Pluricomplex Green function, pluricomplex Poisson kernel and applications

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Abstract. This is a small survey about the pluricomplex Green function of M. Klimek, J.-P. Demailly and L. Lempert and the recently introduced pluricomplex Poisson kernel by the author and G. Patrizio. Aside from reviewing basic properties, we give a new look to iteration theory and semigroups using those tools.

Sommario. Questo è un breve sunto sulla funzione di Green pluricomplessa definita da M. Klimek, J.-P. Demailly e L. Lempert e sul nucleo di Poisson pluricomplesso recentemente introdotto dall'autore e G. Patrizio. Oltre ad un breve ripasso delle proprietà di base, daremo una visione nuova alla teoria della iterazione e dei semigruppi utilizzando tali strumenti.

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1 Introduction

In the past decades the study of homogeneous Monge-Ampère equations with some prescribed singularity played a central role in pluripotential theory, especially due to many applications it has, such as in Ricci's flow, extremal metrics, representation formulas, etc...

In one dimension a homogeneous Monge-Ampère equation is nothing but a Dirichlet-type problem and the solution (when it exists) is a harmonic function with some prescribed singularity. For instance, the Green function in the unit disc can be defined as the solution to a homogeneous Monge–Ampère equation, zero on the boundary and with a logarithmic singularity at a given point. Or, in other terms, using Perron's point of view, it can be described as the maximum of a family of negative subharmonic functions having at least a logarithmic singularity at a given point. This point of view is very useful in iteration theory because a given holomorphic self-map of the unit disc maps such a family into another such family, and hence it contracts the Green function, providing information on the dynamics of the map itself. Moreover, such argument can be used to characterize semicomplete holomorphic vector fields of the unit disc.

Similar arguments work for the Poisson kernel in the unit disc. The Poisson kernel is nothing but the solution to a homogeneous Monge-Ampère equation with a prescribed non-tangential simple pole at a boundary point. Again, it can be seen as the maximal element of a particular family of subharmonic functions (this is in fact the content of the classical Phragmen-Lindelöf theorem) and hence used in iteration theory.

The Green function and the Poisson kernel can be also used to obtain representation formulas for subharmonic functions of the unit disc (and allow to solve non-homogeneous Dirichlet problem).

In higher dimensions, in the case of strongly convex domains in the 80's L. Lempert developed an amazing tool to study the Kobayashi distance. As a result of his theory, he was able to solve a particular homogeneous Monge-Ampère equation with prescribed logarithmic singularity at a given point which resembles completely the one dimensional equation giving rise to the Green function. More or less in the same ages, M. Klimek proposed to name pluricomplex Green function the maximal element of a given family of plurisubharmonic functions. J.-P. Demailly proved that for hyperconvex domains such a function is unique and solves the same homogeneous Monge-Ampère equation as Lempert's (thus being the same in the strongly convex case). Moreover, Demailly proved a representation formula for plurisubharmonic functions, replacing the Poisson kernel with a measure coming from the cut-off Monge-Ampère mass of the pluricomplex Green function.

Recently, the author and G. Patrizio studied a homogeneous Monge-Ampère equation with prescribed non-tangential simple pole in strongly convex domains. The solution shares many properties with the classical Poisson kernel and thus it was named pluricomplex Poisson kernel. Later, the two authors and S. Trapani proved that such a pluricomplex Poisson kernel satisfies a higher dimensional Phragmen-Lindelöf theorem and the Demailly boundary measure in his representation formula is essentially given by such a function.

More recently, the author with M. Contreras and S. Diaz-Madrigal, applied those tools to study iteration theory and semigroups.

The present paper is a small survey along the previously described lines. Starting from the unit disc we review the theory, especially with a view toward iteration theory and semigroups.

The content of this paper is based on the conference the author gave in Bologna in May 2007. The author wishes to sincerely thank prof. Salvatore Coen for both the invitation to give that talk and the opportunity to write this survey.

2 The unit disc

Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc. Let $z \in \mathbb{D}$. The *Green function* with pole at z is defined as

$$G_{\mathbb{D},z}(\zeta) := \log |T_z(\zeta)|,$$

where $T_z(\zeta) = \frac{\zeta - z}{1 - \bar{z}\zeta}$ is an automorphism of $\mathbb{D}$ which maps z to ζ.

The Green function $G_{\mathbb{D},z}$ is a negative subharmonic function in $\mathbb{D}$ (harmonic in $\mathbb{D} \setminus \{z\}$ with a logarithmic singularity at z) which extends continuously up to $\partial\mathbb{D}$ and is identically zero there. Its sub-level sets are exactly the discs for the Poincaré distance of $\mathbb{D}$.

