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COMMUTING HOLOMORPHIC MAPS IN STRONGLY CONVEX DOMAINS

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Abstract. Let D be a bounded strongly convex C^3 domain of $\mathbb{C}^n$. We prove that if $f, g \in \mathrm{Hol}(D,D)$ are commuting holomorphic self-maps of D, then they have a common fixed point in $\overline{D}$ (if it belongs to ∂D, we mean fixed in the sense of K-limits). Furthermore, if f and g have no fixed points in D and $f \circ g = g \circ f$ then f and g have the same *Wolff point*, unless their restrictions to the complex geodesic whose closure contains the Wolff points of f and g, are two commuting (hyperbolic) automorphisms of such geodesic.

0. Introduction

In 1964 A.L. Shields [17] proved that a family of continuous functions mapping the closed unit disk into itself, holomorphic in the open disk and commuting under composition, has a common fixed point. Since then the structure of families of commuting holomorphic self-maps of various cases of domains has been deeply investigated. In particular the result by Shields was generalized to the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ by T.J. Suffridge [19], and M. Abate [1], to the case of strongly convex domains by Abate [1] and finally to the case of convex domains and Riemann surfaces by Abate and J.P. Vigué [4].

On the other hand, a result due to Wolff (which we shall refer to as the *Wolff Lemma* [20], see also A. Denjoy [12]) states that the sequence of iterates $\{f^k\}$ of a holomorphic self-map f of the unit disk Δ converges (if f is not an elliptic automorphism of Δ) to a single point $\tau \in \overline{\Delta}$, the *Wolff point* of f. The Wolff point τ belongs to Δ if and only if τ is the only fixed point of f. In this case, if g is a holomorphic self-maps of Δ which commutes with f, then $g(\tau) = \tau$ and τ is the Wolff point of g, too. If f has no fixed points in Δ, by the classical Julia-Wolff-Carathéodory Theorem, f has non-tangential limit τ at τ. A result by M.H. Heins [13] states that a holomorphic self-map of Δ which commutes with a hyperbolic automorphism of Δ is itself a hyperbolic automorphism of Δ (unless it is the identity). Subsequently, in 1973 D.F. Behan [5] proved that two commuting holomorphic self-maps of Δ with no fixed points in Δ either have the same Wolff point or they are two hyperbolic automorphisms of Δ. As a consequence (the so-called "Behan-Shields Theorem"), two commuting holomorphic self-maps of Δ always have a common fixed point,

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either in the interior of the disk or as non-tangential limit at the boundary.
In the multidimensional case the situation is richer (and more complicated) than in the case of one variable. For instance, C. de Fabritiis and G. Gentili [11] and de Fabritiis [10] showed that in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ $(n > 1)$ there is a large family of holomorphic (non-automorphisms) self-maps which commute with a given hyperbolic automorphism of $\mathbb{B}^n$. Moreover, the fixed points set of a holomorphic self-map of $\mathbb{B}^n$ is (generally) not reduced to one point, and there are couples of commuting holomorphic self-maps of $\mathbb{B}^n$ of which only one has fixed points in $\mathbb{B}^n$ (for instance there are elliptic automorphisms commuting with hyperbolic automorphisms). Another feature of the multidimensional case is that, in a bounded C^2 domain of $\mathbb{C}^n$, the natural admissible regions for the study of boundary behaviours of maps are not cones (i.e. non-tangential limits), but regions which approach the boundary non-tangentially along the normal direction, and tangentially along the complex tangential directions (see E.M. Stein [18], E.M. Čirca [9], J.A. Cima and S.G. Krantz [8], Abate [2], [1]). In this paper we will use Abate's *K-regions* for strongly convex domains (which are however comparable to the admissible approach regions of Stein, Čirka and Cima and Krantz). Then our boundary admissible limits will be the *K-limits* (see §2).
In spite of all the differences, at least in a strongly convex domain D of $\mathbb{C}^n$, the behaviour of the sequence of iterates of $f \in \mathrm{Hol}(D,D)$ -if f has no fixed points- is similar to the behaviour in Δ. In fact $\{f^k\}$ converges to a unique boundary point $\tau(f) \in \partial D$, the *Wolff point* of f; moreover f has K-limit $\tau(f)$ at $\tau(f)$ (see [1],[2] and §1, §2). It is then natural to investigate whether a Behan-Shields-type Theorem holds in strongly convex domains. Indeed we proved (see [6]) that such a result holds in $\mathbb{B}^n$ $(n > 1)$. If $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $f \circ g = g \circ f$, then f and g have a "common fixed point" (possibly at the boundary in the sense of K-limits). Moreover if f and g have no fixed points in $\mathbb{B}^n$ then either f and g have the same Wolff point or, up to conjugation in the group of automorphisms of $\mathbb{B}^n$, there exist f_1 and g_1, commuting hyperbolic automorphisms of Δ, such that $f(\zeta, 0, \ldots, 0) = (f_1(\zeta), 0, \ldots, 0)$ and $g(\zeta, 0, \ldots, 0) = (g_1(\zeta), 0, \ldots, 0)$ for all $\zeta \in \Delta$.

