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# Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces

**Filippo Bracci***, **Francesca Tovena**

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 00133 Roma, Italy (e-mail: fbracci@mat.uniroma2.it / tovena@mat.uniroma2.it)

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**Abstract.** Let $M$ be a two-dimensional complex manifold and let $f : M \to M$ be a holomorphic map that fixes pointwise a (possibly) singular, compact, reduced and globally irreducible curve $C \subset M$. We give a notion of *degeneracy* of $f$ at a point of $C$. It turns out that $f$ is non-degenerate at one point if and only if it is non-degenerate at every point of $C$. When $f$ is non-degenerate on $C$, we define a residual index for $f$ at each point of $C$. Then we prove that the sum of the indices is equal to the self-intersection number of $C$.

## Introduction

In [2], C. Camacho and P. Sad introduced the index of a holomorphic vector field relative to an invariant non-singular curve and proved an index formula. Their result was generalized by A. Lins Neto [6] to the case of an algebraic foliation and a (possibly) singular invariant curve in the complex projective plane. Finally T. Suwa [7] gave a definition of index and proved a formula when the invariant (singular) curve lies in a generic two dimensional complex manifold. Recently M. Abate [1] (cf. also Sect. 1), studying discrete dynamical systems, introduced an index for holomorphic self-maps of a two dimensional complex manifold fixing a smooth compact curve (and *non-degenerate* on it), proving an analogue of the Camacho-Sad Theorem. Here we generalize Abate's result to the case of singular curves, finding an analogue of Suwa's Theorem.

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Let $M$ be a two dimensional complex manifold and $C \subset M$ be a reduced, globally irreducible and compact one dimensional subvariety. We consider a holomorphic map $f : M \to M$, such that $f_{|C} = id_{|C}$. Fixing a point $p \in C$ and localizing our study to the germs, $f_p$ and $C_p$, of $f$ and $C$ at $p$, we introduce a notion of *degeneracy* for $f$ on $C$ at $p$. Roughly speaking, we say that $f$ is degenerate on $C$ at $p \in C$ if one cannot distinguish $C_p$ just looking at "how much" $f_p$ fixes $C_p$ (see Definition 2, 4). It turns out that the non-degeneracy is a local property, *i.e.* if $f$ is non-degenerate on $C$ at $p \in C$ then there exists a neighborhood $U$ of $p$ such that for any $q \in U$, $f_q$ is non-degenerate on $C_q$. Since $C$ is irreducible, this implies that if $f$ is non-degenerate on $C$ at a point $p \in C$, then $f$ is non-degenerate at every point of $C$ (see Proposition 3).

If $f$ is non-degenerate at $p \in C$, we define an index $\mathrm{Ind}(f, C, p)$ (see Definition 5, 6) and we show that the sum of the indices $\mathrm{Ind}(f, C, p)$ is equal to the self-intersection number of $C$ (see Theorem 2). The proof of this result is quite similar to that in [7] for holomorphic vector fields. We take a resolution $\pi : \tilde{M} \to M$ of $C$. The map $f$ induces a holomorphic self-map $\tilde{f}$ of $\tilde{M}$ which fixes the desingularization $\tilde{C}$ of $C$ and is non-degenerate on it. Then we compare the indices of $\tilde{f}$ on $\tilde{C}$ and $f$ on $C$ and apply Abate's index formula for $\tilde{f}$ on $\tilde{C}$.

The actual effort made in this paper is to find a good definition for the index of $f$ on $C$ at $p \in C$ and prove that it is "natural", *i.e.* it is well-behaving under changes of variables and blow-ups. After that, we formally have the same ingredients of [7], and we can argue following the same lines. On the other hand the index here defined is not merely an application of results on vector fields, since in general it is not possible to associate to $f$ a (global) foliation leaving $C$ invariant. In conclusion, the analogy between holomorphic vector fields with an invariant leaf and holomorphic self-maps with a curve of fixed points which seems to come out from [1] and this paper, is pretty far from being really understood.

