INFINITESIMAL GENERATORS ASSOCIATED WITH SEMIGROUPS OF LINEAR FRACTIONAL MAPS

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ABSTRACT. We characterize the infinitesimal generator of a semigroup of linear fractional self-maps of the unit ball in $\mathbb{C}^n$, $n \geq 1$. For the case $n = 1$ we also completely describe the associated Koenigs function and we solve the embedding problem from a dynamical point of view, proving, among other things, that a generic semigroup of holomorphic self-maps of the unit disc is a semigroup of linear fractional maps if and only if it contains a linear fractional map for some positive time.

INTRODUCTION

Let U be an open set in $\mathbb{C}^n$, $n \geq 1$. A continuous semigroup (φ_t) of holomorphic functions in U is a continuous homomorphism from the additive semigroup of non-negative real numbers into the composition semigroup of all holomorphic self-maps of U endowed with the compact-open topology. In other words, the map $[0, +\infty) \ni t \mapsto (\varphi_t) \in \text{Hol}(U, U)$ satisfies the following three conditions:

(1) φ_0 is the identity map id_U in U,
(2) $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
(3) φ_t tends to id_U as t tends to 0 uniformly on compacta of U.

Any element of the above family (φ_t) is called an *iterate* of the semigroup. If the morphism $[0, +\infty) \ni t \mapsto \varphi_t \in \text{Hol}(U, U)$ can be extended linearly to $\mathbb{R}$ (and then necessarily each φ_t is invertible) we have a group of automorphisms of U. It is well known (see, *e.g.*, [1] or [28]) that every fractional iterate of a semigroup is injective and that if for some $t_0 > 0$ the iterate $\varphi_{t_0} \in \mathsf{Aut}(U)$ then $\varphi_t \in \mathsf{Aut}(U)$ for all $t > 0$ and the semigroup is indeed a group of automorphisms of U.

The theory of semigroups in the unit disc $\mathbb{D}$ has been deeply studied and applied in many different contexts. We refer the reader to the excellent monograph by Shoikhet [28] and references therein for more information about this.

For several and natural reasons, those semigroups in $\mathbb{D}$ whose iterates belong to some concrete class of holomorphic functions relatively easy to handle are of special interest. Undoubtedly, the most important example of this kind is that of semigroups of linear fractional self-maps of $\mathbb{D}$ (shortly, LFM$(\mathbb{D}, \mathbb{D})$), namely, those semigroups (φ_t) in $\mathbb{D}$ for which every iterate $\varphi_t \in$ LFM$(\mathbb{D}, \mathbb{D})$. Among those semigroups of linear fractional maps, there are the groups of automorphisms of $\mathbb{D}$.

Semigroups in $\mathbb{D}$ can be first classified looking at their common Denjoy-Wolff fixed point. It can be proved (see [29], [30], [13]) that any semigroup (φ_t) in $\mathbb{D}$ belongs to one and only one of the following five classes :

(1) *trivial-elliptic*: all the iterates are the identity $\mathsf{id}_{\mathbb{D}}$ map.
(2) *Neutral-elliptic*: there exists $\tau \in \mathbb{D}$ with $\varphi_t(\tau) = \tau$ and $|\varphi_t'(\tau)| = 1$ for every $t > 0$ and $\varphi_t \neq \mathsf{id}_{\mathbb{D}}$ for some $t > 0$.
(3) *Attractive-elliptic*: there exists $\tau \in \mathbb{D}$ with $\varphi_t(\tau) = \tau$ and $|\varphi_t'(\tau)| < 1$, for every $t > 0$.
(4) *Hyperbolic*: there exists $\tau \in \partial\mathbb{D}$ with $\lim_{r \to 1^-} \varphi_t(r\tau) = \tau$ and $\lim_{r \to 1^-} \varphi_t'(r\tau) < 1$, for every $t > 0$.
(5) *Parabolic*: there exists $\tau \in \partial\mathbb{D}$ with $\lim_{r \to 1^-} \varphi_t(r\tau) = \tau$ and $\lim_{r \to 1^-} \varphi_t'(r\tau) = 1$, for every $t > 0$.

In cases (2) to (5), the point τ is unique and it is called the *Denjoy-Wolff point* of the semigroup. According to the Julia-Wolff-Carathéodory theorem (see, *e.g.* [1], [27]), in cases (4) and (5) all iterates have non-tangential (or angular) limit τ at τ and their first derivatives have a non-tangential limit at τ given by a real number in $(0, 1]$. As customary, we call *elliptic semigroup* any semigroup in the classes (1) to (3) and *non-elliptic* the semigroups in the classes (4) and (5). An analogous classification is available for semigroups of $\mathbb{B}^n$, the unit ball of $\mathbb{C}^n$ (see, *e.g.*, [1] or [11] or Section two).

To any semigroup in $\mathbb{D}$ there are attached two analytic objects which can be used to describe the dynamical behavior: the infinitesimal generator and the Koenigs map. We are going to quickly recall how they are defined.

Given a semigroup (φ_t) in $\mathbb{D}$, it can be proved (see [28], [7]) that there exists a unique holomorphic function $G : \mathbb{D} \to \mathbb{C}$ such that, for each $z \in \mathbb{D}$, the trajectory

$$\gamma_z : [0, +\infty) \to \mathbb{D}, \qquad t \mapsto \gamma_z(t) := \varphi_t(z)$$

is the solution of the Cauchy problem

$$\begin{cases} \dot{w} = G(w) \\ w(0) = z. \end{cases}$$

Moreover, $G(z) = \lim_{t \to 0^+} (\varphi_t(z) - z)/t$, for every $z \in \mathbb{D}$. The function G is called the *infinitesimal generator* (or the *semi-complete vector field*) of (φ_t). There is a very nice

representation, due to Berkson and Porta [7], of those holomorphic functions of the disc which are infinitesimal generators. Namely:

Theorem 0.1 (Berkson-Porta)**.** *A holomorphic function $G : \mathbb{D} \to \mathbb{C}$ is the infinitesimal generator of a semigroup in $\mathbb{D}$ if and only if there exists a point $b \in \overline{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \to \mathbb{C}$ with $\operatorname{Re} p \geq 0$ such that*

$$G(z) = (z - b)(\bar{b}z - 1)p(z), \quad z \in \mathbb{D}.$$

The point b in Berkson-Porta's theorem is exactly the Denjoy-Wolff point of the semigroup, unless the semigroup is trivial. Other alternative descriptions of infinitesimal generators can be found in [28, Section 3.6] (where a different sign convention is chosen).

As for the Koenigs function, if (φ_t) is a semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \mathbb{D}$ and infinitesimal generator G, then there exists a unique univalent function $h \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ such that $h(\tau) = 0$, $h'(\tau) = 1$ and, for every $t \geq 0$,

$$h \circ \varphi_t(z) = \frac{d\varphi_t}{dz}(\tau)h(z) = e^{G'(\tau)t}h(z).$$

While, if (φ_t) is a semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$, then there exists a unique univalent function $h \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ such that $h(0) = 0$ and, for every $t \geq 0$,

$$h \circ \varphi_t(z) = h(z) + t.$$

In both cases, the function h is called the *Koenigs function* of the semigroup (φ_t).

Infinitesimal generators can be defined also in several variables (see [1], [2], and [28]) even if their characterizations are not so easy to handle as in the one dimensional case. However the construction of Koenigs' functions in several variables is still at a pioneeristic step and it has been successfully developed only for linear fractional semigroups (see [11] and [17] for a different construction but only for a single linear fractional map of $\mathbb{B}^n$) and, in the realm of discrete iteration, for regular hyperbolic self-maps of $\mathbb{B}^n$ (see [10]). See also [18], where C. de Fabritiis characterized infinitesimal generators of a "one dimensional" type of semigroups of the ball using the Koenigs function.

A related problem is that of embedding a given holomorphic self-map of a domain $U \in \mathbb{C}^n$ into a semigroup of holomorphic self-maps of U. Such a problem has been studied since long, see [28] for a good account of available results.

Semigroups of $\mathsf{Aut}(\mathbb{D})$ are quite well-understood (see [8]) in terms of infinitesimal generators and Koenigs functions. Moreover, the embedding problem is completely solved: every automorphism of the unit disc can be embedded in a suitable (semi)group of $\mathsf{Aut}(\mathbb{D})$.

However, strange as it may seem, the situation for semigroups of $\operatorname{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ (even for $n = 1$) is still not completely clear. Despite the fact that convergence questions are basically understood, there is not a full description of their basic theoretical elements such as infinitesimal generators and Koenigs functions. The corresponding embedding problem for $n = 1$ has been treated in the literature and it is known that, in general, the answer is negative. In most cases some analytic criteria for deciding the solvability of this problem

are available (see [28, Sections 4.3 and 5.9]) but the problem in its complete generality seems to be still open.

In this paper, we consider those problems for semigroups of $\mathrm{LFM}(\mathbb{B}^n, \mathbb{B}^n)$, especially for $n = 1$. We completely characterize infinitesimal generators of semigroups of linear fractional self-maps of the ball (in one and several variables). In particular we prove the following result:

Theorem 0.2. *Let (φ_t) be a semigroup of holomorphic self-maps of $\mathbb{B}^n$. Then (φ_t) is a semigroup of linear fractional maps if and only if there exist $a, b \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$ (not all of them zero) such that the infinitesimal generator of (φ_t) is*

$$G(z) = a - \langle z, a \rangle z - [Az + \langle z, b \rangle z]$$

with $|\langle b, u \rangle| \leq \operatorname{Re}\langle Au, u \rangle$, for all $u \in \partial \mathbb{B}^n$.

