PERTURBATION OF BAUM-BOTT RESIDUES

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In memory of Marco Brunella

ABSTRACT. We prove that Baum-Bott residues vary continuously in an appropriate sense under smooth deformations of holomorphic foliations. This provides an effective way of computing residues.

1. INTRODUCTION

A holomorphic foliation $\mathcal{F}$ on a complex manifold M is known to produce a "holomorphic action", as discovered by P. Baum and R. Bott in [4], on the virtual bundle $TM/\mathcal{F}$. Such a partial holomorphic action provides a holomorphic connection for the bundle $TM/\mathcal{F}$ along $\mathcal{F}$ outside the singularities of $\mathcal{F}$ and thus produces localization of sufficiently high degree classes of $TM/\mathcal{F}$ around the singularities of $\mathcal{F}$. Such localizations give rise to the "Baum-Bott residues" (see [4, Thm. 2], [11, Ch.VI, Thm. 3.7]). When the singularity is isolated the Baum-Bott residue can be expressed in terms of a Grothendieck residue (see [4, (0.6)]). When the singular set is non-isolated in some cases some formulas are available (see [4, Thm. 3] and [5]) but, in general, explicit computation of the residues is rather difficult.

The aim of the present paper is to study the behavior of the Baum-Bott residues under smooth deformations. This provides an effective tool for computing residues explicitly.

More in details, we consider a smooth deformation of a complex manifold. This is essentially a smooth fibration over a smooth manifold, whose fibers are complex manifolds (see Section 2). On each such a fiber we consider a holomorphic foliation which varies smoothly (see Section 3). We prove that the Baum-Bott residues (when taken together suitably) vary continuously under smooth deformations.

We state here a simple consequence of our main Theorem 5.4 for the case of classes of top degree, referring the reader to Section 5 for the general case. Thus, let P be a real manifold, the "parameter space". Let $\widetilde{M} := \{M_t\}_{t\in P}$, be a deformation of complex manifolds of dimension n. Let $\widetilde{\mathcal{F}} := \{\mathcal{F}_t\}$ be a deformation of holomorphic foliations on M_t. Then $\widetilde{\mathcal{F}}$ defines naturally a smooth foliation on $\widetilde{M}$ (see Section 3).

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Suppose the singular set S_{t_0} of $\mathcal{F}_{t_0}$ in M_{t_0} is compact and connected. The analytic set S_{t_0} is contained in a connected component in $\widetilde{M}$ of the singular set of the smooth foliation $\widetilde{\mathcal{F}}$, and we denote by S_t the intersection of such component with M_t. The set S_t is contained in the singular set of $\mathcal{F}_t$ but in general may not be connected. Thus, we let $S_t = \cup S_t^\lambda$ be the connected components decomposition of S_t. Under some assumption on $T\widetilde{M}/\widetilde{\mathcal{F}}$, which is always satisfied for instance if $\widetilde{\mathcal{F}}$ is locally generated by a single vector field, we have:

Theorem 1.1. *Suppose that S_t is compact for all $t \in P$. Let φ be a homogeneous symmetric polynomial of degree n and denote by $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda)$ the Baum-Bott residue of $\mathcal{F}_t$ at S_t^λ. Then*

$$\lim_{t\to t_0} \sum_\lambda \mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda) = \mathrm{BB}_\varphi(\mathcal{F}_{t_0}; S_{t_0}).$$

A general version of the previous theorem is Theorem 5.4, whose proof is contained in Sections 4 and 5. The rough idea of the proof is to construct a special connection on the regular part of the virtual bundle $T\widetilde{M}/\widetilde{\mathcal{F}}$ such that on each fiber M_t it induces the special connection given by the Baum-Bott action and to see the residues as the integral of a smooth form on $\widetilde{M}$ along the fibers.

In Section 6 we give explicit examples of the previous result. In particular, aside from explicit computation, the examples show that if the residues in the same connected component of $\widetilde{M}$ are not taken together, continuity is lost.

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2. Deformation of manifolds

The theory of deformation of complex structures was first systematically developed by K. Kodaira and D. C. Spencer [7], here we recall the basic material relevant for our needs.

Definition 2.1. A *deformation of manifolds* is a triple $(\widetilde{M}, P, \pi)$, where P is a C^∞ manifold of real dimension s, called the *parameter space*, $\widetilde{M}$ is a C^∞ manifold of real dimension $2n+s$, called the *ambient manifold*, and $\pi : \widetilde{M} \to P$ is a surjective C^∞ map such that there exists a covering $\{U_\alpha\}$ (called an *adapted deformation coordinates covering*) of $\widetilde{M}$ with the following properties:

(1) for each α, the open set U_α is diffeomorphic to $D \times V$, where D is an open set of $\mathbb{C}^n$ and V is an open set of $\mathbb{R}^s$, with coordinates $(z_1^\alpha, \dots, z_n^\alpha, t_1^\alpha, \dots, t_s^\alpha)$,
(2) $\pi(U_\alpha)$ is diffeomorphic to V and π is compatible with the projection $D \times V \to V$,
(3) on $U_\alpha \cap U_\beta \neq \emptyset$ we may express as

$$\begin{cases} z_i^\beta = z_i^\beta(z^\alpha, t^\alpha) & i = 1, \dots, n \\ t_j^\beta = t_j^\beta(t^\alpha) & j = 1, \dots, s \end{cases} \tag{2.1}$$

and, for each fixed t^α, the map $z^\alpha \mapsto z^\beta(z^\alpha, t^\alpha)$ is holomorphic.

For $t \in P$ we let $M_t := \pi^{-1}(t)$ be the fiber over t. By definition the fibers M_t, for $t \in P$, are complex manifolds. In particular we can define the sheaf $\widetilde{\mathcal{O}}_{\widetilde{M}}$ of C^∞ functions holomorphic along the fibers on $\widetilde{M}$ so that $f \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U)$ if for all $x \in U$, $f|_{U_t} \in \mathcal{O}_{M_t}(U_t)$, where $t = \pi(x)$, $U_t = U \cap M_t$ and $\mathcal{O}_{M_t}$ denotes the sheaf of holomorphic functions on M_t.

Remark 2.2. Let $U_\alpha \subset \widetilde{M}$ be a coordinate chart of an adapted coordinate covering for $\widetilde{M}$. A function f belongs to $\widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha)$ if and only if $f(z_\alpha, t_\alpha)$ is a C^∞ function such that $f(\cdot, t_\alpha)$ is holomorphic (note that this is well defined by (2.1)).

