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ON THE GEOMETRY AT THE BOUNDARY OF HOLOMORPHIC SELF-MAPS OF THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. We introduce two new tools to study a holomorphic self-map f of $\mathbb{B}^n$ (the unit ball of $\mathbb{C}^n$, $n > 1$): the *inner space* $\mathbb{A}(f)$ and the *generalized inner space* $\mathcal{AG}(f)$. After having defined the differential at the boundary for f, k-df_τ, in its Wolff point $\tau \in \partial\mathbb{B}^n$, we prove that the boundary dilatation coefficient $\alpha(f)$ is an eigenvalue for k-df_τ and we define $\mathcal{AG}(f)$ to be the generalized eigenspace associated to $\alpha(f)$; the inner space $\mathbb{A}(f)$ will be the span of the eigenvectors not belonging to the complex tangent space of $\partial\mathbb{B}^n$ at the Wolff point τ and contained in $\mathcal{AG}(f)$. Among other things it turns out that $\mathbb{A}(f)$ is the space of all "directions" of complex geodesics that are mapped into themselves by f, and that the generalized inner space $\mathcal{AG}(f)$ is a direct addend of a boundary Cartan-type decomposition for $\mathbb{C}^n$. Using $\mathbb{A}(f)$ and $\mathcal{AG}(f)$ we obtain several new results on the geometry of holomorphic self-maps of $\mathbb{B}^n$, including some necessary conditions for commutation under composition.

0. Introduction

Since the first years of this century, iteration theory of holomorphic self-maps of Δ, the unit disk of $\mathbb{C}$, and the closely related subject of commutation (under composition) of holomorphic maps have been deeply studied. In particular in these last years necessary and sufficient conditions under which two holomorphic maps commute have been found (see [11] for bibliography and references). In the case of holomorphic self-maps of $\mathbb{B}^n$ ($n > 1$) the analogous problems are still open. Recently (see [1] and [2]), it has been proved that a family of commuting holomorphic self-maps of $\mathbb{B}^n$ extending continuously on $\partial\mathbb{B}^n$, has a common "fixed point"; moreover a complete classification of all holomorphic self-maps of $\mathbb{B}^n$ which commute with a given hyperbolic automorphism of $\mathbb{B}^n$ has been obtained (see [5] and [4]). In this work we study the geometry of a holomorphic self-map of $\mathbb{B}^n$ (with no fixed points) in a neighborhood of its Wolff point. We stress the relationships between the action on the complex geodesics of $\mathbb{B}^n$, the fixed points and the iteration of maps, and we give some necessary conditions in order that two holomorphic maps with no fixed points commute under composition. The plan of this paper is the following. Firstly, we recall some definitions and results such as the definition of