The aim of this paper is to generalize the above result to any bounded strongly convex domain with C^3 boundary.
Firstly, setting $\mathrm{Fix}(f) = \{z \in D : f(z) = z\}$, we prove the following Shields-type Theorem with no assumption of boundary continuity:

Theorem 0.1. *Let D be a bounded strongly convex C^3 domain. Suppose that $f, g \in \mathrm{Hol}(D,D)$ and $f \circ g = g \circ f$.*

(i) *If $\mathrm{Fix}(f) \neq \emptyset$ and $Fix(g) \neq \emptyset$ then $\mathrm{Fix}(f) \bigcap \mathrm{Fix}(g) \neq \emptyset$.*
(ii) *If $\mathrm{Fix}(f) = \emptyset$ and $x \in \partial D$ is the Wolff point of f, then g has K-limit x at x. In particular, f and g have K-limit x at x.*

To state our version of the Behan Theorem we need the "geodesic projection device" of L. Lempert (see [15], [16], and also [2]). We recall that a complex geodesic $\varphi : \Delta \to D$ is a holomorphic isometry with respect to the Poincaré distance on Δ and the Kobayashi distance on D (see §1, §2). Since D is bounded, C^3 and strongly convex, it follows that for every $a \in \partial D$ and $b \in \partial D$, $a \neq b$, there exists a unique (up to parametrizations of Δ) complex geodesic $\varphi : \Delta \to D$ such that $\varphi \in C^1(\overline{\Delta})$ and $\varphi(-1) = a, \varphi(1) = b$ (see [7], [15], [1], [2], [14]). Then our Behan-type Theorem in D can be stated as follows:

Theorem 0.2. *Let D be a bounded strongly convex C^3 domain in $\mathbb{C}^n$. Let $f, g \in$ $\mathrm{Hol}(D,D)$ have no fixed points and let $f \circ g = g \circ f$. Then either f and g have the same Wolff point or there exists a complex geodesic $\varphi : \Delta \to D$ such that $f(\varphi(\Delta)) = \varphi(\Delta)$, $g(\varphi(\Delta)) = \varphi(\Delta)$ and $f|_{\varphi(\Delta)}$, $g|_{\varphi(\Delta)}$ are commuting (hyperbolic) automorphisms of $\varphi(\Delta)$. In the last case, the complex geodesic φ is the unique (up to parametrizations of Δ) such that $\varphi(-1)$ is the Wolff point of g and $\varphi(1)$ is the Wolff point of f.*

The paper is organized as follows. In the first section we recall some facts about iteration theory in strongly convex domains, as developed by Abate (see [1], [2]). We give a (weak) version of the Shields Theorem with no assumption about the continuity at the boundary. That is, using an *ad hoc* version of Julia's Lemma, we shall prove that two commuting holomorphic self-maps of D have a "common fixed point in $\overline{D}$ (if it lies in ∂D, here we mean "fixed" in the sense of non-tangential limit). In the second section we introduce the notions of K-regions, K-limits and we briefly discuss the geodesic projection device of Lempert. Then we state a (maimed) version of the Julia-Wolff-Carathéodory Theorem in D. With these tools we generalize the Shields Theorem in order to obtain Theorem 0.1. The third section is devoted to the proof of Theorem 0.2.

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1. Iteration theory in strongly convex domains

From now on D will denote a bounded strongly convex C^3 domain in $\mathbb{C}^n$. Moreover we fix once for all a point $z_0 \in D$.
As a consequence of a deep result due to Lempert [15] we can define the *Kobayashi distance* on D by

$$k_D(x_1, x_2) := \inf\{\omega(\zeta_1, \zeta_2) : \exists \Phi \in \mathrm{Hol}(\Delta, D), \Phi(\zeta_j) = x_j \ (j = 1, 2)\},$$

where ω is the Poincaré distance in the unit disc Δ.
An useful property of k_D is the following (see [1], [14]):

Lemma 1.1. *Let $d(.,.)$ denote the euclidean distance in $\mathbb{C}^n$. Then there are two constants $C_1 > 0$, $C_2 > 0$ depending only on D and z_0 such that for all $z \in D$*

$$-C_1 - \frac{1}{2} \log d(z, \partial D) \leq k_D(z_0, z) \leq C_2 - \frac{1}{2} \log d(z, \partial D).$$

Through the whole paper we will say that $h \in \mathrm{Hol}(D,D)$ *has no fixed points* to mean that $\mathrm{Fix}(h) = \emptyset$. With this convention we state this type of Wolff-Denjoy Lemma:

Theorem 1.2 (Abate [1], [2])**.** *Let $h \in \mathrm{Hol}(D,D)$. Then $\{h^k\}$ is compactly divergent if and only if h has no fixed points. Moreover if h has no fixed points then $\{h^k\}$ converges uniformly on compacta to a constant $x \in \partial D$.*

Since we have a Wolff Lemma, we can define a "Wolff point":

Definition 1.3. Let $h \in \mathrm{Hol}(D,D)$ be with no fixed points. We call *Wolff point of h* the unique point defined by Theorem 1.2.

