Complex Variables Theory and Appl.
47, 12, (2002), 1115-1119.

HOLOMORPHIC DEPENDENCE OF THE HEINS MAP

FILIPPO BRACCI†

ABSTRACT. Let D be a bounded convex domain and $\mathrm{Hol}_c(D,D)$ the set of holomorphic maps from D to $\mathbb{C}^n$ with image relatively compact in D. Consider $\mathrm{Hol}_c(D,D)$ as a open set in the complex Banach space $H^\infty_n(D)$ of bounded holomorphic maps from D to $\mathbb{C}^n$. We show that the map $\tau : \mathrm{Hol}_c(D,D) \to D$ (called the *Heins map* for D equals to the unit disc of $\mathbb{C}$) which associates to $f \in \mathrm{Hol}_c(D,D)$ its unique fixed point $\tau(f) \in D$ is holomorphic and its differential is given by $d\tau_f(v) = (Id - df_{\tau(f)})^{-1} v(\tau(f))$ for $v \in H^\infty_n(D)$.

1. Introduction

Let $\Delta : \{z \in \mathbb{C} : |z| < 1\}$. By the Schwarz Lemma and the Wolff-Denjoy Theorem (see, *e.g.*, [Ab]) for any $f \in \mathrm{Hol}(\Delta,\Delta)$, $f \neq Id$, there exists a distinguished point $\tau(f) \in \overline{\Delta}$ such that the sequence of iterates of f converges uniformly on compacta to the constant map $z \mapsto \tau(f)$. If $\tau(f) \in \Delta$ then $\tau(f)$ is the only fixed point of f (in Δ). In 1941 M. H. Heins [He] proved that the application—that we call *the Heins map*

$$\tau : \overline{\mathrm{Hol}(\Delta,\Delta)} - \{Id\} \to \overline{\Delta}$$

which maps $f \in \mathrm{Hol}(\Delta,\Delta)$ to $\tau(f)$ and $z \mapsto c \in \partial\Delta$ to c, is continuous when $\mathrm{Hol}(\Delta,\Delta)$ is endowed with the topology of the uniform convergence on compacta (which turns out to be equivalent to the topology induced by $H^\infty(\Delta)$). In 2000 J. E. Joseph and M. H. Kwack [JK] generalized Heins' result to holomorphic self-maps of bounded strongly convex domains with smooth boundary (see also [Br]). In this paper we deal with other properties of regularity of the Heins map (in the unit disc) and its generalization to (not necessarily strongly) convex domains in $\mathbb{C}^n$. Let $D \subset\subset \mathbb{C}^n$ be a convex domain. Let $H^\infty(D)$ be the Banach space of bounded holomorphic function on D (with values in $\mathbb{C}$) and norm given by the supremum norm. We denote by $H^\infty_n(D) := \oplus_n H^\infty(D)$ the Banach space of bounded holomorphic maps from D to $\mathbb{C}^n$. The set

$$\mathrm{Hol}_c(D,D) := \{f \in H^\infty_n(D) : f(D) \subset\subset D\}$$

1991 *Mathematics Subject Classification.* Primary 32H99; secondary30F99, 32H15.
Key words and phrases. fixed points; holomorphic self-maps; Hardy spaces.
† Partially supported by Progetto MURST di Rilevante Interesse Nazionale *Proprietà geometriche delle varietà reali e complesse* and GNSAGA.

is a open set in $H_n^\infty(D)$, dense in the closed set $\overline{\mathrm{Hol}(D,D)}$. A result of J.-P. Vigué [Vi] implies that if $f \in \mathrm{Hol}_c(D,D)$ then there exists a unique point $\tau(f) \in D$ such that $f(\tau(f)) = \tau(f)$. The main result of this paper is the following:

