)d\lambda(z)d\mu_u^U(\zeta) = \int_{\mathbb{C}} \varphi(\zeta)d\mu_u^U(\zeta),\end{aligned}$$

and (1.1) follows. □

Using the Riesz decomposition theorem one can prove the following *Poisson-Jensen formula*:

THEOREM 1.7. *Let $\Omega \subset \mathbb{C}$ be a domain. Let $u \in \mathsf{subh}(\Omega)$ and let $B(a,r)$ is an open ball relatively compact in Ω. If $u(a) > -\infty$ then*

$$u(a) = \frac{1}{2\pi r}\int_{\partial B(a,r)} u(\zeta)d\sigma(\zeta) - \frac{1}{2\pi}\int_0^r \frac{\mu_u(\overline{B(a,s)})}{s}ds.$$

1.1. Regularization and plurisubharmonic functions. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\{\chi_\epsilon\}$ be the sequence of smoothing kernels defined in Section I.3. If $T \in \mathcal{D}is^\infty(\Omega)$ one can define a sequence of C^∞ functions

$$T_\epsilon(x) := T * \chi_\epsilon(x) := T_y(\chi_\epsilon(x-y))$$

such that $T_\epsilon \to T$ in $\mathcal{D}is^\infty(\Omega)$. We have the following generalization of (one side implication of) Proposition III.2.3:

PROPOSITION 1.8. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $T \in \mathcal{D}is^\infty(\Omega)$ be a distribution such that $\mathcal{L}(T) \geq 0$ (namely $T(\mathcal{L}(\varphi)(v;v)) \geq 0$ for all $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ and $v \in \mathbb{C}^n$) then there exists $u \in \mathsf{Psh}(\Omega)$ such that $u = T$ in $\mathcal{D}is^\infty(\Omega)$ (namely, $T(\varphi) = \int_\Omega \varphi u d\lambda$ for all $\varphi \in C_0^\infty(\Omega)$).*

PROOF. Let $u_\epsilon := T * \chi_\epsilon \in C^\infty(\Omega_\epsilon)$. Then $u_\epsilon \to T$ in $\mathcal{D}is(\Omega)$ as $\epsilon \to 0$. By Fubini's theorem $\mathcal{L}(u_\epsilon) = \mathcal{L}(T) * \chi_\epsilon$ as distributions. Thus $\mathcal{L}(u_\epsilon) \geq 0$ and then $u_\epsilon \in \mathsf{Psh}(\Omega_\epsilon) \cap C^\infty(\Omega_\epsilon)$ by Proposition III.1.2. Now, arguing as at the end of Theorem I.4.1 we see that u_ϵ is decreasing in ϵ and thus, by Lemma III.1.10, it follows that $u(z) = \lim_{\epsilon\to 0} u_\epsilon(z) \in \mathsf{Psh}(\Omega)$. By Beppo Levi's theorem $u_\epsilon \to u$ also in the sense of distributions and therefore $T = u$ in the sense of distributions, proving the statement. □

1.2. Sequences of L^1_{loc}-bounded plurisubharmonic functions. The aim of this section is to show that a sequence of plurisubharmonic functions which is bounded in the L^1 norm on compacta is actually uniformly bounded from above on compacta and admits a subsequence converging in L^1_{loc} to a plurisubharmonic function. To start with, we prove the following result:

THEOREM 1.9. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\{u_j\} \subset \mathsf{Psh}(\Omega)$ be a sequence which is locally bounded from above on compacta of Ω. Assume that there exists $T \in \mathcal{D}is^\infty(\Omega)$ such that $u_j \to T$ in $\mathcal{D}is^\infty(\Omega)$. Then there exists $u \in \mathsf{Psh}(\Omega)$ such that $u = T$ in $\mathcal{D}is^\infty(\Omega)$ and $u_j \to u$ in $L^1_{\text{loc}}(\Omega)$.*

PROOF. Since $\mathcal{L}(u_j) \geq 0$ for all j, then $\mathcal{L}(T) \geq 0$. Thus, by Proposition 1.8 there exists $u \in \mathsf{Psh}(\Omega)$ such that $u = T$ in $\mathcal{D}is^\infty(\Omega)$.

It remains to show that $u_j \to u$ in $L^1_{\text{loc}}(\Omega)$. First, since $\{u_j\}$ is locally bounded from above, for any fixed compact subset $K \subset \Omega$, there exists $C = C(K) > 0$ such that $u_j - C \leq 0$ in K for all $j \in \mathbb{N}$. We can thus assume that $u_j \leq 0$ on a fixed compact set K.

Since $u_j \to T$ in $\mathcal{D}is^\infty(\Omega)$ then $\{u_j\}$ cannot converge uniformly on compacta to the constant function $-\infty$. Therefore, there exist a sequence $\{x_k\} \subset \Omega$ such that $x_k \to x_0 \in \Omega$ and a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ such that $|u_{j_k}(x_k)| = -u_{j_k}(x_k)$ is bounded from above by some constant $C > 0$. Let $\mathbb{B} := \mathbb{B}(x_0, r)$ be a small ball centered at x_0 such that $\overline{\mathbb{B}(x_0, 2r)} \subset \Omega$. For $k >> 0$ there exists $r_k > 0$ such that $\mathbb{B}_k := \mathbb{B}(x_k, r_k)$ has the property that $\overline{\mathbb{B}} \subset \overline{\mathbb{B}_k} \subset \overline{\mathbb{B}(x_0, 2r)} \subset \Omega$. Thus, since $u_j, u \in \mathsf{subh}(\Omega)$,

$$\int_{\mathbb{B}} |u_{j_k}| d\lambda \leq \int_{\mathbb{B}_k} |u_{j_k}| d\lambda = -\int_{\mathbb{B}_k} u_{j_k} d\lambda \leq -\mathsf{v}(\mathbb{B}_k) u_{j_k}(x_k)$$
$$= \mathsf{v}(\mathbb{B}_k)|u_{j_k}(x_k)| \leq C\mathsf{v}(\mathbb{B}(x_0, 2r)).$$

Hence, $\{u_{j_k}\}$ is uniformly bounded in $L^1(\mathbb{B})$. By Proposition I.3.4, $\{u_{j_k}\}$ is actually uniformly bounded in $L^1_{\text{loc}}(\Omega)$. Hence, if $\{\chi_\epsilon\}$ is the sequence of smoothing kernels defined in Section I.3 (which are clearly bounded on compacta together with their first derivatives) it follows that for a fixed $\epsilon > 0$, the sequence $\{u_{j_k} * \chi_\epsilon\}$ is equicontinuous and uniformly bounded on compacta of Ω. Therefore—since $u_j \to u$ in $\mathcal{D}is^\infty(\Omega)$ and hence $u_{j_k} * \chi_\epsilon \to u * \chi_\epsilon$ pointwise—it follows from Arzelà-Ascoli's theorem that actually $u_{j_k} * \chi_\epsilon \to u * \chi_\epsilon$ uniformly on compacta.

In order to prove that $u_{j_k} \to u$ in $L^1_{\text{loc}}(\Omega)$, let $K \subset \Omega$ be an open set whose closure is compact in Ω and let $\Psi \geq 0$ be a smooth function which is compactly supported in Ω and such that $\Psi|_K \equiv 1$. By Theorem I.3.1, the sequence $\{u_{j_k} * \chi_\epsilon\}$ (*respectively* $\{u * \chi_\epsilon\}$) decreases to u_{j_k} (*respect.* u) as $\epsilon \to 0^+$. In particular, $u * \chi_\epsilon - u \geq 0$. For $\epsilon, \delta > 0$ small,

$$\lim_{k\to\infty} \int_\Omega (u * \chi_\epsilon + \delta - u_{j_k}) \Psi d\lambda = \int_\Omega (u * \chi_\epsilon + \delta - u) \Psi d\lambda > 0.$$

Thus

$$
\begin{aligned}
\limsup_{k\to\infty}\int_K |u-u_{j_k}|d\lambda &\leq \limsup_{k\to\infty}\int_\Omega |u-u_{j_k}|\Psi d\lambda \\
&\leq \limsup_{k\to\infty}\left[\int_\Omega |u*\phi_\epsilon+\delta-u|\Psi d\lambda + \int_\Omega |-(u*\phi_\epsilon+\delta-u_{j_k})|\Psi d\lambda\right] \\
&\leq 2\int_\Omega |u*\phi_\epsilon+\delta-u|\Psi d\lambda = 2\int_\Omega (u*\phi_\epsilon+\delta-u)\Psi d\lambda,
\end{aligned}
$$

and the last term goes to zero as $\epsilon, \delta \to 0$. Therefore $u_{j_k} \to u$ in $L^1(K)$ and, by arbitrariness of K, $u_{j_k} \to u$ in $L^1_{\text{loc}}(\Omega)$.

Repeating the above argument for all subsequences of $\{u_j\}$, the statement follows. □

COROLLARY 1.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\{u_j\} \subset \mathsf{Psh}(\Omega)$ be a sequence which is locally bounded from above on compacta of Ω. Then either $\{u_j\}$ converges uniformly on compacta to the constant function $-\infty$ or there exist a subsequence $\{u_{j_k}\}$ and a function $u \in \mathsf{Psh}(\Omega)$ such that $u_{j_k} \to u$ in $L^1_{\text{loc}}(\Omega)$.*

PROOF. If $\{u_j\}$ is not uniformly convergent on compacta to the constant function $-\infty$, then, as in the proof of Theorem 1.9 we see that there exists a subsequence $\{u_{j_k}\}$ which has L^1-norm uniformly bounded on compacta. By the Banach-Alaoglu compactness theorem, up to extracting another subsequence, $\{u_{j_k}\}$ is weak* converging to a distribution T. Then Theorem 1.9 applies. □

COROLLARY 1.11. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\{u_j\} \subset \mathsf{Psh}(\Omega)$ be a sequence which is bounded in $L^1_{\text{loc}}(\Omega)$. Then $\{u_j\}$ is uniformly bounded from above on compacta of Ω and there exist a subsequence $\{u_{j_k}\}$ and a function $u \in \mathsf{Psh}(\Omega)$ such that $u_{j_k} \to u$ in $L^1_{\text{loc}}(\Omega)$.*

PROOF. Arguing by contradiction, assume that $\{u_j\}$ is not uniformly bounded from above on compacta. Thus, up to extracting subsequences, there exists a compact set K such that

$$\lim_{j\to\infty}\max_{z\in K} u_j(z) = +\infty. \tag{1.2}$$

By the Banach-Alaoglu theorem there exists a subsequence $\{u_{j_k}\}$ which is weak* converging to some distribution T. Arguing as in the proof of Theorem 1.9, we see that there exists $u \in \mathsf{Psh}(\Omega)$ such that $u_{j_k} \to u$ in $\mathcal{D}is^\infty(\Omega)$. Using the same notations as in the proof of Theorem 1.9, it follows that $u_{j_k} \leq u_{j_k} * \chi_\epsilon$ and $u_{j_k} * \chi_\epsilon \to u * \chi_\epsilon$ uniformly on compacta as $k \to \infty$. Thus, for each compact set $K \subset \Omega$, the sequence $\{u_{j_k}\}$ is uniformly bounded from above, which contradicts (1.2).

The second part of the statement follows from Corollary 1.10 since, being $L^1_{\text{loc}}(\Omega)$ bounded, $\{u_j\}$ cannot converges uniformly on compacta to the constant function $-\infty$. □

THEOREM 1.12. *Let $\Omega \subset \mathbb{C}^n$ be a domain.*

(1) *The real cone $\mathsf{Psh}(\Omega)$ is closed in $L^1_{\text{loc}}(\Omega)$.*

(2) *A subset* $U \subset \mathsf{Psh}(\Omega)$ *is compact in* $\mathsf{Psh}(\Omega)$ *(with respect to the* $L^1_{\mathrm{loc}}(\Omega)$ *topology) if and only if it is bounded and closed in* $L^1_{\mathrm{loc}}(\Omega)$.

PROOF. (1) Let $\{u_j\} \subset \mathsf{Psh}(\Omega)$ be a sequence which converges in $L^1_{\mathrm{loc}}(\Omega)$ to a function $u \in L^1_{\mathrm{loc}}(\Omega)$. In particular $\{u_j\}$ is uniformly bounded in $L^1_{\mathrm{loc}}(\Omega)$ and by Corollary 1.11, up to subsequences, it converges in $L^1_{\mathrm{loc}}(\Omega)$ to a function $v \in \mathsf{Psh}(\Omega)$. Hence $u = v$ almost everywhere and $\mathsf{Psh}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$.

(2) One direction is clear. Conversely, assume $U \subset \mathsf{Psh}(\Omega)$ is bounded and closed in $L^1_{\mathrm{loc}}(\Omega)$. Since $\mathsf{Psh}(\Omega)$ is closed in $L^1_{\mathrm{loc}}(\Omega)$ then U is closed in $\mathsf{Psh}(\Omega)$. Let $\{u_j\} \subset U$ be a sequence. Since it is bounded in $L^1_{\mathrm{loc}}(\Omega)$, by Corollary 1.11, up to subsequences, it is converging in $L^1_{\mathrm{loc}}(\Omega)$ and therefore U is compact in $\mathsf{Psh}(\Omega)$. $\square$

2. Currents. Definition and first properties

Let $\Omega \subset \mathbb{C}^n$ be a domain. We denote by $C^k_0(\Omega, \Lambda^{p,q})$ the space of (complex) (p,q)-forms having C^k coefficients with compact support in Ω. Given

$$\omega = \sum a_{j_1,\ldots,\overline{k}_q} dz_{j_1} \wedge \ldots dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \ldots \wedge d\overline{z}_{k_q} \in C^k_0(\Omega, \Lambda^{p,q})$$

we write $|\omega|_{C_s} < \epsilon$ if $\sup_\Omega |D^\alpha a_{j_1,\ldots,\overline{k}_q}| < \epsilon$ for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq s$.

DEFINITION 2.1. Let $\epsilon > 0$ and let $\omega \in C^k_0(\Omega, \Lambda^{p,q})$. Let K be a compact set in Ω such that $\mathsf{supp}(\omega) \subseteq K$. For $s \leq k$, we denote by C_s the topology on $C^k_0(\Omega, \Lambda^{p,q})$ obtained by declaring open neighborhoods of ω the following sets

$$V(\omega, K, \epsilon) = \{\eta \in C^k_0(\Omega, \Lambda^{p,q}) : \mathsf{supp}(\eta) \subseteq K, |\omega - \eta|_{C_s} < \epsilon\}$$

as $\epsilon > 0$ and K (with $\mathsf{supp}(\omega) \subseteq K$) vary.

Thus a sequence $\{\omega_l\} \subset C^k_0(\Omega, \Lambda^{p,q})$ converges to $\omega \in C^k_0(\Omega, \Lambda^{p,q})$ in the C_s-topology if and only if

(1) $\cup \mathsf{supp}(\omega_l) \cup \mathsf{supp}(\omega)$ is contained in a compact set in Ω and
(2) $\{D^\alpha a^l_{j_1,\ldots,\overline{k}_q}\}_l$ converges uniformly in Ω to $D^\alpha a_{j_1,\ldots,\overline{k}_q}$ for all $|\alpha| \leq s$.

DEFINITION 2.2. A *current of order* k *and bidegree* $(n-p, n-q)$ is a continuous linear functional on $C^k_0(\Omega, \Lambda^{p,q})$ (endowed with the C_s-topology). We denote by $\mathcal{D}^{(n-p,n-q)}_k(\Omega)$ the space of currents of order k and bidegree $(n-p, n-q)$ in Ω.

In what follows we will always consider only the C_k-topology on $C^k_0(\Omega, \Lambda^{p,q})$. Also, we will consider $\mathcal{D}^{(n-p,n-q)}_k(\Omega)$ endowed with the weak*-topology.

When the underlying complex structure has no relevance, we will consider the space of currents of order k and degree m given by

$$\mathcal{D}^m_k(\Omega) = \sum_{s+t=m} \mathcal{D}^{(s,t)}_k(\Omega).$$

DEFINITION 2.3. The *support*, $\mathsf{supp}(T)$, of a current $T \in \mathcal{D}_k^{(n-p,n-q)}(\Omega)$ is the complement in D of the union of all open sets $U \subset D$ such that for all $\varphi \in C_0^k(\Omega, \Lambda^{p,q})$ with $\mathsf{supp}(\varphi) \subset\subset U$ it follows that $T(\varphi) = 0$.

There are two main examples of currents to be kept in mind:

EXAMPLE 2.4. Let $Z \subset \Omega$ be a closed and orientable C^1-submanifold of dimension p. The *current of integration* $[Z] \in \mathcal{D}_0^{n-p}(\Omega)$ is given by

$$[Z](\varphi) := \int_Z i^*(\varphi),$$

for $\varphi \in C_0(\Omega, \Lambda^p)$ and $i : Z \hookrightarrow \Omega$ the natural embedding. It is clear that $\mathsf{supp}[Z] = Z$. If Z is a *complex* submanifold of complex dimension p, then $[Z] \in \mathcal{D}_0^{(n-p,n-p)}(\Omega)$ for $i^*(\varphi) = 0$ for all $2p$-form such that $\varphi \notin C_0(\Omega, \Lambda^{p,p})$.

EXAMPLE 2.5. Let $\psi \in L^1_{\text{loc}}(\Omega, \Lambda^{p,q})$. Define

$$T_\psi(\varphi) := \int_\Omega \psi \wedge \varphi,$$

for $\varphi \in C_0(\Omega, \Lambda^{n-p,n-q})$. Then $T_\psi \in \mathcal{D}_0^{(p,q)}(\Omega)$.