## 1 The smooth case

In this section we recall Abate's work [1].

Let $C$ be a compact one-dimensional (smooth) submanifold of $M$, with $M$ being a two dimensional complex manifold. Suppose $f : M \to M$ is holomorphic and $f_{|C} = id_{|C}$. Let $p \in C$ and choose an *adapted* local chart $U$ with coordinates $(x, y)$ around $p$, *i.e.* such that $p = (0, 0)$, $C \cap U = \{y = 0\}$. In these coordinates we can write

$$f(x,y) = \begin{cases} f_1(x,y) = x + y^m g(x,y) \\ f_2(x,y) = b(x)y + y^{n+1} h(x,y) \end{cases} \tag{1}$$

with $m \geq 1$, $n \geq 1$ and $g(x, 0) \neq 0$ (or $m = \infty$ if $g \equiv 0$) and $h(x, 0) \not\equiv 0$ (or $n = \infty$ if $h \equiv 0$). We avoid considering the case $f \equiv id_M$.

The compactness of $C$ implies that $b(x) \equiv b(f)$ is constant. The map $f$ is said *non-degenerate* on $C$ at $p$ if $m \leq n$. If $b(f) = 1$ or if $M$ is the total space of a line bundle over $C$, then Abate proves that if $f$ is non-degenerate on $C$ at $p$, it is so independently of the adapted chart chosen and of the point $p \in C$. In this situation Abate's residual index is defined as

$$\iota_p(f, C) = \text{Res}(k(x)dx, 0),$$

where

$$k(x) = \lim_{y \to 0} \frac{f_2(x, y) - b(f)y}{y(f_1(x, y) - x)}.$$

This index is independent of the local coordinates chosen, and the Abate's index formula is

**Theorem 1 (Abate).** *Let $C$ be a one dimensional compact submanifold of a two dimensional complex manifold $M$ and let $f : M \to M$ be holomorphic such that $f_{|C} = id_{|C}$. Assume $b(f) = 1$ or that $M$ is the total space of a line bundle $E$ over $C$. Assume moreover that $f$ is non-degenerate on $C$. Then*

$$\sum_{p \in C} \iota_p(f, C) = b(f)\ (C \cdot C).$$

If $C$ has a singularity at $p \in C$, there are no adapted charts available at $p$, and Abate's theory doesn't apply.

## 2 The residual index in the irreducible case

Let $\mathcal{O}_p$ be the ring of germs of holomorphic functions at $p \in M$ and $\mathcal{O}_p^*$ the field of its invertible elements. If $H$ is a holomorphic map defined on a open set $U \subseteq M$, then $H_p \in \mathcal{O}_p$ is the germ defined by $H$ at $p \in U$. If $H \in \mathcal{O}_p$ then $(H)_p$ is the ideal generated by $H$ in $\mathcal{O}_p$ and $\mathbf{V}(H)$ is the germ of the subvariety defined by $H$. If $C_p$ is a germ of a subvariety at $p$, then $\mathcal{I}(C_p)$ is the ideal of $C_p$ (see [4]).

Let $C$ be a (possibly singular) curve whose germ $C_p$ is irreducible at $p \in M$ and let $f$ be a holomorphic self-map of $M$ which pointwise fixes $C$. We avoid considering the case $f \equiv id_M$. Let $U \subset M$ be a open neighborhood of $p$ and $\phi : U \to \mathbb{C}^2$ a local chart such that $\phi(p) = (0, 0)$. If $l \in \mathcal{O}_p$ is a defining function for $C_p$ at $p$, then

$$\phi \circ f \circ \phi^{-1} = Id + (l \circ \phi^{-1})^{\mu} G, \tag{2}$$

for some germ $G = (G_1, G_2)$ of holomorphic self-map of $\mathbb{C}^2$ at $(0, 0)$, $G \not\equiv 0$ on $\phi(S)$ and $\mu \geq 1$. It is easy to see that $\mu$ is independent of $\phi$ and $l$.