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complex geodesics and the Wolff and the Julia-Wolff-Carathéodory Theorems for $\mathbb{B}^n$. We then extend some of the definitions to the multidimensional case (see §1). The Wolff point of a holomorphic self-map without fixed points can be now characterized and used to prove that if a complex geodesic is transformed in itself by a self-map of $\mathbb{B}^n$, then it has to "touch" the Wolff point (see §2). As a spin-off result, we also give a geometric condition which guarantees that two commuting holomorphic self-maps of $\mathbb{B}^n$ have the same "boundary fixed point" (see Theorem 2.4). In §3 we prove some technical results and present a notion of boundary regularity that we request in the next sections: we say that f is K-differentiable at a boundary point if there exists the K-limit of df_z at that point. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be with no fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$, let $k\text{-}df_\tau$ be the K-differential of f at τ and let $\alpha(f) = \liminf_{z\to\tau} \frac{1-\|f(z)\|}{1-\|z\|}$ be the boundary dilatation coefficient of f at τ. In §4 we prove that if v is an eigenvector of $k\text{-}df_\tau$ and $v \notin \mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ (where $\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ means the complex tangent space of $\partial\mathbb{B}^n$ at τ), then $k\text{-}df_\tau(v) = \alpha(f)v$. We then define the *inner space* $\mathbb{A}(f)$ to be the span of the eigenvectors of $k\text{-}df_\tau$ not belonging to $\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$. If $\mathbb{A}(f)\neq \{0\}$ then it turns out that $\mathbb{A}(f)$ is the eigenspace associated to $\alpha(f)$. The inner space $\mathbb{A}(f)$ is closely connected to the complex geodesics of $\mathbb{B}^n$ which are mapped into themselves by f: each of those geodesics (if any) corresponds to an element of $\mathbb{A}(f)$. In the case that f maps complex geodesics into complex geodesics (every automorphism of $\mathbb{B}^n$ has this property) then the correspondence is one to one. We end the section by proving that if two commuting holomorphic maps f, g have the same Wolff point and $\dim\mathbb{A}(f)= 1$ then $\mathbb{A}(f)\subseteq\mathbb{A}(g)$. In §5 we prove that if v is a generalized eigenvector of $k\text{-}df_\tau$ and $v \notin \mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ then v has to belong to the generalized eigenspace associated to $\alpha(f)$ (and hence $\alpha(f)$ is always an eigenvalue of $k\text{-}df_\tau$). We define the *generalized inner space* $\mathcal{AG}(f)$ to be the generalized eigenspace of $\alpha(f)$. Since each "generalized inner eigenvector" is associated to $\alpha(f)$, we obtain a Cartan-type decomposition of $\mathbb{C}^n$ at τ (see Theorem 5.3). By using the decomposition we prove that, if two holomorphic maps f, g having a common Wolff point commute, then $\mathcal{AG}(f)\bigcap\mathcal{AG}(g) \neq \{0\}$. In §6 we study the geometry of a class of boundary K-differentiable maps: the automorphisms of $\mathbb{B}^n$, denoted by $\mathrm{Aut}(\mathbb{B}^n)$. In particular we see that if γ is a hyperbolic automorphism of $\mathbb{B}^n$ then it is equivalent to know the two fixed points of γ, the (only) complex geodesic transformed to itself by γ or the inner space $\mathbb{A}(\gamma)$. Moreover $\mathcal{AG}(\gamma)$ is the "smaller" space on which γ is a well-defined self-map. In the case of a parabolic automorphism η of $\mathbb{B}^n$, we characterize the normal form of η (see [6]) by using $\mathbb{A}(\eta)$. Finally for two commuting parabolic automorphisms η and μ of $\mathbb{B}^n$ we prove that if $\mathbb{A}(\eta)= \{0\}$ then $\dim(\mathcal{AG}(\eta)\bigcap\mathcal{AG}(\mu)) \geq 2$.

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1. Preliminary results

In this section we state some general definitions and theorems. First let us recall some facts about complex geodesics in the simple case of the unit ball $\mathbb{B}^n$; for the general case and references see, *e.g*, [1].

A *m-dimensional affine subset* of $\mathbb{B}^n$ is the intersection of $\mathbb{B}^n$ with an affine m-dimensional subspace of $\mathbb{C}^n$.

Definition 1.1. A *complex geodesic* of $\mathbb{B}^n$ (so called because it is an isometry between the Poincaré metric in Δ and the Bergman metric in $\mathbb{B}^n$) is a injective holomorphic map $\varphi : \Delta \to \mathbb{B}^n$ such that $\varphi(\Delta)$ is a one-dimensional affine subset of $\mathbb{B}^n$.

We will often call "complex geodesic" the image $\varphi(\Delta)$. Using affine maps, it follows that every one-dimensional affine subset of $\mathbb{B}^n$ is a complex geodesic. We can state:

Proposition 1.2.
(1) *If $z_0, z_1 \in \overline{\mathbb{B}^n}$, $z_0 \neq z_1$, then there is one, and only one (up to automorphisms of the unit disk Δ), complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ such that $z_0, z_1 \in \overline{\varphi(\Delta)}$.*
(2) *If $z \in \overline{\mathbb{B}^n}$, $v \in \mathbb{C}^n$, $v \neq 0$, and $\{\xi v + z | \xi \in \mathbb{C}\} \bigcap \mathbb{B}^n \neq \emptyset$, then there is one, and only one (up to automorphisms of the unit disk Δ), complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ such that $z \in \overline{\varphi(\Delta)}$ and that $\varphi(\Delta)$ is parallel to the complex line generated by v.*