Let

$$dV = dx_1 \wedge \ldots \wedge dx_{2n} = \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n \tag{2.1}$$

be the standard volume form on $\Omega \subset \mathbb{C}^n$. Let $\eta \in C_0^k(\Omega, \Lambda^{2n})$. Then there exists $a \in C_0^k(\Omega)$ such that $\eta = adV$. This allows to define a homeomorphism $\mathcal{W} : \mathcal{D}_k^0(\Omega) \to \mathcal{D}_k^{2n}(\Omega)$ given by

$$\mathcal{W}(T)(a) := T(adV)$$

for $T \in \mathcal{D}_k^0(\Omega)$ and $a \in C_0^k(\Omega)$. The inverse is

$$\mathcal{W}^{-1}(S)(\eta) = S(a),$$

for $S \in \mathcal{D}_k^{2n}(\Omega)$, $\eta \in C_0^k(\Omega, \Lambda^{2n})$ and $\eta = adV$ (notice that $\mathcal{W}, \mathcal{W}^{-1}$ are linear isomorphisms and are continuous in the weak* topology).

Let us denote by $\mathcal{Q}_p(m)$ the set of all multi-indices $I = (i_1, \ldots, i_p)$ with $1 \le i_1 < \ldots < i_p \le m$.

THEOREM 2.6. *Let $\Omega \subset \mathbb{C}^n$. Let $T \in \mathcal{D}_k^p(\Omega)$. For each multi-index $I \in \mathcal{Q}_p(2n)$ there exists a unique $T_I \in \mathcal{D}is^k(\Omega)$ such that*

$$T = \sum_{I \in \mathcal{Q}_p(2n)} T_I dx_{i_1} \wedge \ldots \wedge dx_{i_p}.$$

In other words, for $\varphi \in C_0^k(\Omega, \Lambda^{2n-p})$ *it follows*

$$T(\varphi) = \sum_{I \in \mathcal{Q}_p(2n)} \mathcal{W}^{-1}(T_I)(dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge \varphi).$$

PROOF. Let $I = \{i_1, \ldots, i_p\} \in \mathcal{Q}_p(2n)$ and let $I^c = (j_1, \ldots, j_{2n-p}) \in \mathcal{Q}_{2n-p}(2n)$ be its complementary. Let us write $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{j_p}$. Then $dx_I \wedge dx_{I^c} = \sigma_I dV$, where $\sigma_I = \pm 1$ and dV is the standard volume form in $\mathbb{C}^n$. For $a \in C_0^k(\Omega)$, let us define

$$T_I(a) := \sigma_I T(a dx_{I^c}).$$

Let now $\varphi \in C_0^k(\Omega, \Lambda^{2n-p})$. Then $\varphi = \sum_{J \in \mathcal{Q}_{2n-p}(2n)} a_J dx_J$ for some $a_J \in C_0^k(\Omega)$. Thus

$$\begin{aligned} T(\varphi) &= \sum_{J \in \mathcal{Q}_{2n-p}(2n)} T(a_J dx_J) = \sum_{I \in \mathcal{Q}_p(2n)} \sigma_I T_I(a_{I^c}) \\ &= \sum_{I \in \mathcal{Q}_p(2n)} \mathcal{W}^{-1}(T_I)(\sigma_I a_{I^c} dV) = \sum_{I \in \mathcal{Q}_p(2n)} \mathcal{W}^{-1}(T_I)(dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge \varphi), \end{aligned}$$

as wanted. □

According to Theorem 2.6 a current of degree p and order k is a p-form with distributional coefficients of order k. An analogue of Theorem 2.6 holds in the complex category, namely, if $T \in \mathcal{D}_k^{(p,q)}(\Omega)$ (and, say, $q \geq p$) then

$$T = (i/2)^p \sum_{I \in \mathcal{Q}_p(n), J \in \mathcal{Q}_q(n)} T_{I,J} dz_{i_1} \wedge d\overline{z}_{j_1} \wedge \ldots dz_{i_p} \wedge d\overline{z}_{j_p} \wedge \ldots \wedge d\overline{z}_{j_q}, \tag{2.2}$$

with $T_{I,J}$ distribution of order k on Ω. The $T_{I,J}$ are called *coefficients* of T.

Be aware: the term $(i/2)^p$ in (2.2) is clearly asymmetric in (p, q). However, it really makes sense only in case $p = q$ (when discussing positive currents).

3. Operations with currents

Here we consider only few operations among those that can be operated on currents. We refer to [**12**] or [**9**] for the general theory.

3.1. Exterior derivatives. Let $\Omega \subset \mathbb{C}^n$. Let $T \in \mathcal{D}_k^p(\Omega)$. We define $dT \in \mathcal{D}_{k+1}^{p+1}(\Omega)$ as follows:

$$dT(\varphi) := (-1)^{p+1} T(d\varphi) \quad \forall \varphi \in C_0^{k+1}(\Omega, \Lambda^{2n-(p+1)}).$$

Since the operator $d : C_0^{k+1}(\Omega, \Lambda^{2n-(p+1)}) \to C_0^k(\Omega, \Lambda^{2n-p})$ is continuous (with respect to the C_{k+1} and C_k-topologies) then $d : \mathcal{D}_k^p(\Omega) \to \mathcal{D}_{k+1}^{p+1}(\Omega)$ is continuous (with respect to the weak*-topology).

Similarly, in the complex case, one can define the operators $\partial : \mathcal{D}_k^{p,q}(\Omega) \to \mathcal{D}_{k+1}^{(p+1,q)}(\Omega)$ and $\overline{\partial} : \mathcal{D}_k^{p,q}(\Omega) \to \mathcal{D}_{k+1}^{(p,q+1)}(\Omega)$.

PROPOSITION 3.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\psi \in C_0^k(\Omega, \Lambda^p)$. Then $dT_\psi = T_{d\psi}$. Similarly, $\partial T_\psi = T_{\partial\psi}$ and $\overline{\partial} T_\psi = T_{\overline{\partial}\psi}$.*

PROOF. By definition, integrating by parts and by Stokes' theorem it follows

$$dT_\psi(\varphi) = (-1)^{p+1} T_\psi(d\varphi) = (-1)^{p+1} \int_\Omega \psi \wedge d\varphi = \int_\Omega d\psi \wedge \varphi = T_{d\psi}(\varphi),$$

for all $\varphi \in C_0^{k+1}(\Omega, \Lambda^{2n-p-1})$. In case of ∂ (and $\overline{\partial}$) the argument is similar because $d(\psi \wedge \varphi) = \partial(\psi \wedge \varphi)$ for $\psi \in C_0^k(\Omega, \Lambda^{p,q})$ and $\varphi \in C_0^{k+1}(\Omega, \Lambda^{n-p-1,n-q})$ and thus Stokes' theorem applies. □

3.2. Wedge product. . Let $\Omega \subset \mathbb{C}^n$. Let $T \in \mathcal{D}_k^p(\Omega)$ and $\psi \in C_0^k(\Omega, \Lambda^q)$ with $p + q \leq 2n$. We define $T \wedge \psi \in \mathcal{D}_k^{p+q}(\Omega)$ as follows:

$$(T \wedge \psi)(\varphi) := T(\psi \wedge \varphi), \quad \forall \varphi \in C_0^k(\Omega, \Lambda^{2n-p-q}).$$

Be aware: The wedge product is defined (here) only between currents and (smooth) forms and *not* between two currents, that is

$$\wedge : \mathcal{D}_k^p(\Omega) \times C_0^k(\Omega, \Lambda^q) \longrightarrow \mathcal{D}_k^{p+q}(\Omega).$$

It can be easily proved that $d(T \wedge \psi) = dT \wedge \psi + (-1)^{\deg_T} T \wedge d\psi$.

4. Positive forms and positive currents

4.1. Positive forms. We recall that a distribution T is said to be *positive* provided $T(\varphi) \geq 0$ for all test functions $\varphi \geq 0$. In order to define positive currents, we first define positive forms.

DEFINITION 4.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. A form $\omega \in C^k(\Omega, \Lambda^{2p})$ is *real* if $\omega(X) \in \mathbb{R}$ for all $X \in (T\Omega^{\mathbb{R}})^{\otimes p}$.

Notice that ω is real if and only if $\omega = \overline{\omega}$. In particular if a (p, q)-form is real then $p = q$.

PROPOSITION 4.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. A $(1, 1)$-form ω is real if and only if*

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{jk} dz_j \wedge d\overline{z}_k$$

with $(h_{jk}(x))$ being a $n \times n$ Hermitian matrix for all $x \in \Omega$.

PROOF. It is a direct computation from $\omega = \overline{\omega}$. □

DEFINITION 4.3. Let $\Omega \subset \mathbb{C}^n$ be a domain. A $2n$-form ω is *positive* if $\omega = f dV$ with $f \geq 0$ and dV the standard volume form (2.1). If ω is a positive $2n$-form we write $\omega \geq 0$.

DEFINITION 4.4. A (p, p)-form ω is *elementary strongly positive* if $\omega(x) \neq 0$ for all $x \in \Omega$ and there exist $\omega_j \in C^k(\Omega, \Lambda^{(1,0)})$, $j = 1, \ldots, p$ such that

$$\omega = \left(\frac{i}{2}\right)^p \omega_1 \wedge \overline{\omega}_1 \wedge \ldots \omega_p \wedge \overline{\omega}_p. \tag{4.1}$$

Notice that $\omega_1, \ldots, \omega_p$ as in the previous definition are linearly independent at each point of Ω since ω is nowhere zero in Ω.

Let $\mathcal{S}P^{(p,p)}(\Omega)$ be the real cone in $C^k(\Omega, \Lambda^{(p,p)})$ generated by the elementary strongly positive forms (that is, $\eta \in \mathcal{S}P^{(p,p)}(\Omega)$ if there exist $\lambda_j \in C^k(\Omega, \mathbb{R})$ with $\lambda_j \geq 0$ and η_j elementary strongly positive forms such that $\eta = \sum \lambda_j \eta_j$). A form $\omega \in \mathcal{S}P^{(p,p)}(\Omega)$ is said *strongly positive*.

PROPOSITION 4.5. *The $C^k(\Omega, \mathbb{R})$-module $C^k(\Omega, \Lambda^{(p,p)}_{\mathbb{R}})$ of real (p,p)-forms has a basis of strongly positive (p,p)-forms. In particular $\mathcal{S}P^{(p,p)}(\Omega)$ has non-empty interior.*

PROOF. First of all we notice that the *complex* space of (p,p)-forms has a basis of strongly positive forms. To this aim, notice that

$$dz_{j_1} \wedge d\overline{z}_{k_1} \wedge \ldots \wedge dz_{j_p} \wedge d\overline{z}_{k_p} = \pm \bigwedge_{l=1}^{p} dz_{j_l} \wedge d\overline{z}_{k_l}$$

and

$$dz_j \wedge d\overline{z}_k = \frac{i}{4}\{-i(dz_j + dz_k) \wedge (d\overline{z}_j + d\overline{z}_k) - i(dz_j - dz_k) \wedge (d\overline{z}_j - d\overline{z}_k)$$
$$+ (dz_j + idz_k) \wedge (d\overline{z}_j - id\overline{z}_k) - (dz_j - idz_k) \wedge (d\overline{z}_j + id\overline{z}_k)\}.$$

Now let ω be a *real* (p,p)-form. Write ω as linear combination of strongly positive (p,p)-forms (with complex coefficients *a priori*). Since strongly positive forms are real, it follows that the coefficients in such a linear combination are real, proving the statement. □

The cone of strongly positive forms is invariant under holomorphic changes of coordinates:

PROPOSITION 4.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $f : \Omega' \to \Omega$ be a biholomorphism. Then $f^*(\mathcal{S}P^{(p,p)}(\Omega)) = \mathcal{S}P^{(p,p)}(\Omega')$.*

PROOF. First of all notice that if $\eta_1, \ldots, \eta_r$ are $(1,0)$-forms linearly independent at each $z \in \Omega$ then $f^*(\eta_1), \ldots, f^*(\eta_r)$ are $(1,0)$-forms linearly independent at each $x \in \Omega'$ and then $f^*(\eta)$ is elementary strongly positive if and only if η is elementary strongly positive. Therefore $f^*(\mathcal{S}P^{(p,p)}(\Omega)) = \mathcal{S}P^{(p,p)}(\Omega')$. □

DEFINITION 4.7. A (p,p)-form $\omega \in C^k(\Omega, \Lambda^{(p,p)})$ is *positive*, and we write $\omega \geq 0$, if for all $\psi \in \mathcal{S}P^{(n-p,n-p)}(\Omega)$ it follows $\omega \wedge \psi \geq 0$.

REMARK 4.8. Since $\eta \in \mathcal{S}P^{(n-p,n-p)}(\Omega)$ is of the form $\sum \lambda_j \eta_j$ with $\lambda_j \geq 0$ and η_j elementary strongly positive, then a (p,p)-form ω is positive if and only if $\omega \wedge \eta \geq 0$ for all elementary strongly positive $(n-p, n-p)$-forms.

Clearly, one can localize the notions of elementary strongly positivity, strongly positivity and positivity to each fiber of the bundle $\Lambda^{(p,p)}$ on Ω. In other words, a form $\alpha \in \Lambda^{(p,p)}_x$ (here α is an element of the fiber of $\Lambda^{(p,p)}$ at x) is positive if for all elementary strongly positive $(n-p, n-p)$-form $\beta \in \Lambda^{(n-p,n-p)}_x$ it follows $\alpha \wedge \beta = \lambda dV(x)$ with $\lambda \geq 0$.

LEMMA 4.9. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\omega \in C^k(\Omega, \Lambda^{(p,p)})$. Then $\omega \geq 0$ if and only if $\omega(x) \geq 0$ for all $x \in \Omega$.*

PROOF. If $\omega(x) \geq 0$ for all $x \in \Omega$ then $\omega \geq 0$ because if it were $\omega \wedge \eta = f dV \not\geq 0$ for some elementary strongly positive $(n-p, n-p)$-form that there would exists $x \in \Omega$ such that $f(x) < 0$ and thus $\omega(x) \wedge \eta(x) = f(x) dV(x) < 0$ contrarily to our hypothesis.

Conversely, if $\omega(x) \wedge \eta_x = \lambda dV(x)$ with $\lambda < 0$ and $\eta_x \in \Lambda^{(n-p,n-p)}$ elementary strongly positive, then there exists $\eta \in C^k(\Omega, \Lambda^{(n-p,n-p)})$ elementary strongly positive such that $\eta(x) = \eta_x$ and then $\omega \wedge \eta \not\geq 0$. □

Lemma 4.9 allows to check pointwise if a given form is positive. Next result says that positivity is a notion compatible with holomorphic maps:

PROPOSITION 4.10. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\omega \in C^k(\Omega, \Lambda^{(p,p)})$.*

(1) *If $\omega \geq 0$ then for all $\sigma : U \to \Omega$ holomorphic from a domain $U \subset \mathbb{C}^s$ (with $s \leq n$) to Ω it follows $\sigma^*(\omega) \geq 0$.*
(2) *If for all $\sigma : U \to \Omega$ holomorphic from a domain $U \subset \mathbb{C}^p$ to Ω it follows $\sigma^*(\omega) \geq 0$ then $\omega \geq 0$.*

PROOF. (1) First of all notice that if $\sigma : U \to \Omega$ with $U \subset \mathbb{C}^n$ is a biholomorphism then $\sigma^*(\mathcal{S}P^{(p,p)}(\Omega)) = \mathcal{S}P^{(p,p)}(U)$ by Proposition 4.6. Let $\eta \in \mathcal{S}P^{(p,p)}(\Omega)$. By Cauchy-Riemann equations, $\sigma^*(dV) = |\det(\frac{\partial \sigma_k}{\partial z_j})|^2 dV$ and then $\sigma^*(\omega \wedge \eta) \geq 0$ if and only if $\omega \wedge \eta \geq 0$, namely, $\omega \geq 0$ if and only if $\sigma^*(\omega) \geq 0$.

Assume now that $\omega \geq 0$. Let $\sigma : U \to \Omega$ be holomorphic with $U \subset \mathbb{C}^s$. If $s < p$ then $\sigma^*(\omega) = 0$ and there is nothing to prove. Assume that $s \geq p$. Let $\psi = (i/2)^{s-p} \psi_{p+1} \wedge \overline{\psi}_{p+1} \wedge \ldots \psi_s \wedge \overline{\psi}_s$ be an elementary strongly positive $(s-p, s-p)$-form in U (here we take $\psi \equiv 1$ if $s = p$). We have to show that $\sigma^*(\omega) \wedge \psi \geq 0$. According to Lemma 4.9, it is enough to check that $\sigma^*(\omega)(x) \wedge \psi(x) \geq 0$ for all $x \in U$. Moreover, for what we already proved at the beginning, we can compose with biholomorphisms. Thus we can choose holomorphic coordinates $\{z_1, \ldots, z_s\}$ in U near x such that $\psi_j(x) = dz_j(x)$ for $j = p+1, \ldots, s$ (if $s = p$ we do not need this change of coordinates). In such coordinates, we need to show that $\sigma^*(\omega)(x) \wedge (i/2)^{s-p} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots dz_s \wedge d\overline{z}_s \geq 0$.

Let $E_j = d\sigma_x(\frac{\partial}{\partial z_j})$ for $j = 1, \ldots, s$. We can assume that $E_1, \ldots, E_r$ are linearly independent (with $r \leq s$) and generate $d\sigma_x(T_x U)$. Notice that if $r < p$ then $\sigma^*(\omega)(x) = 0$ and there is nothing to prove. We can thus assume that $r \geq p$. Let $\{E_{r+1}, \ldots, E_n\}$ be a completion of $\{E_1, \ldots, E_r\}$ to a basis of $T_{\sigma(x)}\Omega = \mathbb{C}^n$. Let

$$T(z_1, \ldots, z_n) = \sigma(z_1, \ldots, z_s) + \sum_{j=p+1}^{s} E_j(z_j - x_j) + \sum_{j=s+1}^{n} E_j z_j.$$

Then there exist $V \in \mathbb{C}^{n-s}$ an open neighborhood of O and $U' \subset U$ an open neighborhood of x such that $T : U' \times V \to \Omega$ is holomorphic and $T(x, O) = \sigma(x)$. Notice that by construction

$dT_{(x,O)}(\frac{\partial}{\partial z_j}) = \epsilon_j E_j$ with $\epsilon_j = 2$ if $j = p+1, \ldots, r$ and $\epsilon_j = 1$ otherwise. Thus $dT_{x,O}$ is invertible and, up to shrink U', V, we can assume that T is a biholomorphism on its image. Let η be an elementary strongly positive $(n-p, n-p)$-form. Then $T^*(\omega \wedge \eta) \geq 0$. For $j = 1, \ldots, n$, let η_j be $(1,0)$-forms such that $\eta_j(\sigma(x))(E_k) = \delta_j^k$. By construction

$$T^*(\eta_j)(x, O) = \epsilon_j dz_j$$

with $\epsilon_j = 2$ if $j = p+1, \ldots, r$ and $\epsilon_j = 1$ otherwise.