Definition 2.3. Let E be a C^∞ complex vector bundle of rank r over $\widetilde{M}$. We say that E is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$*-(vector) bundle* if there exists a trivializing atlas $\{U_\alpha\}$ for E, with frames $\{e_1^\alpha, \dots, e_r^\alpha\}$ for $E|_{U_\alpha}$, such that the transition matrices with respect to those frames have entries which are local sections of $\widetilde{\mathcal{O}}_{\widetilde{M}}$. Such frames $\{e_1^\alpha, \dots, e_r^\alpha\}$ are called $\widetilde{\mathcal{O}}_{\widetilde{M}}$*-frames*.

Given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E, we denote by $\widetilde{\mathcal{O}}_{\widetilde{M}}(E)$ the $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module of $\widetilde{\mathcal{O}}_{\widetilde{M}}$ sections of E. Namely, $s \in \widetilde{\mathcal{O}}_{\widetilde{M}}(E)(U)$ is a C^∞ section of E over the open set $U \subset \widetilde{M}$ such that in any $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame $\{e_1^\alpha, \dots, e_r^\alpha\}$ over U_α with $U_\alpha \cap U \neq \emptyset$ the section s is given by

$$s(z^\alpha, t^\alpha) = \sum_{j=1}^r f_j^\alpha(z^\alpha, t^\alpha) e_j^\alpha, \quad f_j^\alpha \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha \cap U).$$

Let $T_{\mathbb{R}}\pi := \operatorname{Ker} \pi_*$. Since the fibers of the fibration $\pi : \widetilde{M} \to P$ are holomorphic, we can define the complex vector bundles

$$T\pi := \bigcup_{x \in \widetilde{M}} T_x M_{\pi(x)}, \quad \overline{T}\pi := \bigcup_{x \in \widetilde{M}} \overline{T}_x M_{\pi(x)}.$$

Local frames for $T\pi$ and $\overline{T}\pi$ in an adapted deformation coordinates covering are given respectively by $\{\frac{\partial}{\partial z_i^\alpha}\}_{i=1,\dots,n}$ and $\{\frac{\partial}{\partial \bar{z}_i^\alpha}\}_{i=1,\dots,n}$ and

$$T_{\mathbb{R}}\pi \otimes \mathbb{C} = T\pi \oplus \overline{T}\pi.$$

Using an adapted deformation coordinates covering, by (2.1), it is easy to see that $T\pi$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-vector bundle over $\widetilde{M}$. Moreover, it has a natural structure of $\widetilde{\mathcal{O}}_{\widetilde{M}}$*-Lie algebra*, namely, using local coordinates, one can easily see that if $v, w \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U)$ then

$$[v, w] \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U).$$

3. Deformation of foliations

Deformations of holomorphic foliations, especially from the viewpoint of moduli spaces, have been studied by a number of authors (*e.g.*, [6], [9], [10]). Here we consider C^∞ families of singular holomorphic foliations.

Let $\mathcal{S}$ be an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module. We say that $\mathcal{S}$ is *coherent* if, for each point $x \in \widetilde{M}$ there exists an open neighborhood $U \subset \widetilde{M}$ of x and two integers $p, q \geq 0$ such that

$$\widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{S}|_U \to 0, \tag{3.1}$$

is an exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$-modules, where φ is a suitable $\widetilde{\mathcal{O}}_{\widetilde{M}}$-morphism.

Definition 3.1. Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds . A *deformation of foliations* on $(\widetilde{M}, P, \pi)$ is a coherent $\widetilde{\mathcal{O}}_{\widetilde{M}}$-submodule $\widetilde{\mathcal{F}}$ of $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ such that $[\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}] \subset \widetilde{\mathcal{F}}$.

Given a deformation of foliations $\widetilde{\mathcal{F}}$ on a deformation of manifolds $(\widetilde{M}, P, \pi)$, we denote by $\mathcal{C}_P^\infty$ the sheaf of germs of complex valued smooth functions on P, and for each $t \in P$, by $\mathcal{I}_t := \{f \in \mathcal{C}_P^\infty : f(t) = 0\}$ the ideal sheaf of smooth functions vanishing at t. The set $\mathcal{R} := \pi^*\mathcal{C}_P^\infty$ is the sheaf of smooth functions on $\widetilde{M}$ that are constant along the fibers, and it is naturally a subsheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}$. Noting that $\mathcal{R}/\pi^*\mathcal{I}_t$ is supported on $M_t = \pi^{-1}(t)$, we define

$$\mathcal{F}_t := \widetilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t.$$

Note that $\widetilde{\mathcal{O}}_{\widetilde{M}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t = \mathcal{O}_{M_t}$, the sheaf of holomorphic functions on M_t. Hence, if $\mathcal{E}$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-module over $\widetilde{M}$, then $\mathcal{E} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$ is an $\mathcal{O}_{M_t}$-module over M_t.

In particular, the sheaf $\mathcal{F}_t$ is an $\mathcal{O}_{M_t}$-module. In adapted deformation coordinates, if $X_1, \ldots, X_r$ are local generators of $\widetilde{\mathcal{F}}$, given by

$$X_j(z^\alpha, t^\alpha) = \sum f_{ij}(z^\alpha, t^\alpha) \frac{\partial}{\partial z_i^\alpha},$$

then $\mathcal{F}_{t_0}$ is locally generated by the $X_j(z^\alpha, t_0^\alpha)$'s. Namely it is generated by the vector fields

$$X_j(z^\alpha, t_0^\alpha) = \sum f_{ij}(z^\alpha, t_0^\alpha) \frac{\partial}{\partial z_i^\alpha} \tag{3.2}$$

obtained by evaluating $f_{ij}(z^\alpha, t^\alpha)$ at $t = t_0$. From this remark, it follows easily:

Lemma 3.2. *For all $t \in P$, the sheaf $\mathcal{F}_t$ defines a holomorphic foliation on M_t.*

The normal sheaf $\mathcal{N}_{\widetilde{\mathcal{F}}}$ of $\widetilde{\mathcal{F}}$ is defined by the following exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}$-modules on $\widetilde{M}$:

$$0 \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}} \longrightarrow 0. \tag{3.3}$$

The *singular set* of $\widetilde{\mathcal{F}}$ is by definition

$$S(\widetilde{\mathcal{F}}) := \{x \in \widetilde{M} : \mathcal{N}_{\widetilde{\mathcal{F}},x} \text{ is not } \mathcal{O}_{\widetilde{M},x} - \text{free}\}.$$

Remark 3.3. As in the case of usual singular holomorphic foliations, even if $\widetilde{\mathcal{F}}$ is locally free, it is possible that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is not locally free. On the other hand, if $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is locally free, so is $\widetilde{\mathcal{F}}$, as $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is locally free.