Definition 1.4. The *boundary dilatation coefficient* of h at $\tau \in \partial D$ is the value $\alpha_h(\tau) \in \mathbb{R}\bigcup\{\infty\}$ such that

$$\frac{1}{2}\log \alpha_h(\tau) := \liminf_{w\to\tau} \left[k_D\big(z_0, w\big) - k_D\big(z_0, h(w)\big)\right].$$

Proposition 1.5 (Abate [1], [2]). *Let $h \in \mathrm{Hol}(D,D)$. Then*

(i) $\alpha_h(\tau) > 0$ *for every* $\tau \in \partial D$.
(ii) *If h has no fixed point and $\tau \in \partial D$ is its Wolff point, then*

$$0 < \alpha_h(\tau) \le 1.$$

By the Julia-type Lemma in D (see [1], [2]), and from the uniqueness of the Wolff point, it follows:

Proposition 1.6. *Let $h \in \mathrm{Hol}(D,D)$ have no fixed points. A point $\tau \in \partial D$ is the Wolff point of h if and only if $\alpha_h(\tau) \le 1$ and h has non-tangential limit τ at τ.*

We recall that a continuous curve $\gamma : [0,1) \to D$ (resp. a sequence $\{w_k\} \subset D$) is said to be a *τ-curve* (resp. *τ-sequence*) if $\lim_{t\to 1}\gamma(t) = \tau$ (resp. $\lim_{k\to\infty} w_k = \tau$). The following proposition is a Julia-type Lemma for strongly convex domains. Even if it seems to the author that the Julia's Lemma appears in this form for the first time, the proof is not presented since it is just a re-assembling of the proofs of several versions of Julia-type Lemmas given by Abate (see [1], [2]).

Proposition 1.7. *Let $h \in \mathrm{Hol}(D,D)$, let $\tau \in \partial D$ and let $\gamma(t)$ be a τ-curve. If*

$$\limsup_{t\to 1}\left[k_D\big(z_0,\gamma(t)\big) - k_D\big(z_0, h(\gamma(t))\big)\right] < +\infty, \tag{1.1}$$

then there exists a unique point $\sigma \in \partial D$ such that

$$\lim_{t\to 1} h\big(\gamma(t)\big) = \sigma.$$

Moreover h has non-tangential limit σ at τ.

Of course, in the above statement one can replace the τ-curve $\gamma(t)$ with any τ-sequence.
Proposition 1.7 has the following interpretation. We say that a map $h : D \to D$ has *J-limit* $L \in \overline{D}$ at $\tau \in \partial D$ if $h\big(\gamma(t)\big) \to L$ as $t \to 1$ for any τ-curve γ for which (1.1) holds. Then

Corollary 1.8. *Let $h \in \mathrm{Hol}(D,D)$. Suppose that the boundary dilatation coefficient of h at $\tau \in \partial D$ is finite. Then h has J-limit $\sigma \in \partial D$ at τ.*

Notice that J-limit implies non-tangential limit, since if the boundary dilatation coefficient of h at τ is finite then non-tangential τ-curves have property (1.1) (one can check this directly by means of Lemma 1.1, or by means of one of the versions of Julia-Wolff-Carathéodory Theorem).
With this tools we can state and prove a (weak) version of the Shields Theorem in D:

Theorem 1.9. *Let $f, g \in \mathrm{Hol}(D,D)$. Suppose that f has no fixed points, $x \in \partial D$ is the Wolff point of f and $f \circ g = g \circ f$. Then f and g have non-tangential limit x at x.*

Proof. Let $w_k := f^k(z_0)$. By Theorem 1.2 it follows that $w_k \to x$. By the hypothesis f and g commute, and by Theorem 1.2, we have

$$\lim_{k\to\infty} g(w_k) = \lim_{k\to\infty} f^k\big(g(z_0)\big) = x.$$

Now, thanks to Proposition 1.7 (applied to the x-sequence w_k) we are left to show that

$$\limsup_{k\to\infty} \big[k_D\big(z_0, w_k\big) - k_D\big(z_0, g(w_k)\big)\big] < +\infty.$$

Since holomorphic maps contract the Kobayashi distance, it follows that

$$\begin{aligned} k_D\big(z_0, w_k\big) - k_D\big(z_0, g(w_k)\big) &= k_D\big(z_0, f^k(z_0)\big) - k_D\big(z_0, f^k(g(z_0))\big) \\ &\le k_D\big(f^k(z_0), f^k(g(z_0))\big) \le k_D\big(z_0, g(z_0)\big) < +\infty. \end{aligned}$$

As we claimed. □

2. Admissible tangential limits

In the previous section we showed that two commuting holomorphic self-maps of D (one of which with no fixed points) have a common "boundary fixed point" in the sense of non-tangential limits. A careful reading of the proof of Theorem 1.9 shows that g has finite boundary dilatation coefficient at the Wolff point of f. This will be the key to get complex tangential directions available as admissible limits. Recall that D is a bounded strongly convex C^3 domain and that $z_0 \in D$.

