PARABOLIC ATTITUDE

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Abstract. Being parabolic in complex dynamics is not a state of fact, but it is more an attitude. In these notes we explain the philosophy under this assertion.

1. Introduction

The word "dynamics" is one of the most used in mathematics. Here we use it in the sense of *local discrete holomorphic dynamics*, namely, the study of iterates of a germ of a holomorphic map in $\mathbb{C}^n$, $n \geq 1$ near a fixed point. Aside from its own interest, the study of such dynamics is useful to understand global dynamics of foliations or vector fields (considering the germ as the holonomy on a compact leaf). Although the germ might be non-invertible, here we will concentrate only on holomorphic diffeomorphisms.

Let F denote such a germ of holomorphic diffeomorphism in a neighborhood of the origin 0 in $\mathbb{C}^n$. As expected, the dynamical behavior of the sequence of iterates $\{F^{\circ q}\}_{q\in\mathbb{N}}$ of F in a neighborhood of 0 is described at the first order by the dynamics of its differential dF_0. In fact, depending on the eigenvalues $\lambda_1, \ldots, \lambda_n$ of dF_0, in some cases both dynamics are the same.

The so-called "hyperbolic case" is the generic case, that is, when none of the eigenvalues is of modulus 1. In this case the map is topologically conjugated to its differential (by the Hartman-Grobman theorem [26], [21]) and the dynamics is then completely clear. In case the eigenvalues have either all modulus strictly smaller than one or all strictly greater than one, then the origin is an attracting or respectively repelling fixed point for an open neighborhood of 0. Also, by the stable/unstable manifold theorem, there exists a holomorphic (germ of) manifold invariant under F and tangent to the sum of the eigenspaces of those λ_j's such that $|\lambda_j| < 1$ (*resp.* $|\lambda_j| > 1$) which is attracted to (*resp.* repelled from) 0. However, already in case when all eigenvalues have modulus different from 1, holomorphic linearization is not always possible due to the presence of resonances among the eigenvalues (see, for instance, [7, Chapter IV]).

The "non-hyperbolic case" is the most interesting from a dynamical point of view. In dimension one, $F(z) = \lambda z + \ldots$ and $|\lambda| = 1$, the dynamics depends on the arithmetic properties of λ. Namely, if λ is a root of 1 (the so called "parabolic case") then either F is linearizable (which is the case if and only if $F^{\circ m} = \mathsf{id}$ for some $m \in \mathbb{N}$) or there exist certain F-invariant sets, called "petals", which form a pointed neighborhood of 0 and which are alternatively attracting and repelling (and permuted each other by the multiplicity of λ as a root of unity). On such petals the map F (or F^{-1}) is conjugated to an Abel translation of the type $z \mapsto z+1$ via a change of coordinates which is nowadays known as "Fatou coordinates". This is the content of the famous Leau-Fatou flower theorem, which we will recall in detail in Section 2. In such a parabolic (non linearizable) case, the topological classification is rather simple (see [14] and [27]), in fact, the map is topologically equivalent to $z \mapsto \lambda z(1+z^{mk})$, where $\lambda^m = 1$. While, from the formal point of view, the map F is conjugated to $z \mapsto \lambda z + z^{mk+1} + az^{2mk+1}$ for some "index" $a \in \mathbb{C}$ which can be computed as a residue around the origin. The holomorphic classification is however much more complicated and it is due to Voronin [40] and Écalle [17], [18]. The very rough idea for germs tangent to identity is to consider the changes of Fatou coordinates on the intersection between an attracting petal and the subsequent repelling petal. This provides twice the multiplicity of F of certain holomorphic functions which are known as "sectorial invariants". These invariants, together with the multiplicity and the index, are the searched for complete system of holomorphic invariants.