The *rank* of $\widetilde{\mathcal{F}}$ is defined to be the rank of the locally free part of $\widetilde{\mathcal{F}}$.

Lemma 3.4. *For each point $x \in \widetilde{M}$ there exists an open neighborhood $U \subset \widetilde{M}$ of x and two integers $p, q \geq 0$ such that*

$$\widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}}|_U \to 0, \tag{3.4}$$

is an exact sequence of $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$-modules. Moreover,

$$S(\widetilde{\mathcal{F}})|_U = \{x \in U : \operatorname{rank} \varphi_x \textit{ is not maximal}\}.$$

Proof. Since $\widetilde{\mathcal{F}}$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-coherent and $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-locally free, from (3.3) it follows that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is $\widetilde{\mathcal{O}}_{\widetilde{M}}$-coherent as well, so that (3.4) holds. The final statement follows from (3.4) and standard commutative algebra. □

Lemma 3.5. *For each $t \in P$ such that $M_t \not\subset S(\widetilde{\mathcal{F}})$ the following sequence of $\mathcal{O}_{M_t}$-modules over M_t is exact:*

$$0 \to \widetilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \xrightarrow{\iota} \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to 0. \tag{3.5}$$

Proof. Since taking tensor products is right exact, it suffices to prove that ι is injective.

It is true on the stalk over each $x \in M_t$ such that $x \notin S(\widetilde{\mathcal{F}})$, since $\mathcal{N}_{\mathcal{F},x}$ is $\widetilde{\mathcal{O}}_{\widetilde{M},x}$-free. We note that according to Lemma 3.4, $S(\widetilde{\mathcal{F}})|_{U \cap M_t} = \{x \in U \cap M_t : \operatorname{rank} \varphi_x \text{ is not maximal}\}$. Hence, for t fixed, these equations give rise to an analytic subset $S(\widetilde{\mathcal{F}}) \cap M_t$ of M_t, provided $M_t \not\subset S(\widetilde{\mathcal{F}})$. As a consequence, $S(\widetilde{\mathcal{F}}) \cap M_t$ is thin in M_t. This shows that, since $\widetilde{\mathcal{F}}$ is a subsheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$, ι is also injective on the stalk over $x \in S(\widetilde{\mathcal{F}}) \cap M_t$. □

For each $t \in P$ we have the following exact sequence of $\mathcal{O}_{M_t}$-modules:

$$0 \longrightarrow \mathcal{F}_t \longrightarrow \mathcal{O}_{M_t}(TM_t) \longrightarrow \mathcal{N}_{\mathcal{F}_t} \longrightarrow 0. \tag{3.6}$$

Definition 3.6. Let $t \in P$. If $M_t \subset S(\widetilde{\mathcal{F}})$, we let $S(\mathcal{F}_t) := M_t$. Otherwise we let

$$S(\mathcal{F}_t) := \{x \in M_t : \mathcal{N}_{\mathcal{F}_t,x} \text{ is not } \mathcal{O}_{M_t} - \text{free}\}.$$

Proposition 3.7. *For all $t \in P$ it holds*

$$S(\mathcal{F}_t) = S(\widetilde{\mathcal{F}}) \cap M_t.$$

Proof. If $M_t \subset S(\widetilde{\mathcal{F}})$ there is nothing to prove.

Thus, assume $M_t \not\subset S(\widetilde{\mathcal{F}})$. Since $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t = \mathcal{O}_{M_t}(TM_t)$, comparing (3.5) and (3.6) we see that

$$\mathcal{N}_{\mathcal{F}_t} = \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t, \tag{3.7}$$

from which the statement follows at once. □

4. Relative Bott vanishing for a deformation of foliations

In this section we discuss a Bott type vanishing theorem for deformations of foliations. Thus, we let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. In this section we assume

$$S(\widetilde{\mathcal{F}}) = \emptyset.$$

This means that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ and hence $\widetilde{\mathcal{F}}$ is locally free so that there exists an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-subbundle $\tilde{F}$ of $T\pi$ such that $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}_{\widetilde{M}}(\tilde{F})$.

We refer to [4] for the notion of partial connections (see also [1], [2], [11]). As an example, given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E over $\widetilde{M}$, we can define a "relative $\overline{\partial}$-connection" for E along $\overline{T}\pi$ as follows. We define

$$\overline{\partial}_E : C^{\infty}_{\widetilde{M}}(E) \to C^{\infty}_{\widetilde{M}}(\overline{T}^*\pi \otimes E),$$

imposing that, given an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame $\{\sigma_1^\alpha, \dots, \sigma_r^\alpha\}$, and a C^∞ section of E, $\sigma^\alpha := \sum f_j^\alpha \sigma_j^\alpha$, it holds

$$\overline{\partial}_E(\sigma^\alpha) = \sum_{j=1}^{r} \sum_{i=1}^{n} \frac{\partial f_j^\alpha}{\partial \overline{z}_i^\alpha} d\overline{z}_i^\alpha \otimes \sigma_j^\alpha.$$

Since the transition matrices for E with respect to $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frames contains only entries in $\widetilde{\mathcal{O}}_{\widetilde{M}}$, it is easy to see that such a definition is well posed and it is a partial connection for E along $\overline{T}\pi$.

Definition 4.1. Let E be an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle over $\widetilde{M}$ and let $\mathcal{E}$ be the sheaf of its $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections. A *partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection* for E along $\widetilde{\mathcal{F}}$ is a $\mathbb{C}$-linear map

$$\delta : \mathcal{E} \to \widetilde{\mathcal{F}}^* \otimes \mathcal{E}$$

with the properties that for all $X \in \widetilde{\mathcal{F}}$, $f, g \in \widetilde{\mathcal{O}}_{\widetilde{M}}$ and $\sigma \in \mathcal{E}$

$$\delta_{(fX)}(g\sigma) = f\left(g\delta_X(\sigma) + dg(X)\sigma\right).$$

Moreover, it is said to be *flat* if

$$\delta_X \circ \delta_Y - \delta_Y \circ \delta_X - \delta_{[X,Y]} = 0, \quad \forall X, Y \in \widetilde{\mathcal{F}}.$$

If δ is as above, it induces a (C^∞) partial connection

$$\delta : C^{\infty}_{\widetilde{M}}(E) \to C^{\infty}_{\widetilde{M}}(\tilde{F}^* \otimes E)$$

such that, for $X \in \widetilde{\mathcal{F}}$ and $\sigma \in \mathcal{E}$, we have $\delta_X(\sigma) \in \mathcal{E}$. Thus

$$\delta \oplus \bar{\partial}_E : C^{\infty}_{\widetilde{M}}(E) \to C^{\infty}_{\widetilde{M}}((\tilde{F}^* \oplus \overline{T}^*\pi) \otimes E)$$

is a partial connection. We say that a connection $\nabla : C^{\infty}_{\widetilde{M}}(E) \to C^{\infty}_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E)$ extends $\delta \oplus \bar{\partial}_E$ if $\nabla_X = (\delta \oplus \bar{\partial}_E)_X$ for all sections X of $F \oplus \overline{T}\pi$. Such a connection ∇ always exists (cf. [4]).