From now on we will often say, with an abuse of language, that the complex geodesic φ *passes through* $z \in \overline{\mathbb{B}^n}$ or that the complex geodesic φ *has the direction* v if, respectively, $z \in \overline{\varphi(\Delta)}$ or $\varphi(\Delta)$ is parallel to the complex line generated by v. Here we have another definition:

Definition 1.3. Let be $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and let $\varphi : \Delta \to \mathbb{B}^n$ be a complex geodesic of $\mathbb{B}^n$. If $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$ we call φ a *cut complex geodesic* of f.

It turns out that if $\varphi : \Delta \to \mathbb{B}^n$ is a cut complex geodesic of f, then $\varphi^{-1} \circ f \circ \varphi$ is a well-defined holomorphic self-map of Δ. Now let us state some classical results from the theory of several complex variables (see [10], [1]) and give some new definitions:

Theorem 1.4 (MacCluer [9]). *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points; then there is a unique $\tau \in \partial\mathbb{B}^n$ such that for every $z \in \mathbb{B}^n$*

$$\frac{|1 - \langle f(z), \tau\rangle|^2}{1 - \|f(z)\|^2} \leq \frac{|1 - \langle z, \tau\rangle|^2}{1 - \|z\|^2},$$

where $\langle\ ,\ \rangle$ denotes the hermitian product in $\mathbb{C}^n$.

Definition 1.5. If f is a holomorphic self-map of $\mathbb{B}^n$ without fixed points, we call *the Wolff point of f* the unique $\tau \in \partial\mathbb{B}^n$ defined by Theorem 1.4.

We recall that a *Korányi region* of vertex $\tau \in \partial\mathbb{B}^n$ and amplitude $M > 1$ is given by

$$K(\tau, M) := \left\{ z \in \mathbb{B}^n : \frac{|1 - \langle z, \tau\rangle|}{1 - \|z\|} < M \right\}.$$

If $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ we say that f has *K-limit* σ at $\tau \in \partial\mathbb{B}^n$ and we write $K\text{-}\lim_{z\to\tau} f(z) = \sigma$ if, for each $M > 1$ and for each sequence $\{z_m\} \subset K(\tau, M)$ such that $\lim_{m\to\infty} z_m = \tau$, we get $\lim_{m\to\infty} f(z_m) = \sigma$.

Definition 1.6. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $\tau \in \partial\mathbb{B}^n$. The *boundary dilatation coefficient* of f at τ is the value $\liminf_{z\to\tau}(1 - \|f(z)\|) \cdot (1 - \|z\|)^{-1}$.

If $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ has no fixed points and $\tau \in \partial\mathbb{B}^n$ is its Wolff point, then we denote the boundary dilatation coefficient of f at τ by $\alpha(f)$. The following result holds:

Lemma 1.7. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points with $\tau \in \partial\mathbb{B}^n$ its Wolff point. Then $0 < \alpha(f) \leq 1$.*

Proof. By a straightforward computation, or by the Julia Lemma (see [1] or [10]) it follows that $\alpha(f) > 0$.
Now let $E(\tau, 1/k) := \{z \in \mathbb{B}^n : (|1 - \langle z, \tau\rangle|^2) \cdot (1 - \|z\|^2)^{-1} < 1/k\}$ be a horosphere. The map $z \mapsto \|z\|$ reaches its minimum on $\overline{E(\tau, 1/k)}$ at $z_k := \tau \cdot (k-1) \cdot (k+1)^{-1}$. The sequence $\{z_k\}$ converges to τ as k tends to ∞, and $f(E(\tau, 1/k)) \subseteq E(\tau, 1/k)$ by Theorem 1.4. Then $\|f(z_k)\| \geq \|z_k\|$ and $(1 - \|f(z_k)\|) \cdot (1 - \|z_k\|)^{-1} \leq 1$. Therefore $\alpha(f) \leq 1$. □