In case λ has modulo one but it is not a root of unity, the map is called "elliptic". In such a case the germ is always formally linearizable, but, as strange as it might be, it is holomorphically linearizable if and only if topologically linearizable (and this last condition is related to boundedness of the orbits in a neighborhood of 0). Writing $\lambda = e^{2\pi i\theta}$, first Siegel and later Bruno and Yoccoz [41] showed that holomorphic linearization depends on the arithmetic properties of $\theta \in \mathbb{R}$. In particular they showed that for almost every $\theta \in \mathbb{R}$ the germ is holomorphically linearizable. Later Yoccoz proved that the arithmetic condition for which every map starting with $e^{2\pi i\theta}z + \ldots$ is linearizable can be characterized exactly (in the sequel we will refer to such a condition as the "Bruno condition", but we are not going to write it here explicitly). In particular, the quadratic polynomial $e^{2\pi i\theta}z + z^2$ is holomorphically linearizable if and only if θ satisfies the Bruno condition. Non-linearizable elliptic germs present very interesting dynamics that we are not going to describe in details here, leaving the reader to check the survey papers [2], [8], [9].

In higher dimension, the situation is much more complicated. To be precise, only the definition of "hyperbolic germs" makes really sense. There is not such a clear distinction between parabolic or elliptic germs. And, as we will try to make clear in these notes, this is not just a matter of definition, but it is really a matter of dynamics that, in higher dimension, can mix different types of behaviors without privileging none. We will concentrate on the parabolic behavior. And we will see how, even germs which one would call "elliptic" can have a parabolic attitude.

The first instance of parabolic behavior in higher dimension is clearly a map tangent to the identity. This is the prototype of parabolic dynamics. It has been proved by Écalle [18] and Hakim [25] that generically there exist "petals", also called "parabolic curves", namely, one-dimensional F-invariant analytic discs having the origin in their boundary and on which the dynamics is of "parabolic type", namely, the restriction of the map is a Abel type translation. Later, Abate [1] (see also [3]) proved that such petals always exist in dimension two. Hakim also gave conditions for which the petals are "fat" in the sense that there exist basins of attraction modeled on such parabolic curves. We will describe such results in details in Section 3.

Other examples of parabolic behaviors are when one eigenvalue is 1. However, in such a case it is not always clear that some "parabolic attitude" exists, depending on the other eigenvalues and some invariants. Hakim [24] (based on the previous work by Fatou [19] and Ueda [38], [39] in $\mathbb{C}^2$) studied the *semi-attractive* case, with one eigenvalue equal to 1 and the rest of eigenvalues having modulus less than 1. She proved that either there exists a curve of fixed points or there exist attracting open petals, modeled on parabolic curves. Such a result has been later generalized by Rivi [32] and Rong [36].

The case when one eigenvalue is 1 and the other has modulus equal to one, but not a root of unity, has been studied in [10]—the so-called "quasi-parabolic" case—and it has been proved that, under a certain generic hypothesis called "dynamical separation", there exist petals tangent to the eigenspace of 1, so that, in such a case, there is a parabolic attitude. Such a result has been generalized to higher dimension by Rong [33], [34], [35]. We will describe more in details such results in Section 4.

However, as recently proved in [12], parabolic behavior can appear, maybe unexpected, also in those situations when no eigenvalue is a root of unity. Indeed, the new phenomenon, which generates "parabolic attitude" discovered in [12] can be roughly summarized as follows. Assume for simplicity that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the differential dF_0 have a unique one-dimensional resonance of type $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n} = 1$. Let $\hat{F}$ be the formal Poincaré-Dulac normal form of F. For the moment, assume that $\hat{F}$ and the formal conjugation are converging. Then $\hat{F}$ has an invariant one-codimensional foliation given by $\mathcal{F} := \{z_1^{\alpha_1} \cdots z_n^{\alpha_n} = \text{const}\}$ (and so does F). Considering the map $\varphi : \mathbb{C}^n \to \mathbb{C}$ given by $\varphi(z_1, \ldots, z_n) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, it follows that $\varphi \circ \hat{F}(z) = f(\varphi(z))$, where f is a germ in $\mathbb{C}$ tangent to the identity. In other words, $\hat{F}$ acts as a germ tangent to the identity on the space of leaves of the foliation $\mathcal{F}$. The idea is then that the parabolic dynamics (petals) on such a space can be pulled back to $\mathbb{C}^n$ and creates invariant sets that, under some suitable conditions, are basins of attraction for $\hat{F}$ (and thus for F). In fact, the similar argument is correct also when the conjugation of F to its formal Poincaré-Dulac normal form is not converging, although much more complicated. We will describe more in details that in Section 5, where we also provide explicit examples.