For the case $n = 1$, in Section two we present a more precise statement classifying all possible cases (see Propositions 2.4 and 2.5), and we present a precise description of Koening maps for linear fractional semigroups (see Proposition 2.8). Finally, in the third section we deal with the embedding problem. First we prove the following rigidity result:

Theorem 0.3. *Let (φ_t) be a semigroup in $\mathbb{D}$. If for some $t_0 > 0$ the iterate φ_{t_0} is a linear fractional self-map of $\mathbb{D}$ then φ_t is a linear fractional self-map of $\mathbb{D}$ for all $t \geq 0$.*

Then we settle the embedding problem for a linear fractional self-map of $\mathbb{D}$ proving that it can be embedded in a semigroup of $\mathbb{D}$ if and only if it can be embedded into a semigroup of linear fractional self-maps of $\mathbb{D}$ if and only if it can be embedded into a group of Möbius transformations of the Riemann sphere $\mathbb{C}_\infty$ which for $t \geq 0$ preserves the unit disc (see Theorem 3.3). Finally, we give a simple criterium for embeddability of linear fractional self map of $\mathbb{D}$ (see Theorem 3.4).

1. Infinitesimal Generators in Several Variables

Following [16], we say that a map $\varphi : \mathbb{B}^n \to \mathbb{C}^n$ is a *linear fractional map* if there exist a complex $n \times n$ matrix $A \in \mathbb{C}^{n \times n}$, two column vectors B and C in $\mathbb{C}^n$, and a complex number $D \in \mathbb{C}$ satisfying

$$\text{(i) } |D| > \|C\|; \quad \text{(ii) } DA \neq BC^*,$$

such that

$$\varphi(z) = \frac{Az + B}{\langle z, C \rangle + D}, \qquad z \in \mathbb{B}^n.$$

Condition (i) implies that $\langle z, C \rangle + D \neq 0$ for every $z \in \mathbb{B}^n$ and therefore, φ is actually holomorphic in a neighborhood of the closed ball. In fact, $\varphi \in \text{Hol}(r\mathbb{B}^n; \mathbb{C}^n)$ for some $r > 1$. On the other hand, condition (ii) just says that φ is not constant. If the image $\varphi(\mathbb{B}^n) \subset \mathbb{B}^n$, then we say that φ is a linear fractional self-map of $\mathbb{B}^n$ and write $\varphi \in \text{LFM}(\mathbb{B}^n, \mathbb{B}^n)$.

It is worth recalling that if $\varphi \in \text{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ has no fixed points in $\mathbb{B}^n$, then there exists a unique point $\tau \in \partial\mathbb{B}^n$ such that $\varphi(\tau) = \tau$ and $\langle d\varphi_\tau(\tau), \tau \rangle = \alpha(\varphi)$ with $0 < \alpha(\varphi) \leq 1$ (see, e.g., [9, Theorem 2.2]). We call τ the *Denjoy-Wolff point* of φ and $\alpha(\varphi)$ the *boundary dilatation coefficient* of φ.

A semigroup in $\text{Hol}(\mathbb{B}^n; \mathbb{B}^n)$ is a *semigroup of linear fractional maps* if $\varphi_t \in \text{LFM}(\mathbb{B}^n, \mathbb{B}^n)$ for all $t \geq 0$.

Likewise the one-dimensional case, given a semigroup (φ_t) in $\text{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ there exists a holomorphic map $G \in \text{Hol}(\mathbb{B}^n; \mathbb{C}^n)$, the *infinitesimal generator* of the semigroup, such that

$$\frac{\partial\varphi_t}{\partial t} = G \circ \varphi_t$$

for all $t \geq 0$ (see, e.g., [1, Proposition 2.5.22]).

The following result, essentially due to Abate [1] (see also [4] and [11]) allows to talk about elliptic, hyperbolic, and parabolic semigroups in $\mathbb{B}^n$:

Theorem 1.1. *Let (φ_t) be a semigroup in $\text{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Then, either all the iterates have a common fixed point in $\mathbb{B}^n$ or all the iterates φ_t ($t > 0$) have no fixed points in $\mathbb{B}^n$ and then they share the same Denjoy-Wolff point $\tau \in \partial\mathbb{B}^n$. In this case, there exists $0 < r \leq 1$ such that $\alpha_t = r^t$, where $\alpha_t := \alpha(\varphi_t)$ denotes the boundary dilatation coefficient of φ_t (for $t > 0$) at τ.*

A detail study of semigroups of linear fractional maps in $\text{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ can be found in [11]. In this section, we present a characterization of holomorphic functions $G : \mathbb{B}^n \to \mathbb{C}^n$ which are infinitesimal generators of semigroups of linear fractional maps. Before that, we need the following technical lemma.

Lemma 1.2. *Let (φ_t) be a semigroup of holomorphic self-maps of $\mathbb{B}^n$ with associated infinitesimal generator G. Assume that $e_1 \in \partial\mathbb{B}^n$ is the Denjoy-Wolff point of (φ_t) and that G extend C^1 in a open neighborhood of e_1. Then*

(1) $G(e_1) = 0$.
(2) $\langle dG_{e_1}(e_1), e_1 \rangle \in \mathbb{R}$.
(3) $\langle dG_{e_1}(e_j), e_1 \rangle = 0$ *for all $j = 2, \ldots, n$.*

Proof. ¿From $G(\varphi_t(z)) = \frac{\partial}{\partial t}\varphi_t(z)$ it follows that φ_t are C^1 in a neighborhood of e_1 as well, and then (1) follows. Also for any $v \in \mathbb{C}^n$, we have $\langle dG_{e_1}(d(\varphi_t)_{e_1})v, e_1 \rangle = \frac{\partial}{\partial t}\langle d(\varphi_t)_{e_1}v, e_1 \rangle$ and then (2) and (3) follow from Rudin's version of the classical Julia-Wolff-Carathéodory theorem in $\mathbb{B}^n$ applied to φ_t, see [26] or [1]. $\square$

In the proof of Theorem 0.2 we will also make use of the following generalization of Berkson-Porta's criterion due to Aharonov, Elin, Reich and Shoikhet (see [2, Theorem 4.1], where however a different sign convention is chosen because they look at the problem $\frac{\partial \varphi_t}{\partial t} = -G \circ \varphi_t$):

Theorem 1.3. *Let $F \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{C}^n)$. Then F is the infinitesimal generator of a semigroup of holomorphic self-maps of $\mathbb{B}^n$ fixing the origin if and only if $F(z) = -Q(z)z$ where $Q(z)$ is a $(n \times n)$-matrix with holomorphic entries such that*

$$\mathrm{Re}\langle Q(z)z, z\rangle \geq 0,$$

for all $z \in \mathbb{B}^n$.

And now we can prove Theorem 0.2:

Theorem 1.4. *Let (φ_t) be a semigroup of holomorphic self-maps of $\mathbb{B}^n$. Then (φ_t) is a semigroup of linear fractional maps if and only if there exist $a, b \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$ (not all of them zero) such that the infinitesimal generator of (φ_t) is*

$$G(z) = a - \langle z, a\rangle z - [Az + \langle z, b\rangle z]$$

with

$$(1.1) \qquad\qquad\qquad |\langle b, u\rangle| \leq \mathrm{Re}\langle Au, u\rangle,$$

for all $u \in \partial\mathbb{B}^n$.

Proof. Suppose first that (φ_t) is a semigroup of linear fractional maps given by

$$\varphi_t(z) = \frac{A_t z + B_t}{\langle z, C_t\rangle + 1},$$

for some holomorphic functions $t \mapsto A_t \in \mathbb{C}^{n \times n}$, $t \mapsto B_t, C_t \in \mathbb{C}^n$. Differentiating with respect to t at $t = 0$ and denoting by $A = -\frac{\partial A_t}{\partial t}|_{t=0}$, $B = \frac{\partial B_t}{\partial t}|_{t=0}$, $C = -\frac{\partial C_t}{\partial t}|_{t=0}$ we obtain

$$G(z) := \frac{\partial \varphi_t}{\partial t}|_{t=0} = B - \langle z, B\rangle z - [Az + \langle z, C - B\rangle z].$$

Thus the infinitesimal generator G of (φ_t) has the required expression. We only need to verify (1.1). To this aim, if $B \neq 0$ then $H(z) = -B + \langle z, B\rangle z$ is the infinitesimal generator of a group of hyperbolic automorphisms of $\mathbb{B}^n$ (this follows either by a direct simple computation or by applying [2, Theorem 3.1]). Since the set of infinitesimal generators is a (real) cone (see, for example, [1, Corollary 2.5.59]) it follows that $G + H$ is an infinitesimal generator of a semigroup of holomorphic self-maps of $\mathbb{B}^n$. Let $b = C - B$. By Theorem 1.3 applied to $G + H$, we have

$$(1.2) \qquad\qquad\qquad \mathrm{Re}\langle Az, z\rangle + \|z\|^2 \mathrm{Re}\langle z, b\rangle \geq 0,$$

for all $z \in \mathbb{B}^n$. Now, apply (1.2) to $z = e^{i\theta}ru$ where $u \in \partial\mathbb{B}^n$, $r \in (0, 1)$ and $\theta \in \mathbb{R}$. It follows that

$$\mathrm{Re}\langle Au, u\rangle + r \mathrm{Re}\langle e^{i\theta}u, b\rangle \geq 0,$$

for $u \in \partial \mathbb{B}^n$ and all $r \in (0,1)$, $\theta \in \mathbb{R}$. Therefore for a fixed $u \in \partial \mathbb{B}^n$ and $\theta \in \mathbb{R}$, taking the limit for $r \to 1$ we get

$$(1.3) \qquad \operatorname{Re}\langle Au, u \rangle \geq -\operatorname{Re}\langle e^{i\theta} u, b \rangle.$$

Finally, fix $u \in \partial \mathbb{B}^n$ and write $b = \rho e^{i\beta} u + u^\perp$ with $\rho \geq 0$, $\beta \in \mathbb{R}$ and $u^\perp \in \mathbb{C}^n$ so that $\langle u, u^\perp \rangle = 0$. Set $\theta = \beta + \pi$. Then

$$-\operatorname{Re}\langle e^{i\theta} u, b \rangle = -\rho \cos(\theta - \beta) = \rho = |\langle u, b \rangle|,$$

and from (1.3) we obtain (1.1).