Definition 2.1 (Abate [1], [2])**.** Let $\tau \in \partial D$ and $M > 1$. The K-region $K_{z_0}(\tau, M)$ of center τ, amplitude M and pole z_0 is given by

$$K_{z_0}(\tau, M) := \left\{ z \in D : \lim_{w\to\tau} [k_D(z, w) - k_D(z_0, w)] + k_D(z_0, z) < \log M \right\}.$$

We remark that in general the above limit does not exist if D is not strongly convex. For the properties of K-regions we refer to [1],[2]. As customary we say that a holomorphic self-map h of D has K-limit σ at $\tau \in \partial D$ if $f(w_k) \to \sigma$ as $k \to \infty$ for every τ-sequence $\{w_k\}$ for which there exists $M > 1$ such that $\{w_k\}$ belongs to $K_{z_0}(\tau, M)$.

Before going ahead we need now to recall some facts. A holomorphic map $\varphi : \Delta \to D$ is a *complex geodesic* if

$$\forall \zeta_1, \zeta_2 \in \Delta \quad k_D\big(\varphi(\zeta_1), \varphi(\zeta_2)\big) = \omega\big(\zeta_1, \zeta_2\big).$$

Lempert (see [15],[16]) has shown that the complex geodesics in a strongly convex C^3 domain extend C^1 to $\partial\Delta$ and furthermore that for every $z_0 \in D$ and $\tau \in \partial D$ there exists a unique complex geodesic $\varphi_\tau : \Delta \to D$ such that $\varphi_\tau(0) = z_0$ and $\varphi_\tau(1) = \tau$. Moreover Lempert has constructed a holomorphic map $p_\tau : D \to \varphi_\tau(\Delta)$, such that $p_\tau \circ p_\tau = p_\tau$ and $p_\tau \circ \varphi_\tau = \varphi_\tau$. We set $\tilde{p}_\tau := \varphi_\tau^{-1} \circ p_\tau$ and we call $\tilde{p}_\tau$ the *left inverse* of φ_τ (because of $\tilde{p}_\tau \circ \varphi_\tau = \mathrm{id}_\Delta$).
From now on we shall use the above terminology on complex geodesics.

Remark 2.2. From the very definition of complex geodesic, it follows that $r \mapsto \varphi_\tau(r)$ belongs to any K-region of center τ.

Definition 2.3. Let $\tau \in \partial D$ and let $\gamma : [0, 1) \mapsto D$ be a τ-curve. We set

$$\gamma_\tau(t) := p_\tau\big(\gamma(t)\big),$$

and

$$\tilde{\gamma}_\tau(t) := \tilde{p}_\tau\big(\gamma(t)\big).$$

Now we can state a (maimed) version of the Julia-Wolff-Carathéodory Theorem in strongly convex domains:

Theorem 2.4 (Abate [1], [2])**.** *Let $h \in \mathrm{Hol}(D,D)$ and $\tau \in \partial D$ be such that the boundary dilatation coefficient $\alpha_\tau(h)$ of h at τ is finite. Then there exists a unique $\sigma \in \partial D$ such that h has K-limit σ at τ. Moreover*

(i) $\displaystyle\lim_{r\to 1} \frac{1-\tilde{h}_\sigma(\varphi_\tau(r))}{1-r} = \alpha_\tau(h).$

(ii) $\displaystyle\lim_{r\to 1} \frac{h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))}{\sqrt{1-r}} = 0.$

We can prove Theorem 0.1:

Theorem 2.5. *Let D be a bounded strongly convex C^3 domain. Suppose that $f, g \in \mathrm{Hol}(D,D)$ and $f \circ g = g \circ f$.*

(i) *If* $\mathrm{Fix}(f) \neq \emptyset$ *and* $Fix(g) \neq \emptyset$ *then* $\mathrm{Fix}(f) \bigcap \mathrm{Fix}(g) \neq \emptyset$.

(ii) *If* $\mathrm{Fix}(f) = \emptyset$ *and $x \in \partial D$ is the Wolff point of f, then g has K-limit x at x. In particular, f and g have K-limit x at x.*

Proof. (i)(*sketch, see also* [1] *Prop. 2.5.14*). Since $f \circ g = g \circ f$ then $f(\mathrm{Fix}(g)) \subseteq \mathrm{Fix}(g)$ and (since $\mathrm{Fix}(g)$ is a connected closed submanifold of D) Theorem 1.2 implies that f has a fixed point in $\mathrm{Fix}(g)$.
(ii) Let x be the Wolff point of f. By Theorem 1.9, g has non-tangential limit x at x and the boundary dilatation coefficient of g at x is finite. Then Theorem 2.4 and Proposition 1.6 imply the statement. □

3. A Behan-type Theorem for strongly convex domains

In this section we prove Theorem 0.2. Because of the length of the proof, we first give a sketch and then we examine it in detail.