Let $\mathsf{subh}(\mathbb{D})$ denote the real cone of subharmonic functions in $\mathbb{D}$. It can be proved that

$$G_{\mathbb{D},z} = \max\{u \in \mathsf{subh}(\mathbb{D}) : u < 0, \limsup_{\zeta \to z}[u(\zeta) - \log |\zeta - z|] < +\infty\}. \tag{1}$$

Given a holomorphic self-map of the unit disc, $f : \mathbb{D} \to \mathbb{D}$, it follows that if $u \in \{u \in \mathsf{subh}(\mathbb{D}) : u < 0, \limsup_{\zeta \to z}[u(\zeta) - \log |\zeta - z|] < +\infty\}$ then $f^*u := u \circ f \in \{v \in \mathsf{subh}(\mathbb{D}) : v < 0, \limsup_{\zeta \to f(z)}[v(\zeta) - \log |\zeta - f(z)|] < +\infty\}$. Hence, by the previous result, we obtain the classical Schwarz lemma rephrased in the following way:

Theorem [Potential form of the Schwarz lemma] *For all $f : \mathbb{D} \to \mathbb{D}$ holomorphic it holds*

$$f^*(G_{\mathbb{D},f(z)}) \leq G_{\mathbb{D},z}. \tag{2}$$

Moreover, at some—hence any—point it holds $=$ if and only if f is an automorphism of $\mathbb{D}$.

A strict relative of the Green function is the *Poisson kernel*. Let $z \in \partial\mathbb{D}$. The Poisson kernel with pole at z is defined as

$$P_{\mathbb{D},z}(\zeta) := -\frac{1 - |\zeta|^2}{|z - \zeta|^2}.$$

It is a negative harmonic function on $\mathbb{D}$, which extends smoothly on $\partial\mathbb{D} \setminus \{z\}$ and has a simple pole at z when moving *nontangentially* to z (namely, for any sequence $\{\zeta_k\} \subset \mathbb{D}$ such that $\zeta_k \to z$ and $|\zeta_k - z| < c(1 - |\zeta_k|)$ for some constant $c > 0$ independent of k, it follows that $P_{\mathbb{D},z}(\zeta_k) \approx |z - \zeta_k|^{-1}$ as $k \to \infty$).

The sublevel sets of the Poisson kernel $P_{\mathbb{D},z}$ are discs tangent to $\partial\mathbb{D}$ at z. A direct computation (see, [16] or [12]) shows that the Poisson kernel is the normal derivative of the Green function. In fact, a theorem similar to Schwarz's lemma holds at the boundary. Such a theorem is usually known as the Julia lemma, see, *e.g.*, [1] and it is stated in terms of horodiscs (*i.e.* level sets of the Poisson kernel). Here is its theoretical potential rephrasing:

Theorem [Potential form of the Julia lemma]*Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Suppose that $z_0, z_1 \in \partial\mathbb{D}$ and that f has nontangential limit z_1 at z_0. Then there exists $\lambda \geq 0$ such that*

$$f^*(P_{\mathbb{D},z_1}) \leq \lambda P_{\mathbb{D},z_0}.$$

Moreover, at some—hence any—point it holds $=$ (and necessarily $\lambda > 0$) if and only if f is an automorphism of $\mathbb{D}$.

The Poisson kernel satisfies a maximal condition similar to (1). Indeed, the content of the classical Phragmen-Lindelöf theorem is the following:

$$P_{\mathbb{D},z} = \max\{u \in \mathsf{subh}(\mathbb{D}) : u < 0, \limsup_{\mathbb{R}\ni r \to 1} u(r)(1 - r) \leq -2\}. \tag{3}$$

The Poisson kernel can be used to reproduce harmonic functions on $\mathbb{D}$ which extends continuously up to $\partial\mathbb{D}$ (see [16] or any book of potential theory).