Conversely, suppose $G(z) = a - \langle z, a \rangle z - [Az + \langle z, b \rangle z]$ satisfies (1.1). Then, we have that

$$(1.4) \qquad \operatorname{Re}\langle Az, z \rangle + \|z\|^2 \operatorname{Re}\langle z, b \rangle \geq 0,$$

for all $z \in \mathbb{B}^n$. We can write $G(z) = H(z) + P(z)$ with $H(z) = a - \langle z, a \rangle z$ and $P(z) = -[Az + \langle z, b \rangle z]$. As before, one can prove that H is the infinitesimal generator of a (semi)group of holomorphic maps in $\mathbb{B}^n$. By (1.1) and Theorem 1.3 the function $P(z)$ is an infinitesimal generator of a semigroup of holomorphic self-maps of $\mathbb{B}^n$ as well. Therefore G is an infinitesimal generator of a semigroup (φ_t) of holomorphic self-maps of $\mathbb{B}^n$. The point now is to show that such a semigroup is composed by linear fractional maps. There are two cases: either $G(z_0) = O$ for some $z_0 \in \mathbb{B}^n$ or $G(z) \neq O$ for all $z \in \mathbb{B}^n$.

If $G(z_0) = O$ then $\varphi_t(z_0) = z_0$ for all $t \geq 0$. In this case we first rotate conjugating with a unitary matrix U in order to map z_0 to the point re_1 where $r := \|z_0\| < 1$. In terms of infinitesimal generators this amounts to send G to UGU^*, and (1.4) is preserved. Next, we move re_1 to the origin 0 by means of the transform $T \in \mathsf{Aut}(\mathbb{B}^n)$

$$T(\zeta, w) = \frac{(r - \zeta)e_1 - (1 - r^2)^{1/2}(0, w)}{1 - r\zeta}, \quad (\zeta, w) \in \mathbb{C} \times \mathbb{C}^{n-1}.$$

The automorphism T is an involution, that is, $T(T(z)) = z$ for all $z \in \mathbb{B}^n$ (see [1, Section 2.2.1] for this and others properties of this T). The infinitesimal generator G is sent to $\tilde{G}(z) = dT_{T(z)} G(T(z))$. Let $\delta := (1 - r^2)^{1/2}$. A direct computation shows that (with obvious notations)

$$dT_{T(\zeta, w)} G(T(\zeta, w)) = \delta^{-2} \begin{pmatrix} 1 - r\zeta & 0 \\ -rw & \delta\mathsf{Id} \end{pmatrix} A \begin{pmatrix} r - \zeta \\ -\delta w \end{pmatrix}$$

$$+ \frac{\langle (r - \zeta, -\delta w), b \rangle}{1 - r\zeta} \begin{pmatrix} 1 - r\zeta & 0 \\ -rw & -\delta\mathsf{Id} \end{pmatrix} \begin{pmatrix} r - \zeta \\ -\delta w \end{pmatrix}.$$

Then $\tilde{G}$ is of the form $\tilde{G}(z) = -[Mz + \langle z, c \rangle z]$ for some $M \in \mathbb{C}^{n \times n}$ and $c \in \mathbb{C}^n$ and, since it is an infinitesimal generator, again by Theorem 1.3, it satisfies

$$(1.5) \qquad \operatorname{Re}\langle Mz, z \rangle + \|z\|^2 \operatorname{Re}\langle z, c \rangle \geq 0$$

for all $z \in \mathbb{B}^n$. Notice that $-M$ is dissipative because, looking at degrees in z, equation (1.5) implies that $\mathrm{Re}\langle Mz, z\rangle \geq 0$ for all $z \in \mathbb{B}^n$.

Assume first that $-M$ is not asymptotically stable. Thus there exists $v \in \mathbb{C}^n$, $\|v\| = 1$, such that $Mv = i\alpha v$ for some $\alpha \in \mathbb{R}$. Substituting $z = \zeta v$ for $\zeta \in \mathbb{C}$, $|\zeta| < 1$ in (1.5) we obtain that $\langle v, c \rangle = 0$. Now let $z = \zeta v + \epsilon c$ for $|\zeta| < 1$ and $\epsilon \in \mathbb{R}$ small. Substituting this into (1.5) we obtain

$$(1.6) \qquad \epsilon(\mathrm{Re}\langle Mc, \zeta v\rangle + \|\zeta v\|^2 \|c\|^2) \geq O(\epsilon^2),$$

where $O(\epsilon^2)$ is the Landau symbol for denoting a (polynomial in this case) expression divisible by ϵ^2. If $c \neq 0$, we can take ζ of small modulus such that $d := \mathrm{Re}\langle Mc, \zeta v\rangle + \|\zeta v\|^2 \|c\|^2 \neq 0$. But then, taking $|\epsilon| << 1$ such that $\epsilon \cdot d < 0$ we contradict (1.6). Therefore if $-M$ is not asymptotically stable then $c = 0$. Thus $\varphi_t(z) = e^{-tM}z$ is a semigroup of linear fractional maps from $\mathbb{B}^n$ into $\mathbb{B}^n$ (because $-M$ is dissipative) whose infinitesimal generator is $\tilde{G}$. Hence, also G is an infinitesimal generator of a semigroup of linear fractional self-maps of $\mathbb{B}^n$.

Next, we assume $-M$ is asymptotically stable, that is, all its eigenvalues have negative real part. In particular M and M^* are invertible and there exists a unique $v \in \mathbb{C}^n$ such that $-M^*v = c$. Let $A_t := \exp(-tM)$ and $c_t := (\exp(-tM^*) - I)v$. Notice that both A_t and c_t are defined for all $t \in [0, +\infty)$. Moreover set $\tilde{\varphi}_t(z) = (A_t z)/(\langle z, c_t\rangle + 1)$. A priori since $\|c_t\|$ can be strictly greater than 1 for some $t > 0$, $\tilde{\varphi}_t$ might not be defined in all $\mathbb{B}^n$. However, once fixed $t_0 \in (0, +\infty)$, there exists $r = r(t_0) > 0$ and $\epsilon = \epsilon(t_0) > 0$ such that for all $z \in \mathbb{B}^n_r := \{z \in \mathbb{B}^n : \|z\| < r\}$ and $t \in S(t_0, \epsilon) := \{t \in [0, +\infty) : |t - t_0| < \epsilon\}$, the map $\tilde{\varphi}_t$ is well defined in $\mathbb{B}^n_r \times S(t_0, \epsilon)$, holomorphic in the first variable and holomorphic in the second variable. A direct computation shows that $\tilde{\varphi}_t(z)$ satisfies

$$\frac{\partial}{\partial t}\tilde{\varphi}_t(z) = \tilde{G}(\tilde{\varphi}_t(z)), \quad \tilde{\varphi}_0(z) = z$$

in $\mathbb{B}^n_r \times S(t_0, \epsilon)$. Therefore, by uniqueness of Cauchy-type problem (see, e.g., [22]), if ψ_t is the semigroup of holomorphic self-maps of $\mathbb{B}^n$ whose infinitesimal generator is $\tilde{G}$ it follows that $\psi_t(z) \equiv \tilde{\varphi}_t(z)$ in $\mathbb{B}^n_r \times S(t_0, \epsilon)$. By holomorphic continuation for any fixed $t \in S(t_0, \epsilon)$ it follows that $\psi_t(z) = \tilde{\varphi}_t(z)$ for all $z \in \mathbb{B}^n$. Since ψ_t sends the unit ball into itself, we have that $\|c_t\| < 1$ for all $t > 0$, proving that $\psi_t = \tilde{\varphi}_t$ is actually a linear fractional self-map of $\mathbb{B}^n$. By the arbitrariety of t_0 we find that ψ_t is a linear fractional semigroup of $\mathbb{B}^n$ and then $\tilde{G}$—and hence G—is the infinitesimal generator of a semigroup of linear fractional self-maps of $\mathbb{B}^n$, as wanted.