*Remark 1.* We will omit to write explicitly the local chart $\phi$ in the formulae: for instance we write simply $f = Id + l^\mu G$ instead of (2). We denote by $H'$ the gradient of $H \in \mathcal{O}_p$ in the given local chart and by $< H, G >$ the scalar product of two germs $H, G$ of holomorphic self-map of $\mathbb{C}^2$.

Given $H \in \mathcal{O}_p$, expanding $H \circ f - H$ we find that

$$\frac{H \circ f - H}{l^\mu} \equiv < H', G > \mod \mathcal{I}(C_p).$$

**Definition 1.** *The map $f$ satisfying (2) is said to be* non-degenerate *at $p$ on the locally irreducible curve $C$ if*

$$\frac{l \circ f - l}{l^\mu} \equiv 0 \;\; \mathit{mod}\, \mathcal{I}(C_p),$$

i.e. *if $< l', G > \equiv 0$ on the germ $C_p$.*

*Remark 2.* For $k \in \mathcal{O}_p^*$ we have

$$\frac{(kl) \circ f - kl}{(kl)^\mu} \equiv k^{1-\mu} \frac{l \circ f - l}{l^\mu} \mod \mathcal{I}(C_p).$$

Thus the definition of non-degeneracy of $f$ on $C$ at $p$ is independent of the defining function $l$.

*Remark 3.* Assume that $p$ is a smooth point for $C$. As in the first section, we can choose an adapted local chart with coordinates $(x, y)$ around $p$, so that $f$ satisfies (1) and $y$ is a defining function for $C$. Since $y \circ f - y = (b(x)-1)y + y^{n+1}h(x, y)$, our definition of *non-degeneracy* coincides with Abate's one whenever $b(p) = 1$. In the case $b(p) \neq 1$, the map $f$ is degenerate according to Definition 1 but it could be non-degenerate according to Abate's definition. However, if $C$ has a singularity at some $q \in C$ and $f$ is non-degenerate on $C$ at $q$ then—as a consequence of Lemma 2—it turns out that $f$ is non-degenerate on $C$ at every point $p \in C$ and in particular $b(p) = 1$ at any smooth point of $C$. Therefore, *if $C$ is singular, $f$ is non-degenerate on $C$ according to Definition 1 if and only if it is non-degenerate at one—and hence any—smooth point according to Abate's definition.*

**Definition 2.** *We say that $H \in \mathcal{O}_p$ is* transverse *to $(f, C_p)$ if*

$$\frac{H \circ f - H}{l^\mu} \not\equiv 0 \;\; \mathit{mod}\, \mathcal{I}(C_p),$$

i.e. $< H', G > \not\equiv 0$ *on* $C_p$.

Therefore $f$ is degenerate on $C_p$ at $p$ if and only if a defining function of $C_p$ —and hence any— is transverse to $(f, C_p)$.

**Proposition 1.** *Suppose $f$ is non-degenerate on $C_p$ at $p$ and let $l \in \mathcal{O}_p$ be a defining function for $C_p$. A germ $H \in \mathcal{O}_p$ is transverse to $(f, C_p)$ if and only if* $\det(H', l') \not\equiv 0 \mod \mathcal{I}(C_p)$.

*Proof.* Let $f = I + l^\mu G$, as in (2). We have $\det(H', l') \equiv 0 \mod \mathcal{I}(C_p)$ if and only if $H' \equiv kl' \mod \mathcal{I}(C_p)$ for some $k \in \mathcal{O}_p$. Now if $< H', G > \equiv 0$ on $C_p$ then, since $G \not\equiv 0$ on $C_p$, it follows that $H' = kl$ for some $k \in \mathcal{O}_p$. On the other hand, if $H' \equiv kl' \mod \mathcal{I}(C_p)$ then $< H', G > \equiv k < l', G > \equiv 0$ on $C_p$ for $f$ is non-degenerate on $C_p$.

Note that, if $C_p$ is smooth at $p$, then the regular curves transverse (in the usual sense) to $C_p$ at $p$ are actually transverse to $(f, C_p)$.