3 The pluricomplex Green function in higher dimension

Let $D \subset \mathbb{C}^N$ be a bounded domain. Let $z_0 \in D$. Let $\mathsf{Psh}(D)$ denote the space of plurisubharmonic functions on D, namely subharmonic functions which remain subharmonic under holomorphic changes of variables. We recall here that in dimension one harmonic functions can be characterized as maximal subharmonic functions, namely, if u is subharmonic in $\mathbb{D}$ and $C \subset\subset \mathbb{D}$ is a disc, then u is harmonic if and only if for every subharmonic function v on $\mathbb{D}$ such that $v \leq u$ on ∂C it follows that $v \leq u$ on C. Harmonic functions are smooth by the Caccioppoli-Weil lemma. In higher dimension however, not all pluriharmonic functions are smooth, not even continuous, but definitely they are maximal. Moreover, by a deep result of Bedford and Taylor [3], [4] if $u \in L^\infty_{\mathsf{loc}}(D) \cap \mathsf{Psh}(D)$ then u is maximal (and hence pluriharmonic) if and only if $(dd^c u)^N \equiv 0$, where $d^c := i(\overline{\partial} - \partial)$ and the operator $(dd^c \cdot)^n$ has a meaning (see also, [16]) in the sense of currents. Note that if $u \in C^2(D) \cap \mathsf{Psh}(D)$ then $(dd^c u)^n \equiv 0$ if and only if the complex Hessian matrix of u has determinant zero at any point.

Let us consider the following family

$$\mathcal{F}_{D,z_0} := \{u \in \mathsf{Psh}(D) : u < 0, \limsup_{z \to z_0} [u(z) - \log \|z - z_0\|] < +\infty\}.$$

According to Klimek [16] we define the *pluricomplex Green function* with logarithmic pole at z_0 as

$$G_{D,z_0}(z) := \sup_{u \in \mathcal{F}_{D,z_0}} u(z).$$

Klimek proved that G_{D,z_0} exists, it is actually the maximum of the family $\mathcal{F}_{D,z_0}$, it is plurisubharmonic and *maximal*. Later and independently Demailly [13], [14] and Lempert [17], [18] proved regularity results of the pluricomplex Green's function and its relation to the complex Monge-Ampére equation. Moreover, Guan [15] and Błocki [5] provided finer regularity for strongly pseudoconvex domains D. We collect all the results in the following theorem.

Theorem A

• **(Demailly)** *Let $D \subset \mathbb{C}^N$ be a bounded hyperconvex domain. Then the pluricomplex Green function $G_{D,\cdot}(\cdot)$ is continuous in $\overline{D} \times \overline{D} \setminus \mathsf{Diag}$ and it is the unique plurisubharmonic function $u \in \mathsf{Psh}(D) \cap L^\infty_{\mathsf{loc}}(D)$ such that $(\partial\overline{\partial}u)^N(z) = \delta_{z_0}$ (the Dirac delta), $u(z) = 0$ for $z \in \partial D$ and $u(z) - \log \|p - z\| = O(1)$ as $z \to z_0$.*

• **(Lempert)** *If D is strongly convex with smooth boundary then $G_{D,\cdot}(\cdot)$ is C^∞ in $\overline{D} \times \overline{D} \setminus \mathsf{Diag}$. Moreover, $G_{D,z}(w) = \log \tanh k_D(z, w)$ where k_D denotes the Kobayashi distance of D, and the associated Monge–Ampére foliation consists of complex geodesics passing through z_0.*

• **(Guan/Błocki)** *If D is strongly pseudoconvex then $G_{D,z_0}(\cdot)$ is $C^{1,1}(D \setminus \{z_0\})$ in D.*

It is worth to say that the three previous results are proved with completely different techniques.

We recall that a complex geodesic is a holomorphic isometry between the unit disc endowed with the Poincaré distance and the domain D endowed with the Kobayashi distance.

Demailly [14, Théorème 5.1] proved moreover that for all $F \in \mathsf{Psh}(D) \cap C^0(\overline{D})$ the following representation formula holds:

$$F(z) = \mu_z(F) - \frac{1}{2\pi^n} \int_{w \in D} |L_{D,z}(w)| \, dd^c F(w) \wedge (dd^c L_{D,z})^{n-1}(w), \qquad (4)$$

where μ_z is defined by taking the limit in the sense of distributions as $R \to 0$ of the measures $(dd^c \max\{G_{D,z}(\cdot), R\})^N$ supported on the level set $\{w \in D : G_{D,z}(w) = R\}$.