We have the following "relative Bott vanishing" theorem for actions of deformations of foliations:

Theorem 4.2. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$ of rank p. Assume that $S(\mathcal{F}) = \emptyset$. Let $\mathcal{E}$ be the sheaf of $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections of an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle E over $\widetilde{M}$. Assume there exists a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection δ for $\mathcal{E}$ along $\widetilde{\mathcal{F}}$. Then, for any connection ∇ for E extending $\delta \oplus \bar{\partial}_E$, denoting by $\iota_t : M_t \hookrightarrow \widetilde{M}$ the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all $t \in P$ and all symmetric homogeneous polynomials φ of degree $d > n - p$.

Proof. Let $\tilde{F}$ be the $\widetilde{\mathcal{O}}_{\widetilde{M}}$-bundle whose associated sheaf of sections is $\widetilde{\mathcal{F}}$. Write

$$T\widetilde{M} \otimes \mathbb{C} = \tilde{F} \oplus F_1 \oplus \overline{T}\pi \oplus \pi^*(TP \otimes \mathbb{C}),$$

where F_1 is any C^∞ complement of $\tilde{F}$ in $T\pi$.

Let K be the curvature of ∇. Let $\{s_1, \ldots, s_p\}$ be a local $\widetilde{\mathcal{O}}_{\widetilde{M}}$-frame for $\tilde{F}$, and $\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\}$ the natural frame for $\overline{T}\pi$ in adapted deformation coordinates. Since $\tilde{F}$ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-subbundle of $T\pi$, we can write $s_j = \sum_{k=1}^n a_k(z,t)\frac{\partial}{\partial z_k}$ for $j = 1, \ldots, p$ and $a_k \in \widetilde{\mathcal{O}}_{\widetilde{M}}$. Hence, $[s_j, \frac{\partial}{\partial \bar{z}_k}] = 0$ for $j = 1, \ldots, p$ and $k = 1, \ldots, n$.

Arguing similarly as in the proof of [4, Prop. 3.27] (see also [2, Thm. 6.1]) since $\widetilde{\mathcal{O}}_{\widetilde{M}}$-sections of E generate as $C^\infty_{\widetilde{M}}$-module the sheaf of C^∞-sections of E, one can see that

$$K(s_j, s_k) = K(s_j, \frac{\partial}{\partial \bar{z}_h}) = K(\frac{\partial}{\partial \bar{z}_h}, \frac{\partial}{\partial \bar{z}_l}) = 0$$

for all $j, k = 1, \ldots, p$ and $h, l = 1, \ldots, n$. In fact, for the second term, given σ an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section of E, we have

$$K(s_j, \frac{\partial}{\partial \bar{z}_h})(\sigma) = \nabla_{s_j}(\nabla_{\frac{\partial}{\partial \bar{z}_h}}\sigma) - \nabla_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j}\sigma) - \nabla_{[s_j, \frac{\partial}{\partial \bar{z}_h}]}\sigma = 0,$$

because $\nabla_{\frac{\partial}{\partial \bar{z}_h}}\sigma = (\overline{\partial}_E)_{\frac{\partial}{\partial \bar{z}_h}}\sigma = 0$ by definition, since σ is an $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section; $\nabla_{s_j}\sigma$ is another $\widetilde{\mathcal{O}}_{\widetilde{M}}$-section of E, hence $\nabla_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j}\sigma) = (\overline{\partial}_E)_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j}\sigma) = 0$; and $[s_j, \frac{\partial}{\partial \bar{z}_h}] = 0$. The first and third terms vanish as δ and $\overline{\partial}_E$ are flat.

As a consequence, the entries of the matrix representing K are 2-forms belonging to the ideal generated by a dual basis of F_1 (which has dimension $n - p$) and by $dt_1, \ldots, dt_s$, where these latter are a basis of $\pi^*(T^*P)$. Therefore, if φ has degree d greater than $n - p$, it follows that

$$\varphi(\nabla) = \sum \omega_j \wedge dt_j,$$

for some $(2d-1)$-forms ω_j, hence, $\iota^*(\varphi(\nabla)) = 0$. $\square$

We recall that if M is a complex manifold and $\mathcal{F}$ is a non-singular holomorphic foliation on M then there exists a natural holomorphic partial connection δ for the normal bundle of the foliation $\mathcal{N}_{\mathcal{F}}$ along $\mathcal{F}$ given by the so called *Baum-Bott action* (see [4], [11]). Such a partial connection is *flat*, in the sense similar to the one in Definition 4.1. It is defined as follows:

$$\delta_X(\sigma) := \rho([X, \tilde{\sigma}]) \tag{4.1}$$

where $\sigma \in \mathcal{N}_{\mathcal{F}}$ is a holomorphic section of the normal bundle to the foliation, $\tilde{\sigma} \in \mathcal{O}_M(TM)$ is a holomorphic section of the tangent bundle to M such that $\rho(\tilde{\sigma}) = \sigma$, where $\rho : \mathcal{O}_M(TM) \to \mathcal{N}_{\mathcal{F}}$ is the natural projection, and $X \in \mathcal{F}$.

We are going to show that a deformation of foliations gives rise to a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$ such that its "restriction" to each fiber M_t is the holomorphic flat partial connection for the normal bundle to $\mathcal{F}_t$ given by the Baum-Bott action:

Proposition 4.3. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. Assume that $S(\widetilde{\mathcal{F}}) = \emptyset$. Then there exists a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection $\tilde{\delta}$ for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$. Moreover, if $\iota_t : M_t \hookrightarrow \widetilde{M}$ is the natural embedding, then $\iota_t^*(\tilde{\delta})$ is the holomorphic flat partial connection for $\mathcal{N}_{\mathcal{F}}$ along $\mathcal{F}_t$ given by the Baum-Bott action.*

Proof. Let $\tilde{\rho} : \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \to \mathcal{N}_{\widetilde{\mathcal{F}}}$ be the natural projection. For $X \in \widetilde{\mathcal{F}}$ and $\sigma \in \mathcal{N}_{\widetilde{\mathcal{F}}}$ we define

$$\tilde{\delta}_X(\sigma) := \tilde{\rho}([X, \tilde{\sigma}]), \tag{4.2}$$

where $\tilde{\sigma} \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$ is such that $\tilde{\rho}(\tilde{\sigma}) = \sigma$. Involutivity of $\widetilde{\mathcal{F}}$ shows that $\tilde{\delta}$ is well-defined and flatness follows from the Jacobi identity, so that $\tilde{\delta}$ is a flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ along $\widetilde{\mathcal{F}}$.