Let $\eta = (i/2)^{n-p} \eta_{p+1} \wedge \overline{\eta}_{p+1} \wedge \ldots \wedge \eta_n \wedge \overline{\eta}_n$. Then η is an elementary strongly positive $(n-p, n-p)$-form near x. Thus

$$\begin{aligned} 0 \leq T^*(\omega \wedge \eta)(x, O) &= T^*\omega(x, O) \wedge T^*\eta(x, O) \\ &= T^*\omega(x, O) \wedge (i/2)^{(n-p)} 2^{r-p} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \\ &= 2^{r-p} \sigma^*(\omega)(x) \wedge (i/2)^{(n-p)} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n, \end{aligned}$$

which implies that $\sigma^*(\omega)(x) \wedge (i/2)^{(s-p)} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge dz_s \wedge d\overline{z}_s \geq 0$.

(2) Assume that for all $\sigma : U \to \Omega$ holomorphic it follows $\sigma^*(\omega) \geq 0$. Let η be an elementary strongly positive $(n-p, n-p)$-form. We have to show that $\omega \wedge \eta(x) \geq 0$ for all $x \in \Omega$. Fix $x \in \Omega$. Write $x = (x', x'') \in \mathbb{C}^p \times \mathbb{C}^{n-p}$. By (1) we can choose local holomorphic coordinates near x such that

$$\eta(x) = (i/2)^{n-p} dz_{p+1} \wedge d\overline{z}_{p+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n.$$

Now let $U = \Omega \cap (\mathbb{C}^p \times \{x''\})$ and let $\sigma : U \to \Omega$ be given by

$$\sigma(z_1, \ldots, z_p) = (z_1, \ldots, z_p, x'').$$

By hypothesis $\sigma^*(\omega)(x') = \lambda (i/2)^p dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_p \wedge d\overline{z}_p$ with $\lambda \geq 0$ and $\sigma^*(\omega)(x') \wedge \eta(x'') = \omega \wedge \eta(x)$, from which $\omega \wedge \eta(x) = \lambda dV$ proving that $\omega \geq 0$. □

LEMMA 4.11. *Let $\Omega \subset \mathbb{C}^n$ be a domain. If $\omega \in \mathcal{S}P^{(p,p)}(\Omega)$ then $\omega \geq 0$.*

PROOF. Let $x \in \Omega$. According to Lemma 4.9 and Proposition 4.10 we can prove that ω is positive using any local holomorphic coordinates change. We can thus choose local holomorphic coordinates at x such that $\omega(x) = (i/2)^p dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_p \wedge d\overline{z}_p$. From this it follows easily that $\omega(x) \geq 0$. □

THEOREM 4.12. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\omega = i/2 \sum h_{jk} dz_j \wedge d\overline{z}_k \in C^k(\Omega, \Lambda^{(1,1)})$. Then the following are equivalent:*

(1) $\omega \geq 0$.
(2) $(h_{jk}(x))$ *is a semi-positive definite hermitian matrix for all* $x \in \Omega$.
(3) $\omega \in \mathcal{S}P^{(1,1)}(\Omega)$.

PROOF. Assume (1). By Proposition 4.2 it follows that (h_{jk}) is an hermitian matrix if and only if ω is real. Let $x \in \Omega$ and $w \in \mathbb{C}^n$ and define $\sigma(\zeta) := x + \zeta w$ for $|\zeta| << 1$. Then

$$\sigma^*(\omega)(0) = i/2 \sum h_{jk}(x) d\sigma_j(0) \wedge d\overline{\sigma}_k(0) = i/2 \sum h_{jk}(x) w_j \overline{w}_k d\zeta \wedge d\overline{\zeta}$$

and by Proposition 4.10.(1), $\omega \geq 0$ implies $\sigma^*(\omega)(0) \geq 0$ and then $(h_{jk}(x)) \geq 0$, proving (2).

Assume (2). Then $H = (h_{jk}(x)) \geq 0$. Let $W = (u_{jk})$ be a unitary $(n \times n)$-matrix such that $W^*HW = D$ with D a diagonal matrix with entries $\lambda_j \geq 0$ on the diagonal. Let us consider the following change of coordinates $z = W\tilde{z}$. Then

$$\omega(x) = \frac{i}{2}\sum_{j,k} h_{jk} dz_j \wedge d\overline{z}_k = \frac{i}{2}\sum_{j,k} h_{jk} d(W\tilde{z})_j \wedge d(\overline{W\tilde{z}})_k$$
$$= \sum_{j,k,l,m} h_{jk} u_{jl}\overline{u}_{km} d\tilde{z}_l \wedge d\overline{\tilde{z}}_m = \sum_{m,l} \lambda_l \delta_l^m d\tilde{z}_l \wedge d\overline{\tilde{z}}_m = \sum_m \lambda_m d\tilde{z}_m \wedge d\overline{\tilde{z}}_m.$$

Thus $\omega \in \mathcal{S}P^{(1,1)}(\Omega)$ by Proposition 4.6, and this proves (3).

Finally, if (3) holds, then (1) follows from Lemma 4.11. □

4.2. Positive currents. Now we are in the good shape to define positive currents.

DEFINITION 4.13. Let $\Omega \subset \mathbb{C}^n$ be a domain. A current $T \in \mathcal{D}_k^{(p,p)}(\Omega)$ is a *positive* current of degree p, and we write $T \geq 0$, if $T(\omega) \geq 0$ for all $\omega \in \mathcal{S}P^{(n-p,n-p)}(\Omega)$.

Notice that a positive (p,p)-current T is *real* in the sense that for all real $(n-p,n-p)$-form ω with compact support it follows $T(\omega) \in \mathbb{R}$. Indeed, by Proposition 4.5 it is enough to prove it for strongly positive elementary forms. But if $\omega \in \mathcal{S}P^{(p,p)}(\Omega)$ then $T(\omega) \geq 0$.

Let $x \in \Omega$ be a given point and consider the natural bilinear map $\Lambda^{(p,p)}_{\mathbb{R},x} \times \Lambda^{(n-p,n-p)}_{\mathbb{R},x} \longrightarrow \Lambda^{(n,n)}_{\mathbb{R},x}$ given by $(\eta,\varphi) \mapsto \eta \wedge \varphi$. It is a non-degenerate bilinear application which defines a duality between $\Lambda^{(p,p)}_{\mathbb{R},x}$ and $\Lambda^{(n-p,n-p)}_{\mathbb{R},x}$. In other words, it defines a $\mathbb{R}$-linear isomorphism between $\Lambda^{(p,p)}_{\mathbb{R},x}$ and $(\Lambda^{(n-p,n-p)}_{\mathbb{R},x})^*$ (and in particular $\Lambda^{(p,p)}_{\mathbb{R},x}$ and $\Lambda^{(n-p,n-p)}_{\mathbb{R},x}$ have the same dimension). Thus, if $\{\eta_J\}$ is a basis for $\Lambda^{(p,p)}_{\mathbb{R},x}$, we say that $\{\varphi_J\}$ is a *dual basis* for $\Lambda^{(n-p,n-p)}_{\mathbb{R},x}$ if $\eta_J \wedge \varphi_I = 0$ for $I \neq J$ and $\eta_I \wedge \varphi_I = dV$.

THEOREM 4.14. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $T \in \mathcal{D}_\infty^{(p,p)}(\Omega)$, $T \geq 0$. Then $T \in \mathcal{D}_0^{(p,p)}(\Omega)$. In particular the coefficients of T are positive Radon measures with respect to any basis which is dual for a basis of strongly positive forms of $C^\infty(\Omega, \Lambda^{(n-p,n-p)}_{\mathbb{R}})$.*

PROOF. By Proposition 4.5 the space of (real) $(n-p,n-p)$-forms has a basis of strongly positive forms, say $\{\varphi_1,\ldots,\varphi_M\}$. Let $\{\eta_1,\ldots,\eta_M\}$ be a basis of real (p,p)-forms, dual for $\{\varphi_1,\ldots,\varphi_M\}$. According to Theorem 2.6 we can write

$$T = \sum_{j=1}^M T_j \eta_j,$$

with $T_j \in \mathcal{D}is^\infty(\Omega)$. Fix $t \in \{1,\ldots,M\}$. Then

$$0 \leq T(\varphi_t) = \sum_{j=1}^M \mathcal{W}^{-1}(T_j)(\eta_j \wedge \varphi_t) = \mathcal{W}^{-1}(T_t)(\eta_t \wedge \varphi_t).$$

Since clearly the isomorphism $\mathcal{W}$ maps positive distributions to positive (n,n)-forms, then T_t is a positive functional on $C_0^\infty(\Omega)$ and by Lemma 1.5 it follows that T_t is a positive Radon measure. Another application of Theorem 2.6 implies that $T \in \mathcal{D}_0^{(p,p)}(\Omega)$. □

Now we can relate plurisubharmonic functions to positive currents:

THEOREM 4.15. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in L^1_{\text{loc}}(\Omega)$. The following are equivalent:*

(1) *There exists $v \in \mathsf{Psh}(\Omega)$ such that $u = v$ almost everywhere in Ω.*
(2) *The matrix $(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k})$ is positive semidefinite in the sense of distributions.*
(3) *$dd^c u$ is a positive $(1,1)$-current.*

PROOF. By Proposition III.2.3 (1) is equivalent to (2). Now assume that $u \in C^2(\Omega)$. Then $dd^c u = 4(i/2)\partial\overline{\partial} u$ and thus $u \in \mathsf{Psh}(\Omega)$ if and only if $dd^c u$ is a positive $(1,1)$-current.

If $u \in \mathsf{Psh}(\Omega)$ (no further regularity), let $u_\epsilon \in C^\infty(\Omega_\epsilon) \cap \mathsf{Psh}(\Omega_\epsilon)$ be such that $\{u_\epsilon\}$ pointwise decreases to u (see Theorem III.2.1). Then $dd^c u_\epsilon \geq 0$. Let $\varphi \in \mathcal{S}P^{(n-1,n-1)}(\Omega)$. By Beppo Levi's theorem it follows

$$dd^c u(\varphi) := \int_\Omega u \, dd^c \varphi = \lim_{\epsilon \to 0} \int_\Omega u_\epsilon \, dd^c \varphi = \lim_{\epsilon \to 0} dd^c u_\epsilon(\varphi) \geq 0,$$

thus $dd^c u \geq 0$ showing that (1) implies (3). Conversely, assume $dd^c u \geq 0$. Let $u_\epsilon = u * \chi_\epsilon \in C^\infty(\Omega_\epsilon)$. Then u_ϵ converges to u in $L^1_{\text{loc}}(\Omega)$ (and almost everywhere). In particular for all $f \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} f dV := \int_\Omega u \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} dV = \lim_{\epsilon \to 0} \int_\Omega u_\epsilon \frac{\partial^2 f}{\partial z_j \partial \overline{z}_k} dV.$$

Therefore $dd^c u \geq 0$ implies that $dd^c u_\epsilon \geq 0$ and then $u_\epsilon \in \mathsf{Psh}(\Omega_\epsilon)$ for all ϵ. Now, arguing as at the end of Theorem I.4.1 we see that u_ϵ is decreasing in ϵ and thus, by Lemma III.1.10, it follows that $v(z) = \lim_{\epsilon \to 0} u_\epsilon(z) \in \mathsf{Psh}(\Omega)$. Since $u = v$ almost everywhere then (1) follows. □

REMARK 4.16. By the previous theorem, if $u \in \mathsf{Psh}(\Omega)$ then $dd^c u$ is a d-*closed* positive $(1,1)$-current, namely, $d(dd^c u) = 0$.

REMARK 4.17. According to Theorem 4.15 and Theorem 4.12, if $u \in \mathsf{Psh}(\Omega) \cap C^\infty(\Omega)$ then $dd^c(u) \in SP^{(1,1)}(\Omega)$.

Conversely we state and sketch a proof of the following result which says that locally every $(1,1)$-positive current has a potential.

THEOREM 4.18. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $T \in \mathcal{D}^{(1,1)}(\Omega)$ be a positive current such that $dT = 0$. For any $z \in \Omega$ there exist an open neighborhood $U_z \subset \Omega$, $z \in U_z$ and $u \in \mathsf{Psh}(U_z)$ such that $T = dd^c u$ in $\mathcal{D}^{(1,1)}(U_z)$. Moreover, if $T = T_\omega$ is the current associated to $\omega \in C^\infty(\Omega, \Lambda^{(1,1)})$ then $u \in C^\infty(U_z)$ for all z.*

PROOF. One can define a cohomology on Ω given by d-closed currents over d-exact currents. As in the smooth case, such a cohomology is locally exact, in the sense that an analogue of Poincaré's Lemma holds for currents (see [**12**], [**9**]). Namely, since $dT = 0$, for any $z \in \Omega$ there exist a convex open neighborhood $U \subset \Omega$, $z \in U$ and $S \in \mathcal{D}^1(U)$, S real, such that $dS = T$ in $\mathcal{D}^2(U)$. Now $S = S^{(1,0)} + S^{(0,1)}$ with $S^{(1,0)} \in \mathcal{D}^{(1,0)}(U)$ and $S^{(0,1)} = \overline{S^{(1,0)}} \in \mathcal{D}^{(0,1)}(U)$. Now,

$$dS = \partial S + \overline{\partial} S = \partial S^{(1,0)} + \partial S^{(0,1)} + \overline{\partial} S^{(1,0)} + \overline{\partial} S^{(0,1)}$$

and because T is of bidegree $(1,1)$ it follows that $\partial S^{(1,0)} = \overline{\partial} S^{(0,1)} = 0$. By the Dolbeault lemma for currents (similar to the one for smooth forms, see [**9**]) there exists $\varphi \in \mathcal{D}^0(U) \simeq \mathcal{D}is(U)$ such that $\overline{\partial}\varphi = S^{(0,1)}$. Therefore $S = \overline{\overline{\partial}\varphi} + \overline{\partial}\varphi = \partial\overline{\varphi} + \overline{\partial}\varphi$ and

$$T = dS = d(\partial\overline{\varphi} + \overline{\partial}\varphi) = \overline{\partial}\partial\overline{\varphi} + \partial\overline{\partial}\varphi = \partial\overline{\partial}(\varphi - \overline{\varphi}) = 2i\partial\overline{\partial} v = dd^c v,$$

where $v = -i(\varphi - \overline{\varphi})/2$ is a real distribution. Thus we have a real distribution v such that $T = dd^c v$. If $T \geq 0$ then $dd^c v \geq 0$, which implies that $\mathcal{L}(v) \geq 0$, and by Proposition 1.8 it follows that v is associated to a function $u \in \mathsf{Psh}(U)$.

Finally, assume that ω is a smooth $(1,1)$-form and $T = T_\omega$. Then $T = -2i\overline{\partial}(\partial u)$. Since T is C^∞ then $\partial u \in C^\infty(U, \Lambda^{(1,0)})$ because $\overline{\partial}$ is a hypoelliptic operator in degree $(p,0)$. Indeed, $\overline{\partial}\omega = 0$ because $\overline{\partial} T = 0$ and $\overline{\partial} T = T_{\overline{\partial}\omega}$ and thus (by the Poincaré lemma for the $\overline{\partial}$ operator—recall that U is convex and thus pseudoconvex) there exists $\theta \in C^\infty(U, \Lambda^{(1,0)})$ such that $\overline{\partial}\theta = \omega$. Therefore (identifying forms with currents as usual) $\overline{\partial}(\theta + 2i\partial u) = 0$ and then $\theta + 2i\partial u$ is holomorphic which implies that $\partial u \in C^\infty(U, \Lambda^{(1,0)})$ and, since $\overline{\partial} u = \overline{\partial u}$ because u is real, $du \in C^\infty(U, \Lambda^1)$. From this it follows that $u \in C^\infty(U)$. ☐

EXAMPLE 4.19. Let $Z \subset \Omega$ be a complex submanifold of (complex) dimension p (with no boundary in Ω). Then the integration current $[Z]$ is a $(n-p, n-p)$-positive d-closed current. Indeed, let $i : Z \hookrightarrow \Omega$ be the natural holomorphic embedding. If $\varphi \in \mathcal{S}P^{(p,p)}(\Omega)$ then $i^*(\varphi) \geq 0$ by Proposition 4.10 and thus $[Z](\varphi) := \int_Z i^*(\varphi) \geq 0$. Finally, for $\varphi \in C^0(\Omega, \Lambda^{2n-2p-1})$ and by Stokes' theorem

$$d[Z](\varphi) := [Z](d\varphi) = \int_Z i^*(d\varphi) = \pm \int_{\partial Z} i^*(\varphi) = 0.$$

5. Integration over analytic sets

In this section we sketch how to define currents of integration along analytic subsets of $\Omega \subset \mathbb{C}^n$. We refer to [**9**] for details.