If $D = \mathbb{D}$ the unit disc of $\mathbb{C}$ then $G_{D,z}(w)$ is nothing but the Green function. By the very definition of the pluricomplex Green function as maximal element of the family $\mathcal{F}_{D,z_0}$ and using complex geodesics and Lempert's theorem one can prove (see [6]) the following Schwarz'type lemma:

Theorem *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary and let $z_0 \in D$. Let $h : D \to D$ be holomorphic. Then $h(z_0) = z_0$ if and only if for all $z \in D$*

$$G_D(z_0, h(z)) \leq G_D(z_0, z). \qquad (5)$$

Moreover, if equality holds in (5) for some $z \neq z_0$ and $\varphi : \mathbb{D} \to D$ is the complex geodesic such that $\varphi(0) = z_0$ and $\varphi(t) = z$ for some $t \in (0,1)$, it follows that $h \circ \varphi : \mathbb{D} \to D$ is a complex geodesic and $h : \varphi(\mathbb{D}) \to h(\varphi(\mathbb{D}))$ is an automorphism.

Such a theorem is the bulk for an interesting application to the theory of semigroups. Given a holomorphic vector field X on D, one is interested in studying the behavior of its trajectories. By the holomorphic flow box theorem, for any $z \in D$ there exists a compact subset $K \subset\subset D$ and $\delta > 0$ and a map $\phi : [0, \delta) \times K \to D$ holomorphic in $w \in K$ and real analytic in $t \in [0, \delta)$ such that $\varphi(0, w) = w$ for all $w \in K$ and

$$\frac{\partial \varphi(t, w)}{\partial t} = X(\varphi(t, w)).$$

The vector field X is said to be *semicomplete* provided $\delta = +\infty$ for all $z \in D$. Obviously, this holds also for negative times and a vector field is said to be *complete* if it is semicomplete both for positive and negative times. In case a vector field is semicomplete, for each $t \geq 0$ the map $z \mapsto \varphi_t(z) := \varphi(t, z)$ is a holomorphic self-map of D. Moreover, the family (φ_t) forms a *semigroup* in the sense that the map $(\mathbb{R}^+, +) \ni t \mapsto \varphi_t \in (\mathsf{Hol}(D, D), \circ)$ is a morphism of semigroup, continuous with respect to the Euclidean topology of $\mathbb{R}^+$ and with the topology of uniform convergence on compacta of $\mathsf{Hol}(D, D)$.

The semigroup (φ_t) fixes a point $z_0 \in D$ if and only if $X(z_0) = 0$. And, by the previous theorem, in case D is strongly convex, this is the case if and only if $G_{D,z_0}(\varphi_t(z)) \leq G_{D,z_0}(z)$ for all $t \geq 0$. Differentiating in t, it follows that if X is semicomplete and $X(z_0) = 0$ then $dG_{D,z_0}(X)(z) \leq 0$ for all $z \in D$. It can be shown that also the converse holds, also, dropping the hypothesis of a singular point, indeed, we have

Theorem [6]. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let X be a holomorphic vector field in D. Then X is semicomplete if and only if for all $z, w \in D$ with $z \neq w$ it follows $d(G_D)|_{(z,w)} \cdot (F(z), F(w)) \leq 0$ (where here we consider $G_D : D \times D \to \mathbb{R}$).*

Such a condition can be computed in specific cases such as the unit disc, the polydisc or the unit ball, giving rise to some handleable conditions. In particular, in the unit disc case, it reduces to the infamous Berkson-Porta formula (see, *e.g.*, [1]).

In some very recent papers [7], [8] such a condition (and a similar one related to the Kobayashi distance on manifolds) has been used to characterize evolution families and Herglotz vector fields in the Loewner equation in one and several complex variables.

4 The pluricomplex Poisson kernel in higher dimension

Let $D \subset\subset \mathbb{C}^N$ be a strongly convex domain with smooth boundary, $z_0 \in D$ and let $p \in \partial D$. In the paper [9], G. Patrizio and the author introduced a smooth plurisubharmonic function $u_{D,p} : D \to (-\infty, 0)$ which extends smoothly on $\overline{D} \setminus \{p\}$ such that $d(u_{D,p})_z \neq 0$ for all $z \in D$, $u_{D,p}(q) = 0$ for all $q \in \partial D \setminus \{p\}$ and $u_{D,p}$ has a simple pole at p along non-tangential directions. Up to a real positive multiple, we assume here that $u_{D,p}(z_0) = -1$. The function $u_{D,p}$ solves the following homogeneous Monge-Ampère equation:

$$\begin{cases} u \text{ plurisubharmonic in } D \\ (dd^c u)^N = 0 \text{ in } D \\ u < 0 \text{ in } D \\ u(w) = 0 \text{ for all } w \in \partial D \setminus \{p\} \\ u(w) \approx \|w - p\|^{-1} \text{ as } w \to p \text{ non-tangentially.} \end{cases} \tag{6}$$