A final warning about the paper. These notes are by no means intended to be a survey paper on parabolic dynamics in higher dimensions, so that we are not going to cite all the results proved in this direction so far: for this, we refer the readers to the papers [8] and, mainly, [2]. These notes are intended to be a hint of what the word "parabolic" should mean in higher dimensional local holomorphic dynamics.

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2. Parabolic dynamics in one variable

Definition 2.1. Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. We say that $v \in \partial\mathbb{D}$ is an *attracting direction* if $\frac{A}{|A|}v^k = -1$.

We say that $v \in \partial\mathbb{D}$ is a *repelling direction* if $\frac{A}{|A|}v^k = 1$.

Clearly there exist exactly k attracting and k repelling directions.

Remark 2.2. The attracting directions of f are the repelling directions of f^{-1} and conversely the repelling directions of f are the attracting directions of f^{-1}.

Definition 2.3. Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. An *attracting petal* centered at an attracting direction v is a simply connected open set P_v such that

(1) $O \in \partial P_v$,
(2) $f(P_v) \subseteq P_v$,
(3) $\lim_{n\to\infty} f^{\circ n}(z) = O$ and $\lim_{n\to\infty} \frac{f^{\circ n}(z)}{|f^{\circ n}(z)|} = v$ for all $z \in P_v$.

A *repelling petal* centered at a repelling direction v is an attracting petal for f^{-1} centered at the attracting direction v (for f^{-1}).

As a matter of notation, let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $a_{k+1} \neq 0$. We write $v_1^+, \ldots, v_k^+$ for the attracting directions of f and $v_1^-, \ldots, v_k^-$ for the repelling directions of f, ordered so that starting from 1 and moving counterclockwise on $\partial\mathbb{D}$ the first point we meet is v_1^+, then v_1^-, then v_2^+ and so on.

Here we state the following version of the Leau-Fatou flower theorem:

Theorem 2.4 (Leau-Fatou)**.** *Let $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ be a holomorphic germ in $\mathbb{C}$ at 0 with $a_{k+1} \neq 0$. Let $\{v_1^+, \ldots, v_k^+, v_1^-, \ldots, v_k^-\}$ be the ordered attracting and repelling directions of f. Then*

(1) *For any $v_j^\pm$ there exists an attracting/repelling petal $P_{v_j^\pm}$ centered at $v_j^\pm$.*
(2) *$P_{v_j^+} \cap P_{v_l^+} = \emptyset$ and $P_{v_j^-} \cap P_{v_l^-} = \emptyset$ for $j \neq l$.*
(3) *For any attracting petal $P_{v_j^+}$ the function $f|_{P_{v_j^+}}$ is holomorphically conjugated to $\zeta \mapsto \zeta + 1$ defined on $\{\zeta \in \mathbb{C} : \mathsf{Re}\,\zeta > C\}$ for some $C > 0$.*
(4) *$|f^{\circ m}(z)|^k \sim \frac{1}{m}$ for all $z \in P_{v_j^+}$, $j = 1, \ldots, k$.*

Proof. The proof can be found, e.g. in [16] or [28]. We only mention here how (4) is obtained. Up to a dilation one can assume $a_{k+1} = -1/k$ and $v_1^+ = 1$. By the Leau-Fatou construction, if $C > 0$ is sufficiently large, then setting $H := \{w \in \mathbb{C} : \mathsf{Re}\,w > C\}$,

$\Psi(w) := w^{-1/k}$ for $w \in H$ with the kth root chosen such that $1^{1/k} = 1$ and $P := \Psi(H)$, the conjugated map $\varphi := \Psi^{-1} \circ f \circ \Psi \colon H \to H$ satisfies

$$\varphi(w) = w + 1 + O(|w|^{-1}), \quad w \in H.$$

From this, (4) follows at once. □

3. Germs tangent to the identity

Definition 3.1. Let F be a germ of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let $F(X) = X + P_h(X) + \ldots$, $h \geq 2$ be the expansion of F in homogeneous polynomials, $P_h(X) \neq 0$. The polynomial $P_h(X)$ is called the *Hakim polynomial* and the integer h the *order* of F at O.