For $\tau \in \partial\mathbb{B}^n$, a *$\tau$-curve* is a curve $\gamma : [0, 1) \to \mathbb{B}^n$ such that $\gamma(t) \to \tau$ as $t \to 1$. To every τ-curve we associate its orthogonal projection $\gamma_\tau = \langle \gamma, \tau\rangle\tau$ into $\mathbb{C}\tau$. A τ-curve is said to be *special* if $\lim_{t\to 1}(\|\gamma(t) - \gamma_\tau(t)\|^2) \cdot (1 - \|\gamma_\tau(t)\|^2)^{-1} = 0$, and *restricted* if it is special and moreover there is $A < \infty$ such that for all $t \in [0, 1)$ we get $(\|\gamma_\tau(t) - \tau\|)(1 - \|\gamma_\tau(t)\|)^{-1} \leq A$. In a natural way, we say that a holomorphic self-map f of $\mathbb{B}^n$ has *restricted K-limit* σ at $\tau \in \partial\mathbb{B}^n$ if $f(\gamma(t)) \to \sigma$ as $t \to 1$ for any restricted τ-curve γ (see [10]). Let us now state the Julia-Wolff-Carathéodory Theorem for $\mathbb{B}^n$:

Theorem 1.8 (Rudin [10], p.177)**.** *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $\tau \in \partial\mathbb{B}^n$ be such that*

$$\liminf_{z\to\tau} \frac{1 - \|f(z)\|}{1 - \|z\|} = \beta < \infty.$$

Then f has K-limit $\sigma \in \partial\mathbb{B}^n$ at τ and the following functions are bounded in every Korányi region:

(1) $\dfrac{1 - \langle f(z), \sigma\rangle}{1 - \langle z, \tau\rangle}$
(2) $\langle df_z\tau, \sigma\rangle$
(3) $\dfrac{\langle df_z\tau^\perp, \sigma\rangle}{(1 - \langle z, \tau\rangle)^{\frac{1}{2}}}$

where $\tau^\perp$ is any non-zero vector orthogonal to τ. Moreover the function 1) and 2) have restricted K-limit β at τ and the function 3) has restricted K-limit 0 at τ.

Remark 1.9. We recall that the point $\sigma \in \partial\mathbb{B}^n$ in Theorem 1.8 is the unique point of $\partial\mathbb{B}^n$ such that

$$\frac{|1 - \langle f(z), \sigma\rangle|^2}{1 - \|f(z)\|^2} \leq \beta \frac{|1 - \langle z, \tau\rangle|^2}{1 - \|z\|^2}$$

for every $z \in \mathbb{B}^n$.

Remark 1.10. By Lemma 1.7 we can apply Theorem 1.8 at the Wolff point of a holomorphic self-map f of $\mathbb{B}^n$ with no fixed points, replacing β with $\alpha(f)$ and σ with τ.

2. The Wolff point and the cut complex geodesics

In this section we prove three interesting facts related to the Wolff point, to the cut complex geodesics and to the common fixed points of commuting holomorphic maps.

Theorem 2.1. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points. A point $\tau \in \partial\mathbb{B}^n$ is the Wolff point of f if and only if*

(1) $\lim_{r\to 1} f(r\tau) = \tau$.
(2) $\limsup_{r\to 1} |\langle df_{r\tau}\tau, \tau\rangle| \leq 1$.