Now we are left with the case $G(z) \neq O$ for all $z \in \mathbb{B}^n$. In this case φ_t has no fixed points in $\mathbb{B}^n$ for all $t > 0$ and there exists a unique common Denjoy-Wolff point that, up to rotations, we may assume to be e_1. We transfer our considerations to the Siegel half-plane $\mathbb{H}^n := \{(\zeta, w) \in \mathbb{C} \times \mathbb{C}^n : \mathrm{Re}\,\zeta > \|w\|^2\}$ by means of the Cayley transform $C : \mathbb{B}^n \to \mathbb{H}^n$ given by $C(z_1, z'')) = (\frac{1+z_1}{1-z_1}, \frac{z''}{1-z_1})$. The infinitesimal generator G is mapped to $dC_{C^{-1}(\zeta,w)}(G(C^{-1}(\zeta, w))$. For a vector $v \in \mathbb{C}^n$ we will write $v = (v_1, v'') \in \mathbb{C} \times \mathbb{C}^{n-1}$.

A direct computation shows that

$$dC_{C^{-1}(\zeta,w)}(G(C^{-1}(\zeta,w))) = \frac{\zeta+1}{2}\begin{pmatrix} \zeta+1 & 0 \\ w & \mathsf{Id} \end{pmatrix}\begin{pmatrix} a_1 \\ a'' \end{pmatrix}$$

$$-\frac{\left\langle \begin{pmatrix} \zeta-1 \\ 2w \end{pmatrix}, a\right\rangle}{2(\zeta+1)}\begin{pmatrix} \zeta+1 & 0 \\ w & \mathsf{Id} \end{pmatrix}\begin{pmatrix} \zeta-1 \\ 2w \end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix} \zeta+1 & 0 \\ w & \mathsf{Id} \end{pmatrix} A \begin{pmatrix} \zeta-1 \\ 2w \end{pmatrix}$$

$$-\frac{\left\langle \begin{pmatrix} \zeta-1 \\ 2w \end{pmatrix}, b\right\rangle}{2(\zeta+1)}\begin{pmatrix} \zeta+1 & 0 \\ w & \mathsf{Id} \end{pmatrix}\begin{pmatrix} \zeta-1 \\ 2w \end{pmatrix}.$$

Therefore $dC_{C^{-1}(\zeta,w)}(G(C^{-1}(\zeta,w)))$ is of the form $N + Mz + Q\langle z,z\rangle$ with $N \in \mathbb{C}^n$, $M \in \mathbb{C}^{n\times n}$ and Q a quadratic form on $\mathbb{C}^n \times \mathbb{C}^n$. We examine the quadratic terms. As a matter of notation, we write $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, for $A_{11} \in \mathbb{C}$, $A_{12}^t, A_{21} \in \mathbb{C}^{n-1}$ and $A_{22} \in \mathbb{C}^{(n-1)\times(n-1)}$. Moreover, if v,w are vector in $\mathbb{C}^m$ for $m \geq 1$ we write $vw := \sum_{j=1}^m v_j w_j$. Thus we have

$$Q((\zeta,w),(\zeta,w)) = \frac{a_1\zeta}{2}\begin{pmatrix} \zeta \\ w \end{pmatrix} - \frac{\overline{a_1}\zeta + 2w\overline{a''}}{2}\begin{pmatrix} \zeta \\ w \end{pmatrix} - \frac{1}{2}\begin{pmatrix} A_{11}\zeta^2 + 2\zeta wA_{12} \\ A_{11}\zeta w + 2(wA_{12})w \end{pmatrix}$$

$$-\frac{\overline{b_1}\zeta + 2w\overline{b''}}{2}\begin{pmatrix} \zeta \\ w \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\zeta^2}{2}(a_1 - \overline{a_1} - A_{11} - \overline{b_1}) - \zeta w(\overline{a''} + A_{12} + \overline{b''}) \\ \frac{\zeta w}{2}(a_1 - \overline{a_1} - A_{11} - \overline{b_1}) - [w(\overline{a''} + A_{12} + \overline{b''})]w \end{pmatrix}.$$

By Lemma 1.2.(1) applied to $G(z)$ we obtain the following equality:

$$a_1 - \overline{a_1} - A_{11} - \overline{b_1} = 0,$$

while, applying Lemma 1.2.(3) to $G(z)$ we obtain

$$\overline{a''} + A_{12} + \overline{b''} = O,$$

and therefore $Q \equiv 0$.

Thus $\tilde{G}(z) = Mz + N$ for some $n \times n$ matrix and some vector $N \in \mathbb{C}^n$. Let ψ_t be the semigroup of holomorphic self-maps of $\mathbb{H}^n$ of whom $\tilde{G}$ is the infinitesimal generator. Let

$$\tilde{\varphi}_t(z) := e^{tM}z + \left(\int_0^t e^{sM}ds\right)N.$$

Then $\tilde{\varphi}_t$ is defined for all $z \in \mathbb{C}^n$ and all $t \in [0, +\infty)$. A direct computation shows that

$$\frac{\partial}{\partial t}\tilde{\varphi}_t(z) = \tilde{G}(\tilde{\varphi}_t(z))$$

for all z and t. Thus by the uniqueness of solution of partial differential equations, it follows that $\tilde{\varphi}_t = \psi_t$ for all t and in particular the semigroup associated to $\tilde{G}$ is linear, which, going back to the ball, means that G is the infinitesimal generator of a linear fractional semigroup of $\mathbb{B}^n$ as wanted. $\square$

As a corollary we have the following result of which we give a proof just for the sake of completeness (for other proofs see [5], [19], [32] and see also [3] for a slightly improved result).

Corollary 1.5. *Let (φ_t) be a semigroup of holomorphic self-maps of $\mathbb{B}^n$ with infinitesimal generator $G \in Hol(\mathbb{B}^n, \mathbb{C}^n)$. The following are equivalent.*

   (1) *(φ_t) is a group of holomorphic self-maps of $\mathbb{B}^n$.*
   (2) *There exist $a \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n \times n}$ (not all of them zero) such that $G(z) = a - \langle z, a \rangle z - Az$ with*

$$(1.7) \qquad \qquad \text{Re}\langle Az, z \rangle = 0,$$

   for all $z \in \mathbb{C}^n$.

Proof. If (φ_t) is a group of holomorphic self-maps of $\mathbb{B}^n$, then the family of functions $(\psi_t) := (\varphi_{-t})$ is a semigroup of linear fractional maps of $\mathbb{B}^n$ and its infinitesimal generator is $-G$. Therefore, bearing in mind that G and $-G$ are infinitesimal generator of linear fractional maps, the above theorem implies that we have

$$(1.8) \qquad \qquad \text{Re}\langle Au, u \rangle = 0$$

for all u and (1.1) implies that $b = 0$. Conversely, if $G(z) = a - \langle z, a \rangle z - Az$ with $\text{Re}\langle Az, z \rangle = 0$ for all $z \in \mathbb{C}^n$, the above theorem shows that both G and $-G$ are infinitesimal generators and the Cauchy problems

$$\frac{\partial g}{\partial t} = \pm G \circ g, \quad \text{with } g(0) = z$$

have solutions in $[0, +\infty)$ for all $z \in \mathbb{B}^n$. Therefore G is the infinitesimal generator of a group of automorphisms. $\square$

2. INFINITESIMAL GENERATORS IN DIMENSION ONE AND KOENIGS FUNCTIONS

Koenigs functions and infinitesimal generators are related in a very concrete way. An explicit reference for the following result is not really available but essentially the idea of the proof is given in [29] (also, cfr. [21]).

Proposition 2.1. *Let (φ_t) be a non-trivial semigroup in $\mathbb{D}$ with infinitesimal generator G and Koenigs function h.*

   (1) *If (φ_t) has Denjoy-Wolff point $\tau \in \mathbb{D}$ then h is the unique holomorphic function from $\mathbb{D}$ into $\mathbb{C}$ such that*
       (i) *$h'(z) \neq 0$, for every $z \in \mathbb{D}$,*

    (ii) $h(\tau) = 0$ *and* $h'(\tau) = 1$,
    (iii) $h'(z)G(z) = G'(\tau)h(z)$, *for every* $z \in \mathbb{D}$.
  (2) *If* (φ_t) *has Denjoy-Wolff point* $\tau \in \partial\mathbb{D}$ *then* h *is the unique holomorphic function from* $\mathbb{D}$ *into* $\mathbb{C}$ *such that:*
    (i) $h(0) = 0$,
    (ii) $h'(z)G(z) = 1$, *for every* $z \in \mathbb{D}$.

For further reference we state here the following simple fact.

Lemma 2.2. *Let* $p(z) = mz + n$ *be a complex polynomial. Then* $\operatorname{Re} p(z) \geq 0$ *for every* $z \in \mathbb{D}$ *if and only if* $\operatorname{Re}(n) \geq |m|$.

Our first result is a translation to one variable of the result contained in the previous section. This result extends and, somehow, completes [28, Proposition 3.5.1].

Theorem 2.3. *Let* $G : \mathbb{D} \to \mathbb{C}$ *be a holomorphic function. Then, the following are equivalent:*

  (1) *The map* G *is the infinitesimal generator of a semigroup of* $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$.
  (2) *The map* G *is a polynomial of degree at most two and satisfies the following boundary flow condition*

$$\operatorname{Re}(G(z)\overline{z}) \leq 0, \ \text{for all } z \in \partial\mathbb{D}.$$

  (3) *The map* G *is a polynomial of the form* $G(z) = \alpha z^2 + \beta z + \gamma$ *with* $\operatorname{Re}(\beta) + |\alpha + \overline{\gamma}| \leq 0$.
  (4) *The map* G *is a polynomial of the form*

$$G(z) = a - \overline{a}z^2 - z(mz + n), \ z \in \mathbb{D}$$

    with $a, m, n \in \mathbb{C}$ *and* $\operatorname{Re}(n) \geq |m|$.
  (5) *The map* G *is the infinitesimal generator of a semigroup in* $\mathbb{D}$ *and it is a polynomial of degree at most two.*

Moreover, $\operatorname{Re}(n) = m = 0$ *in statement* (4) *if and only if* $\operatorname{Re}(\beta) = |\alpha + \overline{\gamma}| = 0$ *in statement* (3) *if and only if equality holds for all* $z \in \partial\mathbb{D}$ *in statement* (2) *if and only if* G *is the infinitesimal generator of a semigroup of* $\mathsf{Aut}(\mathbb{D})$.