Let $v \in \mathbb{C}^n$ be a nonzero vector such that $P_h(v) = \alpha v$ for some $\alpha \in \mathbb{C}$. Then v is called a *characteristic direction* for F. If moreover $\alpha \neq 0$ then v is said a *nondegenerate characteristic direction*.

A *parabolic curve* for a map F tangent to the identity is a holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^n$ from the unit disc to $\mathbb{C}^n$, continuous up to the boundary and such that $F(\varphi(\mathbb{D})) \subset \varphi(\mathbb{D})$ and $F^{\circ m}(\varphi(\zeta)) \to 0$ for all $\zeta \in \mathbb{D}$. Moreover, the parabolic curve φ is tangent to a direction $v \in \mathbb{C}^n \setminus \{0\}$ if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}$ as $\mathbb{D} \ni \zeta \to 0$ (here $[v]$ denotes the class of v in $\mathbb{P}^{n-1}$).

It can be proved that if P is a parabolic curve for F at O tangent to v then v is a characteristic direction. However there exist examples of germs tangent to the identity with a parabolic curve not tangent to a single direction (that is with tangent cone spanning a vector space of dimension greater than one).

Theorem 3.2 (Écalle, Hakim)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O with order h. If v is a nondegenerate characteristic direction for F then there exist (at least) $h - 1$ parabolic curves tangent to v.*

Hakim's proof relies essentially on a finite number of blow-ups and changes of coordinates in such a way that the map assumes a good form and one can define an operator (which is a contraction) on a suitable space of curves. The fixed point of such an operator is the wanted curve.

Actually Hakim's work provides the existence of basins of attraction or lower dimensional invariant manifolds which are attracted to the origin, called *parabolic manifolds*, according to other invariants related to any nondegenerate characteristic direction. Let v be a nondegenerate characteristic direction for F and let P_h be the Hakim polynomial. We denote by $A(v) := d(P_h)_{[v]} - \mathrm{id} : T_{[v]}\mathbb{CP}^{n-1} \to T_{[v]}\mathbb{CP}^{n-1}$. Then we have

Theorem 3.3 (Hakim)**.** *Let F be a germ of holomorphic diffeomorphism of $\mathbb{C}^n$ fixing O and tangent to the identity at O. Let v be a nondegenerate characteristic direction. Let $\beta_1, \ldots, \beta_{n-1} \in \mathbb{C}$ be the eigenvalues of $A(v)$. Moreover assume $\mathsf{Re}\,\beta_1, \ldots, \mathsf{Re}\,\beta_m > 0$ and $\mathsf{Re}\,\beta_{m+1}, \ldots, \mathsf{Re}\,\beta_{n-1} \leq 0$ for some $m \leq n-1$ and let E be the sum of the eigenspaces associated to $\beta_1, \ldots, \beta_m$. Then there exists a parabolic manifold M of dimension $m + 1$*

tangent to $\mathbb{C}v \oplus E$ *at* O *such that for all* $p \in M$ *the sequence* $\{F^{\circ k}(p)\}$ *tends to* O *along a trajectory tangent to* v*.*

In particular if all the eigenvalues of $A(v)$ have positive real part then there exists a basin of attraction for F at O.