Theorem 1.1. *The map $\tau : Hol_c(D,D) \to D$, defined as $\tau : f \to \tau(f)$, is holomorphic. The differential of τ at $f \in Hol_c(D,D)$ for $v \in H_n^\infty(D)$ is given by*

$$d\tau_f(v) = (Id - df_{\tau(f)})^{-1} v(\tau(f)).$$

Note that $\|df_{\tau(f)}\| < 1$ as a consequence of the *Cartan-Carathéodory Theorem* (see, *e.g.*, [Ab]) and therefore $I - df_{\tau(f)}$ is invertible. The proof of Theorem 1.1 is based on the Cauchy formula and on the continuity of the map $\tau : \mathrm{Hol}_c(D,D) \to D$, which we prove for bounded (not necessarily strongly) convex domain.

2. The proof of Theorem 1.1

First we need to prove the following result:

Lemma 2.1. *The map $\tau : Hol_c(D,D) \to D$ is continuous.*

Proof. Let $\{f_k\} \subset \mathrm{Hol}_c(D,D)$ be such that $f_k \to f \in \mathrm{Hol}_c(D,D)$. We have to show that $\tau(f_k) \to \tau(f)$. Let x be an accumulation point of $\{\tau(f_k)\}$. Since the limit f has image relatively compact in D then $x \in D$. Now let $\tau(f_{k_j}) \to x$. Then

$$f(x) = \lim_{j\to\infty} f_{k_j}(\tau(f_{k_j})) = \lim_{j\to\infty} \tau(f_{k_j}) = x.$$

Since by [Vi] the map f has only one fixed point in D then $\tau(f) = x$ and actually $\tau(f_k) \to x$. $\square$

Let $f \in \mathrm{Hol}_c(D,D)$. Now let $h \in H_n^\infty(D)$ be such that $f + h \in \mathrm{Hol}_c(D,D)$. Let us denote by $p := \tau(f) \in D$ and $q := \tau(f+h) \in D$. In what follows if $x \in \mathbb{C}^n$ we write x_j for the j-th coordinate of x. Let P be a polydisc such that $\overline{P} \subset D$ and $p, q \in P$ (such a polydisc exists for $\|h\|_\infty$ small since τ is continuous by Lemma 2.1). Since $f(p) = p$ and $(f+h)(q) = q$ from the Cauchy formula we get

$$q - p = \frac{1}{(2\pi i)^n} \int_{\partial P} \left[\frac{f(\zeta)}{\zeta - q} - \frac{f(\zeta)}{\zeta - p} \right] d\zeta + h(q), \tag{2.1}$$

where, as usual, $(\zeta - x) = \prod_{j=1}^n (\zeta_j - x_j)$, $d\zeta = d\zeta_1 \wedge \dots d\zeta_n$ and the integrals are vector valued. Let us examine the integrand in (2.1):

$$\frac{f(\zeta)}{\zeta - q} - \frac{f(\zeta)}{\zeta - p} = \frac{\left[\prod_{j=1}^n (\zeta_j - p_j) - \prod_{j=1}^n (\zeta_j - q_j)\right] f(\zeta)}{(\zeta - p)(\zeta - q)}.$$

For calculating this last expression we need to define some sets of indices. For $m \in \{0, \dots, n\}$ let $I_m := \{(k_1, \dots, k_m) \in \mathbb{N}^m | 1 \le k_1 < \dots < k_m \le n\}$. If $\mathbf{k} \in I_m$ we say $j \notin \mathbf{k}$ if $\mathbf{k} = (k_1, \dots, k_m)$ and $j \neq k_l$ for $l = 1, \dots, m$. For $\mathbf{k} \in I_m$ we let $I'_{\mathbf{k}} := \{l_1, \dots, l_{n-m} | l_j \notin \mathbf{k}\}$. Moreover we write $\zeta_{\mathbf{k}} := \zeta_{k_1} \cdots \zeta_{k_m}$. With this notation