Proof. By Theorem 0.2, statement (1) is equivalent to (4). Statement (4) is then equivalent to (2), (3) and (5) by direct computations using Lemma 2.2. Finally, the last assertion follows from the fact that if G is the infinitesimal generator of a semigroup in $\mathbb{D}$, then this semigroup is composed of automorphisms of $\mathbb{D}$ if and only if $-G$ is the infinitesimal generator of a semigroup in $\mathbb{D}$ as well. $\qquad\square$

This theorem clearly implies that not every polynomial of degree two can be realized as an infinitesimal generator of a semigroup in $\mathbb{D}$. At the same time, it also suggests to analyze carefully those complex polynomials G of degree zero, one, and two which are infinitesimal generators of semigroups in $\mathbb{D}$.

Trivially, if G is of degree zero, then G is an infinitesimal generator of a semigroup in $\mathbb{D}$ if and only if the constant is zero and, in this case, the corresponding semigroup is the trivial one. The next two propositions describe what happens when the degree is one and two.

Proposition 2.4. *Let $G(z) = \lambda(z - c)$, $\lambda \neq 0$, be a complex polynomial of degree one. Then G is the infinitesimal generator of a semigroup in $\mathbb{D}$ (which is necessarily a semigroup of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$) if and only if $\operatorname{Re}\lambda + |\lambda c| \leq 0$. Moreover, if G is an infinitesimal generator the associated semigroup is given by*

$$\varphi_t(z) = e^{\lambda t} z + c(1 - e^{\lambda t}), \qquad z \in \mathbb{D},$$

and $c \in \overline{\mathbb{D}}$ is its Denjoy-Wolff point. Furthermore,

(1) *if $\operatorname{Re}\lambda = 0$ then $c = 0$ and the semigroup is a neutral-elliptic semigroup of $\mathsf{Aut}(\mathbb{D})$.*

(2) *if $0 < |\operatorname{Re}\lambda| < |\lambda|$ then $|c| < 1$, $|\operatorname{Re}\lambda| \geq |\operatorname{Im}\lambda| \dfrac{|c|}{\sqrt{1 - |c|^2}}$ with $\operatorname{Re}\lambda < 0$ and the semigroup is attractive-elliptic.*

(3) *if $|\operatorname{Re}\lambda| = |\lambda|$ then $|c| = 1$, $\lambda \in (-\infty, 0)$ and the semigroup is hyperbolic.*

Proof. Theorem 2.3 implies that $G(z) = \lambda(z - c)$ is an infinitesimal generator of a semigroup in $\mathbb{D}$ (which is necessarily a semigroup of linear fractional maps) if and only if $\operatorname{Re}\lambda + |\lambda c| \leq 0$. Note that the latter inequality implies that $c \in \overline{\mathbb{D}}$. Moreover, $G(c) = 0$ and thus in case G is the infinitesimal generator of a semigroup in $\mathbb{D}$, the point c is the Denjoy-Wolff point of the associated semigroup (this follows for instance from Berkson-Porta's Theorem 0.1 which shows that, unless $G \equiv 0$ then it has a unique zero in $\mathbb{D}$ which is exactly the Wolff-Denjoy point of the associated semigroup).

Now assume that G is the infinitesimal generator of the semigroup (φ_t) in $\mathbb{D}$. If $|\operatorname{Re}\lambda| < |\lambda|$, then the inequality $\operatorname{Re}\lambda + |\lambda c| \leq 0$ implies $c \in \mathbb{D}$. By Proposition 2.1, the Koenigs function h of the semigroup satisfies

$$h'(z)(z - c) = h(z), \qquad z \in \mathbb{D},$$

$h(c) = 0$ and $h'(c) = 1$. That is, $h(z) = z - c$ and $\varphi_t(z) = h^{-1}(e^{G'(c)t} h(z)) = e^{\lambda t} z + c(1 - e^{\lambda t})$ for all z. Now (1) and (2) follow easily from direct computations.

If $|\operatorname{Re}\lambda| = |\lambda|$ then $\lambda = \operatorname{Re}\lambda < 0$ and $c \in \partial\mathbb{D}$ again by the inequality $\operatorname{Re}\lambda + |\lambda c| \leq 0$ and the semigroup is hyperbolic. $\qquad\square$

The previous proposition implicitly says that linear infinitesimal generators of semigroups in $\mathbb{D}$ are in one-to-one correspondence with affine semigroups. In other words, for seeing non-linear phenomena, we have to deal with polynomials of degree two or more.

Proposition 2.5. *Let $G(z) = \lambda(z - c_1)(z - c_2)$ be a complex polynomial of degree two with $\lambda = |\lambda| e^{i\theta}$, $\lambda \neq 0$. Then, G is the infinitesimal generator of a semigroup in $\mathbb{D}$ (necessarily a semigroup of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$) if and only if*

(2.1) $\operatorname{Re}(e^{i\theta} c_1 + e^{i\theta} c_2) \geq 0$ *and* $(|c_1|^2 - 1)(1 - |c_2|^2) \geq [\operatorname{Im}(e^{i\theta} c_1 - e^{i\theta} c_2)]^2$.

Moreover if G is the infinitesimal generator of a semigroup in $\mathbb{D}$ then $c_1 \in \overline{\mathbb{D}}$ is the Denjoy-Wolff point of (φ_t) and the following are the only possible cases:

(1) *if $c_1 = c_2 \in \partial\mathbb{D}$ then $\operatorname{Re}(e^{i\theta}c_1) \geq 0$ and (φ_t) is a parabolic semigroup. Moreover in this case, (φ_t) is a parabolic semigroup of $\mathsf{Aut}(\mathbb{D})$ if and only if $\operatorname{Re}(e^{i\theta}c_1) = 0$.*

(2) *if $c_1 \in \partial\mathbb{D}$ and $c_2 \in \mathbb{C} \setminus (\mathbb{D} \cup \{c_1\})$ then $e^{i\theta}(c_2 - c_1) \in (0, +\infty)$ and (φ_t) is a hyperbolic semigroup. Moreover in this case, (φ_t) is a hyperbolic semigroup of $\mathsf{Aut}(\mathbb{D})$ if and only if $c_2 \in \partial\mathbb{D}$ if and only if $e^{i2\theta}c_1c_2 = -1$.*

(3) *if $c_1 \in \mathbb{D}$ then $c_2 \in \mathbb{C} \setminus \mathbb{D}$ and (φ_t) is an elliptic semigroup. Moreover:*

    a) *if $c_2 \in \partial\mathbb{D}$ then the semigroup is attractive-elliptic with two fixed points in $\overline{\mathbb{D}}$ and $e^{i\theta}(c_2 - c_1) \in (0, +\infty)$.*

    b) *if $c_2 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ and $c_2\overline{c_1} \neq 1$ then the semigroup is attractive-elliptic with only one fixed point in $\overline{\mathbb{D}}$ and $\operatorname{Re}(e^{i\theta}(c_1 + c_2)) \in (0, +\infty)$, $\operatorname{Im}(e^{i\theta}(c_1 - c_2)) \in [-\beta, \beta]$, where $\beta := \sqrt{(|c_1|^2 - 1)(1 - |c_2|^2)} > 0$.*

    c) *if $c_1 \neq 0$ and $c_2\overline{c_1} = 1$ then the semigroup is a neutral-elliptic semigroup of $\mathsf{Aut}(\mathbb{D})$ and $\operatorname{Re}(e^{i\theta}c_1) = 0$.*

Proof. First of all, a direct computation from Theorem 2.3 shows that G is the infinitesimal generator of a semigroup of linear fractional maps of $\mathbb{D}$ if and only if (2.1) holds.

Now assume that G is the infinitesimal generator of a semigroup (φ_t) of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$ and then (2.1) holds. We first recall that (as a consequence of Berkson-Porta's Theorem 0.1) either c_1 or c_2 is the Denjoy-Wolff point of (φ_t). Without loss of generality we can assume that c_1 is the Denjoy-Wolff point of (φ_t). From Theorem 0.1 one sees that $c_2 \notin \mathbb{D}$. Furthermore, as a consequence of the Schwarz lemma or the Julia-Wolff-Caratheodory theorem (see [14, Theorem 1] or [28]) one can show that $\operatorname{Re} G'(c_1) \leq 0$ and, if $c_1 \in \partial\mathbb{D}$ then $\operatorname{Re} G'(c_1) = 0$ if and only if (φ_t) is a parabolic semigroup of linear fractional maps. Taking these into account, direct computations from (2.1) give statements (1), (2), and (3), with the possible exception of the characterization of semigroups of automorphisms.