Since $C_p$ is irreducible, it admits a local uniformization (see [5]). Namely, there exists a homeomorphism $\varphi : \Delta \to C_p$ (where $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$) such that $\varphi(0) = p$, $\varphi$ is holomorphic on $\Delta - \{0\}$ and $\varphi'(\zeta) \neq 0$ for all $\zeta \in \Delta - \{0\}$ ($\varphi'(0) \neq 0$ if and only if $C_p$ is smooth at $p$). We denote by $\Gamma \in H_1(C_p - \{p\}, \mathbb{Z})$ the class of $\varphi(\partial\Delta)$, where $\partial\Delta$ is given by $\theta \mapsto \rho e^{i\theta}$ for $\theta \in [0, 2\pi]$ and a fixed $0 < \rho < 1$.

**Definition 3.** *Suppose $f$ is non-degenerate on $C$ at $p$, with $C_p$ reduced and irreducible at $p$. Let $l \in \mathcal{O}_p$ be a defining function for $C_p$ and $\tau \in \mathcal{O}_p$ be transverse to $(f, C_p)$. The* residual index *of $f$ with respect to $C_p$ is*

$$\mathit{Ind}(f, C, p) := \frac{1}{2\pi i} \int_\Gamma \frac{l \circ f - l}{l \cdot (\tau \circ f - \tau)} d\tau.$$

*Remark 4.* If $C_p$ is smooth at $p$ and $f$ is non-degenerate on $C_p$, then $\mathrm{Ind}(f, C, p) = \iota_p(f, C_p)$, the Abate's index. Indeed, choosing adapted local coordinates such that $C_p = \mathbf{V}(y)$, we can take $l = y$, $\tau = x$. The cycle $\Gamma$ is given by the curve $y = 0$, $x = \rho e^{i\theta}$, for some $\rho > 0$ small and $\theta \in [0, 2\pi]$. Therefore

$$\mathrm{Ind}(f, C, p) = \frac{1}{2\pi i} \int_\Gamma k(x) dx = \iota_p(f, C_p).$$

**Lemma 1.** *The index $\mathit{Ind}(f, C, p)$ is well-defined,* i.e. *it is independent of the defining function $l \in \mathcal{O}_p$ and the transverse $\tau \in \mathcal{O}_p$.*

*Proof.* Let $f = I + l^\mu G$, as in (2) and let $\varphi : \Delta \to C_p$ be a local uniformization of $C_p$. Since $f$ is non degenerate on $C_p$ then $< l', G > \equiv 0 \mod \mathcal{I}(C_p)$. Namely $< (l' \circ \varphi), (G \circ \varphi) > \equiv 0$ on $\Delta$. Together with $< (l' \circ \varphi), \varphi' > \equiv 0$ on $\Delta$, and since we may assume $l' \circ \varphi \not\equiv 0$ on $\Delta - \{0\}$, this implies that there exists $\gamma \in \mathcal{O}^*(\Delta - \{0\})$ such that

$$G \circ \varphi(\zeta) = \gamma(\zeta)\varphi'(\zeta) \quad \forall \zeta \in \Delta - \{0\}. \tag{3}$$

Let $h(\zeta) := \frac{l\circ f - l}{l^{\mu+1}} \circ \varphi(\zeta)$ for $\zeta \in \Delta$ and let $k \in \mathcal{O}_p^*$. Using (3) we get