Comparing (4.2) with (4.1), it is easy to see that $\iota_t^*(\tilde{\delta})$ is the flat partial $\mathcal{O}_{M_t}$-connection for $\mathcal{N}_{\mathcal{F}_t}$ along $\mathcal{F}_t$ given by the Baum-Bott action. □

In particular, Theorem 4.2 and Proposition 4.3 imply the following:

Corollary 4.4. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$. Assume that $S(\widetilde{\mathcal{F}}) = \emptyset$. Then there exists a connection ∇ for $\mathcal{N}_{\widetilde{\mathcal{F}}}$ such that, denoting by $\iota_t : M_t \hookrightarrow \widetilde{M}$ the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all $t \in P$ and all symmetric homogeneous polynomials φ of degree $d > n - p$.

5. Residues of Baum-Bott type on deformations of manifolds

In this section we assume $(\widetilde{M}, P, \pi)$ is a deformations of manifolds and $\widetilde{\mathcal{F}}$ is a deformation of foliations on $\widetilde{M}$. We also assume that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ admits a C^∞ *locally free resolution*, namely, there exists an exact sequence of $\mathcal{C}^\infty_{\widetilde{M}}$-modules:

$$0 \to \mathcal{E}_q \to \cdots \to \mathcal{E}_0 \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}} \to 0, \tag{5.1}$$

such that each $\mathcal{E}_j$ is locally $\mathcal{C}^\infty_{\widetilde{M}}$-free.

Remark 5.1. Every coherent $\mathcal{O}_M$-module on a complex manifold M admits a real analytic locally free resolution (see [3]). This fact is used in the original construction of the Baum-Bott residues in [4]. What we need is a relative version of this. In practice, a resolution as above often arises naturally with a given $\widetilde{\mathcal{F}}$. The simplest is the case where $\widetilde{\mathcal{F}}$ is locally $\widetilde{\mathcal{O}}_{\widetilde{M}}$-free; we may let $q = 1$ and $\mathcal{E}_1 = \widetilde{\mathcal{F}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}$, $\mathcal{E}_0 = \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}$. This applies in particular to the case where $\widetilde{\mathcal{F}}$ is generated locally by a single vector field.

Let E_j be the vector bundle over $\widetilde{M}$ whose sheaf of C^∞ sections is $\mathcal{E}_j$. Then $\mathcal{N}_{\widetilde{\mathcal{F}}}$ is a virtual bundle in the K-group $K(\widetilde{M})$ and its total Chern class is defined as

$$c(\mathcal{N}_{\widetilde{\mathcal{F}}}) = \prod_{i=0}^{q} c(E_i)^{(-1)^i}.$$

We briefly sketch here the theory we need for localizing characteristic classes and obtaining the associated residues in the framework of the Chern-Weil theory adapted to the Čech-de Rham cohomology, and refer the reader to [4, Section 4], [8] and [11, Ch.II, 8] for details.

Let $\tilde{U}_1$ be an open neighborhood of $S(\widetilde{\mathcal{F}})$ and let $\tilde{U}_0 := \widetilde{M} \setminus S(\widetilde{\mathcal{F}})$. We denote by $(\nabla_0^\bullet, \nabla_1^\bullet)$ the family of $q+1$ connections compatible with (5.1) and adapted to the covering $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$ of $\widetilde{M}$. Namely, $\nabla_l^\bullet = (\nabla_l^q, \dots, \nabla_l^0)$, $l = 0, 1$ is a family such that ∇_l^j is a connection for $E_j|_{\tilde{U}_l}$, $j = 0, \dots, q$, $l = 0, 1$ and the following diagram is commutative for $i = 1, \dots, q$ and $l = 0, 1$:

$$\begin{array}{ccc} E_i|_{\tilde{U}_l} & \xrightarrow{\nabla_l^i} & C^\infty_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E_i|_{\tilde{U}_l}) \\ \downarrow & & \downarrow \\ E_{i-1}|_{\tilde{U}_l} & \xrightarrow{\nabla_l^{i-1}} & C^\infty_{\widetilde{M}}((T^*\widetilde{M} \otimes \mathbb{C}) \otimes E_{i-1}|_{\tilde{U}_l}). \end{array} \tag{5.2}$$

Moreover, let $N_{\tilde{F}}$ be the vector bundle on $\tilde{U}_0$ whose sheaf of sections is $\mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}}|_{\tilde{U}_0}$. Let ∇ be an extension of the flat partial $\widetilde{\mathcal{O}}_{\widetilde{M}}$-connection $\tilde{\delta}$ for $\mathcal{N}_{\widetilde{\mathcal{F}}}|_{\tilde{U}_0}$ along $\widetilde{\mathcal{F}}$ given by Proposition 4.3. It is then possible to choose $\nabla_0^\bullet$ to be compatible with ∇ (in the sense explained before).

Now, we let φ be a homogeneous symmetric polynomial of degree $d > n-p$. One can define the class $\varphi(\mathcal{N}_{\widetilde{\mathcal{F}}})$ in the Čech-de Rham cohomology $\check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}})$ which is represented by

$$\varphi(\nabla_*^\bullet) := (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)),$$

where, by the compatibility condition, $\varphi(\nabla_0^\bullet) = \varphi(\nabla)$ is a $2d$-form on $\tilde{U}_0$, $\varphi(\nabla_1^\bullet)$ is the $2d$-form on $\tilde{U}_1$ associated to the family $\nabla_1^\bullet$ and $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$ is a $(2d-1)$-form on $\tilde{U}_0 \cap \tilde{U}_1$ such that $d\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet)$. The Čech-de Rham cohomology $\check{\mathrm{H}}^*(\tilde{\mathcal{U}})$ is naturally isomorphic to the de Rham cohomology $H^*_{\mathrm{dR}}(\widetilde{M}, \mathbb{C})$.