In order to obtain the characterization of semigroup of automorphisms, it is enough to realize that (φ_t) is a semigroup of automorphisms if and only if $-G$ is an infinitesimal generator for a semigroup in $\mathbb{D}$ and apply (2.1) to $-G$. $\qquad\square$

Remark 2.6. Assume G is as in Proposition 2.5.(3).(c) and conjugate the semigroup (φ_t) with the automorphism $T(z) = (c_1 - z)(1 - \overline{c_1}z)^{-1}$. The new semigroup $(T \circ \varphi_t \circ T)$ has infinitesimal generator given by $\tilde{G}(z) = T'(T(z)) \cdot G(T(z))$. A direct computation shows that $\tilde{G}(z) = \lambda(\overline{c_1})^{-1}(1 - |c_1|^2)z$ and thus we are in the case of Proposition 2.4.(1).

Since an infinitesimal generator G generates a semigroup of automorphisms of $\mathbb{D}$ if and only if $-G$ is an infinitesimal generator of a semigroup in $\mathbb{D}$, in the Berkson-Porta representation of an infinitesimal generator $G(z) = (z - \tau)(\overline{\tau}z - 1)p(z)$, parabolic semigroups of $\mathsf{Aut}(\mathbb{D})$ appear exactly when $\tau \in \partial\mathbb{D}$ and $p = i\beta$ for some real $\beta \neq 0$ and elliptic (necessary neutral-elliptic) semigroups of $\mathsf{Aut}(\mathbb{D})$ are exactly generated when $\tau \in \mathbb{D}$ and

$p = i\beta$ for some real $\beta \neq 0$. Using Theorem 2.3, in the following example we explain the hyperbolic case.

Example 2.7. Recall that if $\mathbb{H} := \{w \in \mathbb{C} : \mathrm{Re}\, w > 0\}$ is the right half plane, a *Cayley transform* with pole $\tau \in \partial\mathbb{D}$ is any biholomorphic map $C : \mathbb{D} \to \mathbb{H}$ such that $\lim_{\mathbb{D} \ni z \to \tau} |C(z)| = \infty$. It is well known that every Cayley transform is a linear fractional map. Let $G : \mathbb{D} \to \mathbb{C}$ be a holomorphic function. Then G *is the infinitesimal generator of a hyperbolic semigroup of* $\mathsf{Aut}(\mathbb{D})$ *if and only if* $G(z) = (z - \tau)(\overline{\tau}z - 1)p(z)$, *with* $\tau \in \partial\mathbb{D}$ *and* p *is a Cayley transform with pole* τ.

Indeed, if G generates a semigroup (φ_t) of hyperbolic automorphisms then there exists $\sigma \in \partial\mathbb{D} \setminus \{\tau\}$ such that $G(\sigma) = 0$ (such a point σ is the second common fixed point for (φ_t)). According to Theorem 2.3 we have then $G(z) = \lambda(z - \tau)(z - \sigma)$. From this we get $p(z) = \lambda(z - \sigma)(\overline{\tau}z - 1)^{-1}$. Now $p(\sigma) = 0$ and since linear fractional maps on the Riemann sphere maps circles onto circles, we see that $p(\partial\mathbb{D}) = \partial\mathbb{H}$ proving that p is a Cayley transform with pole τ.

Conversely, using Theorem 2.3, we see that G is the infinitesimal generator of a nontrivial semigroup of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$. It is enough to show that the semigroup is composed of automorphisms of $\mathbb{D}$. Indeed p is not constant, and then G cannot generate parabolic or elliptic groups. To this aim, using again Theorem 2.3 and writing $p(z) = \dfrac{az + b}{\overline{\tau}z - 1}$, we have only to check that $\overline{a} = \tau b$ and $\mathrm{Re}(b - a\tau) = 0$. Since p is bijective, $p(\partial\mathbb{D}) = \partial\mathbb{H}$ and, in particular, $\mathrm{Re}\, p(-\tau) = 0$ thus $\mathrm{Re}(a(-\tau) + b) = 0$. Therefore, it only remains to prove the other condition. For every $z \in \partial\mathbb{D}$, we have

$$0 = \mathrm{Re}((az + b)(1 - \overline{z}\tau)) = \mathrm{Re}(b - a\tau) + \mathrm{Re}((\overline{a} - b\tau)\overline{z}) = \mathrm{Re}((\overline{a} - b\tau)\overline{z}).$$

Therefore $|\overline{a} - b\tau| = 0$ as needed.

We end this section looking at relationships between Koenigs functions and semigroups of linear fractional maps. As customary, ∞^{-1} means 0.

Proposition 2.8. *Let* $h : \mathbb{D} \to \mathbb{C}$ *be a holomorphic function.*

(1) *The map* h *is the Koenigs function associated to a non trivial-elliptic semigroup of* $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$ *if and only if*

$$h(z) = (1 - \beta^{-1}\tau)\frac{z - \tau}{1 - \beta^{-1}z}$$

   for some $\tau \in \mathbb{D}$ *and* $\beta \in \mathbb{C}_\infty \setminus \mathbb{D}$.

(2) *The map* h *is the Koenigs function associated to a hyperbolic semigroup of* $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$ *if and only if there exists* $\alpha \in (-\infty, 0)$ *such that*

$$h(z) = \alpha \log\left(\frac{1 - \overline{\tau}z}{1 - \beta^{-1}z}\right)$$

   where $\tau \in \partial\mathbb{D}$, $\beta \in \mathbb{C}_\infty \setminus \mathbb{D}$ *and* $\log$ *denotes the principal branch of the logarithm.*

(3) *The map h is the Koenigs function associated to a parabolic semigroup of* $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$
   if and only if there exists $\alpha \neq 0$ *with* $\mathrm{Re}\,\alpha \leq 0$ *such that*

$$h(z) = \alpha \frac{z}{z - \tau}$$

where $\tau \in \partial\mathbb{D}$.

Proof. (1) Let $\eta_\tau(z) := (\tau - z)(1 - \bar{\tau}z)^{-1}$. Notice that $\eta_\tau \in \mathsf{Aut}(\mathbb{D})$ and $\eta_\tau^{-1} = \eta_\tau$. Conjugating (φ_t) with η_τ we obtain a semigroup (ϕ_t) with Wolff-Denjoy point 0. By uniqueness of the Koenigs function, if h is the Koenigs function of (φ_t) and g is the Koenigs function of (ϕ_t), we have

(2.2) $$h(z) = (|\tau|^2 - 1)g(\eta_\tau(z)).$$

Since the infinitesimal generator F of (ϕ_t) is given by $F(z) = \eta_\tau'(z)G(\eta_t(z))$ and, by Theorem 2.3, it is of the form $-Mz^2 - Nz$ (note that $F(0) = 0$ because $\phi_t(0) = 0$ for all t). Now, a direct computation using Proposition 2.1.(1.iii) shows that g is of the claimed form and, by (2.2), so is h.

   (2) Let G be the infinitesimal generator of (φ_t); and let $\tau \in \partial\mathbb{D}$ be the Denjoy-Wolff point of (φ_t). Using Propositions 2.1.(2.iii), 2.4.(3), and 2.5.(2) a direct computation yields that the Koenigs function h is of the form

$$\frac{1}{G'(\tau)}\left(\log(1 - \bar{\tau}z) - \log(1 - \frac{1}{\beta}z)\right),$$

where $\beta \in \mathbb{C}_\infty \setminus \mathbb{D}$ and $\log$ is the principal branch of the logarithm. Now, we recall that $G'(\tau) \in (-\infty, 0)$ [14] and since $1 - \bar{\tau}z, 1 - \beta^{-1}z \in \mathbb{H}$, for all $z \in \mathbb{D}$,

$$\mathrm{Arg}(1 - \bar{\tau}z) - \mathrm{Arg}(1 - \frac{1}{\beta}z) = \mathrm{Arg}\left(\frac{1 - \bar{\tau}z}{1 - \beta^{-1}z}\right),$$

where Arg denotes the principal argument.

   (3) Once more, a direct computation from Proposition 2.5.(1) and Proposition 2.1.(2.iii), implies that Koenigs functions associated to parabolic semigroups of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$ are exactly those of the form $\alpha\dfrac{z}{z - \tau}$, with $\tau \in \partial\mathbb{D}$ the Denjoy-Wolff of the semigroup and α a given number which, by Proposition 2.1.(2), is $\alpha \neq 0$ and $\mathrm{Re}\,\alpha \leq 0$.

   Finally, we point out that proving the converse in each of the three cases is just a direct (and lengthy) computation. $\qquad\square$

   It is clear from the proof that the point τ in the statement of the previous theorem is the Denjoy-Wolff point of the corresponding semigroup, while the point β in cases 1 and 2 is exactly the repulsive fixed point of each iterate, seen as a Möbius transform of the Riemann sphere.

3. The Embedding Problem

The abstract embedding problem is a rather classical one and has been treated from the times of Abel. In its most general form the embedding problem can be stated as follows: given a space X in some category (topological, differential, holomorphic) and a map $f : X \to X$ in the same category, determine whether it is possible to construct a semigroup (f_t) over X, continuous in t with iterates f_t in the same category of f, such that $f = f_1$.

When $X = \mathbb{D}$ and f is a linear fractional map, the embedding problem has been treated by several authors (see [23] and references therein) and it is known that, in general, it has a negative answer.

As stated in the introduction, in this paper we prove that a linear fractional self-map of $\mathbb{D}$ can be embedded in a semigroup of $\mathbb{D}$ if and only if it can be embedded into a semigroup of linear fractional self-maps of $\mathbb{D}$, obtaining a precise characterization of those linear fractional map which can be embedded.