In [1] Abate proved the following:

Theorem 3.4 (Abate)**.** *Let* F *be a germ of holomorphic diffeomorphism of* $\mathbb{C}^2$ *having the origin as an isolated fixed point and tangent to the identity at* O *with order* h*. Then there exist (at least)* $h-1$ *parabolic curves.*

The original proof of Abate is rather involved. Other simpler or clearer proofs have been discovered later. In [3], a very general construction explaining the essence of the argument in Abate's proof has been developed. In [13] Brochero, Cano and Hernanz gave a proof of such a theorem which relies on foliations. Namely, they associate to the germ F a formal vector field X such that $\exp(X) = F$. Then they perform a number of blow-ups to "solve" the singularities of X using the Seidenberg theorem (which holds in the formal category) and then use the Camacho-Sad construction [15] proving that, on a smooth component of the exceptional divisor, there exists a suitable "good" singularity for X. Such a good singularity corresponds to the existence of a nondegenerate characteristic direction for the blow-up of F, hence applying Theorem 3.2 one gets parabolic curves that can be projected downstairs.

This method of taking a germ tangent to the identity and associating a formal vector field whose time one flow is the germ itself is a good way to transfer the result from the better known theory of vector fields to the study of germs tangent to the identity. However, there is a disadvantage with respect to more direct methods: since the vector field is in general only formal, deep problems of resurgence can occur. For instance, it is likely that one can get a proof of Abate's theorem directly in the category of formal vector fields (namely, without using Theorem 3.2) by proving using resurgence that given a formal vector field and one of its formal separatrix, the latter is convergent on pieces which are exactly the parabolic curves for the map.

In the same direction, in the recent paper [6], Abate and Tovena studied *real* dynamics of *complex* homogeneous vector fields. Besides its intrinsic interest, this is an useful problem to study because the discrete dynamics of the time 1-map is encoded in the real integral curves of the vector field, and time 1-maps of homogeneous vector fields are prototypical examples of holomorphic maps tangent to the identity at the origin. The main idea here is that, roughly speaking, integral curves for homogeneous vector fields are geodesics for a meromorphic connection on a projective space.

More generally, thanks to a result by Takens [37] (see also [23, Chapter 1]), in case of diffeomorphisms with *unipotent* linear part, one can embed such germs in the flow of formal vector fields, so that this type of argument might be used also in such cases. On the other hand, when the linear part of the diffeomorphism is not unipotent, the author is not aware of any general result about embedding such a diffeomorphism into the flow of a formal vector field. In fact, one encounters somewhat unexpected differences between the dynamics of diffeomorphisms and that of vector fields, see Raissy [31].

In [3] (see also [11], [4], [5]) a different point of view, more abstract but more intrinsic, has been adopted. Blowing up the origin one obtains a new germs of diffeomorphism pointwise fixing the exceptional divisor. Thus the situation is that of considering a holomorphic self-map of a complex manifold having a (hyper)surface of fixed points. Roughly speaking, the differential of the map acts on the normal bundle of such a hypersurface in a natural way and thus creates a meromorphic connection, whose singularities essentially rule the dynamics of the map.

4. Semiattractive and Quasi-parabolic germs

4.1. **Semiattractive germs.** We say that a parabolic germ F is *semi-attractive* if 1 is an eigenvalue of dF_O and all the other eigenvalues have modulus strictly less than 1 (if all the other eigenvalues have modulus strictly greater than 1 we argue on F^{-1}). There are essentially two cases to be distinguished here: either F has or has not a submanifold of fixed points.

In case F has a submanifold of fixed points (of the right dimension) there is a result due to Nishimura [29] which roughly speaking says that, in absence of resonances, F is conjugated along S to its action on the normal bundle N_S to S in $\mathbb{C}^n$.

In case F has no curves of fixed points, Hakim [24] (based on the previous work by Fatou [19] and Ueda [38], [39] in $\mathbb{C}^2$) proved that, under suitable generic hypotheses, there exist "fat petals" (called *parabolic manifolds* or *basins of attraction* when they have dimension n) for F at O. That is

Theorem 4.1 (Hakim)**.** *Let F be a semi-attractive parabolic germ at O, with* 1 *as eigenvalue of dF_O of (algebraic) multiplicity* 1*. If O is an isolated fixed point of F then there exist k disjoint basins of attraction for F at O, where $k+1 \geq 2$ is the "order" of $F - \mathrm{id}$ at O.*

It is worth noticing that if F is an automorphism of $\mathbb{C}^2$ then each basin of attraction provided by Theorem 4.1 is biholomorphic to $\mathbb{C}^2$ (the existence of proper subsets of $\mathbb{C}^n$ biholomorphic to $\mathbb{C}^n$ for $n > 1$ is known as the *Fatou-Bierbach phenomenon*).