In the papers [9] and [10], the authors prove that $u_{D,p}$ shares many properties with the classical Poisson kernel for the unit disc, so it deserves the name *pluricomplex Poisson kernel*. In case $D = \mathbb{D}$ the unit disc in $\mathbb{C}$, the function

$u_{\mathbb{D},p}$ (normalized so that $u_{\mathbb{D},p}(0) = -1$) is in fact the classical (negative) Poisson kernel. In case $D = \mathbb{B}^n$, the pluricomplex Poisson kernel (normalized so that $u_{\mathbb{B}^n,p}(0) = -1$) is given by

$$u_{\mathbb{B}^n,p}(z) = -\frac{1 - \|z\|^2}{|\langle p - z, p\rangle|^2}.$$

The level sets of $u_{D,p}$ are exactly boundaries of Abate's horospheres [1]. Recall that a *horosphere* $E_D(p, R)$ of center $p \in \partial D$ and radius $R > 0$ (with respect to z_0) is given by

$$E_D(p, R) = \{z \in D : \lim_{w \to p}[k_D(z, w) - k_D(z_0, w)] < \frac{1}{2} \log R\}.$$

Notice that the existence of the limit in the definition of $E_D(p, R)$ is a characteristic of smooth strongly convex domains and follows again from Lempert's theory (see [1, Theorem 2.6.47]). Thanks to our normalization $u_{D,p}(z_0) = -1$, it follows that

$$E_D(p, R) = \{z \in D : u_{D,p}(z) < -1/R\}. \tag{7}$$

In particular horospheres are smooth strongly pseudoconvex domains, but one can prove that if D is strongly convex they are also strongly convex [10].

For the unit disc, these level sets are boundaries of horocycles and, in case $D = \mathbb{B}^n$, these are boundaries of horospheres in $\mathbb{B}^n$ with center p, whose explicit expression is

$$E_{\mathbb{B}^n}(p, R) = \{z \in \mathbb{B}^n : \frac{|1 - \langle z, p\rangle|^2}{1 - \|z\|^2} < R\}.$$

In the paper [10] it is shown that the pluricomplex Poisson kernel is the maximum of a special family, thus generalizing the classical Phragmen-Lindelöf theorem. More precisely, let $D \subset \mathbb{C}^N$ be a bounded strongly convex domain with smooth boundary. We let Γ_p be the set of all C^∞ curves $\gamma : [0, 1] \to D \cup \{p\}$ such that $\gamma(1) = p$ and $\gamma'(1) \notin T_p\partial D$ (notice that, if ν_p is the unit outward normal to ∂D at p then $\gamma'(1) \notin T_p\partial D$ if and only if $\mathsf{Re}\,\langle \gamma'(1), \nu_p\rangle > 0$).

Theorem PL. *Let $D \subset \mathbb{C}^N$ be a bounded strongly convex domain with smooth boundary and let $p \in \partial D$. Let ν_p be the unit outward normal to ∂D at p. Consider the following family $\mathcal{S}_p(D)$:*

$$\begin{cases} u \in \mathsf{Psh}(D) \\ \limsup_{z \to x} u(z) \leq 0 \quad \text{for all } x \in \partial D \setminus \{p\} \\ \liminf_{t \to 1} |u(\gamma(t))(1 - t)| \geq 2\mathsf{Re}\,(\langle \gamma'(1), \nu_p\rangle^{-1}) \quad \text{for all } \gamma \in \Gamma_p, \end{cases} \tag{8}$$

Then $u_{D,p} \in \mathcal{S}_p(D)$ and $u \leq u_{D,p}$ for all $u \in \mathcal{S}_p(D)$.

The function $u_{D,p}$ is unique as the maximal solution to (6), and has some other uniqueness features, although an "absolute" uniqueness (such as the pluricomplex Green function) is not known because the Monge-Ampère problem (6) only establishes non-tangential behavior at p.