If $M_t \not\subset S(\widetilde{\mathcal{F}})$, tensorizing (5.1) with $\mathcal{R}/\pi^*\mathcal{I}_t$ we obtain the following exact sequence of $\mathcal{C}^\infty_{M_t}$-modules (cf. the proof of Lemma 3.5):

$$0 \to \mathcal{E}_q \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \cdots \to \mathcal{E}_0 \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to \mathcal{N}_{\widetilde{\mathcal{F}}} \otimes_{\widetilde{\mathcal{O}}_{\widetilde{M}}} \mathcal{C}^\infty_{\widetilde{M}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \to 0, \tag{5.3}$$

where $\mathcal{E}_j \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$ is the sheaf of C^∞ sections of the restriction of the bundle E_j to M_t. By (3.7), it is then easy to see the following:

Lemma 5.2. *Let $t \in P$ and let $\iota_t : M_t \to \widetilde{M}$ be the natural embedding. If $M_t \not\subset S(\widetilde{\mathcal{F}})$ then $(\iota_t^*(\nabla_0^\bullet), \iota_t^*(\nabla_1^\bullet))$ is a family of connections for the virtual bundle $\mathcal{N}_{\mathcal{F}_t}$ compatible with* (5.3).

By Corollary 4.4 and by the compatibility condition, it follows that for all homogeneous symmetric polynomials φ of degree $d > n-p$, the class $\varphi(\mathcal{N}_{\mathcal{F}_t})$ is represented in the Čech-de Rham cohomology associated to the covering $\tilde{\mathcal{U}} \cap M_t$ of M_t by the cocyle

$$\begin{aligned}\varphi(\iota_t^*\nabla_*^\bullet) &= (\iota_t^*\varphi(\nabla_0^\bullet), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) = (\iota_t^*\varphi(\nabla), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) \\ &= (0, \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)).\end{aligned}$$

Suppose that $M_t \not\subset S(\widetilde{\mathcal{F}})$ and that $S(\mathcal{F}_t)$, which is $S(\widetilde{\mathcal{F}}) \cap M_t$ by Proposition 3.7, is compact. Since $\tilde{U}_0 \cap M_t = M_t \setminus S(\mathcal{F}_t)$, the above cocycle $\varphi(\iota_t^*\nabla_*^\bullet)$ defines a localization of $\varphi(\mathcal{N}_{\mathcal{F}_t})$, call it $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$, in the relative Čech-de Rham cohomology $\check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t))$. The *Baum-Bott residue* is the image of $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$ by the Alexander homomorphism

$$A : \check{\mathrm{H}}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t)) \to H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^*.$$

If $S(\mathcal{F}_t)$ is made of k connected components and $\tilde{U}_1$ is small enough, then $H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^*$ is a direct sum of k addends, and we can consider the Baum-Bott residue at each connected component of $S(\mathcal{F}_t)$. Note that if $\tilde{U}_1 \cap M_t$ is a regular neighborhood of $S(\mathcal{F}_t)$, we have $H^{2n-2d}_{\mathrm{dR}}(\tilde{U}_1 \cap M_t)^* \xrightarrow{\sim} H_{2n-2d}(S(\mathcal{F}_t), \mathbb{C})$ and the above Alexander homomorphism is an isomorphism. Thus in this case the above residue, as well as the ones corresponding to the connected components of $S(\mathcal{F}_t)$, does not depend on $\tilde{U}_1$.

Now, let $S'(\widetilde{\mathcal{F}}) \subseteq S(\widetilde{\mathcal{F}})$ be a connected component. We assume that

$$S_t := S'(\widetilde{\mathcal{F}}) \cap M_t \text{ is compact} \quad \forall t \in P.$$

Note that S_t may not be connected. Let $\tilde{U}_1'$ be a neighborhood of $S'(\widetilde{\mathcal{F}})$, small enough so that it does not intersect with any other components of $S(\widetilde{\mathcal{F}})$, and $\tilde{R}$ a real manifold of dimension $2n+s$ with boundary in $\tilde{U}_1'$ such that $S'(\widetilde{\mathcal{F}})$ is contained in the interior of $\tilde{R}$ and that $\partial\tilde{R}$ is transverse to M_t for all $t \in P$. Moreover, we can take $\tilde{R}$ in such a way that $R_t := \tilde{R} \cap M_t$ is compact for all $t \in P$.

We let $U_t := \tilde{U}_1' \cap M_t$. By the previous construction, we can express the *Baum-Bott residue* $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) \in H^{2n-2d}_{\mathrm{dR}}(U_t)^*$ as follows:

$$\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) : H^{2n-2d}_{\mathrm{dR}}(U_t) \ni [\tau] \mapsto \int_{R_t} \iota_t^*\varphi(\nabla_1^\bullet) \wedge \tau - \int_{\partial R_t} \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet) \wedge \tau. \tag{5.4}$$

Remark 5.3. 1. If $d = n$, the Baum-Bott residue is a complex number given by

$$\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) = \int_{R_t} \iota_t^* \varphi(\nabla_1^\bullet) - \int_{\partial R_t} \iota_t^* \varphi(\nabla_0^\bullet, \nabla_1^\bullet).$$

2. As mentioned above, if U_t is a regular neighborhood of S_t, $H^{2n-2d}_{\mathrm{dR}}(U_t)^* \xrightarrow{\sim} H_{2n-2d}(S_t, \mathbb{C})$ and one can remove the dependence on $\tilde{U}_1'$ or $\tilde{R}$ in this construction.

Now we are in good shape to prove our main result:

Theorem 5.4. *Let $(\widetilde{M}, P, \pi)$ be a deformation of manifolds and $\widetilde{\mathcal{F}}$ a deformation of foliations on $\widetilde{M}$ of rank p. Suppose that $\mathcal{N}_{\widetilde{\mathcal{F}}}$ admits a C^∞ locally free resolution. Let $S'(\widetilde{\mathcal{F}}) \subseteq S(\widetilde{\mathcal{F}})$ be a connected component of the singular set of $\widetilde{\mathcal{F}}$ and let $S_t := S'(\widetilde{\mathcal{F}}) \cap M_t$. Assume that for all $t \in P$ the set S_t is compact and $S_t \neq M_t$. Let φ be a homogeneous symmetric polynomial of degree $d > n - p$. Under these assumptions, the Baum-Bott residue $\mathrm{BB}_\varphi(\mathcal{F}_t; S_t)$ is continuous in $t \in P$. Namely, for any C^∞ $(2n-2d)$-form $\tilde{\tau}$ on $\widetilde{M}$ such that $\iota_t^*(\tilde{\tau})$ is closed for all $t \in P$,*

$$\lim_{t \to t_0} \mathrm{BB}_\varphi(\mathcal{F}_t; S_t)\left(\iota_t^*(\tilde{\tau})\right) = \mathrm{BB}_\varphi(\mathcal{F}_{t_0}; S_{t_0})\left(\iota_{t_0}^*(\tilde{\tau})\right).$$

Proof. From the previous construction and (5.4) it follows that the Baum-Bott residues on M_t are expressed by means of smooth forms on $\widetilde{M}$. Hence, they vary continuously. □

Note that, if S_t is not connected and $S_t = \cup_\lambda S_t^\lambda$ is its connected components decomposition, then

$$\mathrm{BB}_\varphi(\mathcal{F}_t; S_t) = \sum_\lambda \mathrm{BB}_\varphi(\mathcal{F}_t; S_t^\lambda).$$

6. Examples

Let $\mathbb{P}^3$ denote the three dimensional complex projective space with homogeneous coordinates $[x_1 : x_2 : x_3 : x_4]$.