$$\int_\Gamma \frac{(kl)\circ f - kl}{kl(\tau\circ f - \tau)} d\tau = \int_\Gamma \frac{l\circ f}{l} \frac{k\circ f - k}{k(\tau\circ f - \tau)} d\tau + \int_\Gamma \frac{l\circ f - l}{l(\tau\circ f - \tau)} d\tau$$
$$= \int_\Gamma \frac{< k', G >}{k < \tau', G >} d\tau + \int_\Gamma \frac{l\circ f - l}{l^{\mu+1}} \frac{d\tau}{< \tau', G >}$$
$$= \int_{\partial\Delta} \frac{< (k'\circ\varphi), (G\circ\varphi) >}{(k\circ\varphi) < (\tau'\circ\varphi), (G\circ\varphi) >} d(\tau\circ\varphi)$$
$$+ \int_{\partial\Delta} \frac{h}{< (\tau'\circ\varphi), (G\circ\varphi) >} d(\tau\circ\varphi) = \int_\Gamma \frac{dk}{k} + \int_{\partial\Delta} \frac{h}{\gamma} d\zeta = \int_{\partial\Delta} \frac{h}{\gamma} d\zeta,$$

as wanted.

*Example 1.* Let $C_p = \mathbf{V}(l_{(0,0)})$, with $l(x,y) = x^2 - y^3$, and let $f(x,y) = (x + 3x(x^2 - y^3), y + 2y(x^2 - y^3))$. Then $C_p$ is a irreducible germ of a curve with singularity at $(0,0)$, and it is easy to see that $f$ is non-degenerate on $C_p$. A local uniformization of $C_p$ is given by $\varphi(\zeta) = (\zeta^3, \zeta^2)$. Therefore a straightforward calculation gives

$$\mathrm{Ind}(f, C, (0,0)) = \frac{1}{2\pi i} \int_\Gamma \frac{6 - 3y^3}{3x} dx = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{6}{\zeta} d\zeta = 6.$$

## 3 The residual index in the reducible case

We let $C = \cup_{\alpha=1}^N C_\alpha$ be a germ of a reduced curve at $p$, $C_\alpha$ irreducible in $\mathcal{O}_p$ for $\alpha = 1, \ldots, N$ and let $f$ be a germ of holomorphic map at $p$ fixing $C$.

**Definition 4.** *We say that $f$ is* non-degenerate *on $C$ at $p$ if it is non-degenerate on $C_\alpha$ for $\alpha = 1, \ldots, N$.*

Let $l = l_1 \cdots l_N$ be a defining function of $C$ with $\mathbf{V}(l_\alpha) = C_\alpha$. Let $\Gamma_\alpha$ be the cycle for $C_\alpha$ given by the local uniformization.

**Definition 5.** *If $f$ is non-degenerate on $C$ then we define the* residual index *of $f$ on $C$ with respect to $C_\alpha$ as*

$$\mathit{Ind}(f, C_\alpha, C, p) := \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{l\circ f - l}{l \cdot (\tau\circ f - \tau)} d\tau,$$

*where $\tau \in \mathcal{O}_p$ is transverse to $(f, C_\alpha)$.*

Recall that one of many equivalent definitions of the *(local) intersection number* at $p$ of $C_\alpha$ and $C_\beta$ for $\alpha \neq \beta$ is

$$(C_\alpha \cdot C_\beta)_p = \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{dl_\beta}{l_\beta}.$$

The following relation between $\mathrm{Ind}(f, C_\alpha, p)$ and $\mathrm{Ind}(f, C_\alpha, C, p)$—from which it follows that $\mathrm{Ind}(f, C_\alpha, C, p)$ is well defined—holds (cfr. *Proposition (1.4) of [7]*):

**Proposition 2.** *If $f$ is non-degenerate on $C$ at $p$ then*

$$\mathit{Ind}(f, C_\alpha, C, p) = \mathit{Ind}(f, C_\alpha, p) + \sum_{\beta \neq \alpha} (C_\alpha \cdot C_\beta)_p \tag{4}$$

*Proof.* Let $f = Id + l^\mu G$ as in (2). Up to reordering, we can assume $\alpha = 1$. Then

$$\begin{aligned}\frac{l \circ f - l}{l} &= \frac{l_1 \circ f}{l_1} \cdots \frac{l_{N-1} \circ f}{l_{N-1}} \cdot \frac{l_N \circ f - l_N}{l_N} \\ &\quad + \frac{l_1 \circ f}{l_1} \cdots \frac{l_{N-2} \circ f}{l_{N-2}} \cdot \frac{l_{N-1} \circ f - l_{N-1}}{l_{N-1}} + \ldots + \frac{l_1 \circ f - l_1}{l_1}.\end{aligned}$$