The proof of this result requires several tools, most of them based on model theory and some of them quite recent. In fact, we will need a new representation, with strong uniqueness, for hyperbolic semigroups in $\mathbb{D}$ which could have some interest in its own. Such a model is a continuous version of the classical Valiron's construction, (see, *e.g.* [12]).

Proposition 3.1. *Let (φ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and associated Koenigs function h. Then there exists a univalent function $\sigma : \mathbb{D} \to \mathbb{H}$ such that $|\sigma(0)| = 1$ and*

$$\sigma \circ \varphi_t = \varphi_t'(\tau)\sigma, \ \text{for every } t \geq 0.$$

Indeed, there exists $\alpha \in (-\infty, 0)$ such that $\sigma = \sigma(0)e^{\alpha h}$.

Moreover, if a non-constant holomorphic (a priori, not necessary univalent) function $\rho \in \mathrm{Hol}(\mathbb{D}, \mathbb{H})$ satisfies $|\rho(0)| = 1$ and $\rho \circ \varphi_t = \varphi_t'(\tau)\rho$, for every $t \geq 0$, then $\rho = \sigma$.

Proof. Since every iterate φ_t ($t > 0$) is hyperbolic, we can consider the Valiron normalization (see, *e.g.* [12]) with respect to 0 and obtain a non-constant holomorphic function $\sigma_t \in \mathrm{Hol}(\mathbb{D}, \mathbb{H})$ such that $|\sigma_t(0)| = 1$ and

$$\sigma_t \circ \varphi_t = \varphi_t'(\tau)\sigma_t.$$

Such a map σ_t is univalent for all $t \geq 0$ because φ_t is. Now recall that by the chain-rule for non-tangential derivatives, $[0, +\infty) \ni t \mapsto \varphi_t'(\tau) \in [0, 1]$ is a measurable algebraic homomorphism from $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$. Therefore, there exists $\alpha \in (-\infty, 0)$ such that $\varphi_t'(\tau) = e^{\alpha t}$.

Set $\sigma = \sigma_1$. Then, for every $n \in \mathbb{N}$, we have $\sigma \circ \varphi_n = e^{n\alpha t}\sigma = \varphi_n'(\tau)\sigma$. Hence, both σ, σ_n are intertwining functions for φ_n and, according to the strong uniqueness result proved in [12, Proposition 6], we deduce that there exists $c_n > 0$ such that $\sigma = c_n\sigma_n$. From the choice of our normalization, $|\sigma(0)| = |\sigma_n(0)| = 1$, and so $\sigma = \sigma_n$. A similar argument

also shows that $\sigma = \sigma_t$ for every positive and rational t and, finally, continuity in t of the semigroup implies that $\sigma = \sigma_t$ for every $t > 0$.

Since $\sigma(\mathbb{D}) \subset \mathbb{H}$, we can consider $\log \sigma(z)$, where $\log$ denotes the principal branch of the logarithm and, by definition, $\log(\sigma \circ \varphi_t) = \log \sigma + \alpha t$ for all $t \geq 0$. Therefore, the function $\widehat{\sigma} := \log \sigma(z) - \log \sigma(0) \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfies $\widehat{\sigma}(0) = 0$ and $\widehat{\sigma} \circ \varphi_t = \widehat{\sigma} + \alpha t$. Differentiating with respect to t and evaluating at $t = 0$, we find that

$$\widehat{\sigma}'(z)G(z) = \alpha, \text{ for every } z \in \mathbb{D},$$

where G denotes the infinitesimal generator of (φ_t). According to Proposition 2.1, we conclude that $\dfrac{1}{\alpha}\widehat{\sigma} = h$. Thus $\sigma = \sigma(0)e^{\alpha h}$.

Finally, the uniqueness assertion follows from the corresponding one for intertwining mappings. Indeed, if $\rho \in \mathrm{Hol}(\mathbb{D}, \mathbb{H})$ is non constant and satisfies $\rho \circ \varphi_t = \varphi_t'(\tau)\rho$ for all $t \geq 0$, then in particular $\rho \circ \varphi_1 = \varphi_1'(\tau)\rho$ and according to [12, Proposition 6] then $\rho = c\sigma$ for some $c > 0$. The further normalization $|\rho(0)| = 1$ implies that $\rho = \sigma$. $\qquad \square$

Now we can prove the following rigidity result:

Theorem 3.2. *Let (φ_t) be a semigroup in $\mathbb{D}$. If for some $t_0 > 0$ the iterate φ_{t_0} is a linear fractional self-map of $\mathbb{D}$ then φ_t is a linear fractional self-map of $\mathbb{D}$ for all $t \geq 0$.*

Proof. Up to rescaling we can assume that $t_0 = 1$. First of all, if $\varphi_1 \in \mathsf{Aut}(\mathbb{D})$ then $\varphi_t \in \mathsf{Aut}(\mathbb{D})$ for all $t \geq 0$ and the result holds. Thus we assume φ_1 is not surjective and we consider the possible dynamical types of φ_1:

(Attractive-Elliptic case) Since composition of linear fractional maps is linear fractional, up to conjugation with a suitable automorphism of $\mathbb{D}$, we can assume that (φ_t) has Denjoy-Wolff point 0. Thus

$$\varphi_1(z) = \frac{az}{cz + 1}, \quad z \in \mathbb{D}$$

for $a = \varphi_1'(0)$ (thus $0 < |a| < 1$) and $c \in \mathbb{C}$. Hence for all $n \in \mathbb{N}$

$$(3.1) \qquad \varphi_n(z) = \frac{a^n z}{c\frac{1-a^n}{1-a}z + 1}.$$

Now let $\sigma : \mathbb{D} \to \mathbb{C}$ be the Schröder function of φ_1 (see, e.g., [27]). This is the unique univalent (because φ is) holomorphic function such that $\sigma(0) = 0$, $\sigma'(0) = 1$ and $\sigma \circ \varphi_1 = \varphi_1'(0)\sigma$; it is defined as the limit of the sequence $\{\varphi_n/\varphi_n'(0)\}$. Thus a direct computation from (3.1) shows that σ is the linear fractional map given by

$$(3.2) \qquad z \mapsto \frac{z}{\frac{c}{1-a}z + 1}.$$

If $h : \mathbb{D} \to \mathbb{C}$ is the Koenigs function of the semigroup (φ_t) then $h(0) = 1$, $h'(0) = 1$ and $h \circ \varphi_t = \varphi_t'(0)h$ for all $t \geq 0$. In particular $h \circ \varphi_1 = \varphi_1'(0)h$ and by uniqueness $h = \sigma$, which implies $\varphi_t(z) = h^{-1}(\varphi_t'(0)h(z)) \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$ for all $t \geq 0$.

(Hyperbolic case) The semigroup (φ_t) is hyperbolic with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$. Let $\sigma : \mathbb{D} \to \mathbb{H}$ be the function defined in Proposition 3.1. By the very construction and uniqueness of the Valiron intertwining function (see [12]) it follows that

$$(3.3) \qquad \sigma(z) = \lim_{\mathbb{N} \ni n \to \infty} \frac{1 - \overline{\tau}\varphi_n(z)}{|1 - \overline{\tau}\varphi_n(0)|}, \ z \in \mathbb{D}.$$

Now, $\varphi_n \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$ for all $n \in \mathbb{N}$, so the Schwarzian derivates $S_{\varphi_n} \equiv 0$ in $\mathbb{D}$ for all $n \in \mathbb{N}$. Since the limit in (3.3) holds uniformly on compacta in $\mathbb{D}$, then $S_\sigma \equiv 0$ in $\mathbb{D}$ implying that $\sigma \in \mathrm{LFM}(\mathbb{D}, \mathbb{H})$. Finally $\varphi_t(z) = \sigma^{-1}(\varphi_t'(\tau)\sigma(z)) \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$ for every $t \geq 0$.

(Parabolic case) The semigroup (φ_t) is parabolic with Denjoy-Wolff point $\tau \in \partial\mathbb{D}$ and $\varphi_t'(\tau) = 1$. Let h be the associated Koenigs function. Thus, $h \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is univalent, verifies $h(0) = 0$ and $h \circ \varphi_t = h + t$. Since $\varphi_1 \in \mathrm{LFM}(\mathbb{D}, \mathbb{D}) \setminus \mathsf{Aut}(\mathbb{D})$ and it is parabolic,

$$\lim_{\mathbb{N} \ni n \to \infty} k_\mathbb{D}(\varphi_n(z), \varphi_{n+1}(z)) = 0,$$

for every $z \in \mathbb{D}$, where $k_\mathbb{D}$ denotes the hyperbolic metric in $\mathbb{D}$. Thus we can apply Baker-Pommerenke's normalization (see [6]) and obtain a univalent map $\sigma \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$, which is a uniform limit on compacta of $\mathbb{D}$ of linear fractional combinations of iterates of φ_1 such that $\sigma(0) = 1$ and $\sigma \circ \varphi = \sigma + 1$. Arguing as before, $\sigma \in \mathrm{LFM}(\mathbb{D}, \mathbb{C})$. By [15, Theorem 3.1] there exists $\lambda \in \mathbb{C}$ such that $h = \sigma + \lambda \in \mathrm{LFM}(\mathbb{D}, \mathbb{C})$. Thus $\varphi_t(z) = h^{-1}(h(z) + t) \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$, for every $t \geq 0$. $\qquad\square$

Now the following result is a direct consequence of Theorem 3.2 and the fact that the automorphisms of $\mathbb{C}_\infty$ are exactly linear fractional maps.