Example 6.1. On $\mathbb{P}^3$ we consider the vector field which is defined in the affine chart $x_4 \neq 0$ with coordinates $x = x_1/x_4, y = x_2/x_4, z = x_3/x_4$ by

$$X(x, y, z) := x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

The singularities are the line L given by $x_1 = x_2 = 0$ and the point at infinity given by $Q := [1 : 1 : 1 : 0]$ (see the next expression (6.2)).

The vector field X generates a one-dimensional foliation $\mathcal{F}$ given by $X : \mathbb{P}^3 \times \mathbb{C} \to T\mathbb{P}^3$ on $\mathbb{P}^3$. By the Baum-Bott theorem, we can localize $\varphi(T\mathbb{P}^3/\mathcal{F})$ for homogeneous symmetric polynomials φ of degree 3. Such polynomials are essentially given by c_1^3, c_1c_2 and c_3. Moreover, since $\mathcal{F}$ is trivial, we see that $\varphi(T\mathbb{P}^3/\mathcal{F}) = \varphi(T\mathbb{P}^3)$. Let $\mathcal{O}(1)$ be the hyperplane bundle on $\mathbb{P}^3$ and let $\xi := c_1(\mathcal{O}(1)) \in H^2_{\mathrm{dR}}(\mathbb{P}^3)$. From the Euler exact sequence, it follows that $c(T\mathbb{P}^3) = (1+\xi)^4$, from which

$$\int_{\mathbb{P}^3} c_1^3(T\mathbb{P}^3) = 64, \quad \int_{\mathbb{P}^3} c_1c_2(T\mathbb{P}^3) = 24, \quad \int_{\mathbb{P}^3} c_3(T\mathbb{P}^3) = 4. \tag{6.1}$$

Changing coordinates, in the affine chart $x_3 \neq 0$ with coordinates $\tilde{x} = x_1/x_3, \tilde{y} = x_2/x_3, \tilde{z} = x_4/x_3$ the vector field X has the expression:

$$X(\tilde{x},\tilde{y},\tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y})\frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2)\frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z}\frac{\partial}{\partial \tilde{z}}. \tag{6.2}$$

From this it follows that the first jet of X at Q is given by the non-degenerate matrix

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence since Q is a non-degenerate isolated singularity for X it follows (see, *e.g.* [4, (0.7)] or [11])

$$\mathrm{BB}_\varphi(X;Q) = \frac{\varphi(A)}{\det A}, \tag{6.3}$$

that is

$$\mathrm{BB}_{c_1^3}(X;Q) = 27 \quad \mathrm{BB}_{c_1c_2}(X;Q) = 9 \quad \mathrm{BB}_{c_3}(X;Q) = 1. \tag{6.4}$$

By the Baum-Bott theorem,

$$\int_{\mathbb{P}^3} \varphi(T\mathbb{P}^3) = \mathrm{BB}_\varphi(X;Q) + \mathrm{BB}_\varphi(X;L).$$

From this and by (6.1) and (6.4) we obtain

$$\mathrm{BB}_{c_1^3}(X;L) = 37 \quad \mathrm{BB}_{c_1c_2}(X;L) = 15 \quad \mathrm{BB}_{c_3}(X;L) = 3. \tag{6.5}$$

However, it sometimes happens that we need to compute such residues only from the local data near the singularity, without using the Baum-Bott theorem, and it is usually very complicated to do so particularly if the singular set is non-isolated.

We present now a deformation procedure which allows to compute the previous residues and explain in practice how our Theorem 1.1 works.

Let $\widetilde{M} := \mathbb{P}^3 \times (-1,1)$ and let $\widetilde{\mathcal{F}}$ be the deformation of foliations defined by the vector fields X_t, $t \in (-1,1)$, which on the chart $x_4 \neq 0$ are defined as

$$X_t(x,y,z) = (x + tz)\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

On the chart $x_3 \neq 0$ the vector field X_t is given by

$$X_t(\tilde{x},\tilde{y},\tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y} + t)\frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2)\frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z}\frac{\partial}{\partial \tilde{z}}.$$

The singularities of X_t for $t \neq 0$ are given by $O := [0:0:0:1]$ and $P_i(t) := [u_{t,i}^2 : u_{t,i} : 1 : 0]$ for $i = 1, 2, 3$, where the $u_{t,i}$'s are the three roots of the equation $\lambda^3 - \lambda^2 - t = 0$.

At the point O the first jet of X_t, $t \neq 0$, is non-degenerate and it is given by the matrix

$$\begin{pmatrix} 1 & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

From this and (6.3),

$$\mathrm{BB}_{c_1^3}(X_t; O) = \frac{1}{t} \quad \mathrm{BB}_{c_1c_2}(X_t; O) = 0 \quad \mathrm{BB}_{c_3}(X_t; O) = 1. \tag{6.6}$$

Remark 6.2. It is interesting to note that $\lim_{t\to 0} \mathrm{BB}_{c_1^3}(X_t; O) = \infty$, namely the residue by itself may not be continuous. Only the sum of the residues for all the singularities belonging to one connected component in the ambient space $\widetilde{M}$ is guaranteed to be continuous.

At the point $P_i(t)$ the vector field X_t has first jet given by the matrix

$$B(t,i) := \begin{pmatrix} 1 - u_{t,i} & -u_{t,i}^2 & 0 \\ 1 & -2u_{t,i} & 0 \\ 0 & 0 & -u_{t,i} \end{pmatrix},$$

with determinant $\det B(t,i) = u_{t,i}^2(2 - 3u_{t,i})$. Thus, for $t \to 0$, $t \neq 0$ the points $P_i(t)$ are isolated non-degenerate singularities for X_t and one can use (6.3) to compute the residues:

$$\mathrm{BB}_{c_1^3}(X_t; P_i(t)) = \frac{(4u_{t,i} - 1)^3}{u_{t,i}^2(3u_{t,i} - 2)} \quad \mathrm{BB}_{c_1c_2}(X_t; P_i(t)) = \frac{3(2u_{t,i} - 1)(4u_{t,i} - 1)}{u_{t,i}(3u_{t,i} - 2)}$$
$$\mathrm{BB}_{c_3}(X_t; P_i(t)) = 1.$$

Now, as $t \to 0$, two of the roots of of the equation $\lambda^3 - \lambda^2 - t = 0$ tend to 0 and one tends to 1. We assume that $u_{t,1}, u_{t,2} \to 0$ and $u_{t,3} \to 1$. Hence, if $S'(\widetilde{\mathcal{F}})$ is the connected component which contains the line L in the manifold deformation $M \times (-1,1)$, the intersection of $S'(\widetilde{\mathcal{F}})$ with $M \times \{t\}$ is given by the points $O, P_1(t), P_2(t)$. While, the connected component in $M \times (-1,1)$ which contains Q contains all the points $P_3(t)$.