$$\prod_{j=1}^{n}(\zeta_j-p_j)-\prod_{j=1}^{n}(\zeta_j-q_j)=\sum_{m=0}^{n}(-1)^{n-m}\sum_{\mathbf{k}\in I_m}\zeta_{\mathbf{k}}\Big(\prod_{j\in I'_{\mathbf{k}}}p_j-\prod_{j\in I'_{\mathbf{k}}}q_j\Big) \tag{2.2}$$

$$=\sum_{m=0}^{n}(-1)^{n-m}\sum_{\mathbf{k}\in I_m}\zeta_{\mathbf{k}}\left(\sum_{r\in I'_{\mathbf{k}}}(p_r-q_r)\prod_{s\in I'_{\mathbf{k}},s>r}p_s\prod_{t\in I'_{\mathbf{k}},t<r}q_t\right)$$

$$=\sum_{\mathbf{k}\in I_{n-1},j\notin I_{\mathbf{k}}}\zeta_{\mathbf{k}}(q_j-p_j)+\sum_{m=0}^{n-2}(-1)^{n-m}\sum_{\mathbf{k}\in I_m}\zeta_{\mathbf{k}}\left(\sum_{r\in I'_{\mathbf{k}}}(p_r-q_r)\prod_{s\in I'_{\mathbf{k}},s>r}p_s\prod_{t\in I'_{\mathbf{k}},t<r}q_t\right)$$

$$=\sum_{l=1}^{n}\zeta_1\cdots\widehat{\zeta_l}\cdots\zeta_n(q_l-p_l)+\sum_{l=1}^{n}\tilde{C}_l^h(q_l-p_l),$$

where we set

$$\tilde{C}_l^h=\sum_{m=0}^{n-2}(-1)^{n-m+1}\sum_{\mathbf{k}\in I_m:l\notin\mathbf{k}}\zeta_{\mathbf{k}}\prod_{r\in I'_{\mathbf{k}},r<l}p_r\prod_{s\in I'_{\mathbf{k}},s>l}q_s.$$

Therefore for $j=1,\ldots,n$

$$q_j-p_j=\sum_{l=1}^{n}B_{l,j}^h(q_l-p_l)+\sum_{l=1}^{n}C_{l,j}^h(q_l-p_l)+h_j(q),$$

where

$$B_{l,j}^h:=\frac{1}{(2\pi i)^n}\int_{\partial P}\frac{\zeta_1\cdots\widehat{\zeta_l}\cdots\zeta_n f_j(\zeta)d\zeta}{(\zeta-p)(\zeta-q)},$$

and

$$C_{l,j}^h:=\frac{1}{(2\pi i)^n}\int_{\partial P}\frac{\tilde{C}_l^h f_j(\zeta)d\zeta}{(\zeta-p)(\zeta-q)}.$$

Note that the $C_{l,j}^h$'s are sums of terms of the type

$$p_{j_1}\ldots p_{j_s}q_{l_1}\ldots q_{l_t}\frac{1}{2\pi i}\int_{\partial P}\frac{\zeta_{r_1}\ldots\zeta_{r_u}f_j(\zeta)d\zeta}{(\zeta-p)(\zeta-q)}.$$

Therefore by Lemma 2.1 and by the Cauchy formula for $\|h\|_\infty\to 0$ the $B_{l,j}^h$'s and $C_{l,j}^h$'s tend to sums of terms of the form

$$p_{j_1}\ldots p_{j_s}\frac{\partial^m f_j(p)}{\partial\zeta_{k_1}\ldots\partial\zeta_{k_r}}.$$