Sketch of the proof. Let $f, g \in \mathrm{Hol}(D,D)$ be such that f, g have no fixed points and $f \circ g = g \circ f$. Denote by x the Wolff point of f and by y the Wolff point of g. If $x \neq y$ let $\varphi : \Delta \to D$ be the complex geodesic such that $\varphi(-1) = y$ and $\varphi(1) = x$ and let $\tilde{p}$ be the left-inverse of φ. Proposition 1.6 and Theorem 2.4 imply that g has K-limit y at y and Theorem 0.1 implies that g has K-limit x at x (The same holds for f, of course). So g has two boundary fixed points (in the sense of K-limits) with finite boundary dilatation coefficients at each point. Then the key to the whole proof is to show that the product of the boundary dilatation coefficients of g at x and at y is less than or equal to 1. Once we do this it turns out that $p \circ g \circ \varphi$ is a (hyperbolic) automorphism of Δ thanks to the following:

Lemma 3.1 (Behan [5])**.** *Let $\eta \in Hol(\Delta, \Delta)$ be such that $\lim_{r\to 1} \eta(r) = 1$ and $\lim_{r\to -1} \eta(r) = -1$. If $d_\eta(\mu)$ is the boundary dilatation coefficient of η at $\mu \in \partial\Delta$, i.e. $d_\eta(\mu) := \liminf_{\xi\to\mu} \frac{1-|\eta(\xi)|}{1-|\xi|}$, we have*

$$d_\eta(1) \cdot d_\eta(-1) \geq 1. \tag{3.1}$$

Moreover the equality holds in (3.1) *if and only if η is a (hyperbolic) automorphism of Δ.*

Then we shall show that $p \circ g \circ \varphi = g \circ \varphi$, and since the same holds for f and $f \circ g = g \circ f$ then Theorem 0.2 will follow.

The most difficult part of the proof is the estimate of the boundary dilatation coefficient of g at the Wolff point of f. We begin by recalling the following (see [1], [2]):

Definition 3.2. Let $\sigma \in \partial D$. A σ-curve $\gamma : [0,1) \to D$ is *special* if

$$\lim_{t\to 1} k_D(\gamma(t), \gamma_\sigma(t)) = 0,$$

and it is *restricted* if $\gamma_\sigma(t) \to \sigma$ non-tangentially as $t \to 1$.

The relationship among K-regions, special curves and restricted curves is the following (see [2] and [1], Prop. 2.7.11):

Lemma 3.3. *Let $\sigma \in \partial D$ and let $\gamma : [0,1) \to D$ be a σ-curve in D.*

(i) *If $\gamma(t) \in K_{z_0}(\sigma, M)$ eventually for some $M > 1$, then γ is restricted.*
(ii) *If γ is restricted, if*

$$\lim_{t\to 1} \frac{\|\gamma(t) - \gamma_\sigma(t)\|^2}{d(\gamma_\sigma(t), \partial D)} = 0, \tag{3.2}$$

and if there exists an euclidean ball $B \subset D$ tangent to ∂D at σ such that $\gamma(t) \in B$ eventually, then γ is special.

We remind that if $h \in \mathrm{Hol}(D,D)$ and $\sigma \in \partial D$ then $h_\sigma := p_\sigma \circ h$, where p_σ is the holomorphic retraction associated to the complex geodesic φ_σ. With this notation we prove:

Proposition 3.4. *Let $h \in \mathrm{Hol}(D,D)$. Suppose that $\alpha_h(\tau) < \infty$ at $\tau \in \partial D$ and let $\sigma \in \partial D$ be the point defined by Theorem 2.4 such that $K\text{-}\lim_{z\to\tau} h(z) = \sigma$. Then*

$$\lim_{r\to 1} k_D\big(h(\varphi_\tau(r)), h_\sigma(\varphi_\tau(r))\big) = 0. \tag{3.3}$$

Remark 3.5. The curve $r \mapsto h(\varphi_\tau(r))$ is a well-defined σ-curve by Remark 2.2 and by Theorem 2.4. The above Proposition 3.4 asserts that such a curve is *special.* Furthermore, since h maps K-regions with center τ into K-regions with center σ (see [2], Cor.1.8), then by Lemma 3.3(i), it follows that $r \mapsto h_\sigma(\varphi_\tau(r))$ is *non-tangential.*

Proof of Prop. 3.4. Our aim is to apply Lemma 3.3(ii) to the σ-curve $r \mapsto h(\varphi_\tau(r))$. Such a curve is restricted as pointed out in Remark 3.5. Firstly we will show that

$$\lim_{r\to 1} \frac{\|h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))\|^2}{\mathrm{d}(h_\sigma(\varphi(r)), \partial D)} = 0. \tag{3.4}$$

Up to dilatation and traslation we can suppose that $D \subset \mathbb{B}^n$ and $\mathbb{B}^n$ is tangent to ∂D at σ. Moreover since $r \mapsto h_\sigma(\varphi_\tau(r))$ is non-tangential by Remark 3.5, we can change the denominator of (3.4) with $1 - \|h_\sigma(\varphi_\tau(r))\|$. Then we are led to prove that

$$\lim_{r\to 1} \frac{\|h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))\|^2}{1 - \|h_\sigma(\varphi_\tau(r))\|} = 0. \tag{3.5}$$

But now

$$\frac{\|h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))\|^2}{1 - \|h_\sigma(\varphi_\tau(r))\|} = \left[\frac{\|h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))\|}{\sqrt{1-r}}\right]^2 \cdot \frac{1-r}{1 - \|h_\sigma(\varphi_\tau(r))\|}. \tag{3.6}$$