Let $T \in \mathcal{D}_0^{(p,q)}(\Omega)$ be a (p,q)-currents in Ω. According to Theorem 2.6 we can write T as in (2.2), with $T_{I,J}$ Radon measure. Let us define the *mass* $\|T\|$ of T as

$$\|T\| := \sum_{I,J} |T_{I,J}|,$$

where $|T_{I,J}|$ is the measure total variation of $T_{I,J}$ (see Remark 1.4). Since by construction $T_{I,J}$ is absolutely continuous with respect to $\|T\|$ then the Radon-Nykodim theorem implies that there exists a locally $\|T\|$-measurable complex function $f_{I,J} \in L^1_{\text{loc}}(\Omega, \|T\|)$ such that $T_{I,J} = f_{I,J}\|T\|$ (according to Remark 1.4, such a function $f_{I,J}$ is defined on a relatively compact Borel subset E of Ω by applying the Radon-Nykodim theorem to $T_{I,J}|_E$). Since

$$\|T\| = \sum |T_{I,J}| = \sum |f_{I,J}|\|T\|,$$

it follows that $\sum |f_{I,J}| = 1$. Thus if we set $f := (i/2)^p \sum f_{I,J} dz_{i_1} \wedge d\overline{z}_{j_1} \wedge \dots$ it follows that

$$T = \|T\| f.$$

LEMMA 5.1. *The current $T \in \mathcal{D}_0^{(p,p)}(\Omega)$ is positive if and only if the form f is positive at $\|T\|$-almost all points of Ω.*

PROOF. If $f \geq 0$ (for $\|T\|$-almost everywhere) then for all $\varphi \in \mathcal{S}P^{(n-p,n-p)}(\Omega)$, we have

$$T(\varphi) = \|T\| f(\varphi) := \int_\Omega f \wedge \varphi \|T\| \geq 0.$$

Conversely, if $T \geq 0$ then for all $\varphi \in \mathcal{S}P^{(n-p,n-p)}(\Omega)$ we have

$$0 \leq T(\varphi) = \|T\| f(\varphi) := \int_\Omega f \wedge \varphi \|T\|$$

thus $f \wedge \varphi \geq 0$ but at most zero $\|T\|$-measure sets. Letting φ varying, we see that $f \geq 0$ for $\|T\|$-almost all points. □

DEFINITION 5.2. A *complete pluripolar set* $E \subset \Omega$ is a subset such that for each $x \in E$ there exists an open neighborhood $V_x \subset \Omega$ and a function $v \in \mathsf{Psh}(V_x)$ such that $E \cap V_x = \{v = -\infty\}$.

The following theorem is due to Skoda and El Mir:

THEOREM 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $E \subset \Omega$ be a closed complete pluripolar set. Let $T \in \mathcal{D}^{(p,p)}(\Omega \setminus E)$ be a positive (p,p)-current, $dT = 0$. Suppose that $\|T\|$ is bounded near each point of E. Let $\tilde{T}$ be the trivial extension of T to E obtained by extending $T_{I,J}$ to zero on E. Then $\tilde{T}$ is positive and closed on Ω.*

Now let $X \subset \Omega$ be a (possibly singular) complex subvariety (with no boundary) of pure dimension p. Then X defines an element of $(H^{2n-2p}(\Omega, \mathbb{R}))^*$ as follows. Given any $(2n-2p)$-form φ such that $d\varphi = 0$, one can define $\int_X \varphi$ by taking any C^∞-smooth submanifold $X' \subset \Omega$ which is homologous to X and defining $\int_X \varphi = \int_{X'} i^*_{X'}(\varphi)$, with $i_{X'} : X' \hookrightarrow \Omega$ the natural embedding. Since $d\varphi = 0$, Stokes theorem implies that such a definition is independent of the cycle X' homologous to X which has been chosen (see, *e.g.*, [**13**]).

However, this definition does not allow to define a current of integration on X (the problem being how to defining integration of non-closed test forms).

We can thus try to define the current of integration $[X]$ by integrating over the regular part X^r of X:

$$[X^r](\varphi) := \int_{X^r} i^*(\varphi)$$

where $i : X \hookrightarrow \Omega$ is the natural embedding (and $\varphi \in C_0(\Omega, \Lambda^{(p,p)})$. It can be proved that $[X^r]$ is a current of bidegree $(n-p, n-p)$ on $\Omega \setminus \mathsf{Sing(X)}$. It is also clearly positive and, suitably using Stokes theorem for subvariety, one can even shows that it is closed. The following result of Lelong implies that such a definition is the good one:

THEOREM 5.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $X \subset \Omega$ be a complex subvariety (with no boundary) of pure dimension p. The current $[X^r]$ has finite mass near every point of* $\mathsf{Sing(X)}$*. Thus its trivial extension $[X]$ is a closed positive $(n-p, n-p)$-current of order 0 on Ω.*

Notice that, by the previous theorem and since $\mathsf{Sing(X)}$ has zero measure in X, the current $[X]$ defined as extension of the current of integration $[X^r]$, coincides *on closed test forms* with the integration on cycles homologous to X .

CHAPTER 5

The Complex Monge-Ampère operator

1. Maximal plurisubharmonic functions

Consider the unit ball $\mathbb{B} \subset \mathbb{R}^m$ and let $\varphi \in C^0(\partial\mathbb{B})$. The unique solution $u \in C^0(\overline{\mathbb{B}}) \cap \mathsf{harm}(\mathbb{B})$ to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{B} \\ u|_{\partial\mathbb{B}} = \varphi \end{cases} \tag{1.1}$$

can be characterized as

$$u(x) = \sup\{v(x) : v \in \mathsf{subh}(\mathbb{B}), \limsup_{\mathbb{B}\ni x\to p} v(x) \leq \varphi(p) \forall p \in \partial\mathbb{B}\}. \tag{1.2}$$

Thus, harmonic functions can be characterized as the *maximal* functions among subharmonic functions. In other words:

PROPOSITION 1.1. *Let $\Omega \subset \mathbb{R}^m$ be a domain and let $u \in C^0(\Omega) \cap \mathsf{subh}(\Omega)$. Then $u \in \mathsf{harm}(\Omega)$ if and only if for all $G \subset\subset \Omega$ open and $v \in \mathsf{subh}(G)$ such that $\limsup_{x\to p} v(x) \leq u(p)$ for all $p \in \partial G$ it follows that $v \leq u$ in G.*

PROOF. The necessity of the condition follows from the maximum principle. Conversely, suppose that $G \subset\subset \Omega$ is an open ball. Let $v \in \mathsf{harm}(G) \cap C^0(\overline{G})$ be such that $v(p) = u(p)$ for all $p \in \partial G$. Then by the subharmonicity $u \leq v$ in G and by hypothesis $v \leq u$ which implies that $u = v$ and thus $u \in \mathsf{harm}(G)$, proving that $u \in \mathsf{harm}(\Omega)$. □

Since pluriharmonic functions are harmonic, and plurisubharmonic functions are subharmonic, pluriharmonic functions are maximal among plurisubharmonic functions. However, considering the unit ball $\mathbb{B} \subset \mathbb{C}^n$ and given $\varphi \in C^0(\partial\mathbb{B})$, there is, in general, no $u \in \mathsf{Ph}(\mathbb{B}) \cap C^0(\overline{\mathbb{B}})$ such that $u|_{\partial\mathbb{B}} = \varphi$ and maximal plurisubharmonic functions are not necessarily pluriharmonic.

EXAMPLE 1.2. Let us consider the unit ball $\mathbb{B} \subset \mathbb{C}^2$. Let $f : \mathbb{D} \to (-\infty, \infty)$ be any continuous subharmonic function which is not C^1 at some points of $\mathbb{D}$. Let $\varphi(z) = f(z_1)$ for $z \in \partial\mathbb{B}$. Then $\varphi \in C^0(\partial\mathbb{B})$. Let us define $u(z) := f(z_1)$ for $z \in \overline{\mathbb{B}}$. Then $\mathbb{D} \ni \zeta \mapsto u(a+\zeta b) = f(a_1+\zeta b_1)$ is subharmonic for all $\{a+\zeta b : \zeta \in \mathbb{D}\} \subset\subset \mathbb{B}$. Hence $u \in \mathsf{Psh}(\mathbb{B}) \cap C^0(\overline{\mathbb{B}})$. Since u is not C^∞ by construction, $u \notin \mathsf{Ph}(\mathbb{B})$. However, if $v \in \mathsf{Psh}(\mathbb{B})$ is such that $\limsup_{z\to p} v(z) \leq \varphi(p)$ for all $p \in \partial\mathbb{B}$ then $v(z) \leq u(z)$ for all $z \in \mathbb{B}$ because u is harmonic on $z_1 =$ constant.

This implies that there are no $V \in \mathbb{C}^0(\overline{\mathbb{B}}) \cap \mathsf{Ph}(\mathbb{B})$ such that $V|_{\partial\mathbb{B}} = \varphi$, because otherwise the maximum principle would imply $V = u$ forcing u to be of class C^∞.

DEFINITION 1.3. Let $\Omega \subset \mathbb{C}^n$ be a domain. A function $u \in \mathsf{Psh}(\Omega)$ is said to be *maximal* (according to Sadullaev) if for any open set $G \subset\subset \Omega$ and $v \in \mathsf{Psh}(G)$ such that $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$ it follows that $v \leq u$ in G.

The function u in Example 1.2 is an example of maximal plurisubharmonic function which is not pluriharmonic. More generally:

PROPOSITION 1.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega)$. Suppose that for all $z \in \Omega$ there exists a proper holomorphic map $\varphi : \mathbb{D} \to \Omega$ such that $z \in \varphi(\mathbb{D})$ and $u \circ \varphi \in \mathsf{harm}(\mathbb{D})$. Then u is maximal.*

PROOF. Let $G \subset\subset \Omega$ be an open set. Let $v \in \mathsf{Psh}(G)$ be such that $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$. Let $z \in G$ and let $\varphi : \mathbb{D} \to \Omega$ holomorphic and proper such that $\varphi(\zeta) = z$ for some $\zeta \in \mathbb{D}$ and $u \circ \varphi \in \mathsf{harm}(\mathbb{D})$. Since φ is proper then $\varphi^{-1}(G)$ is an open set relatively compact in $\mathbb{D}$ and we can assume, without loss of generality, that it is connected. Now, $u \circ \varphi$ is harmonic in $\mathbb{D}$. Also, by Proposition III.3.2 either $v \circ \varphi \equiv -\infty$ or $v \circ \varphi \in \mathsf{subh}(\varphi^{-1}(G))$. In the latter case, since by hypothesis the upper semicontinuous extension of $v \circ \varphi$ to the boundary of $\varphi^{-1}(G)$ is less then or equal to $u \circ \varphi$ on $\varphi^{-1}(\partial G)$ it follows by the very definition of subharmonic functions that $u \circ \varphi \geq v \circ \varphi$ on $\varphi^{-1}(G)$ and hence $u(z) \geq v(z)$ proving that u is maximal. $\square$

Proposition 1.4 gives a geometric criterion for maximality. In particular one can use such a criterion to construct a maximal plurisubharmonic function by giving a foliation on Ω whose leaves are properly embedded holomorphic discs and a plurisubharmonic function on Ω whose restriction on each leaf is harmonic.

PROPOSITION 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega)$. The following are equivalent:*

(1) *u is maximal.*
(2) *For each open set $G \subset\subset \Omega$ and $v \in \mathsf{Psh}(G)$ such that $\liminf_{z\to p}[u(z) - v(z)] \geq 0$ for all $p \in \partial G$ it follows $u \geq v$ in G.*
(3) *For each open set $G \subset\subset \Omega$ and $v \in \mathsf{Psh}(\Omega)$ such that $\liminf_{G\ni z\to p}[u(z) - v(z)] \geq 0$ for all $p \in \partial G$ it follows $u \geq v$ in G.*
(4) *For each $v \in \mathsf{Psh}(\Omega)$ which has the property that for each $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that $u - v \geq -\epsilon$ in $\Omega \setminus K$ then $u \geq v$ in Ω.*
(5) *For each open set $G \subset\subset \Omega$ and $v \in \mathsf{Psh}(\Omega)$ such that $v(p) \leq u(p)$ for all $p \in \partial G$ it follows $u \geq v$ in G.*

PROOF. Assume (1) and let v, G as in (2). Then $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$. Indeed, let $\{z_k\} \subset G$ be such that $z_k \to p$ and $L := \limsup_{z\to p} v(z) = \lim_{k\to\infty} v(z_k)$. Then

$$0 \leq \liminf_{z\to p}[u(z) - v(z)] \leq \liminf_{k\to\infty}[u(z_k) - v(z_k)] \leq \limsup_{k\to\infty} u(z_k) - L \leq u(p) - L.$$

Thus $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$ and by (1) $v \leq u$ in G, which proves (2).

Clearly (2) implies (3) because $\mathsf{Psh}(\Omega)|_G \subset \mathsf{Psh}(G)$.

Now assume (3) holds. Let $v \in \mathsf{Psh}(\Omega)$ with the property that for each $\epsilon > 0$ there exists a compact set $K \subset \Omega$ such that $u - v \geq -\epsilon$ in $\Omega \setminus K$. Seeking for a contradiction, we assume that there exists $a \in \Omega$ such that $u(a) < v(a) - \delta$ for some $\delta > 0$. By hypothesis, there exists a compact set $K \subset \Omega$ such that $u(z) - v(z) \geq -\delta/2$ for all $z \in \Omega \setminus K$. Notice that $a \in K$. Let $G \subset\subset \Omega$ be an open set such that $K \subset G$. Then $\liminf_{G\ni z\to p}[u(z) - v(z) + \delta/2] \geq 0$ for all $p \in \partial G$. Since $(v - \delta/2) \in \mathsf{Psh}(\Omega)$ then (3) implies that $u \geq v - \delta/2$ in G and in particular then $u(a) \geq v(a) - \delta/2$, absurd. Thus (3) implies (4).

Assume (4) holds. Let $G \subset\subset \Omega$ be an open set and let $v \in \mathsf{Psh}(\Omega)$ be such such $v(p) \leq u(p)$ for all $p \in \partial G$. Let us define

$$w(z) := \begin{cases} u(z) & \text{for } z \in \Omega \setminus G \\ \max\{u(z), v(z)\} & \text{for } z \in G \end{cases} \tag{1.3}$$

By the analogous of Proposition I.5.1 for plurisubharmonic functions, $w \in \mathsf{Psh}(\Omega)$. By construction, for all $\epsilon > 0$ it follows that $0 = u(z) - w(z) \geq -\epsilon$ for all $z \in \Omega \setminus \overline{G}$. By (4) it follows that $u \geq w$ in Ω and thus $u \geq v$ in G, proving (5).

Finally, if (5) holds, given $G \subset\subset \Omega$ an open set and $v \in \mathsf{Psh}(G)$ such that $\limsup_{z\to p} v(z) \leq u(p)$ for all $p \in \partial G$ we define w has in (1.3). Then $w \in \mathsf{Psh}(\Omega)$, $w \leq u$ on ∂G and by (5) it follows that $w \leq u$ in G, proving that $v \leq u$ in G and then (1). □

2. Characterization of maximal plurisubharmonic functions of class C^2

In this section we characterize maximal plurisubharmonic functions of class C^2 by means of their Levi form. Let $\Omega \subset \mathbb{C}^n$ and let $u \in C^2(\Omega)$. Then

$$(dd^c u)^n := \underbrace{dd^c u \wedge \ldots \wedge dd^c u}_{n} = 4^n n! \det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right) dV \tag{2.1}$$

where dV is the volume form (IV.2.1)

LEMMA 2.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega) \cap C^2(\Omega)$. Then $(dd^c u)^n \geq 0$.*

PROOF. It follows directly from Theorem IV.4.15 and (2.1). [Alternatively, by Theorem IV.4.15 the $(1,1)$-form (with continuous coefficients) $dd^c u$ is positive. By Theorem IV.4.12.(2) it is actually strongly positive. Therefore $(dd^c u)^n$ is a positive (n,n)-form.] □

THEOREM 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega) \cap C^2(\Omega)$. Then u is maximal in Ω if and only if $(dd^c u)^n = 0$ in Ω.*

PROOF. Assume first that $(dd^c u)^n = 0$. Let $G \subset\subset \Omega$ be an open set and let $v \in \mathsf{Psh}(\Omega)$ be such that $v(p) \leq u(p)$ for all $p \in \partial G$. We want to show that $v \leq u$ in G which, by Proposition 1.5 and by the arbitrariness of v implies that u is maximal. Seeking for a contradiction we assume that there exists $a \in G$ such that $0 < v(a) - u(a) = \sup_{z\in G}(v - u)(z)$. Let $\delta > 0$ be such that $v(a) - \delta > u(a)$. Then $v(z) - \delta \in \mathsf{Psh}(\Omega)$ and $v(p) - \delta < u(p)$ for all $p \in \partial G$. Thus, if $\{v_\epsilon\}$ is the decreasing sequence of regularizing plurisubharmonic functions for $v - \delta$, there exists $\epsilon > 0$ such that $G \subset\subset \Omega_\epsilon$, $v_\epsilon \in C^\infty(\Omega_\epsilon) \cap \mathsf{Psh}(\Omega_\epsilon)$, $v_\epsilon(a) > u(a)$ and $v_\epsilon(p) \leq u(p)$ for all $p \in \partial G$.

Let $M = \max_{z \in \overline{G}} \|z\|^2$. Let $\lambda > 0$ be such that $v_\epsilon(a) + \lambda(\|a\|^2 - M) > u(a)$ and let $w(z) := v_\epsilon(z) + \lambda(\|z\|^2 - M)$. Then $w \in \mathsf{Psh}(\Omega_\epsilon)$, $w(p) \leq u(p)$ for all $p \in \partial G$, $w(a) > u(a)$ and $\mathcal{L}_z(w(z))(b; b) > 0$ for all $z \in G$ and $b \in \mathbb{C}^n \setminus \{0\}$.

Let $x \in G$ be a local maximum of $w - u$. Since $w(a) - u(a) > 0$ and $w - u \leq 0$ on ∂G, such a point does exist.