Let $\tau \in \mathcal{O}_p$ be transverse to $(f, C_1)$. Since $\frac{l_j \circ f}{l_j} = 1$ on $C_1$, we get

$$\begin{aligned}\frac{1}{2\pi i} \int_{\Gamma_1} \frac{l \circ f - l}{l \cdot (\tau \circ f - \tau)} d\tau = \sum_{j=2}^{N} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{l_j \circ f - l_j}{l_j \cdot (\tau \circ f - \tau)} d\tau &+ \\ \frac{1}{2\pi i} \int_{\Gamma_1} \frac{l_1 \circ f - l_1}{l_1 \cdot (\tau \circ f - \tau)} d\tau&.\end{aligned}$$

By definition the last term is $\mathrm{Ind}(f, C_\alpha, p)$. As for the other terms, note first that $\det(l'_j, l'_1) \not\equiv 0$ on $C_1$ for $j \neq 1$, and therefore $l_j$ is transverse to $(f, C_1)$ for $j \neq 1$ by Proposition 1. Thus $< l'_j, G > \not\equiv 0$ on $C_1$. Arguing as in the proof of Lemma 1 for the calculation of the integral $\int_\Gamma \frac{k \circ f - k}{k(\tau \circ f - \tau)} d\tau$, we find

$$\frac{1}{2\pi i} \int_{\Gamma_1} \frac{l_j \circ f - l_j}{l_j \cdot (\tau \circ f - \tau)} d\tau = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{dl_j}{l_j} = (C_1 \cdot C_j)_p.$$

From this we get the formula.

**Definition 6.** *We let* $\mathit{Ind}(f, C, p) := \sum_{\alpha=1}^{N} \mathit{Ind}(f, C_\alpha, C, p)$.

## 4 The index formula

Let $f : M \to M$ be a holomorphic map of a two dimensional complex manifold $M$. Let $C \subset M$ be a connected compact reduced and globally irreducible curve such that $f_{|C} = id_{|C}$. We say that $f$ is *non-degenerate* on $C$ if $f$ is non-degenerate on every $p \in C$.

**Lemma 2.** *Let $M$ be a two dimensional complex manifold and $C$ a connected and locally irreducible curve in $M$. If $f : M \to M$ is a holomorphic map such that $f_{|C} = id_{|C}$ then $f$ is non-degenerate on $C$ (*i.e. *at any point of $C$) if and only if it is non-degenerate at just one point.*

*Proof.* Suppose $f$ is non-degenerate on $C$ at $q \in C$. Let $U$ be a (small) neighborhood of $q$ and $l \in \mathcal{O}(U)$ be a defining function for $C$ on $U$, *i.e.* $\mathcal{I}(C)_p = (l)_p$ for any $p \in U \cap C$ (this is possible for the sheaf of ideals of $C$ is coherent, see [4]). Suppose $f = Id + l^\mu G$ in $U$, with $\mu, G$ as in (2). By hypothesis $< l', G > \equiv 0 \bmod \mathcal{I}(C)_q$, thus $< l', G > \equiv 0 \bmod \mathcal{I}(C)_p$ for any $p \in C \cap U$ by the identity principle. Therefore $f$ is non-degenerate on $C$ at $p$ for any $p \in C \cap U$. Since $C$ is connected, it follows that $f$ is non-degenerate on all of $C$.

Since $C$ globally irreducible means that $C - \mathrm{Sing}(C)$ is pathwise connected, the previous lemma implies:

**Proposition 3.** *Let $M$ be a two dimensional complex manifold, $f : M \to M$ a holomorphic map and $C \subset M$ a connected, reduced, globally irreducible curve. Then $f$ is non-degenerate on $C$ if and only if it is non-degenerate at just one point of $C$.*

Now we can state our main theorem.