By Theorem 2.4 the first factor on the right-hand side of (3.6) tends to zero as r goes to 1. Then equation (3.5) holds whenever we show that the second factor on the right-hand side of (3.6) is bounded as r goes to 1. Let k_n be the Kobayashi distance on $\mathbb{B}^n$ (ω is the Poincaré distance on Δ). Then, for $r \approx 1$

$$\frac{1}{2} \log \frac{1-r}{1-\|h_\sigma(\varphi_\tau(r))\|} \approx k_n\big(0, h_\sigma(\varphi_\tau(r))\big) - \omega\big(0,r\big).$$

Since $D \subset \mathbb{B}^n$ then $k_D \geq k_n$, hence

$$k_n\big(0, h_\sigma(\varphi_\tau(r))\big) - \omega\big(0,r\big) \leq k_D\big(z_0, h_\sigma(\varphi_\tau(r))\big) - k_D\big(z_0, \varphi_\tau(r)\big) + k_n\big(0, z_0\big).$$

So we have to show that $k_D\big(z_0, h_\sigma(\varphi_\tau(r))\big) - k_D\big(z_0, \varphi_\tau(r)\big) < +\infty$ as $r \to 1$. Keeping in mind that $p_\sigma : D \to D$ is a contraction for k_D and $p_\sigma(z_0) = z_0$, we have

$$\begin{aligned}
\limsup_{r\to 1} \big[k_D\big(z_0, h_\sigma(\varphi_\tau(r))\big) - k_D\big(z_0, \varphi_\tau(r)\big)\big] &\leq \\
\limsup_{r\to 1} \big[k_D\big(z_0, h(\varphi_\tau(r))\big) - k_D\big(z_0, \varphi_\tau(r)\big)\big] &= \\
-\liminf_{r\to 1} \big[k_D\big(z_0, \varphi_\tau(r)\big) - k_D\big(z_0, h(\varphi_\tau(r))\big)\big] &\leq \\
-\liminf_{w\to \tau} \big[k_D\big(z_0, w\big) - k_D\big(z_0, h(w)\big)\big] &= -\frac{1}{2}\log \alpha_h(\tau).
\end{aligned}$$

Now it is left to show that there exists an euclidean ball $B \subset D$ tangent to ∂D at σ and such that $h(\varphi_\tau(r)) \in B$ for r close to 1.
Let $\mathbf{n}_\sigma$ be the outer unit normal vector to ∂D at σ. Let $\pi_\sigma : D \to \sigma + \mathbb{C}\mathbf{n}_\sigma$ be the euclidean projection given by $\pi_\sigma(z) := \sigma + \langle z-\sigma, \mathbf{n}_\sigma\rangle \mathbf{n}_\sigma$. Since $h(\varphi_\tau(r)) \subset K_{z_0}(\sigma, M)$ for some $M > 1$, then the σ-curve $r \mapsto \pi_\sigma\big(h(\varphi_\tau(r))\big)$ is non-tangential (see [3] Lemma 1.3(ii)). It is then easy to see that if it holds:

$$\lim_{r\to 1} \frac{\|h(\varphi_\tau(r)) - \pi_\sigma(h(\varphi_\tau(r)))\|^2}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))), \partial D)} = 0, \tag{3.7}$$

then the curve $r \mapsto h(\varphi_\tau(r))$ is belonging eventually to a ball contained in D and tangent to ∂D at σ. Therefore we will prove equation (3.7).
Notice that if D is balanced then (up to choose $z_0 = 0$) the assertion follows from $p_\sigma = \pi_\sigma$, and if ∂D has constant real sectional curvatures at σ then the "ball-condition" is clearly satisfied.
Firstly we claim that for every $r \in [0, 1)$:

$$\frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)), \partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))), \partial D)} \leq C < +\infty. \tag{3.8}$$

By Lemma 1.1, if $u \in \mathrm{Hol}(D,D)$ then there exists $c_0 > 0$ such that for each $w \in D$

$$-c_0 + \frac{1}{2}\log\frac{\mathrm{d}(u(w), \partial D)}{\mathrm{d}(w, \partial D)} \leq [k_D(z_0, w) - k_D(z_0, u(w))] \leq c_0 + \frac{1}{2}\log\frac{\mathrm{d}(u(w), \partial D)}{\mathrm{d}(w, \partial D)}. \tag{3.9}$$

Since the boundary dilatation coefficient is strictly positive at every boundary point (see Prop. 1.5), then by (3.9), with $u = \pi_\sigma \circ h$ and $w = \varphi_\tau(r)$, we get (for every r)

$$\frac{\mathrm{d}(\varphi_\tau(r), \partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))), \partial D)} \leq C_1 < +\infty. \tag{3.10}$$

On the other hand, since (see [1], [2])

$$\lim_{r\to 1} \big[k_D\big(z_0, \varphi_\tau(r)\big) - k_D\big(z_0, h_\sigma(\varphi_\tau(r))\big)\big] = 1/2 \log \alpha_h(\tau),$$

then (3.9) implies that (for every r):

$$\frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}{\mathrm{d}(\varphi_\tau(r),\partial D)} \le C_2 < +\infty. \tag{3.11}$$