Notice that $\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)(x) = 0$ is equivalent to the existence of a vector $b \in \mathbb{C}^n \setminus \{0\}$ such that $\mathcal{L}_x(u)(b; b) = 0$. Let $f(\zeta) := (w - u)(x + \zeta b)$ for $\zeta \in \mathbb{C}$, $|\zeta| << 1$. Since $\zeta = 0$ is a local maximum and f is of class C^2 then $\Delta f(0) \leq 0$. Therefore

$$0 \geq \Delta f(0) = 4\mathcal{L}_x(w - u)(b; b) = 4\mathcal{L}_x(w)(b; b) > 0,$$

a contradiction. Therefore u is maximal.

Conversely, assume that $u \in C^2(\Omega) \cap \mathsf{Psh}(\Omega)$ is maximal. Assume by contradiction that there exists $a \in \Omega$ such that $\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}\right)(a) \neq 0$. This implies that $\mathcal{L}_a(u)$ is positive definite. Since u is of class C^2 one can find a ball $\mathbb{B}(a, r) \subset\subset \Omega$ and $C > 0$ such that $\mathcal{L}_z(u)(b; b) \geq C$ for all $b \in \mathbb{C}^n$ such that $\|b\| = 1$ and $z \in \mathbb{B}(a, r)$. Then $u(z) + C(r^2 - \|z\|^2) \in \mathsf{Psh}(\mathbb{B}(a, r))$ because $\mathcal{L}(u(z) + C(r^2 - \|z - a\|^2))(b; b) = \mathcal{L}(u)(b; b) - C\|b\|^2 \geq 0$ by construction. Let

$$v(z) = \begin{cases} u(z) & \text{for } z \in \Omega \setminus \overline{\mathbb{B}(a, r)} \\ u(z) + C(r^2 - \|z - a\|^2) & \text{for } z \in \mathbb{B}(a, r) \end{cases}$$

By the analogous of Proposition I.5.1 for plurisubharmonic functions, $v \in \mathsf{Psh}(\Omega)$. Moreover, $v(p) = u(p)$ for all $p \in \partial\mathbb{B}(a, r)$ and $v(a) = u(a) + Cr^2 > u(a)$ against the maximality of u. □

REMARK 2.3. With some more effort (see, *e.g.*, [**16**, Proposition 3.1.7]) one can prove the following generalization of the previous theorem: let $u, v \in C^2(\Omega) \cap \mathsf{Psh}(\Omega)$. Let $G \subset\subset \Omega$ be an open set. If $v \leq u$ on ∂G and $(dd^c u)^n \leq (dd^c v)^n$ in G then $v \leq u$ in G.

REMARK 2.4. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $u \in C^2(\Omega)$. The condition $(dd^c u)^n = 0$ is equivalent to the fact that the rank of the Levi form $\mathcal{L}(u)$ is $\leq n - 1$ in Ω. In other words, $(dd^c u)^n = 0$ is equivalent to the existence for every $z \in \Omega$ of vector $v \in \mathbb{C}^n \setminus \{0\}$ (depending on z) such that $\mathcal{L}(u)_z(v; v) = 0$.

3. Maximal plurisubharmonic functions and foliations

In this section we relate maximal (regular) plurisubharmonic functions to complex foliations (in Riemann surfaces).

Let $\Omega \subset \mathbb{C}^n$ be a domain. A real *foliation* of class C^k and dimension $2m$ on Ω is a map $\mathcal{F} : \Omega \to (T\Omega)^{\mathbb{R}}$ such that for each $p \in \Omega$ there exist an open set $U \subset \mathbb{C}^{n-m}$, $D \subset \mathbb{C}^m$ and $\Phi : U \times D \to \Omega$ a C^k-diffeomorphism onto its image such that $d_w\Phi(\{x\} \times T_w D) = \mathcal{F}_{\Phi(x,\zeta)}$ for each $x \in U$. Moreover, if the map $\mathbb{D} \ni w \mapsto \Phi(x, w)$ is holomorphic for each $x \in U$ the foliation $\mathcal{F}$ is said to be a *complex foliation* of dimension m. Notice that if $\mathcal{F}$ is a real (*respectively* complex) foliation of dimension m then for each $p \in \Omega$ there exists a real (*respect.* complex) manifold $M(p) \subset \Omega$, $p \in M(p)$ such that $(T_z M(p))^{\mathbb{R}} = \mathcal{F}_z$ for all $z \in M$. We call such a manifold $M(p)$ the *leaf* for $\mathcal{F}$ at p. We refer the interested reader to **[7]** for details on foliations.

By definition, a foliation is a distribution of $(T\Omega)^{\mathbb{R}}$, namely, a map $\Omega \to (T\Omega)^{\mathbb{R}}$. It is clear that if $\mathcal{F}$ is a foliation of class C^1 then $[\mathcal{F}, \mathcal{F}] \subseteq \mathcal{F}$, that is, $\mathcal{F}$ is *involutive*. The converse is contained in the well known Frobenius' theorem:

THEOREM 3.1 (Frobenius). *Let $\Omega \subset \mathbb{C}^n$ be a domain. A C^k ($k \geq 1$) distribution $\mathcal{F} \subset (T\Omega)^{\mathbb{R}}$ is a foliation of class C^{k+1} if and only if it is involutive.*

If one is interested in the complex side of the story, then one can consider (complex) distributions $\mathcal{F} \subset T\Omega$. For later reference we prove here only the following complex version of Frobenius's theorem.

PROPOSITION 3.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $\mathcal{F} \subset T\Omega$ be a C^k ($k \geq 1$) distribution of complex rank* 1. *Then $\mathcal{F}$ is a complex foliation of class C^{k+1} and (complex) dimension* 1.

PROOF. Consider the associated distribution $\mathcal{F}^{\mathbb{R}} \subset (T\Omega)^{\mathbb{R}}$. Then $\mathcal{F}^{\mathbb{R}}$ is a C^k distribution of rank 2. Let U be an open set in Ω on which $\mathcal{F}$ is trivial and let $z \mapsto Z(z)$ be a generator for $\mathcal{F}$ on U. Then Z, JZ (here J is the complex structure coming from the multiplication by i in $T\Omega$) generate $\mathcal{F}^{\mathbb{R}}$ on U. Now

$$0 = [Z, JZ] = [Z, iZ] = i[Z, Z] = 0,$$

and then $\mathcal{F}^{\mathbb{R}}$ is involutive. By Frobenius's theorem $\mathcal{F}^{\mathbb{R}}$ is a real foliation of dimension 2 and class C^{k+1}. Let $p \in \Omega$ and let $M(p) \subset \Omega$ be a real two dimensional submanifold such that $T_z M(p) = \mathcal{F}^{\mathbb{R}}_z$ for all $z \in M(p)$. To see that $\mathcal{F}$ is a complex foliation it is enough to prove that $M(p)$ is a complex submanifold of Ω. By construction $T_z M(p) = JT_z M(p)$ for all $z \in M(p)$ and therefore $T_z M(p)$ has a structure of complex subspace of $T_z\Omega$. Since a C^1-submanifold of $\mathbb{C}^n$ is a complex manifold if and only if its real tangent space at every point is a complex space, $M(p)$ is a complex curve and thus $\mathcal{F}$ is a complex foliation of dimension one. □

We begin with the following result which generalizes Proposition 1.4.

THEOREM 3.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in C^2(\Omega) \cap \mathsf{Psh}(\Omega)$. If there exists a one dimensional complex foliation $\mathcal{F}$ of class C^1 of Ω such that the restriction of u to each leaf is harmonic then u is maximal.*

PROOF. Let $\Phi : U \times \mathbb{D} \to \Omega$ be a local foliation chart for $\mathcal{F}$, namely, $U \subset \mathbb{C}^{n-1}$ is an open set, Φ is a diffeomorphism (onto its image) of class C^1, for each $x \in U$ the image $\Phi(x, \mathbb{D})$

is contained in a leaf of $\mathcal{F}$ and the map $\zeta \mapsto \Phi(x,\zeta)$ is holomorphic. By hypothesis for each $x \in U$ fixed, the map $\zeta \mapsto u \circ \Phi(x,\zeta)$ is harmonic on Δ and therefore

$$0 = \Delta_\zeta u \circ \Phi(x,\zeta) = 4\mathcal{L}(u)(\frac{\partial \Phi}{\partial \zeta}(x,\zeta), \frac{\partial \Phi}{\partial \zeta}(x,\zeta)).$$

Since $\frac{\partial \Phi}{\partial \zeta}(x,\zeta) \neq 0$ because Φ is a diffeomorphism, then $\mathcal{L}(u)$ has rank $\leq n-1$. In particular it follows that $(dd^c u)^n = 0$ at $\Phi(x,\zeta)$. By the arbitrariness of $x \in U$ and Φ and according to Remark 2.4, it follows that $(dd^c u)^n \equiv 0$ in Ω and then u is maximal by Theorem 2.2. □

The converse of the previous theorem is given in the following form:

THEOREM 3.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in C^3(\Omega) \cap \mathsf{Psh}(\Omega)$. Suppose u is maximal and $(dd^c u)^{n-1}_z \neq 0$ for all $z \in \Omega$. Then there exists a one dimensional complex foliation $\mathcal{F}$ of class C^2 of Ω such that the restriction of u to each leaf of $\mathcal{F}$ is harmonic.*

PROOF. Since u is maximal (and of class C^3) Theorem 2.2 and Remark 2.4 imply that the rank of $\mathcal{L}(u)$ is $\leq n-1$ and by hypothesis it is exactly $n-1$ at each $z \in \Omega$. This implies that for each $z \in \Omega$ there exists a vector $Z(z) \in T_z\Omega \setminus \{0\}$ unique up to complex multiples, such that $\mathcal{L}_z(u)(Z(z), Z(z)) = 0$. Let $\mathcal{F}_z = \mathsf{span}_{\mathbb{C}}\{Z(z)\} \subset T_z\Omega$. Thus we have a well defined distribution $\mathcal{F} : \Omega \ni z \mapsto \mathcal{F}_z \subset T\Omega$. Notice that, if $Z = (Z_1, \ldots, Z_n)$ then Z is the only solution (up to complex multiples) of the system

$$\sum_{j=1}^n \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) Z_j(z) = 0, \quad k = 1, \ldots, n.$$

Thus, for every $p \in \Omega$ there exists a neighborhood U_p of p and $j_p \in \{1, \ldots, n\}$ such that $Z_j(z) = P_j(z) Z_{j_p}$ where $P_j(z)$ is a polynomial combination of $\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)$ for $j \in \{1, \ldots, n\} \setminus \{j_p\}$ and $z \in U_p$. Since by hypothesis u is of class C^3, one can perform a choice (for instance $Z_{j_p} \equiv 1$) which makes the map $z \mapsto Z(z)$ of class C^1, showing that $z \mapsto \mathcal{F}_z$ is a C^1 distribution. Thus $\mathcal{F} \subset T\Omega$ is a C^1 distribution of complex rank one and by Proposition 3.2 it is a complex one dimensional foliation of class C^2.

It remains to show that the restriction of u to every leaf of $\mathcal{F}$ is harmonic. Let $p \in \Omega$ and let $\varphi : \mathbb{D} \to \Omega$ be holomorphic such that $\varphi(0) = p$, $\varphi(\mathbb{D})$ is contained in a leaf of $\mathcal{F}$ and $\dot{\varphi}(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}$. Then $\dot{\varphi}(\zeta) = \lambda(\zeta) Z(\varphi(\zeta))$ for some C^1-function $\lambda(\zeta)$. Hence

$$\Delta_\zeta (u \circ \varphi)(\zeta) = 4\mathcal{L}(u)_{\varphi(\zeta)}(\dot{\varphi}(\zeta); \dot{\varphi}(\zeta)) = 4|\lambda(\zeta)|^2 \mathcal{L}(u)_{\varphi(\zeta)}(Z(\varphi(\zeta)); Z(\varphi(\zeta))) = 0$$

proving that the restriction of u to each leaf of $\mathcal{F}$ is harmonic. □

4. The generalized Dirichlet problem

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\varphi \in C^0(\partial\Omega)$. The generalized Dirichlet problem on Ω is the following:

$$\begin{cases} u \text{ upper semicontinuous in } \overline{\Omega} \\ u \text{ maximal in } \mathsf{Psh}(\Omega) \\ u|_{\partial\Omega} = \varphi \end{cases} \tag{4.1}$$

Notice that, if the requirement $u \in \mathsf{Psh}(\Omega)$ is changed with $u \in \mathsf{subh}(\Omega)$ then the problem (4.1) turns out to be equivalent to the classical Dirichlet problem which has a unique solution in case Ω is *bounded* with boundary of class C^2.

Looking at (1.2), one is tempted to define the following function:

$$M_{\Omega,\varphi}(x) = \sup\{v(x) : v \in \mathsf{Psh}(\Omega), \limsup_{\Omega \ni x \to p} v(x) \le \varphi(p) \forall p \in \partial\Omega\}. \tag{4.2}$$

The function $M_{\Omega,\varphi}$ is called the *Perron-Bremermann function* for Ω.

As a matter of notation, we say that a point $p \in \partial\Omega$ is a *plurisubharmonic peak point* if there exists an open neighborhood U of $\overline{\Omega}$ and $\Phi_p \in \mathsf{Psh}(U)$ such that $\Phi_p(p) = 0$ and $\Phi_p(z) < 0$ for all $z \in \overline{\Omega} \setminus \{p\}$. If this is the case, we say that the function Φ_p *peaks* at p in Ω.

REMARK 4.1. If $\Omega \subset \mathbb{C}^n$ is a strongly convex domain and $p \in \partial\Omega$, there exists a real hyperplane H_p such that $\Omega \cap H_p = \{p\}$. Such a hyperplane can be written as $H_p = \{z \in \mathbb{C}^n : \mathsf{Re}\,\langle z - p, \nu_p\rangle = 0\}$, for some complex vector $\nu_p \in \mathbb{C}^n$. Up to replace ν_p with $-\nu_p$, the strong convexity of Ω implies that $\mathsf{Re}\,\langle z - p, \nu_p\rangle < 0$ for all $z \in \overline{\Omega} \setminus \{p\}$. Thus the function $\Phi_p(z) := \mathsf{Re}\,\langle z - p, \nu_p\rangle$ is a pluri(sub)harmonic function which peaks at p in Ω and then each point of $\partial\Omega$ is a plurisubharmonic peak point. In particular each point of the boundary of the unit ball $\mathbb{B}^n$ is a plurisubharmonic peak point. More generally, it is known that if $\Omega \subset\subset \mathbb{C}^n$ is a strongly pseudoconvex domain, for each $p \in \partial\Omega$ there exists a holomorphic function f_p (defined in a neighborhood of $\overline{\Omega}$) such that $|f_p(p)| = 1$ and $|f_p(z)| < 1$ for all $z \in \overline{\Omega} \setminus \{p\}$. The plurisubharmonic function $\Phi_p := \log|f_p|$ peaks at p in Ω and thus each point of $\partial\Omega$ is a plurisubharmonic peak point.

THEOREM 4.2 (Bremermann-Walsh). *Suppose $\Omega \subset\subset \mathbb{C}^n$ has boundary of class C^2 and assume that every $p \in \partial\Omega$ is a plurisubharmonic peak point. Then the Perron-Bremermann function $M_{\Omega,\varphi}$ is a solution of the generalized Dirichlet problem* (4.1). *Moreover, $M_{\Omega,\varphi} \in C^0(\overline{\Omega})$.*

PROOF. Let us denote by

$$\mathcal{P}_{\Omega,\varphi} := \{v \in \mathsf{Psh}(\Omega), \limsup_{\Omega \ni x \to p} v(x) \le \varphi(p) \forall p \in \partial\Omega\}.$$

Let $H \in \mathsf{harm}(\Omega) \cap C^0(\overline{\Omega})$ be the solution of the classical Dirichlet problem, so that $\Delta H = 0$ in Ω and $H|_{\partial\Omega} = \varphi$. By the maximum principle in Corollary I.1.6 applied to $v - H$, $v \le H$

in Ω for all $v \in \mathcal{P}_{\Omega,\varphi}$. This implies that $M_{\Omega,\varphi} \leq H$ in Ω and by (the analogous for plurisubharmonic functions of) Proposition I.5.3, its upper semicontinuous regularization $(M_{\Omega,\varphi})^*$ is plurisubharmonic in Ω. By the very definition, $M_{\Omega,\varphi} \leq (M_{\Omega,\varphi})^*$ and $(M_{\Omega,\varphi})^* \leq F$ for any upper semicontinuous F such that $M_{\Omega,\varphi} \leq F$. Thus $(M_{\Omega,\varphi})^* \leq H$ in Ω which implies that $(M_{\Omega,\varphi})^* \in \mathcal{P}_{\Omega,\varphi}$. Thus $M_{\Omega,\varphi} = (M_{\Omega,\varphi})^*$ and then $M_{\Omega,\varphi} \in \mathsf{Psh}(\Omega)$.

By construction $\limsup_{\Omega \ni x \to p} M_{\Omega,\varphi}(x) \leq \varphi(p)$ for all $p \in \partial\Omega$. In order to show that $\lim_{\Omega \ni z \to p} M_{\Omega,\varphi}(z) = \varphi(z)$ for all $p \in \partial\Omega$, we will prove that, for all $p \in \partial\Omega$, $\epsilon > 0$ there exists a function $u_{\epsilon,p} \in C^0(\overline{\Omega}) \cap \mathcal{P}_{\Omega,\varphi}$ such that $u_{\epsilon,p}(p) = \varphi(p) - \epsilon$. Assuming such a function $u_{\epsilon,p}$ exists, since $M_{\Omega,\varphi} \geq u_p$ in Ω, it follows that

$$\liminf_{\Omega \ni x \to p} M_{\Omega,\varphi}(x) \geq \liminf_{\Omega \ni x \to p} u_{\epsilon,p}(x) = \varphi(p) - \epsilon,$$

and thus, by the arbitrariness of ϵ, $\liminf_{\Omega \ni x \to p} M_{\Omega,\varphi}(x) \geq \varphi(p)$ showing that $M_{\Omega,\varphi}$ is continuous at p and $M_{\Omega,\varphi}(p) = \varphi(p)$. The function $u_{\epsilon,p}$ can be defined by taking a plurisubharmonic function Φ_p which peaks at p in Ω and defining $u_{\epsilon,p}(z) := c\Phi_p(z) + \varphi(p) - \epsilon$ for $c > 0$ chosen so that $u_{\epsilon,p} \leq \varphi$ on $\partial\Omega$.