A direct computation, taking into account that $u_{t,1} + u_{t,2} + u_{t,3} = 1$, $u_{t,1}u_{t,2} + u_{t,1}u_{t,3} + u_{t,2}u_{t,3} = 0$ and $u_{t,1}u_{t,2}u_{t,3} = t$, shows that

$$\mathrm{BB}_\varphi(X_t; P_1(t)) + \mathrm{BB}_\varphi(X_t; P_2(t)) = \begin{cases} 37 - \frac{1}{t}, & \varphi = c_1^3 \\ 15, & \varphi = c_1c_2 \\ 2, & \varphi = c_3. \end{cases} \tag{6.7}$$

By Theorem 1.1, we have

$$\mathrm{BB}_\varphi(X; L) = \lim_{t\to 0}[\mathrm{BB}_\varphi(X_t; O) + \mathrm{BB}_\varphi(X_t; P_1(t)) + \mathrm{BB}_\varphi(X_t; P_2(t))]$$

and we recover (6.5) from (6.6) and (6.7).

We note that the residues at $P_3(t)$ remain the same for $\varphi = c_1^3, c_1c_2, c_3$:

$$\mathrm{BB}_\varphi(X_t; P_3(t)) = \mathrm{BB}_\varphi(X; Q).$$

We may also apply our method to the following example in [5], where the residues are computed by a rather involved way. We thank D. Lehmann for drawing our attention to this.

Example 6.3. Again on $\mathbb{P}^3$ we consider the vector field

$$X(x, y, z) := z\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The singularities are the line L given by $x_2 = x_3 = 0$ and the point $Q := [0 : 1 : 0 : 0]$. The residues at Q are the same as (6.4). To compute the residues at L, we consider the deformation

$$X_t(x, y, z) = z\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + tx\frac{\partial}{\partial z}.$$

On the chart $x_1 \neq 0$ with coordinates $x' = x_2/x_1, y' = x_3/x_1, z' = x_4/x_1$ the vector field X_t is given by

$$X_t(x', y', z') = x'(1 - y')\frac{\partial}{\partial x'} + (t - y'^2)\frac{\partial}{\partial y'} - y'z'\frac{\partial}{\partial z'}.$$

Also on the chart $x_2 \neq 0$ with coordinates $x'' = x_1/x_2, y'' = x_3/x_2, z'' = x_4/x_2$ it is given by

$$X_t(x'', y'', z'') = (y'' - x'')\frac{\partial}{\partial x''} + (tx'' - y'')\frac{\partial}{\partial y''} - z''\frac{\partial}{\partial z''}.$$

The singularities of X_t for $t \neq 0$ are the four points given by $O := [0 : 0 : 0 : 1]$, Q and $P_i(t) := [1 : 0 : u_{t,i} : 0]$ for $i = 1, 2$, where the $u_{t,i}$'s are the roots of the equation $\lambda^2 - t = 0$.

At the point O the first jet of X_t, $t \neq 0$, is non-degenerate and it is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ t & 0 & 0 \end{pmatrix}.$$

From this and (6.3),

$$\mathrm{BB}_{c_1^3}(X_t; O) = -\frac{1}{t} \quad \mathrm{BB}_{c_1c_2}(X_t; O) = 1 \quad \mathrm{BB}_{c_3}(X_t; O) = 1. \tag{6.8}$$

At the point $P_i(t)$ the vector field X_t has first jet given by the matrix

$$B(t, i) := \begin{pmatrix} 1 - u_{t,i} & 0 & 0 \\ 0 & -2u_{t,i} & 0 \\ 0 & 0 & -u_{t,i} \end{pmatrix}.$$

Thus, for $t \to 0$, $t \neq 0$ one can use (6.3) to compute the residues:

$$\mathrm{BB}_{c_1^3}(X_t; P_i(t)) = \frac{(1 - 4u_{t,i})^3}{2t(1 - u_{t,i})} \quad \mathrm{BB}_{c_1c_2}(X_t; P_i(t)) = \frac{(1 - 4u_{t,i})(5t - 3u_{t,i})}{2t(1 - u_{t,i})}$$

$$\mathrm{BB}_{c_3}(X_t; P_i(t)) = 1.$$

If $S'(\widetilde{\mathcal{F}})$ is the connected component which contains the line L in the manifold deformation $M \times (-1,1)$, the intersection of $S'(\widetilde{\mathcal{F}})$ with $M \times \{t\}$ is given by the points $O, P_1(t), P_2(t)$. While, the connected component in $M \times (-1,1)$ which contains Q equals $Q \times (-1,1)$.

A direct computation, taking into account that $u_{t,1} + u_{t,2} = 0$ and $u_{t,1}u_{t,2} = -t$, shows that

$$\mathrm{BB}_\varphi(X_t; P_1(t)) + \mathrm{BB}_\varphi(X_t; P_2(t)) = \begin{cases} \frac{-64t^2+36t+1}{t(1-t)}, & \varphi = c_1^3 \\ \frac{2(7-10t)}{1-t}, & \varphi = c_1c_2 \\ 2, & \varphi = c_3. \end{cases} \tag{6.9}$$

By Theorem 1.1, we have

$$\mathrm{BB}_\varphi(X; L) = \lim_{t\to 0}[\mathrm{BB}_\varphi(X_t; O) + \mathrm{BB}_\varphi(X_t; P_1(t)) + \mathrm{BB}_\varphi(X_t; P_2(t))]$$

and using (6.8) and (6.9) we see that we have the same values as (6.5) for the residues at L.

The residues at Q are given

$$\mathrm{BB}_{c_1^3}(X_t; Q) = \frac{27}{1-t} \quad \mathrm{BB}_{c_1c_2}(X_t; Q) = \frac{3(3-t)}{1-t} \quad \mathrm{BB}_{c_3}(X_t; Q) = 1.$$

Note that they depend on t and as the limits as $t \to 0$, we have the same values as (6.4).

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