Hence formulae (3.10) and (3.11) imply that

$$\frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)} = \frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}{\mathrm{d}(\varphi_\tau(r),\partial D)} \cdot \frac{\mathrm{d}(\varphi_\tau(r),\partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)} \le C < +\infty$$

for each $r \in [0,1)$, as we claimed.
And now we are able to prove (3.7):

$$\begin{aligned}\frac{\|h(\varphi_\tau(r)) - \pi_\sigma(h(\varphi_\tau(r)))\|}{\sqrt{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)}} &\le \\ &\frac{\|h(\varphi_\tau(r)) - h_\sigma(\varphi_\tau(r))\|}{\sqrt{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}} \cdot \sqrt{\frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)}} + \\ &+ \frac{\|h_\sigma(\varphi_\tau(r)) - \sigma\|}{\sqrt{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}} \cdot \sqrt{\frac{\mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)}{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)}} + \frac{\|\sigma - \pi_\sigma(h(\varphi_\tau(r)))\|}{\sqrt{\mathrm{d}(\pi_\sigma(h(\varphi_\tau(r))),\partial D)}}.\end{aligned}$$

By the estimates (3.8) and (3.4) the first addend goes to zero as $r \to 1$. The second addend tends to zero as r goes to 1 by estimate (3.8) and since $r \mapsto h_\sigma(\varphi_\tau(r))$ is non-tangential, then $\|h_\sigma(\varphi_\tau(r)) - \sigma\| \approx \mathrm{d}(h_\sigma(\varphi_\tau(r)),\partial D)$. Finally the third addend tends to zero as r goes to 1 since $r \mapsto \pi_\sigma(h(\varphi_\tau(r))$ is non-tangential. □

Lemma 3.6 (Estimate at the Wolff point)**.** *Let $f, g \in \mathrm{Hol}(D,D)$ have no fixed points and let $f \circ g = g \circ f$. Let x be the Wolff point of f and let y be the Wolff point of g. Then*

$$\alpha_g(x) \le \frac{1}{\alpha_g(y)}.$$

Proof. Recall that $\frac{1}{2}\log \alpha_g(x) := \liminf_{z\to x}\,[k_D(z_0,z) - k_D(z_0,g(z))]$. By setting $L := \alpha_g(y)$ for clarity, we claim that

$$\liminf_{z\to x}\,\big[k_D\big(z_0,z\big) - k_D\big(z_0,g(z)\big)\big] \le -\frac{1}{2}\log L.$$

By Theorem 1.2, $f^k(w) \to x$ as $k \to \infty$ for all $w \in D$. Since $f \circ g = g \circ f$, we have:

$$\begin{aligned}\liminf_{z\to x}\,\big[k_D\big(z_0,z\big) - k_D\big(z_0,g(z)\big)\big] &\le \\ &\lim_{k\to\infty}\big[k_D\big(z_0,f^k(w)\big) - k_D\big(z_0,g(f^k(w))\big)\big] \le k_D\big(w,g(w)\big),\end{aligned}$$

for all $w \in D$. We evaluate $k_D\big(w,g(w)\big)$ for $w = \varphi_y(r)$ as r tends to 1. By Proposition 1.6 and Proposition 3.4 we have

$$0 \le k_D\big(\varphi_y(r), g(\varphi_y(r))\big) - k_D\big(\varphi_y(r), g_y(\varphi_y(r))\big) \le k_D\big(g(\varphi_y(r)), g_y(\varphi_y(r))\big) \xrightarrow{r\to 1} 0,$$

where the first inequality follows from $p_x \circ \varphi_x = \varphi_x$ and since holomorphic maps are contractions for k_D. Then we are led to evaluate $k_D(\varphi_y(r), g_y(\varphi_y(r)))$ as $r \to 1$. If $\Phi_r(\zeta)$ is a Möebius transformation of Δ which brings r to 0 we get

$$\begin{aligned}k_D(\varphi_y(r), g_y(\varphi_y(r))) &= \omega(r, \tilde g_y(\varphi_y(r))) \\ &= \omega(0, \Phi_r(\tilde g_y(\varphi_y(r)))) = \frac{1}{2}\log\frac{1 + |\Phi_r(\tilde g_y(\varphi_y(r)))|}{1 - |\Phi_r(\tilde g_y(\varphi_y(r)))|}.\end{aligned}$$

Now we have

(3.12)

$$\Phi_r(\tilde{g}_y(\varphi_y(r))) = \frac{r - \tilde{g}_y(\varphi_y(r))}{1 - r\overline{\tilde{g}_y(\varphi_y(r))}} = \left(\frac{1 - \tilde{g}_y(\varphi_y(r))}{1-r} + \frac{r-1}{1-r}\right) \cdot \frac{1-r}{1 - r\overline{\tilde{g}_y(\varphi_y(r))}}.$$