To show maximality of $M_{\Omega,\varphi}$, let $G \subset\subset \Omega$ be a open set and let $u \in \mathsf{Psh}(\Omega)$ be such that $u(z) \leq M_{\Omega,\varphi}(z)$ for all $z \in \partial G$. Define

$$v(z) = \begin{cases} M_{\Omega,\varphi}(z) & z \in \Omega \setminus G \\ \max\{u(z), M_{\Omega,\varphi}(z)\} & z \in \overline{G} \end{cases}$$

By the analogous of Proposition I.5.1, $v \in \mathsf{Psh}(\Omega)$. Moreover, by construction $v \in \mathcal{P}_{\Omega,\varphi}$. Thus $v \leq M_{\Omega,\varphi}$ which implies that $u \leq M_{\Omega,\varphi}$ in G, proving maximality of $M_{\Omega,\varphi}$.

It remains to prove that $M_{\Omega,\varphi}$ is continuous. We already know that it is upper semicontinuous, so it is enough to prove that it is lower semicontinuous. To this aim, we define a new function as follows. Fix $\epsilon > 0$. Let $y \in \mathbb{C}^n$ be such that $\|y\| < \delta$ (with $\delta = \delta(\epsilon) > 0$ small to be chosen later) and let

$$u_y(z) := \begin{cases} \max\{M_{\Omega,\varphi}(z), M_{\Omega,\varphi}(z+y) - \epsilon\} & z \in \overline{\Omega} \cap (\overline{\Omega} - y) \\ M_{\Omega,\varphi}(z) & z \in \overline{\Omega} \setminus (\overline{\Omega} - y) \end{cases}$$

If we can show that $u_y \in \mathcal{P}_{\Omega,\varphi}$ then $u_y \leq M_{\Omega,\varphi}$ in Ω, proving that for $\|z - w\| < \delta$ then

$$M_{\Omega,\varphi}(z) \geq u_{w-z}(z) \geq M_{\Omega,\varphi}(z + w - z) - \epsilon = M_{\Omega,\varphi}(w) - \epsilon,$$

proving that $M_{\Omega,\varphi}$ is lower semicontinuous.

Let, as usual, $\Omega_\delta := \{z \in \Omega : \mathsf{dist}(z, \partial\Omega) > \delta\}$. Since $\Omega_\delta \subset \overline{\Omega} \cap (-y + \overline{\Omega})$, by the analogous of Proposition I.5.1 for plurisubharmonic functions, $u_y \in \mathsf{Psh}(\Omega_\delta)$. In order to prove that $u_y \in \mathcal{P}_{\Omega,\varphi}$, it is then enough to prove that, for a suitable choice of δ, $u_y = M_{\Omega,\varphi}$ on $\Omega \setminus \Omega_{2\delta}$, for then $u_y \in \mathsf{Psh}(\Omega)$ and $\lim_{\Omega \ni z \to p} u_y(z) = \varphi(p)$ for all $p \in \partial\Omega$.

If $z \in \Omega \setminus (\overline{\Omega} - y)$ then $u_y(z) = M_{\Omega,\varphi}(z)$ by definition. If $z \in (\Omega \cap (\overline{\Omega} - y)) \setminus \Omega_{2\delta}$, let $w \in \partial\Omega$ be such that $\|z - w\| \leq 2\delta$. Now, since $\lim_{\Omega \ni z \to p} M_{\Omega,\varphi}(z) = \varphi(p)$ for all $p \in \partial\Omega$ and

$M_{\Omega,\varphi}|_{\partial\Omega} = \varphi$ is uniformly continuous on $\partial\Omega$, we can choose $\delta > 0$ in such a way that for all $\zeta \in \Omega$ and $\eta \in \partial\Omega$ with $\|\zeta - \eta\| < 4\delta$ it follows

$$|M_{\Omega,\varphi}(\zeta) - M_{\Omega,\varphi}(\eta)| < \epsilon/2.$$

Since $\|z - w\| \leq 2\delta$ and $\|z + y - w\| \leq \|z - w\| + \|y\| \leq 3\delta$, hence

$$|M_{\Omega,\varphi}(z) - M_{\Omega,\varphi}(w)| < \epsilon/2, \quad |M_{\Omega,\varphi}(z+y) - M_{\Omega,\varphi}(w)| < \epsilon/2.$$

Thus

$$M_{\Omega,\varphi}(z) > M_{\Omega,\varphi}(w) - \epsilon/2 > M_{\Omega,\varphi}(z+y) - \epsilon/2 - \epsilon/2 = M_{\Omega,\varphi}(z+y) - \epsilon,$$

as needed. □

REMARK 4.3. The Perron-Bremermann function $M_{\Omega,\varphi}$ is also the *unique* solution in the L^∞-class of problem (4.1). However, uniqueness does not follow at once from tools such as the maximum principle, but it is a particular instance of the so-called *comparison principle* of Bedford and Taylor, see Remark 6.7. From the proof of Theorem 4.2 it follows only that $M_{\Omega,\varphi}$ is the maximum among other solutions of (4.1). Indeed, if u is any solution of (4.1) then $u \in \mathcal{P}_{\Omega,\varphi}$ and therefore $u \leq M_{\Omega,\varphi}$ in Ω.

REMARK 4.4. In [**1**] Bedford and Taylor proved that if $\varphi \in C^2(\partial\Omega)$ then $M_{\Omega,\varphi} \in C^{1,1}(\overline{\Omega})$ (namely it is C^1 with Lipschitz first derivatives) and $M_{\Omega,\varphi} \in W^{2,\infty}(\Omega)$ (that is $M_{\Omega,\varphi}$ has weak second order derivatives which are in $L^\infty_{\text{loc}}(\Omega)$).

REMARK 4.5. It is worth noticing that if $\Omega \subset \mathbb{C}^n$ is a domain for which the generalized Dirichlet problem (4.1) has a continuous solution for each $\varphi \in C^0(\partial\Omega)$ then every point of $\partial\Omega$ is a plurisubharmonic peak point (just solve (4.1) with $\varphi(z) = -\|z - p\|$ for $p \in \partial\Omega$).

5. The complex Monge-Ampère operator on locally bounded plurisubharmonic functions

The aim of this section is to extend the definition of the complex Monge-Ampère operator $(dd^c)^n$ to locally bounded plurisubharmonic functions, according to Bedford and Taylor [**1**].

First of all, notice that if $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ then u is actually locally bounded (because it is upper semicontinuous and does not assume the value $+\infty$ be hypothesis).

LEMMA 5.1. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $T = \sum T_J \eta_J \in \mathcal{D}^k_0(\Omega)$ (with η_J smooth k-forms and T_J Radon measures) and let $u \in L^\infty_{\text{loc}}(\Omega)$. If $\varphi \in C_0(\Omega, \Lambda^{2n-k})$ define $a^J_\varphi \in C_0(\Omega)$ by means $a^J_\varphi dV = \varphi \wedge \eta_J$ (with dV the volume form on $\mathbb{C}^n$). The functional uT defined on $C_0(\Omega, \Lambda^{2n-k})$ as*

$$uT(\varphi) := \sum_J \int_\Omega (u a^J_\varphi) T_J, \quad \forall \varphi \in C_0(\Omega, \Lambda^{2n-k}) \tag{5.1}$$

is a current of degree k and order 0. In particular, if $u \in C^0(\Omega)$ then $uT(\varphi) = T(u\varphi)$ for all $\varphi \in C_0(\Omega, \Lambda^{2n-k})$.

PROOF. Since u is locally bounded then it is T_J-integrable on the support of any $\varphi \in C_0(\Omega, \Lambda^{2n-k})$ for any J. Thus the integrals in (5.1) are well defined and finite. It is then clear that the definition does not depend on the choice of the decomposition $T = \sum T_J \eta_J$ and thus uT is a current of degree k and order 0. □

Now we need the following result:

PROPOSITION 5.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $T \in \mathcal{D}_\infty^{(p,p)}(\Omega)$ be a d-closed positive current. Let $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Define*

$$(dd^c u \wedge T)(\varphi) = dd^c(uT)(\varphi) := (uT)(dd^c\varphi),$$

for $\varphi \in C_0^\infty(\Omega, \Lambda^{(n-p-1,n-p-1)})$. Then $dd^c u \wedge T$ is a d-closed positive current of degree $(p+1, p+1)$.

PROOF. It is clear that $dd^c u \wedge T$ is a d-closed current of degree $(p+1, p+1)$. We have only to show positivity. To this aim, let $\{u_\epsilon\}$ be the sequence of smooth regularizing plurisubharmonic functions given by Theorem III.2.1. Since $\{u_\epsilon\}$ pointwisely decreases to u, by Lebesgue dominated convergence theorem and the very definition (5.1) of uT, it follows that $u_\epsilon T \to uT$ in the weak* topology. Therefore, if we can show that $u_\epsilon T(dd^c\varphi) \geq 0$ for all $\epsilon > 0$ and $\varphi \in SP^{n-p-1,n-p-1}(\Omega)$, it will follow that $uT(dd^c\varphi) \geq 0$ and thus uT is positive.

Let $\varphi \in SP^{n-p-1,n-p-1}(\Omega)$. Since u_ϵ is smooth, then $(dd^c u_\epsilon \wedge T)(\varphi) = T(u_\epsilon dd^c\varphi)$. By hypothesis $dT = 0$, that is, $T(d\Phi) = 0$ for all $\Phi \in C_0^\infty(\Omega, \Lambda^{2n-2p-1})$. In particular if we let $\Phi := u_\epsilon d^c\varphi$ we obtain

$$0 = T(d\Phi) = T(d(u_\epsilon d^c\varphi)) = T(du_\epsilon \wedge d^c\varphi) + T(u_\epsilon dd^c\varphi), \tag{5.2}$$

while, if we let $\Phi := d^c u_\epsilon \wedge \varphi$ we obtain

$$0 = T(d\Phi) = T(d(d^c u_\epsilon \wedge \varphi)) = T(dd^c u_\epsilon \wedge \varphi) - T(d^c u_\epsilon \wedge d\varphi). \tag{5.3}$$

Moreover, recalling that T is a (p,p)-current and thus $T(\psi) = 0$ for any $(2n-2p)$-form of type different from $(n-p, n-p)$, it follows that

$$\begin{aligned} T(du_\epsilon \wedge d^c\varphi) &= T((\partial + \overline{\partial})u_\epsilon \wedge i(\overline{\partial} - \partial)\varphi) = T(\partial u_\epsilon \wedge i\overline{\partial}\varphi + \overline{\partial}u_\epsilon \wedge i\partial v) \\ &= T(i(\partial - \overline{\partial})u_\epsilon \wedge (\partial + \overline{\partial})\varphi) = -T(d^c u_\epsilon \wedge d\varphi) \end{aligned} \tag{5.4}$$

Putting together (5.2), (5.3), (5.4) we have

$$T(u_\epsilon dd^c\varphi) = -T(du_\epsilon \wedge d^c\varphi) = T(d^c u_\epsilon \wedge d\varphi) = T(dd^c u_\epsilon \wedge \varphi). \tag{5.5}$$

Now, since u_ϵ is plurisubharmonic (and smooth) then $dd^c u_\epsilon$ is a strongly positive $(1,1)$-form by Remark IV.4.17 and then $dd^c u \wedge \varphi \in SP^{(n-p,n-p)}(\Omega)$. But $T \geq 0$ by hypothesis and therefore $T(dd^c u_\epsilon \wedge \varphi) \geq 0$, which, by (5.5) implies that $T(u_\epsilon dd^c\varphi) \geq 0$ as needed. □

Now we can define the complex Monge-Ampère operator for locally bounded plurisubharmonic function by induction as follows.

Let $\Omega \subset \mathbb{C}^n$. Let $u_1, \ldots, u_k \in \mathsf{Psh}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$. By Proposition 5.2 we can define by induction

$$(dd^c u_1 \wedge \ldots \wedge dd^c u_k) := T \wedge dd^c u_k,$$

where $T = (dd^c u_1 \wedge \ldots \wedge dd^c u_{k-1})$ is the positive d-closed $(k-1, k-1)$-current defined inductively on k. More explicitly, for $\varphi \in C_0^\infty(\Omega, \Lambda^{(n-k,n-k)})$

$$(dd^c u_1 \wedge \ldots \wedge dd^c u_k)(\varphi) := u_k (dd^c u_1 \wedge \ldots \wedge dd^c u_{k-1})(dd^c \varphi).$$

The functional $dd^c u_1 \wedge \ldots \wedge dd^c u_k$ is then a d-closed positive (k,k)-current (of order zero).

The previous definition is coherent with the case $u_1, \ldots, u_k$ are $C^2(\Omega)$. Indeed, for all $\varphi \in C_0^\infty(\Omega, \Lambda^{(n-k,n-k)})$ it follows

$$\int_\Omega dd^c u_1 \wedge \ldots \wedge dd^c u_k \wedge \varphi = \int_\Omega u_k (dd^c u_1 \wedge \ldots \wedge dd^c u_{k-1} \wedge dd^c \varphi) \tag{5.6}$$

and thus, in case $u_1, \ldots, u_k \in C^2(\Omega)$ the current $dd^c u_1 \wedge \ldots dd^c u_k$ defined as before coincides with the natural current associated to continuous forms. Formula (5.6) can be proved by using Stokes' theorem and division into types (see [**16**, p. 111]).

It is known that the complex Monge-Ampère operator cannot be defined on a generic plurisubharmonic function. Demailly (see [**9**]) extended the domain of definition of the complex Monge-Ampère operator to plurisubharmonic functions which are bounded outside compact sets. Other generalization are in Cegrell [**8**]. Jus recently, Z. Błocki [**3**] characterized completely the domain of definition of the complex Monge-Ampère operator.

6. Properties of the complex Monge-Ampère operator

We collect here some basic properties of the Monge-Ampère operator referring the reader to [**1**], [**2**] and [**16**] for those stated without proof.

THEOREM 6.1 (Chern-Levine-Nirenberg estimate). *Let Ω be a domain in $\mathbb{C}^n$ and let $K \subset\subset \Omega$ be a compact set. There exists a constant $C = C(\Omega, K) > 0$ and a compact set $H \subset\subset \Omega \setminus K$ such that for all $u_1, \ldots, u_n \in \mathsf{Psh}(\Omega) \cap L^\infty(\Omega)$ it follows*

$$\int_K dd^c u_1 \wedge \ldots \wedge dd^c u_n \leq C \|u_1\|_{L^\infty(H)} \cdots \|u_n\|_{L^\infty(H)}.$$

Such an estimate can be first proved for plurisubharmonic functions of class C^2 and then, using the next approximation theorem, extended to bounded plurisubharmonic functions.

THEOREM 6.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. For $k = 1, \ldots, m \leq n$ let $\{u_{k,j}\}_{j \in \mathbb{N}}$ be a decreasing sequence of plurisubharmonic functions of class L^∞_{loc}. Let $u_k = \lim_j u_{k,j}$ and assume that $u_k \in \mathsf{Psh}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ for $k = 1, \ldots, m$. Then*

$$\lim_{j \to \infty} dd^c u_{1,j} \wedge \ldots \wedge dd^c u_{m,j} = dd^c u_1 \wedge \ldots \wedge dd^c u_m \quad \textit{in } \mathcal{D}_0^{(m,m)}(\Omega).$$

The previous theorem, together with the regularization theorem, allows to pass all algebraic properties of the Monge-Ampère operator from C^∞-plurisubharmonic functions to locally bounded ones.

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. The (n,n)-current $(dd^c u)^n$ can be seen as a Radon measure on Ω, the *Monge-Ampère mass* of u. We already saw that if u is of class C^2 then u is maximal if and only if its Monge-Ampère mass is zero. The same is true for less regular functions, and it follows from the following result:

THEOREM 6.3 (Comparison Theorem). *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain. Let $u, v \in \mathsf{Psh}(\Omega) \cap L^\infty(\Omega)$. Suppose that for all $p \in \partial\Omega$ it holds*

$$\liminf_{\Omega \ni z \to p} (u(z) - v(z)) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

As a consequence we have:

COROLLARY 6.4. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. If $(dd^c u)^n = 0$ in Ω then u is maximal.*

PROOF. Let $G \subset\subset \Omega$ be a connected open set. Let $v \in \mathsf{Psh}(G)$ be such that $\liminf_{z\to p}[u(z) - v(z)] \geq 0$ for all $p \in \partial G$. We have to show that $v \leq u$ in G. First we claim that we can assume $v \in L^\infty(\Omega)$. Indeed, if this is not the case, we can replace v with the plurisubharmonic function $v' := \max\{u|_G, v\}$. The function v' has the properties that $v \leq v'$ in G and $\liminf_{z\to p}[u(z) - v'(z)] \geq 0$. Moreover, since v is bounded from above in $\overline{G}$ (by definition of plurisubharmonic functions and the hypothesis on the behavior near ∂G) and $u \in L^\infty(G)$ then $v' \in L^\infty(G)$. Thus if we prove that $u \geq v'$ in G then it will follow also that $u \geq v$ in G. We can thus assume $v \in L^\infty(\Omega)$.