The pluricomplex Poisson kernel is related to the pluricomplex Green function as follows [10]

Theorem B. *Let D be a bounded strongly convex domain in $\mathbb{C}^N$ with smooth boundary. Let $z_0 \in D$ and $p \in \partial D$. Let ν_p be the outer normal of ∂D at p. Then*

$$u_{D,p}(z_0) = -\frac{\partial G_{D,z_0}}{\partial \nu_p}(p).$$

Moreover, denoting by $\omega_{\partial D}$ the normalized Levi form of ∂D,

$$d\mu_z(p) = |u_{D,p}(z)|^N \omega_{\partial D}(p).$$

The last equation allows to give an "explicit" representation formula for plurisubharmonic functions in D, namely, putting together Theorem A and Theorem B we obtain

Corollary. *Let D be a strongly convex domain with smooth boundary. Let $F \in \mathsf{Psh}(D) \cap C^0(\overline{D})$. Then for all $z \in D$*

$$\begin{array}{c} F(z) = \int_{p\in\partial D} |u_{D,p}(z)|^n F(p)\omega_{\partial D}(p) \\ -\frac{1}{2\pi^n}\int_{w\in D} |G_{D,z}(w)|\, dd^cF(w) \wedge (dd^cL_{D,z})^{n-1}(w). \end{array}$$

In particular if F is pluriharmonic then

$$F(z) = \int_{p\in\partial D} |u_{D,p}(z)|^n F(p)\omega_{\partial D}(p).$$

In case of domains which are not strongly convex, one can try to use Theorem PL to define a candidate for the pluricomplex Poisson kernel. This is done is [11], and in fact, in case of a strongly pseudoconvex domain D contained in a Stein manifold, one can show that supremum of the family $\mathcal{S}_p(D)$ exists, it is continuous up to the boundary and shares many property with the one-dimensional Poisson kernel.

We end up this little survey by presenting some applications of the pluricomplex Poisson kernel for strongly convex domains to the theory of semigroups. First, we give the following generalization to the Julia's Lemma [6].

Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary, $p \in \partial D$, and $h : D \to D$ holomorphic. The *boundary dilatation coefficient* $\alpha_h(p) \in (0, +\infty]$ is defined as

$$\alpha_h(p) = \inf_{q \in \partial D} \{\sup_{z \in D} \frac{u_{D,p}(z)}{u_{D,q}(h(z))}\}.$$

Theorem. *Let $D \subset\subset \mathbb{C}^N$ be a strongly convex domain with smooth boundary, $p \in \partial D$ and $h : D \to D$ holomorphic. If the boundary dilatation coefficient $\alpha_h(p) < +\infty$, then there exists a unique point $q \in \partial D$ such that h has non-tangential limit q at p and, for all $z \in D$,*

$$u_{D,q}(h(z)) \leq \frac{1}{\alpha_h(p)} u_{D,p}(z). \tag{9}$$

Moreover, if equality holds in (9) for some $z \in D$ and $\varphi : \mathbb{D} \to D$ is the complex geodesic such that $\varphi(1) = p$ and $\varphi(0) = z$, it follows that $h \circ \varphi : \mathbb{D} \to D$ is a complex geodesic and $h : \varphi(\mathbb{D}) \to h(\varphi(\mathbb{D}))$ is an automorphism.

Equation (9) has a geometric meaning, namely, the map h maps any horosphere $E_D(p, R)$ into the horosphere $E_D(q, \alpha_h(p)R)$, exactly as in the unit disc case.

Again, this theorem can be used to characterizes "boundary sinks" for semigroups. Let (Φ_t) be a one-parameter semigroup of holomorphic self-maps of D. A point $p \in \partial D$ is called a *sink for* (Φ_t), if p is a fixed point for Φ_t as non-tangential limit and $\alpha_{\Phi_t}(p) < +\infty$ for all $t \geq 0$. It can be proved that if $p \in \partial D$ is a sink for (Φ_t) then there exists $\beta \in \mathbb{R}$ such that $\alpha_{\Phi_t}(p) = e^{\beta t}$ for all $t \geq 0$.

The question is how to characterize sinks in terms of the associated semicomplete vector field. The answer is obtained by means of the pluricomplex Poisson kernel [6]:

Theorem. *Let $D \subset\subset \mathbb{C}^n$ be a strongly convex domain with smooth boundary. Let $F : D \to \mathbb{C}^n$ be a semicomplete holomorphic vector field generating a semigroup (Φ_t). Let $\beta \in \mathbb{R}$ and $p \in \partial D$. The following are equivalent:*

1. *The semigroup (Φ_t) has a sink at p with boundary dilatation coefficients $\alpha_t(p) \leq e^{\beta t}$ for all $t \geq 0$.*
2. *$d(u_{D,p})_z \cdot F(z) + \beta u_{D,p}(z) \leq 0$ for all $z \in D$.*

Moreover, if p is a sink for (Φ_t) then the boundary dilatation coefficient of Φ_t is $\alpha_t(p) = e^{-tb}$ with $b = \inf_{z \in D} d(u_{D,p})_z \cdot F(z)/u_{D,p}(z)$.

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