The first factor on the right-hand side of (3.12) tends to $L-1$ by Theorem 2.4. The second factor, again by Theorem 2.4, is

$$\frac{1-r}{1 - r\overline{\tilde{g}_y(\varphi_y(r))}} = \frac{1-r}{1 - \overline{\tilde{g}_y(\varphi_y(r))}} \cdot \frac{1}{r + \frac{1-r}{1-\overline{\tilde{g}_y(\varphi_y(r))}}} \stackrel{r\to 1}{\longrightarrow} \frac{1}{L} \cdot \frac{1}{1+\frac{1}{L}}.$$

This implies that

$$\left|\Phi_r\big(\tilde{g}_y(\varphi_y(r))\big)\right| \to \frac{1-L}{1+L},$$

and $\liminf_{z\to x}\left[k_D(z_0,z) - k_D(z_0,g(z))\right] \le -\frac{1}{2}\log L$. □

And now we can prove our main theorem (Th.0.2):

Theorem 3.7. *Let D be a bounded strongly convex C^3 domain in $\mathbb{C}^n$. Let $f, g \in$ Hol(D,D) have no fixed points and let $f \circ g = g \circ f$. Then either f and g have the same Wolff point or there exists a complex geodesic $\varphi : \Delta \to D$ such that $f(\varphi(\Delta)) = \varphi(\Delta)$, $g(\varphi(\Delta)) = \varphi(\Delta)$ and $f|_{\varphi(\Delta)}$, $g|_{\varphi(\Delta)}$ are commuting (hyperbolic) automorphisms of $\varphi(\Delta)$. In the last case, the complex geodesic φ is the unique (up to parametrizations of Δ) such that $\varphi(-1)$ is the Wolff point of g and $\varphi(1)$ is the Wolff point of f.*

Proof. Suppose f and g have different Wolff points, respectively x and y belonging to ∂D. Let $\varphi : \Delta \to D$ be the unique (up to parametrization of Δ) complex geodesic such that $\varphi(1) = x$ and $\varphi(-1) = y$. Let $\tilde{p}$ be the left inverse of φ and let p be the associated holomorphic retraction. Set $\eta(\zeta) := \tilde{p} \circ g \circ \varphi(\zeta)$ for $\zeta \in \Delta$. Then $\eta : \Delta \mapsto \Delta$ is holomorphic. Moreover $\lim_{r\to 1}\eta(r) = 1$ and $\lim_{r\to -1}\eta(r) = -1$ by Theorem 0.1 and Theorem 2.4. By Theorem 2.4 and Lemma 3.6 it holds

$$\liminf_{\zeta\to 1}\frac{1-|\eta(\zeta)|}{1-|\eta|} \le \liminf_{r\to 1}\frac{1-|\tilde{p}\circ g(\varphi(r))|}{1-r} \le$$
$$\lim_{r\to 1}\frac{|1-\tilde{p}\circ g(\varphi(r))|}{1-r} = \alpha_g(x) \le \frac{1}{\alpha_g(y)}.$$

In the same way

$$\liminf_{\zeta\to -1}\frac{1-|\eta(\zeta)|}{1-|\eta|} \le \alpha_g(y).$$

Hence Lemma 3.1 implies that η -and then $\tilde{p}\circ g\circ\varphi$- is a (hyperbolic) automorphism of Δ. And hence $(p\circ g)|_{\varphi(\Delta)}$ is a (hyperbolic) automorphism of $\varphi(\Delta)$. Now we want to prove that, by setting $g_\varphi := p \circ g$, the equality $g_\varphi(\varphi(\zeta)) = g(\varphi(\zeta))$ holds for any $\zeta \in \Delta$. Let $\tilde{g}_\varphi(z) := \tilde{p}\circ g$. Since we have just proved that $\tilde{g}_\varphi \circ \varphi$ is an automorphism of Δ, then we have, for all $\zeta_1, \zeta_2 \in \Delta$

$$k_D\big(g(\varphi(\zeta_1)), g(\varphi(\zeta_2))\big) \le \omega\big(\zeta_1,\zeta_2\big) = \omega\big(\tilde{g}_\varphi(\varphi(\zeta_1)), \tilde{g}_\varphi(\varphi(\zeta_2))\big)$$
$$= k_D\big(g_\varphi(\varphi(\zeta_1)), g_\varphi(\varphi(\zeta_2))\big) \le k_D\big(g(\varphi(\zeta_1)), g(\varphi(\zeta_2))\big).$$

Therefore $k_D(g(\varphi(\zeta_1)), g(\varphi(\zeta_2))) = \omega(\zeta_1,\zeta_2)$ for all $\zeta_1,\zeta_2 \in \Delta$, and $g\circ\varphi : \Delta \to D$ is a complex geodesic with the properties that $g(\varphi(1)) = \varphi(1)$ and $g(\varphi(-1)) = \varphi(-1)$. By the uniqueness (up to parametrization) of complex geodesics with prescribed boundary data, this means that $g(\varphi(\Delta)) = \varphi(\Delta)$. Moreover, since $p \circ \varphi = \varphi$, we have $p \circ g \circ \varphi \equiv g \circ \varphi$ and then $g|_{\varphi(\Delta)}$ is a (hyperbolic) automorphism of $\varphi(\Delta)$. Since the same holds for f and $f \circ g = g \circ f$ the assertion is proved. □

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