Assume by contradiction that the set $\{z \in G : u(z) < v(z)\}$ is not empty. Let $v_{\epsilon,\delta}(z) := v + \epsilon\|z\|^2 - \delta$ and choose $\epsilon > 0, \delta > 0$ so that $v_{\epsilon,\delta} < v$ in $\overline{G}$. Since the set $\{z \in G : u(z) < v(z)\}$ is not empty we can choose ϵ, δ in such a way that the set $\{z \in G : u(z) < v_{\epsilon,\delta}(z)\}$ is not empty as well. The set $\{z \in G : u(z) < v_{\epsilon,\delta}(z)\}$ has positive Lebesgue measure, because otherwise the plurisubharmonic function $\max\{u|_G, v_{\epsilon,\delta}\}$ would be almost everywhere equal to $u|_G$ and thus by Corollary I.3.2 it would be equal to $u|G$ everywhere in G, implying $u \geq v_{\epsilon,\delta}$ in G.

Now we claim that for all $w_1, w_2 \in \mathsf{Psh}(G) \cap L^\infty_{\mathrm{loc}}(G)$

$$(dd^c(w_1 + w_2))^n \geq (dd^c w_1)^n + (dd^c w_2)^n. \tag{6.1}$$

If $w_1, w_2 \in \mathsf{Psh}(G) \cap C^2(G)$ then, taking into account that $(dd^c w_1)^k \wedge (dd^c w_2)^{n-k} \geq 0$ for all $k = 0, \ldots, n$,

$$\begin{aligned}(dd^c(w_1 + w_2))^n &= (dd^c w_1)^n + (dd^c w_2)^n + \sum_{j=1}^{n-1} \binom{n}{j} (dd^c w_1)^j \wedge (dd^c w_2)^{n-j} \\ &\geq (dd^c w_1)^n + (dd^c w_2)^n.\end{aligned}$$

Formula (6.1) for general $w_1, w_2 \in \mathsf{Psh}(G) \cap L^\infty_{\mathrm{loc}}(G)$ follows now from Theorem 6.2 by approximating w_1, w_2 with decreasing sequences of smooth plurisubharmonic functions.

From (6.1) and Theorem 6.3 we have

$$\begin{aligned}\int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c v)^n + \int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c(\epsilon\|z\|^2 - \delta))^n &\leq \int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c v_{\epsilon,\delta})^n \\ &\leq \int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c u)^n = 0.\end{aligned}$$

But $(dd^c v)^n \geq 0$ and $(dd^c(\epsilon\|z\|^2 - \delta))^n = 4^n \epsilon^n n! dV$, thus

$$\int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c v)^n + \int_{\{u|_G < v_{\epsilon,\delta}\}} (dd^c(\epsilon\|z\|^2 - \delta))^n \geq 4^n \epsilon^n n! \int_{\{u|_G < v_{\epsilon,\delta}\}} dV > 0,$$

giving a contradiction. □

REMARK 6.5. An argument similar to that used in the proof of Corollary 6.4 shows that if Ω is a bounded domain, $u, v \in \mathsf{Psh}(\Omega) \cap L^\infty(\Omega)$ are such that $u = v$ on $\partial\Omega$ and $(dd^c u)^n = (dd^c v)^n = 0$ in Ω then $u \equiv v$ in Ω.

To end up the discussion about the complex Monge-Ampère operator, we state the following very deep result of Bedford and Taylor:

THEOREM 6.6. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$ in Ω.*

One implication of this theorem is contained in Corollary 6.4. For the other implication, namely if u is maximal then $(dd^c u)^n = 0$, the hard part is to show that the Perron-Bremermann function $M_{\mathbb{B},\varphi}$ for the ball $\mathbb{B}$ satisfies the Monge-Ampère equation $(dd^c M_{\mathbb{B},\varphi})^n = 0$ in $\mathbb{B}$ (apart the original source, see, *e.g.*, [**16**, Theorem 4.4.1]).

REMARK 6.7. Theorem 6.6 and Remark 6.5 imply that the Perron-Bremermann solution $M_{\Omega,\varphi}$ is the unique solution of the generalized Dirichlet problem (4.1) in the class $L^\infty_{\mathrm{loc}}(\Omega)$. It also satisfies $(dd^c M_{\Omega,\varphi})^n = 0$.

7. The pluricomplex Green function for bounded domains

One of the main object in classical potential theory is the *Green function*. Such a function (and its normal derivative, the *Poisson kernel*) allows to reproduce smooth functions and harmonic functions (see, *e.g.*, [**9**], [**16**]). To be more concrete, in the unit disc $\mathbb{D} \subset \mathbb{C}$ let

$$G_{\mathbb{D}}(z,\zeta) := \log |T_z(\zeta)| \tag{7.1}$$

where $T_z(\zeta) := (z-\zeta)(1-\overline{z}\zeta)^{-1}$ is an automorphism of $\mathbb{D}$ which maps ζ to O and such that $T^2 = \mathsf{id}$. Then $G_{\mathbb{D}} : \overline{\mathbb{D}} \times \overline{\mathbb{D}} \to [-\infty, 0]$ enjoys the following properties:

(1) $G_{\mathbb{D}}$ is of class C^∞ in $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \setminus \mathsf{Diag}_{\mathbb{D}}$, where $\mathsf{Diag}_{\mathbb{D}} = \{(z,\zeta) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}} : z = \zeta\}$.
(2) $G_{\mathbb{D}}(\zeta, z) = G_{\mathbb{D}}(z,\zeta)$ for all $\zeta, z \in \overline{\mathbb{D}}$.
(3) $G_{\mathbb{D}}(\zeta, z) < 0$ in $\mathbb{D} \times \mathbb{D}$ and $G_{\mathbb{D}}(\zeta, z) = 0$ on $\partial\mathbb{D} \times \mathbb{D}$.
(4) $\mathbb{D} \ni \zeta \mapsto G_{\mathbb{D}}(\zeta, z)$ is harmonic in $\mathbb{D} \setminus \{z\}$ for all fixed $z \in \mathbb{D}$.
(5) $\mathbb{D} \ni \zeta \mapsto (G_{\mathbb{D}}(\zeta, z) - \log|z-\zeta|) = O(1)$ for all fixed $z \in \mathbb{D}$.

It can be shown (see, *e.g.* [**9**, Theorem 4.3]) that for all $\varphi \in C_0^\infty(\mathbb{D})$ it follows

$$\varphi(z) = \frac{1}{2\pi}\int_{\mathbb{D}} G_{\mathbb{D}}(\zeta, z)\Delta\varphi(\zeta)\frac{i}{2} d\zeta \wedge d\overline{\zeta},$$

namely, $\Delta_\zeta G_{\mathbb{D}}(\zeta, z) = 2\pi\delta_z$ in $\mathcal{D}is^\infty(\mathbb{D})$.

As we already saw, in several variables the notion of harmonic functions is not invariant by biholomorphisms, thus when working in higher complex dimensions, one is tempted to define and study “pluricomplex Green functions”.

DEFINITION 7.1. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $x \in \Omega$. The *(Klimek) pluricomplex Green function* of Ω with logarithmic pole at x is defined as

$$K_{\Omega,x}(z) := \sup\{u(z) : u \in \mathsf{Psh}(\Omega), u < 0, \limsup_{z\to x}(u(z) - \log\|z-x\|) < +\infty\}$$

with the convention that $\sup\emptyset = -\infty$.

We state here some properties of the pluricomplex Green function:

PROPOSITION 7.2. *Let $\Omega, \Omega' \subset \mathbb{C}^n$ be domains and let $x \in \Omega$.*

(1) *If $\Omega \subset \Omega'$ then $K_{\Omega,x}(z) \geq K_{\Omega',x}(z)$ for all $z \in \Omega$.*
(2) *If $\Omega = \mathbb{B}(x,r)$ then $K_{\mathbb{B}(x,r)}(z) = \log(\|z-x\|/r)$.*
(3) *If $R > 0$ and $\Omega \subset \mathbb{B}(x,R)$ then for all $z \in \Omega$ it follows*

$$\log\left(\frac{\|z-x\|}{R}\right) \leq K_{\Omega,x}(z).$$

(4) *If $r > 0$ and $\mathbb{B}(x,r) \subset \Omega$ then for all $z \in \mathbb{B}(x,r)$ it follows*

$$K_{\Omega,x}(z) \leq \log\left(\frac{\|z-x\|}{r}\right).$$

(5) *If $f:\Omega\to\Omega'$ is holomorphic then $f^*(K_{\Omega',f(x)})\leq K_{\Omega,x}$. In particular the pluricomplex Green function is invariant for biholomorphisms.*
(6) *If Ω is bounded then $K_{\Omega,x}$ is maximal in $\Omega\setminus\{x\}$ (i.e. it is plurisubharmonic in Ω and maximal) and $(dd^cK_{\Omega,x})^n\equiv 0$ in $\Omega\setminus\{x\}$.*

PROOF. Let us denote by

$$\mathcal{K}_{\Omega,x}=\{u\in \mathsf{Psh}(\Omega) : u<0, u(z)-\log\|z-x\|\leq O(1) \text{ as } z\to x\}. \tag{7.2}$$

Then (1) follows directly from the very definition since every if $u\in\mathcal{K}_{\Omega',x}$ then its restriction $u|_\Omega\in\mathcal{K}_{\Omega,x}$.

(2) If $\Omega=\mathbb{B}(x,r)$, let $u\in\mathcal{K}_{\mathbb{B}(x,r),x}$ and let $z\in\mathbb{B}(x,r)\setminus\{x\}$. Fix $v\in\mathbb{C}^n$ with $\|v\|=r$ and such that $x+\zeta_0 v=z$ for some $\zeta_0\in\mathbb{D}$. Consider the function $\tilde{u}:\mathbb{D}\ni\zeta\mapsto u(\zeta v+x)-\log|\zeta|$. Such a function is subharmonic in $\mathbb{D}\setminus\{0\}$ and bounded from above in $\mathbb{D}$ (since $\log|\zeta|=\log(\|\zeta v+x-x\|/r)$ and u has a logarithmic pole at x) thus, by Corollary I.5.5, $\tilde{u}$ extends to a subharmonic function in $\mathbb{D}$. Since $\limsup_{\zeta\to q}\tilde{u}(\zeta)\leq 0$ for all $q\in\partial\mathbb{D}$ by construction, by the maximum principle $\tilde{u}\leq 0$ in $\mathbb{D}$. Therefore for all $\zeta\in\mathbb{D}$ it follows $u(\zeta v+x)\leq\log|\zeta|$, proving that $u(z)\leq\log(\|z-x\|/r)$. Thus $K_{\mathbb{B}(x,r)}(z)\leq\log(\|z-x\|/r)$, but since $\log(\|z-x\|/r)\in\mathcal{K}_{\mathbb{B}(x,r),x}$, then $K_{\mathbb{B}(x,r)}(z)=\log(\|z-x\|/r)$.

(3) and (4) follow from (1) and (2).

(5) follows from the fact that if $u\in\mathcal{K}_{\Omega',f(x)}$ then $u\circ f=f^*(u)\in\mathcal{K}_{\Omega,x}$ because clearly $f^*(u)<0$ in Ω and

$$\begin{aligned} u(f(z))-\log\|z-x\| &= u(f(z))-\log\|f(z)-f(x)\|+\log\frac{\|f(z)-f(x)\|}{\|z-x\|}\\ &= u(f(z))-\log\|f(z)-f(x)\|+O(1). \end{aligned}$$

(6) If Ω is bounded then by (3) the pluricomplex Green function $K_{\Omega,x}(z)>-\infty$ for all $z\in\Omega\setminus\{x\}$. According to (the analogous for plurisubharmonic functions of) Proposition I.5.3 the upper semicontinuous regularization $u:=(K_{\Omega,x})^*$ is plurisubharmonic in Ω. We claim that $u\in\mathcal{K}_{\Omega,x}$. Indeed, by (4) u has a logarithmic singularity at x. Also, clearly $u\leq 0$ in Ω. If it were $u(z)=0$ for some $z\in\Omega$, then by the maximum principle $u\equiv 0$, contradicting the fact that u has a logarithmic singularity at x. Thus $u<0$ in Ω and therefore $u\in\mathcal{K}_{\Omega,x}$. Hence $u=K_{\Omega,x}$ which is then plurisubharmonic (and strictly negative) in Ω.

To show maximality, let $G\subset\subset\Omega\setminus\{x\}$ and let $v\in\mathsf{Psh}(G)$ be such that $\limsup_{G\ni z\to p}v(z)\leq K_{\Omega,x}(p)$ for all $p\in\partial G$. Define

$$u(z)=\begin{cases}\max\{v(z),K_{\Omega,x}\}(z) & z\in G\\ K_{\Omega,x}(z) & z\in\Omega\setminus G\end{cases}$$

Then $u\in\mathcal{K}_{\Omega,x}$ and by definition $u\leq K_{\Omega,x}$, proving that $K_{\Omega,x}$ is maximal in $\Omega\setminus\{x\}$. Finally notice that by (4) $K_{\Omega,x}$ is locally bounded in $\Omega\setminus\{x\}$ and thus Theorem 6.6 implies $(dd^cK_{\Omega,x})^n\equiv 0$ in $\Omega\setminus\{x\}$. □

Recall that a domain $\Omega \subset \mathbb{C}^n$ is called *hyperconvex* if there exists $\rho \in \mathsf{Psh}(\Omega) \cap C^0(\overline{\Omega})$ such that for all $r > 0$ the open set $\{z \in \Omega : \rho(z) < -r\}$ is relatively compact in Ω.

THEOREM 7.3. *Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain and let $x \in \Omega$. Then the pluricomplex Green function $K_{\Omega,x} : \Omega \to [-\infty, 0]$, extended to be 0 on $\partial\Omega$ is plurisubharmonic and continuous. Moreover, the function $\Omega \times \overline{\Omega} \ni (x, z) \mapsto K_{\Omega,x}(z) \in [-\infty, 0]$ is continuous.*

SKETCH OF THE PROOF. First of all we can prove that $\lim_{\Omega \ni z \to p} K_{\Omega,x}(z) = 0$ for all $p \in \partial\Omega$. To this aim, by hypothesis there exists $\rho \in \mathsf{Psh}(\Omega) \cap C^0(\overline{\Omega})$ such that $\{z \in \Omega : \rho(z) < -r\}$ is relatively compact in Ω for all $r > 0$. Let $\mathbb{B}(x, r) \subset \Omega \subset \overline{\Omega} \subset \mathbb{B}(x, R)$ for some $r, R > 0$ and define

$$v(z) = \begin{cases} \max\{C\rho(z), \log(\|z - x\|/R)\} & z \in \Omega \setminus \mathbb{B}(x, r) \\ \log(\|z - x\|/R) & z \in \mathbb{B}(x, r) \end{cases}$$

where $C > 0$ is chosen so that $C\rho(z) < \log(r/R)$ on $\partial\mathbb{B}(x, r)$. Then $v \in \mathcal{K}_{\Omega,x}$ (the family defined in (7.2)). Moreover, since $\rho(z) \to 0$ as $z \to \partial\Omega$, then $v(z) \to 0$ as $z \to \partial\Omega$ and since $v \leq K_{\Omega,x}$, the same holds for $K_{\Omega,x}$.

In order to show continuity, it is enough to prove that $K_{\Omega,x}$ is lower semi-continuous. To this aim, Demailly (see, [**11**]) constructs a sequence of continuous $\{u_k\} \subset \mathcal{K}_{\Omega,x}$ such that $K_{\Omega,x} = \sup u_k$ (and the result follows since the supremum of lower semicontinuous functions is lower semicontinuous). The rather explicit construction of such a family (which requires the use of ρ) is omitted.

Finally, to show continuity of $K_{\Omega,x}(z)$ with respect to (z, x), one can show that for all $a \in \Omega$, $\epsilon > 0$ and open neighborhood U of a, there exists an open set $V \subset\subset U$, $a \in V$ such that for all $(x, z), (y, z) \in V \times (\overline{\Omega} \setminus U)$ it follows

$$(1 + \epsilon)^{-1} \leq \frac{K_{\Omega,x}(z)}{K_{\Omega,y}(z)} \leq 1 + \epsilon. \tag{7.3}$$

Formula (7.3) implies that $K_{\Omega,x}(z)$ is continuous in x locally uniformly in $z \in \overline{\Omega}$ outside the diagonal $\mathsf{Diag}(\Omega \times \Omega)$ in $\Omega \times \Omega$. From this it follows that $K_{\Omega,x}(z)$ is continuous in $\Omega \times \overline{\Omega} \setminus \mathsf{Diag}(\Omega \times \Omega)$. Continuity on the diagonal follows from Proposition 7.2.(4).

Formula (7.3) follows from modification of the pluricomplex Green function, see [**11**] or [**16**, p. 227]. □

Some remarks are in order.

(1) According to Demailly [**11**], every bounded pseudoconvex domain with Lipschitz boundary is hyperconvex.
(2) Although the pluricomplex Green function $K_{\Omega,x}(z)$ (for Ω bounded and hyperconvex) is continuous in (x, z), it is in general not symmetric in (x, z). Moreover, it can be proved that it is symmetric in (x, z) if and only if $x \mapsto K_{\Omega,x}(z)$ is plurisubharmonic for all $z \in \Omega$ fixed.

The following characterization of the pluricomplex Green function is due to Demailly:

THEOREM 7.4. *Let $\Omega \subset\subset \mathbb{C}^n$ be a hyperconvex domain and let $x \in \Omega$. The pluricomplex Green function $K_{\Omega,x}$ is the unique solution of the problem:*

$$\begin{cases} u \in \mathsf{Psh}(\Omega) \cap L^\infty_{loc}(\overline{\Omega} \setminus \{x\}) \\ (dd^c u)^n \equiv 0 \textit{ in } \Omega \setminus \{x\} \\ u(z) - \log \|z - x\| = O(1) \textit{ for } z \to x \\ \lim_{z \to p} u(z) = 0 \textit{ for all } p \in \partial\Omega \end{cases} \tag{7.4}$$

PROOF. We already know that $K_{\Omega,x}(z)$ is plurisubharmonic in Ω, continuous in $z \in \overline{\Omega}$, $K_{\Omega,x}|_{\partial\Omega} \equiv 0$ and $(dd^c K_{\Omega,x})^n \equiv 0$ in $\Omega \setminus \{x\}$ (see Proposition 7.2 and Theorem 7.3). Moreover, according to Proposition 7.2.(3) and (4),

$$K_{\Omega,x}(z) - \log \|z - x\| = O(1) \text{ for } z \to x.$$

Therefore $K_{\Omega,x}$ is a solution of (7.4).