Theorem 4.1 is a special case of the procedure described in the Introduction and in Section 5. Indeed, write the map F as

$$F(z, w) = (z + az^{k+1} + O(\|zw\|, \|w\|^2, |z|^{k+2}), \lambda w + O(\|z\|^2, \|zw\|, \|w\|^2),$$

where $w \in \mathbb{C}^{n-1}$, λ is a $(n-1) \times (n-1)$ matrix with eigenvalues of modulus strictly less than one, $k \in \mathbb{N} \cup \{\infty\}$ and $a \in \mathbb{C} \setminus \{0\}$. The case $k = \infty$ (namely, no pure terms in z are present in the first component) corresponds to the existence of a curve of fixed points. So we assume $k \in \mathbb{N}$. The monomials in w are not resonant in the first component, thus they can be killed by means of a Poincaré-Dulac formal change of coordinates. Hence, the foliation $\{z = \text{const}\}$ is invariant by the formal normal form of F, and the action on the "space of leaves" (which is nothing but $\mathbb{C}$) is given exactly by $z \mapsto z + az^{k+1} + \ldots$. If we perform the Poincaré-Dulac procedure solving only finitely many homological equations to kill terms in w in the first component, the map F has the form:

$$F(z, w) = (z + az^{k+1} + O(z^{k+2}) + O(\|w\|^l), \lambda w + \ldots)$$

with $l >> 1$ as big as we want. Hence, the map F does not preserve the foliation $\{z = \text{const}\}$, but it moves it slowly at order l in w. This implies that the preimage under the map $(z, w) \mapsto z$ of a (suitable) sector S contained in some petals of $z \mapsto z + az^{k+1}$ is in fact invariant for F. Taking open sets of the form $\{(z, w) : \|w\| < |z|^\beta, z \in S\}$ for a suitable $\beta > 0$, it can be then proved that such open sets are invariant and, via the dynamics "downstair", they are actually basins of attraction for F.

4.2. **Quasi-parabolic germs.** We call *quasi-parabolic* a germ if all eigenvalues of dF_O have modulus 1 and at least one, but not all, is a root of unity—and, up to replace the germ with some of its iterates, we can assume all roots of unity are 1.

Let us write the spectrum of dF_O as $R \cup E$, where R contains the roots of unity and E the other unimodular eigenvalues. In case the eigenvalues in E have no resonances and satisfy a Bruno-type condition, a result of Pöschel [30] assures the existence of a complex manifold M tangent to the eigenspace associated to E at O which is F-invariant and such that the restriction of F to M is holomorphically conjugated to the rotation $z \mapsto Ez$.

We describe here the "parabolic attitude" of quasi-parabolic germs in $\mathbb{C}^2$. For results in $\mathbb{C}^n$ we refer to [33], [34]. As strange as it may seem, it is not known whether all quasi-parabolic germs have "parabolic attitude"!

Using Poincaré-Dulac theory, since all resonances are of the type $(1, (m, 0)), (2, (m, 1))$—namely, in the first coordinates the resonant monomials are just z^m and in the second coordinates the resonant monomials are $z^m w$, the map F can be formally conjugated to a map of the form

$$\hat{F}(z, w) = (z + \sum_{j=\nu}^{\infty} a_j z^j, e^{2\pi i\theta} w + \sum_{j=\mu}^{\infty} b_j z^j w), \tag{4.1}$$

where we assume that either $a_\nu \neq 0$ or $\nu = \infty$ if $a_j = 0$ for all j. Similarly for b_μ.