Moreover a careful hunting from the very definition of $B_{l,j}^h$ and $C_{l,j}^h$ shows that for $\|h\|_\infty\to 0$

$$B_{l,j}^h+C_{l,j}^h\to\frac{\partial f_j(p)}{\partial\zeta_l}.$$

Let $A_{l,j}^h:=B_{l,j}^h+C_{l,j}^h$ and let A^h be the $n\times n$ matrix with entries given by $A_{l,j}^h$. Then

$$(Id-A^h)(q-p)=h(q). \tag{2.3}$$

Moreover, since $A^h\to df_p$ as $\|h\|_\infty\to 0$ and $\|df_p\|<1$ by the Cartan-Carathéodory Theorem (see, *e.g.*, [Ab]), for $\|h\|_\infty$ small $(Id-A^h)$ is invertible. Hence

$$(q-p)=(Id-A^h)^{-1}h(q).$$

Therefore

$$\lim_{\|h\|_\infty \to 0} \frac{\|\tau(f+h)-\tau(f)\|}{\|h\|_\infty} = C < \infty. \tag{2.4}$$

It is well known that in order to prove that τ is holomorphic on $\mathrm{Hol}_c(D,D)$ we need only to show that it is complex differentiable at each point. For this—and to get the claimed form for $d\tau$—we need to show that

$$\lim_{\|h\|_\infty \to 0} \frac{\|\tau(f+h)-\tau(f)-(Id-df_{\tau(f)})^{-1}h(\tau(f))\|}{\|h\|_\infty} = 0.$$

Since $Id - df_{\tau(f)}$ is injective, this is clearly equivalent to

$$\lim_{\|h\|_\infty \to 0} \frac{\|(Id-df_{\tau(f)})(\tau(f+h)\tau(f))-h(\tau(f))\|}{\|h\|_\infty} = 0.$$

By (2.3) we get

$$\begin{aligned}
&\frac{\|(Id-df_{\tau(f)})(\tau(f+h)-\tau(f))-h(\tau(f))\|}{\|h\|_\infty}\\
&\quad= \frac{\|[(A^h-df_{\tau(f)})+(Id-A^h)](\tau(f+h)-\tau(f))-h(\tau(f))\|}{\|h\|_\infty}\\
&\quad\leq \|A^h-df_{\tau(f)}\|\frac{\|\tau(f+h)-\tau(f))\|}{\|h\|_\infty}+\frac{\|h(\tau(f+h))-h(\tau(f))\|}{\|h\|_\infty}.
\end{aligned}$$

By (2.4) and since $A^h \to df_{\tau(f)}$ it follows that the first summand in the last line of the previous equation goes to zero as $\|h\|_\infty \to 0$. It remains to show that

$$\|h(\tau(f+h))-h(\tau(f))\| = o(\|h\|_\infty). \tag{2.5}$$

Again by the Cauchy formula, and arguing as in (2.2), we have

$$\begin{aligned}
\|h(\tau(f+h))-h(\tau(f))\| &\leq \frac{\|h\|_\infty}{2\pi}\int_{\partial P}\left|\frac{1}{(\zeta-\tau(f))}-\frac{1}{(\zeta-\tau(f+h))}\right| d|\zeta|\\
&= \|h\|_\infty \sum_{l=1}^n |\tau(f+h)_l-\tau(f)_l|C_l \leq \|h\|_\infty\|\tau(f+h)-\tau(f)\|C,
\end{aligned}$$

with C_l's, C bounded for $\|h\|_\infty \to 0$. By Lemma 2.1 this proves (2.5) and we are done.

References

[Ab] M. Abate *Iteration Theory of holomorphic maps on taut manifolds.* Mediterranean Press, Rende 1989.

[Br] F. Bracci *Dilatation and order of contact for holomorphic self-maps of strongly convex domains.* Preprint 2000.

[He] M. H. Heins *On the iteration of functions which are analytic and single-valued in a given multiply-connected region.* Amer. J. Math. 63 (1941), 461-480.

[JK] J. E. Joseph, M. H. Kwack *A generalization of a theorem of Heins.* Proc. Amer. Math. Soc. 128,6, (2000), 1697-1701.

[Vi] J.-P. Vigué *Points fixes d'applications holomorphes dans un domaine borné convexe de $\mathbb{C}^n$.* Trans. Amer. Math. Soc. 289 (1985), 345-353.

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica 00133 Roma, Italy.

E-mail address: `fbracci@mat.uniroma2.it`