We are left to show uniqueness. Notice that if u is any solution of (7.4) then

$$\lim_{z \to x} \frac{u(z)}{\log \|z - x\|} = 1. \tag{7.5}$$

Let u be a solution of (7.4). Notice that by Corollary 6.4 the function u is maximal in $\Omega \setminus \{x\}$. By the maximum principle, $u < 0$ in Ω and thus $u \in \mathcal{K}_{\Omega,x}$ (where $\mathcal{K}_{\Omega,x}$ is the family defined in (7.2)). Therefore $K_{\Omega,x} \geq u$ in Ω. Seeking for a contradiction we assume that there exists $a \in \Omega$ such that $u(a) < K_{\Omega,x}(a)$. Thus there exist $\delta > 0$ and $0 < c < 1$ such that the set

$$E_{\delta,c} := \{z \in \Omega : K_{\Omega,x}(z) > cu(z) + \delta\}$$

is not empty. Since u is upper semicontinuous (and $K_{\Omega,x}$ is continuous), the set $E_{\delta,c}$ is open. We claim that $E_{\delta,c}$ is relatively compact in $\Omega \setminus \{x\}$. Assume we proved the claim. Then $K_{\Omega,x}(z) \leq cu(z) + \delta$ in $\partial E_{\delta,c}$ which would imply $K_{\Omega,x}(z) \leq cu(z) + \delta$ in $E_{\delta,c}$ by maximality of u, contradiction.

To prove that $E_{\delta,c}$ is relatively compact in $\Omega \setminus \{x\}$, let $\{z_k\} \subset E_{\delta,c}$ be such that $z_k \to q \in \partial E_{\delta,c}$. If $q \in \partial\Omega$ then it would follow that $\lim_{z \to q} K_{\Omega,x}(z) \geq \delta > 0$, a contradiction. If $q = x$ then $K_{\Omega,x}(z_k) \to -\infty$ by Proposition 7.2.(4). Thus, from $K_{\Omega,x}(z_k) > cu(z_k) + \delta$ and (7.5) we obtain

$$\begin{aligned} 1 <& c\frac{u(z_k)}{K_{\Omega,x}(z_k)} + \frac{\delta}{K_{\Omega,x}(z_k)} \\ &= c\frac{u(z_k)}{\log \|x - z_k\|} \cdot \frac{\log \|x - z_k\|}{K_{\Omega,x}(z_k)} + \frac{\delta}{K_{\Omega,x}(z_k)} \to c \text{ as } k \to \infty, \end{aligned}$$

from which $c \geq 1$ against our choice $c < 1$. Thus $E_{\delta,c}$ is relatively compact in $\Omega \setminus \{x\}$ as claimed. □

REMARK 7.5. Demailly [**9**] extended the definition of $(dd^c)^n$ to plurisubharmonic functions which are locally bounded in a domain Ω outside some points (actually he extended such a definition to plurisubharmonic functions which are locally bounded outside bigger subsets). For instance (see, *e.g.*, [**16**, p. 228-229]) fix $x \in \Omega$. If $u \in \mathsf{Psh}(\Omega) \cap L^\infty_{loc}(\Omega \setminus \{x\})$, one can prove that the regularizing sequence $\{u_\epsilon\}$ of smooth plurisubharmonic functions which pointwise decreases to u is such that the Monge-Ampére masses $(dd^c u_\epsilon)^n$ converges in the weak* topology of currents to a unique positive Borel measure denoted $(dd^c u)^n$ (which of course coincides with the already defined mass $(dd^c u)^n$ if u is locally bounded near x as well). Also, it can be shown that $(dd^c K_{\Omega,x})^n = (2\pi)^n \delta_x$, where δ_x is the Dirac delta.

REMARK 7.6. If one changes the requirement that $u(z) - \log \|z - x\| = O(1)$ in (7.4) with another condition of the type $u(z) - \log(|z_1 - x_1|^{\alpha_1} + \ldots + |z_n - x_n|^{\alpha_n}) = O(1)$ with $\sum \alpha_j = 1$ then the previous construction works entirely (changing the type of singularity in x) and gives a unique solution $u_{\Omega,x}$. Even such a solution satisfies $(dd^c u_{\Omega,x})^n = (2\pi)^n \delta_x$.

In case the domain Ω is strongly convex with smooth boundary, Lempert [**17**] proved that $K_{\Omega,x}(z)$ is actually $C^\infty(\overline{\Omega} \times \Omega \setminus \mathsf{Diag}(\Omega \times \Omega))$ and that $(dd^c K_{\Omega,x})^{n-1}(z) \neq 0$ for all $z \in \Omega$. Moreover, the foliation in $\Omega \setminus \{x\}$ associated to $K_{\Omega,x}$ according to Theorem 3.4 is formed by *complex geodesics*, namely, any leaf is the image of a biholomorphic map $\varphi : \mathbb{D} \to D$ which is an isometry between the Poincaré distance of $\mathbb{D}$ and the Kobayashi distance of Ω.

Demailly [**10**], [**11**] used the pluricomplex Green function to prove the following representation formula

THEOREM 7.7. *Let $\Omega \subset\subset \mathbb{C}^n$ be a hyperconvex domain. Let $u \in \mathsf{Psh}(\Omega) \cap C^0(\overline{\Omega})$. Then for all $z \in \Omega$*

$$u(z) = \mu_z(u) - \frac{1}{(2\pi)^n} \int_\Omega |K_{\Omega,z}(w)| (dd^c u)(w) \wedge (dd^c K_{\Omega,z})^{n-1}(w)$$

where μ_z is a suitable positive measure supported on $\partial\Omega$ and depending on $K_{\Omega,z}$.

The measure μ_z, which is called the *pluricomplex Poisson kernel*, is defined as follows. For $r < 0$ let $B_r = \{z \in \Omega : K_{\Omega,x}(z) < r\}$. This set is relatively compact in Ω. Let $u_r(z) = \max\{K_{\Omega,x}(z), r\}$. The Monge-Ampère mass $(dd^c u_r)^n$ is supported on ∂B_r. The positive Borel measure μ_z is thus defined as the weak* limit of $(dd^c u_r)^n$ as $r \to 0$.

If Ω is a strongly convex domain with smooth boundary, it can be proved [**5**], [**6**] that the pluricomplex Poisson kernel is given by $d\mu_z(p) = |P_{\Omega,p}(z)|^n \omega_{\partial\Omega}(p)$, where $p \in \partial\Omega$, $\omega_{\partial\Omega}$ is a volume form on $\partial\Omega$ and $P_{\Omega,p} \in C^\infty(\overline{\Omega} \setminus \{p\}) \cap C^0(\overline{\Omega})$ is the solution of the following problem:

$$\begin{cases} u \in \mathsf{Psh}(D) \\ (\partial\overline{\partial} u)^n = 0 \quad \text{in } D \\ u < 0 \quad \text{in } D \\ u(z) = 0 \quad \text{for } z \in \partial D \setminus \{p\} \\ u(z) \approx \|z - p\|^{-1} \quad \text{as } z \to p \text{ non-tangentially} \end{cases}$$

8. Invariant distances and the pluricomplex Green function

Recall that the *Poincaré distance* ω on $\mathbb{D}$ is defined as

$$\omega(\zeta, z) := \tanh^{-1} |T_z(\zeta)| = \frac{1}{2} \log \frac{1 + |T_z(\zeta)|}{1 - |T_z(\zeta)|}. \tag{8.1}$$

where $T_z(\zeta) := (z - \zeta)(1 - \overline{z}\zeta)^{-1}$ is an automorphism of $\mathbb{D}$ which maps z to 0. Such a distance is complete. By (7.1) it follows

$$G_{\mathbb{D}}(z, \zeta) = \log \tanh \omega(z, \zeta), \quad z, \zeta \in \mathbb{D}$$

and by Proposition 7.2 (or directly from the Schwarz lemma) it follows that $f^*\omega \leq \omega$ for all $f : \mathbb{D} \to \mathbb{D}$ holomorphic.

DEFINITION 8.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. The *Carathéodory pseudodistance* $C_\Omega : \Omega \times \Omega \to \mathbb{R}^+$ is defined as

$$C_\Omega(z, w) := \sup\{\omega(f(z), f(w)) : f : \Omega \to \mathbb{D} \text{ is holomorphic}\}.$$

The *Lempert function* $\delta_\Omega : \Omega \times \Omega \to \mathbb{R}^+$ is defined as

$$\delta_\Omega(z, w) = \inf\{\omega(\zeta_1, \zeta_2) : \exists f : \mathbb{D} \to \Omega \text{ holomorphic with } f(\zeta_1) = z, f(\zeta_2) = w\},$$

with the convention that $\delta_\Omega(z, w) = +\infty$ if there do not exist holomorphic functions $\varphi : \mathbb{D} \to \Omega$ with $\varphi(\zeta) = z$ and $\varphi(\eta) = w$.

The *Kobayashi distance* $d_\Omega : \Omega \times \Omega \to \mathbb{R}^+$ is defined as

$$d_\Omega(z, w) = \inf \sum_{j=1}^{m} \delta_\Omega(z_j, z_{j+1}),$$

where the infimum is taken over all finite chains of points $z_1, \ldots, z_m \in \Omega$ such that $z_1 = z$ and $z_m = w$.

Notice that d_Ω is the biggest pseudodistance smaller than δ_Ω. It is not too difficult to see that

$$C_\Omega(z, w) \leq d_\Omega(z, w) \leq \delta_\Omega(z, w).$$

Both the Carathéodory and the Kobayashi pseudodistances are continuous, but in general the induced distance topology is less finer than the euclidean topology of Ω. The topology induced by the Kobayashi pseudodistance is equivalent to the euclidean topology if and only if d_Ω is a distance (namely, $d_\Omega(z, w) = 0$ if and only if $z = w$). The topology induced by the Carathéodory pseudodistance is equivalent to the euclidean topology if the *inner pseudodistance* associated to C_Ω is a distance (the inner pseudodistance $C^i_\Omega(z, w)$ is defined to be the infimum of the C_Ω-length of piecewise smooth curves joining z to w. Notice that $C_\Omega \leq C^i_\Omega$). However, if Ω is bounded then C_Ω induces a topology equivalent to the euclidean one. For all these properties and much more see [**15**].

PROPOSITION 8.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then for all $z, w \in \Omega$*

$$\log \tanh C_\Omega(z, w) \leq K_{\Omega,z}(w) \leq \log \tanh \delta_\Omega(z, w).$$

PROOF. Let $f : \mathbb{D} \to \Omega$ be holomorphic and such that $f(\zeta) = z, f(\eta) = w$. Then

$$K_{\Omega,f(\zeta)}(f(\eta)) \leq K_{\mathbb{D},\zeta}(\eta) = \log |T_\zeta(\eta)|.$$

Therefore $\exp(K_{\Omega,f(\zeta)}(f(\eta))) \leq |T_\zeta(\eta)|$ which implies

$$\tanh^{-1} \exp(K_{\Omega,f(\zeta)}(f(\eta))) \leq \tanh^{-1} |T_\zeta(\eta)| = \omega(\zeta, \eta).$$

For the arbitrariness of f we obtain $\tanh^{-1} \exp(K_{\Omega,z}(w)) \leq \delta_\Omega(z, w)$ from which the second inequality follows.

As for the other inequality, the argument is similar. Let $f : \Omega \to \mathbb{D}$. Then $K_{\Omega,z}(w) \geq K_{\mathbb{D},f(z)}(f(w)) = \log |T_{f(z)}(f(w))|$. Arguing as before this implies that $\tanh^{-1} \exp(K_{\Omega,z}(w)) \geq \omega(f(z), f(w))$. By arbitrariness of f the first inequality follows. □

In [**17**] Lempert showed that for a convex domain $C_\Omega = d_\Omega = \delta_\Omega$ and therefore the previous proposition implies that *if* Ω *is a convex domain then*

$$K_{\Omega,z}(w) = \log \tanh \delta_\Omega(z, w) = \log \tanh d_\Omega(z, w) = \log \tanh C_\Omega(z, w).$$

In particular in this case the pluricomplex Green function is symmetric.

COROLLARY 8.3. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain and* $z \in \Omega$. *Suppose that* $\Omega \ni w \mapsto \log \tanh \delta_\Omega(z, w)$ *is plurisubharmonic. Then*

$$K_{\Omega,z}(w) = \log \tanh \delta(z, w) \quad \forall w \in \Omega.$$

PROOF. Let $u(w) = \log \tanh \delta(z, w)$. Then $u < 0$ in Ω and $u \in \mathsf{Psh}(\Omega)$. If we show that u has a logarithmic singularity at z then $u \in \mathcal{K}_{\Omega,z}$ which implies that $u \leq K_{\Omega,z}$ and by Proposition 8.2 we have the result. Now let $r > 0$ be such that $\mathbb{B}(z, r) \subset \Omega$. Let $w \in \mathbb{B}(z, r) \setminus \{z\}$ and define

$$\varphi(\zeta) := z + \zeta r \frac{w - z}{\|w - z\|}$$

for $\zeta \in \mathbb{D}$. Clearly $\varphi : \mathbb{D} \to \Omega$ is holomorphic, $\varphi(0) = z$ and $\varphi(\|w - z\|/r) = w$. Thus by definition $\delta_\Omega(z, w) \leq \omega(0, \|w - z\|/r)$. Therefore by (8.1)

$$\log \tanh \delta(z, w) \leq \log \tanh \omega(0, \|w - z\|/r) = \log \|z - w\|/r,$$

as needed. □

9. Some further geometrical directions

The works [**5**] and [**6**] show that it is possible to define a maximal plurisubharmonic function in strongly convex domains such that it solves a complex homogeneous Monge-Ampère equation with a simple pole at the boundary. Such a function is the normal derivative of the pluricomplex Green function and it is called the pluricomplex Poisson kernel. It is strongly related to the invariant geometry of the domain because its level sets are *horospheres* of the domain (namely, limits of Kobayashi balls) and the associated Monge-Ampère foliation is made of *complex geodesics* (namely, isometries between the Poincaré metric of the disc and the Kobayashi metric of the domain). Roughly speaking, similarly to what have been done for the pluricomplex Green function, the pluricomplex Poisson kernel can be characterized as the maximum of the family

$$\mathcal{F}_p = \{u \in \mathsf{Psh}(\Omega) : u < 0, \mathsf{K}\text{-}\limsup_{z \to p} u(z)\|z - p\| < -1\},$$

where $\mathsf{K}\text{-}\limsup$ means non-tangential limit, Ω is a bounded strongly convex domain in $\mathbb{C}^N$ with smooth boundary and $p \in \partial\Omega$. However, the proof involves the use of fine properties of complex geodesics and Lempert's theory, and such tools are not available in other domains.

Thus, a geometrically relevant problem is to understand whether the family $\mathcal{F}_p$ has a maximum (and which are its regularity properties) when Ω is not strongly convex, for instance if Ω is strongly pseudoconvex or weakly convex or hyperconvex. Also, it would be interesting to know whether the Demailly measure μ_z introduced in Theorem 7.7 can be expressed in terms of the maximal element (if any) of the family $\mathcal{F}_p$.

Bibliography

[1] E. Bedford, B.A. Taylor, *The Dirichlet problem for the complex Monge-Ampère equation.* Invent. Math. 37, (1976), 1-44.

[2] E. Bedford, B.A. Taylor, *A new capacity for plurisubharmonic functions.* Acta. Math. 149, (1982), 1-41.

[3] Z. Błocki, *The domain of definition of the complex Monge-Ampère operator.* Amer. J. Math. 128 (2006), no. 2, 519–530.

[4] R. Bott, L. W. Tu, *Differential forms in algebraic topology*, Springer 1982.

[5] F. Bracci, G. Patrizio, *Monge-Ampère foliations with singularities at the boundary of strongly convex domains*, Math. Ann., 332, 3, (2005) 499-522.

[6] F. Bracci, G. Patrizio, S. Trapani, *The pluricomplex Poisson kernel for strongly convex domains*, to appear Trans. Amer. Math. Soc. (available on ArXiV).

[7] C. Camacho, A. Lins Neto, *Geometric theory of foliations.* Birkhäuser 1985.

[8] U. Cegrell, *The general definition of the complex Monge-Ampère operator*, Ann. Inst. Fourier 54 (2004), 159–179.

[9] J.-P. Demailly, *Complex analytic and differential geometry*. Grenoble 1997.

[10] J.-P. Demailly, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines* Mém. Soc. Math. France (N.S.) No. 19 (1985).

[11] J.-P. Demailly, *Mesures de Monge-Ampère et mesures pluriharmoniques.* Math. Z. 194, (1987), 519-564.

[12] G. de Rham, *Variétés différentiables.* Hermann, Paris, 1954.

[13] P. Griffiths, J. Harris, *Principles of algebraic geometry*. Wiley Classic Library 1994.

[14] L. Hörmander, *Notions of convexity*. Progress in Mathematics, 127. Birkhuser Boston, Inc., Boston, MA, 1994.

[15] S. Kobayashi, *Hyperbolic complex spaces.* Springer, Grundlehren der mathematischen Wissenschaften 318.

[16] M. Klimek, *Pluripotential Theory*. London Math. Soc. Monographs, New Series, Academic Press, 1991.

[17] L. Lempert, *La métrique de Kobayashi et la representation des domaines sur la boule.* Bull. Soc. Math. Fr. 109 (1981), 427-474.

[18] W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1974.