As it is proved in [10], the number $\nu(F) := \nu$ is a formal invariant of F. Moreover, it is proved that, in case $\nu < +\infty$, the sign of $\Theta(F) := \nu - \mu - 1$ is a formal invariant. The map F is said *dynamically separating* if $\nu < +\infty$ and $\Theta(F) \leq 0$.

The next proposition is proved in [12] and its proof is a simple argument based on the implicit function theorem:

Proposition 4.2. *Let F be a quasi-parabolic germ of diffeomorphism of $\mathbb{C}^2$ at 0. Then $\nu(F) = +\infty$ if and only if there exists a germ of (holomorphic) curve through 0 that consists of fixed points of F.*

In case $\nu(F) < +\infty$, the following result is proved in [10]:

Theorem 4.3. *Let F be a quasi-parabolic germ of diffeomorphism of $\mathbb{C}^2$ at 0. If F is dynamically separating then there exist $\nu(F) - 1$ parabolic curves for F at 0.*

The argument in [10] is based on a series of blow-ups and changes of coordinates which allow to write F into a suitable form so that one can write a similar operator to the one defined by Hakim and prove that its fixed points in a certain Banach space of curves are exactly the sought parabolic curves.

As in the semi-attractive case, one can argue using the invariant formal foliation $\{z = \text{const}\}$. Contractiveness in the w-variable is however not for free here. Indeed, in [12] (see also Section 5) it is proved

Proposition 4.4. *Let F be a dynamically separating quasi-parabolic germ, formally conjugated to* (4.1)*. If*

$$\mathsf{Re}\left(\frac{b_{\nu-1}}{e^{2\pi i\theta}a_\nu}\right) > 0.$$

then there exist $\nu(F) - 1$ disjoint connected basins of attraction for F at 0.

It should be remarked that the condition in the previous proposition is destroyed under blow-ups.

The non-dynamically separating case is still open. In such a case there is still a formal foliation $\{z = \text{const}\}$ which is invariant, but the w-variable cannot be controlled appropriately by the z variable.

F. Fauvet [20] told me that using Écalle's resurgence theory it is possible to prove that parabolic curves exist also in the non-dynamically separating case when the other eigenvalue satisfies a Bruno-type condition.

5. One resonant germs

Let F be a germ of holomorphic diffeomorphism in $\mathbb{C}^n$ fixing 0. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the linear part. We say that F is *one-resonant with respect to the first m eigenvalues* $\{\lambda_1, \ldots, \lambda_m\}$ $(1 \leq m \leq n)$ (or *partially one-resonant*) if there exists a fixed multi-index $\alpha = (\alpha_1, \ldots, \alpha_m, 0, \ldots, 0) \neq 0 \in \mathbb{N}^n$ such for $s \leq m$, the resonances $\lambda_s = \prod_{j=1}^n \lambda_j^{\beta_j}$ are precisely of the form $\lambda_s = \lambda_s \prod_{j=1}^m \lambda_j^{k\alpha_j}$, where $k \geq 1 \in \mathbb{N}$ is arbitrary.

This notion has been introduced in [12]. The main advantage of such a notion of partial one-resonance is that it can be applied to the subset of all eigenvalues of modulus equal to 1 that is natural to be treated differently from the rest of the eigenvalues.

In case of partial one-resonance, the classical Poincaré-Dulac theory implies that, whenever F is not formally linearizable in the first m components, F is formally conjugated to a map whose first m components are of the form $\lambda_j z_j + a_j z^{\alpha k} z_j + R_j(z)$, $j = 1, \ldots, m$, where, the number $k \in \mathbb{N}$ is an invariant, called *the order of F with respect to* $\{\lambda_1, \ldots, \lambda_m\}$, the vector $(a_1, \ldots, a_m) \neq 0$ is invariant up to a scalar multiple and the R_j's contain only resonant terms. The fact that the number

$$\Lambda(F) := \sum_{j=1}^{m} \frac{a_j}{\lambda_j} \alpha_j$$

is equal or not to zero is an invariant, and the map F is said to be *non-degenerate* provided $\Lambda(F) \neq 0$. In fact, one can always rescaling the map to make $\Lambda(F) = 1$ provided it is not zero.