Theorem 3.3. *Let $\varphi \in \mathrm{LFM}(\mathbb{D}, \mathbb{D})$. Then, the following are equivalent:*

    (1) *The map φ can be embedded into a semigroup in $\mathbb{D}$.*
    (2) *The map φ can be embedded into a semigroup of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$.*
    (3) *The map φ, thought of as an element of $\mathsf{Aut}(\mathbb{C}_\infty)$, can be embedded into a group $(\varphi_t)_{t \in \mathbb{R}}$ of $\mathsf{Aut}(\mathbb{C}_\infty)$ with the property that $\varphi_t(\mathbb{D}) \subseteq \mathbb{D}$ for all $t \geq 0$.*

In the last result of this section, we provide a purely dynamical analysis of the embedding problem for linear fractional self-maps of the unit disc. Probably, we are giving new information only for what concerns the case (2) but, for the sake of completeness, we deal with all cases. It is interesting to compare our approach to this embedding problem with the one given in [28, Section 5.9].

Given a point $a \in \mathbb{D} \setminus \{0\}$ there exists a unique $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda < 0$, $\mathrm{Im}\,\lambda \in (-\pi, \pi]$ and $e^\lambda = a$. The *canonical spiral associated to a* is the curve

$$\gamma_a : [1, \infty) \longrightarrow \mathbb{D}, \quad \gamma_a(t) := e^{\lambda t}.$$

Note that γ_a is a spiral (actually a segment if $a \in (0,1)$) which goes from a to zero. The curve γ_a has finite length $\ell(\gamma_a)$ given by

$$\ell(\gamma_a) = \int_1^{+\infty} |\gamma_a'(t)| dt = \frac{|\lambda a|}{\text{Re}(-\lambda)} \geq |a|.$$

Theorem 3.4. *Let φ be an arbitrary element of* $\text{LFM}(\mathbb{D}, \mathbb{D})$.

(1) *If φ is trivial, neutral-elliptic, hyperbolic or parabolic, then φ can be always embedded into a semigroup in $\mathbb{D}$.*

(2) *If φ is attractive-elliptic with Denjoy-Wolff point $\tau \in \mathbb{D}$ and repulsive fixed point $\beta \in \mathbb{C}_\infty \setminus \mathbb{D}$, let ℓ be the length of the canonical spiral associated to $\varphi'(\tau) \in \mathbb{D} \setminus \{0\}$. Then, φ can be embedded into a semigroup in $\mathbb{D}$ if and only if*

(3.4)
$$\left| \overline{\tau} - \frac{1}{\beta} \right| \ell \leq |\varphi'(\tau)| \left| 1 - \frac{\tau}{\beta} \right|.$$

[Again, ∞^{-1} means 0].

Proof. (1) If φ is trivial or neutral-elliptic, then $\varphi \in \text{Aut}(\mathbb{D})$ and, up to conjugation with an automorphism of $\mathbb{D}$ which maps the Denjoy-Wolff point of φ to 0, by the Schwarz lemma $\varphi(z) = e^{ia}z$ for some $a \in \mathbb{R}$ and clearly φ can be embedded into the semigroup $(z,t) \mapsto e^{ita}z$. In the hyperbolic or parabolic case, we can conjugate φ with a Cayley transform from $\mathbb{D}$ to $\mathbb{H}$ which maps the Denjoy-Wolff point of φ to ∞. Thus we obtain a linear fractional self-map ϕ of $\mathbb{H}$ of the form $w \mapsto e^a w + b$ with $a \in (0, +\infty)$ and $b \in \mathbb{C}$, $\text{Re}\, b \geq 0$ in the hyperbolic case; and $a = 0$, $b \in \mathbb{C}$ with $\text{Re}\, b \geq 0$ in the parabolic case. Thus in the hyperbolic case ϕ belongs to the semigroup $(w,t) \mapsto e^{at}w + \frac{b}{e^a - 1}(e^{at} - 1)$ while in the parabolic case ϕ belongs to the semigroup $(w,t) \mapsto w + bt$. Going back to the unit disc this implies that φ can be embedded into a semigroup in $\mathbb{D}$.

(2) Let $\phi := \alpha_\tau \circ \varphi \circ \alpha_\tau$, with $\alpha_\tau(z) = \dfrac{\tau - z}{1 - \overline{\tau}z}$. The map ϕ is an attractive-elliptic linear fractional map with Denjoy-Wolff point 0 and repulsive fixed point $\widehat{\beta} = \alpha_\tau(\beta) \in \mathbb{C}_\infty \setminus \mathbb{D}$ given by

$$\phi(z) = \frac{az}{cz + 1},$$

where $a := \varphi'(\tau) = \phi'(0)$, $0 < |a| < 1$ and $c = \dfrac{1}{\widehat{\beta}}(1 - a) \in \mathbb{C}$. Let $\lambda \in \mathbb{C}$ be such that $\text{Re}\, \lambda < 0$, $\text{Im}\, \lambda \in (-\pi, \pi]$ and $e^\lambda = a$. An easy computation shows that

(3.5)
$$\left| \overline{\tau} - \frac{1}{\beta} \right| \ell \leq |\varphi'(\tau)| \left| 1 - \frac{\tau}{\beta} \right| \iff \left| \lambda \frac{c}{1 - a} \right| \leq \text{Re}(-\lambda).$$

Now, assume (3.4) holds. By Theorem 2.3 and (3.5), we deduce that

$$F(z) := \lambda \frac{c}{1 - a} z^2 + \lambda z, \ z \in \mathbb{D}$$

is the infinitesimal generator of a semigroup (ϕ_t) of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$. Since $F(0) = 0$ then (ϕ_t) is a non trivial-elliptic semigroup in $\mathbb{D}$. Using Proposition 2.1(1), one can find that the Koenigs function of the semigroup is $h(z) = \frac{z}{\frac{c}{1-a}z+1}$ and $\phi_t = h^{-1}(e^{\lambda t}h)$. From this one can check that $\phi_1 = \phi$. Thus ϕ can be embedded in a semigroup in $\mathbb{D}$ and so does φ.

Conversely, if φ can be embedded in a semigroup in $\mathbb{D}$, so does ϕ. Let (ϕ_t) be a semigroup in $\mathbb{D}$ such that $\phi_1 = \phi$. According to Theorem 3.2, ϕ_t is a linear fractional self-map of $\mathbb{D}$ for all $t \geq 0$. If h is the Koenigs function of (ϕ_t) then $\phi_t(z) = h^{-1}(e^{\alpha t}h(z))$ with $\mathrm{Re}\,\alpha < 0$ and $h(z) = \dfrac{z}{\frac{c}{1-a}z + 1}$ by (3.2). Using Proposition 2.1, one can deduce that the infinitesimal generator of (ϕ_t) is

$$G(z) = \alpha \frac{c}{1-a}z^2 + \alpha z, \ z \in \mathbb{D}.$$

By Theorem 2.3

$$\left| \alpha \frac{c}{1-a} \right| \leq \mathrm{Re}(-\alpha).$$

Since $\phi_1 = \phi$, $e^\alpha = \phi_1'(0) = \phi'(0) = a = e^\lambda$. Therefore, $\lambda - \alpha = 2k\pi i$, for some $k \in \mathbb{Z}$. Bearing in mind that $\lambda \in (-\pi, \pi]$, we find that $\mathrm{Re}(\alpha) = \mathrm{Re}(\lambda)$ and $|\alpha| \geq |\lambda|$ and by (3.5) inequality (3.4) holds.   □

The above inequality explains dynamically well-known phenomena concerning the embedding problem (shortly, EP). For instance, when $\varphi'(\tau) \in (0, 1)$, trivially the length of the associated spiral is exactly $\varphi'(\tau)$. Since always $\left| \bar\tau - \dfrac{1}{\beta} \right| \leq \left| 1 - \dfrac{\tau}{\beta} \right|$, we see that the answer to (EP) for those attractive-elliptic maps is always positive. However, when $\varphi'(\tau) \in (-1, 0)$, the length of the spiral is strictly bigger than $|\varphi'(\tau)|$, so we can have positive and negative answers. The inequality also says that, for an attractive-elliptic element of $\mathrm{LFM}(\mathbb{D}, \mathbb{D})$ the (EP) can be positively solved whenever the repulsive fixed point is close enough to $\frac{1}{\bar\tau}$, namely when the map is similar enough to a neutral-elliptic map.

Moreover, if φ is a hyperbolic or parabolic linear fractional self-map of $\mathbb{D}$ and β denote the repelling fixed point of φ in the hyperbolic case and $\beta = \tau$ the Denjoy-Wolff point of φ in the parabolic case, it follows

$$\left| \bar\tau - \frac{1}{\beta} \right| = \left| 1 - \frac{\tau}{\beta} \right| \text{ and } \ell(\gamma_a) = |\varphi'(\tau)| = \varphi'(\tau) \in (0, 1],$$

and thus (3.4) always holds for φ hyperbolic or parabolic.

Finally, if φ is neutral-elliptic, since $\tau \in \mathbb{D}$ and the repulsive fixed point is $\frac{1}{\bar\tau}$ it follows

$$0 = |\bar\tau - \bar\tau| \, \ell(\gamma_a) < |\varphi'(\tau)| \, (1 - |\tau|^2),$$

and even in this case (3.4) always holds.

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