**Theorem 2.** *Let $M$ be a two dimensional complex manifold, $f : M \to M$ a holomorphic map and $C \subset M$ a compact, reduced, globally irreducible curve of fixed points of $f$. If $f$ is non-degenerate on $C$ then*

$$\sum_{p \in C} \mathit{Ind}(f, C, p) = C \cdot C \ ,$$

*where $C \cdot C$ is the self-intersection number of $C$.*

Note that $\sum_{p \in C} \mathrm{Ind}(f, C, p)$ is actually a finite sum since $C$ is compact.

In order to prove the theorem we need some local analysis. Suppose $U$ is a (small) open neighborhood of a singular point $p \in C$ and $C$ is irreducible at $p$. Let $V \subseteq U$ be a open neighborhood of $p$ such that $f(V) \subseteq U$. Let $\pi : \tilde{U} \to U$ be a quadratic transformation of the point $p$. That is $\tilde{U}$ is a two dimensional complex manifold, $\pi : \tilde{U} \to U$ is a proper holomorphic map, $D := \pi^{-1}(p)$ is a projective line and $\pi : \tilde{U} - D \to U - \{p\}$ is a biholomorphism. We let $\tilde{C} := \overline{\pi^{-1}(C - \{p\})}$ be the strict transform of $C$. Thus $\tilde{C} \cap D = \{\tilde{p}\}$ and the total transform of $C$ under $\pi$ is given by $\pi^{-1}(p) = \tilde{C} + mD$, for $m = (\tilde{C} \cdot D)_{\tilde{p}}$, the *multiplicity* of $C$ at $p$. The map $f$ naturally induces a holomorphic map $\tilde{f} : \pi^{-1}(V) \to \tilde{U}$ such that $\tilde{f}_{|D} = Id_{|D}$ and $\pi \circ \tilde{f} = f \circ \pi$ (see [1]). In particular $\tilde{f}_{|\tilde{C}} = Id_{|\tilde{C}}$. Note that since $f$ is non-degenerate on $C \cap U$ and $\pi$ is a biholomorphism out of $D$ then $\tilde{f}$ is non-degenerate on $\tilde{C}$.

**Lemma 3.** *In the above situation we have*

$$\mathit{Ind}(f, C, p) = \mathit{Ind}(\tilde{f}, \tilde{C}, \tilde{p}) + m^2. \tag{5}$$

*Proof.* Let $\{l = 0\}$ be a local defining function for $C$ at $p$. We can choose local coordinates $(x, y)$ on $U$ and $(u, v)$ on $\tilde{U}$ such that $l(0, y) \not\equiv 0$, $p = (0, 0)$, $\tilde{p} = (0, 0)$ and $\pi(u, v) = (u, uv)$. Then $\tilde{l} := l \circ \pi$ is a defining function of the total transform $\pi^{-1}(C)$ and $\tilde{l}(u, v) = l(u, uv) = u^m t(u, v)$, where $t(0, 0) = 0$, $t(0, v) \not\equiv 0$ and $t$ is a defining function of $\tilde{C}$. In particular $t(u, v) = u^{-m}(l \circ \pi)$. Note that $u$ is transverse to $(\tilde{f}, \tilde{C})$ at $\tilde{p}$ and that if $\tilde{\Gamma}$ is the cycle for $\tilde{C}$ at $\tilde{p}$ given by the local uniformization then $\pi_*(\tilde{\Gamma}) = \Gamma$, the cycle for $C$ at $p$. We are now ready to calculate $\mathrm{Ind}(\tilde{f}, \tilde{C}, \tilde{p})$:

$$\begin{aligned}
2\pi i\ \mathrm{Ind}(\tilde{f}, \tilde{C}, \tilde{p}) &= \int_{\tilde{\Gamma}} \frac{t \circ \tilde{f} - t}{t(u \circ \tilde{f} - u)} du \\
&= \int_{\tilde{\Gamma}} \frac{(l \circ f \circ \pi)}{(l \circ \pi)} \frac{u^{-m} \circ \tilde{f} - u^{-m}}{u^{-m}(u \circ \tilde{f} - u)} du \\
&\quad + \int_{\pi_*(\tilde{\Gamma})} (\pi^{-1})^* \left( \frac{l \circ \pi \circ \tilde{f} - l \circ \pi}{l \circ \pi(u \circ \tilde{f} - u)} du \right) \\
= -m \int_{\tilde{\Gamma}} \frac{du}{u} &+ \int_{\Gamma} \frac{l \circ f - l}{l(x \circ f - x)} dx = 2\pi i(-m^2 + \mathrm{Ind}(f, C, p)).
\end{aligned}$$

Now we are in the position to prove Theorem 2.

*Proof of Theorem 2.* Take a global resolution $\pi : \tilde{M} \to M$ of the singularities of $C$. The total transform of $C$ can be written as

$$\tilde{C} + \sum_{j=1}^{N} m_j D_j,$$

where (1) $\tilde{C}$ is a compact connected non-singular curve and $\pi_{|\tilde{C}} : \tilde{C} \to C$ is a resolution of singularities of $C$, (2) each $D_j$ is a projective line and $m_j$ a positive integer, (3) $\pi$ is biholomorphic out of the exceptional divisor $D := \cup_{j=1}^{N} D_j$, (4) $\tilde{C}$ intersects $D$ at a finite number of points which are non-singular points of $D$ and each intersection is transverse. The map $f$ induces a holomorphic map $\tilde{f}$ on $\tilde{M}$ with $\tilde{f}_{|\tilde{C}} = id_{|\tilde{C}}$ and $\tilde{f}$ non-degenerate on $\tilde{C}$. Using Proposition 2 and Lemma 3 it is easy to see that if $p \in C$ is a singularity of $C$, $U$ a (small) open neighborhood of $p$ such that $C \cap U = \cup_{\alpha=1}^{N} C_\alpha$ with $C_\alpha$ irreducible, then

$$\mathrm{Ind}(f, C_\alpha, C, p) = \mathrm{Ind}(\tilde{f}, \tilde{C}_\alpha, q_\alpha) + \sum_{j=1}^{M} m_j (D_j \cdot \tilde{C}_\alpha)_{q_\alpha}, \tag{6}$$

where $\tilde{C}_\alpha$ is the strict transform of $C_\alpha$, and $(D_j \cdot \tilde{C}_\alpha)_{q_\alpha} = 0$ or $1$ according whether $D_j$ intersects $\tilde{C}_\alpha$ at the point $q_\alpha \in \pi^{-1}(p)$ or not. By (6) the sum $\sum_{p\in C} \mathrm{Ind}(f, C, p)$ equals $\sum_{\tilde{p}\in\tilde{C}} \mathrm{Ind}(\tilde{f}, \tilde{C}, \tilde{p}) + \sum m_j(D_j \cdot \tilde{C})$. By Theorem 1 and the projection formula we get the result.

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## References

1. M. Abate: The residual index and the dynamics of holomorphic maps tangent to the identity. Duke Math. J. (1) **107** (2001), 173–207
2. C. Camacho, P. Sad: Invariant varieties through singularities of holomorphic vector fields. Ann. of Math. (2) **115** (1982), 579–595
3. W. Fulton: Intersection Theory, 2nd edn. Springer, 1998
4. R. Gunning, H. Rossi: Analytic functions of several complex variables. Prentice-Hall, 1965
5. H. B. Laufer: Normal two dimensional singularities. Ann. Math. Stud. **71** (1971)
6. A. Lins Neto: Algebraic solutions of polynomial differential equations and foliations in dimension two. Holomorphic Dynamics (Mexico 1986), Lecture Notes in Math., vol. 1345, pp. 192–232. Springer, 1988
7. T. Suwa: Indices of holomorphic vector fields relative to invariant curves on surfaces. Proc. Amer. Math. Soc. **123** (1995), 2989–2997