In [12] it is proved that a partially one-resonant non-degenerate germ F has a simple formal normal form $\hat{F}$ such that

$$\hat{F}_j(z) = \lambda_j z_j + a_j z^{k\alpha} z_j + \mu \frac{\alpha_j}{\lambda_j} z^{2k\alpha} z_j, \quad j = 1, \ldots, m.$$

Although none of the eigenvalues λ_j, $j = 1, \ldots, m$, might be roots of unity, such a normal form is the exact analogue of the formal normal form for parabolic germs in $\mathbb{C}$. In fact, a one-resonant germ acts as a parabolic germ on the space of leaves of the formal invariant foliation $\{z^\alpha = \text{const}\}$ and that is the reason for this parabolic-like behavior.

Let F be a one-resonant non-degenerate diffeomorphism with respect to the eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. We say that F is *parabolically attracting* with respect to $\{\lambda_1, \ldots, \lambda_m\}$ if

$$|\lambda_j| = 1, \quad \mathsf{Re}\left(\frac{a_j}{\lambda_j} \frac{1}{\Lambda(F)}\right) > 0, \quad j = 1, \ldots, m.$$

Again, such a condition is invariant, indeed, conjugating the map, both $\Lambda(F)$ and the a_j's vary suitably and the sign of the previous expression persists invariant. Such a condition is vacuous in dimension 1 or whenever $m = 1$ since in that case $\alpha = (\alpha_1, 0, \ldots, 0)$ with $\alpha_1 > 0$. In [12] it is proved the following:

Theorem 5.1. *Let F be a holomorphic diffeomorphism germ at 0 that is one-resonant, non-degenerate and parabolically attracting with respect to $\{\lambda_1, \ldots, \lambda_m\}$. Suppose that $|\lambda_j| < 1$ for $j > m$. Let $k \in \mathbb{N}$ be the order of F with respect to $\{\lambda_1, \ldots, \lambda_m\}$. Then F has k disjoint basins of attraction having 0 on the boundary.*

The different basins of attraction for F (that may or may not be connected) project via the map $z \mapsto u = z^\alpha$ into different petals of the germ $u \mapsto u + \Lambda(F)u^{k+1}$.

A semi-attractive germ is always one-resonant, non-degenerate and parabolically attracting, thus the previous theorem is a generalization of Hakim's result.

It should be noticed that the conditions about non-degeneracy and parabolically attractiveness are sharp. In [12] examples are given of one-resonant germs for which such conditions are not satisfied and that have no basins of attractions.

More interesting for the purpose of these notes is the following example (still from [12]). Let $\lambda = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$F(z, w) = (\lambda z + az^2 w + \ldots, \lambda^{-1} w + bzw^2 + \ldots),$$

with $|a| = |b| = 1$. Then F is one-resonant with index of resonance $(1,1)$ and for each choice of (a,b) such that the germ is non-degenerate (i.e. $a\lambda^{-1} + b\lambda \neq 0$), there exists a basin of attraction for F at 0. Indeed, it can be checked that the non-degeneracy condition implies that F is parabolically attracting with respect to $\{\lambda, \lambda^{-1}\}$ and hence Theorem 5.1 applies.

A similar argument can be applied to F^{-1}, producing a basin of repulsion for F at 0. Hence we have a parabolic type dynamics for F.

On the other hand, suppose further that θ satisfies a Bruno condition. Since $\lambda^q \neq \lambda$ for all $q \in \mathbb{N}$, it follows from Pöschel's theorem [30, Theorem 1] that there exist two analytic discs through 0, tangent to the z-axis and to the w-axis respectively, which are F-invariant and such that the restriction of F on each such a disc is conjugated to $\zeta \mapsto \lambda\zeta$ or $\zeta \mapsto \lambda^{-1}\zeta$ respectively. Thus, in such a case, the elliptic and parabolic dynamics mix, although the spectrum of dF_0 is only of elliptic type.

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