Proof. The necessity of the assertion follows directly from Theorem 1.8. Suppose the two conditions hold. If $\tau \neq e_1$, then there is an unitary matrix U such that $U\tau = e_1$. It is easy to see that by setting $\tilde{f} = Uf\overline{U}^t$, then $\tilde{f}$ is a holomorphic self-map of $\mathbb{B}^n$ without fixed points, and such that $\lim_{r\to 1}\tilde{f}(re_1) = e_1$, $\limsup_{r\to 1}|\langle df_{re_1}e_1, e_1\rangle| \leq 1$ and $\alpha(\tilde{f}) = \alpha(f)$. Hence we can suppose $\tau = e_1$, w.l.o.g. Since $\limsup_{r\to 1}|\langle df_{re_1}e_1, e_1\rangle| \leq 1$, for any fixed $\epsilon > 0$ there exists $\overline{r} \geq 0$ such that for every $r > \overline{r}$ we get $|\frac{\partial f_1}{\partial z_1}(re_1)| < 1+\epsilon$. The function $\Psi(r) := f_1(re_1)$ is a differentiable map from $[0,1)$ to Δ such that $\Psi'(r) = \frac{\partial f_1}{\partial z_1}(re_1)$; then

$$\Psi(r_2) - \Psi(r_1) = \int_{r_1}^{r_2} \Psi'(r)dr \quad \text{for all} \quad r_1, r_2 \in [0,1).$$

Whence

$$|f_1(r_2e_1) - f_1(r_1e_1)| \leq \int_{r_1}^{r_2} |\frac{\partial f_1}{\partial z_1}(re_1)|dr < (1+\epsilon)(r_2 - r_1),$$

for all $r_2, r_1 > \overline{r}$. By hypotesis 1), by allowing $r_2 \to 1$ we get $|1 - f_1(re_1)| < (1+\epsilon)(1-r)$, for all $r > \overline{r}$. Therefore, by setting $\beta := \liminf_{z\to e_1}\frac{1-\|f(z)\|}{1-\|z\|}$, we have:

$$\beta \leq \liminf_{r\to 1}\frac{1-\|f(re_1)\|}{1-r} \leq \liminf_{r\to 1}\frac{1-|f_1(re_1)|}{1-r} < (1+\epsilon).$$

Since this last inequality is in force for an arbitrary $\epsilon > 0$ we get $\beta \leq 1$. Now we can apply Theorem 1.8, and by the following Remark 1.9 we have:

$$\frac{|1-\langle f(z),\tau\rangle|^2}{1-\|f(z)\|^2} \leq \beta\frac{|1-\langle z,\tau\rangle|^2}{1-\|z\|^2} \leq \frac{|1-\langle z,\tau\rangle|^2}{1-\|z\|^2}.$$

Then the assertion holds by the uniqueness of the Wolff point (Theorem 1.4). □

Remark 2.2. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $\tau, \sigma \in \partial\mathbb{B}^n$. Arguing as in the proof of Theorem 2.1 it is easy to see that if $\lim_{r\to 1} f(r\tau) = \sigma$ and $\limsup_{r\to 1}|\langle df_{r\tau}\tau, \sigma\rangle| \leq k$ for some $k > 0$ then $\liminf_{z\to\tau}\frac{1-\|f(z)\|}{1-\|z\|} \leq k$.

Theorem 2.3. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and let $\tau \in \partial\mathbb{B}^n$ be the Wolff point of f. If $\varphi : \Delta \to \mathbb{B}^n$ is a cut complex geodesic of f then $\tau \in \overline{\varphi(\Delta)}$.*

Proof. Define $\tilde{\varphi} : \Delta \to \mathbb{B}^n$ to be $\tilde{\varphi}(\xi) = (\xi, 0, \ldots, 0)$. Since $\mathrm{Aut}(\mathbb{B}^n)$ acts doubly transitively on $\partial\mathbb{B}^n$ and transforms complex geodesics onto complex geodesics, then there exists $\Phi \in \mathrm{Aut}(\mathbb{B}^n)$ such that $\Phi \circ \varphi(\Delta) = \tilde{\varphi}(\Delta)$. Set $\tilde{f} := \Phi \circ f \circ \Phi^{-1}$. It is easy to see that $\tilde{f}$ has no fixed points and that $\Phi(\tau)$ is the Wolff point of $\tilde{f}$. Furthermore $\tilde{f}(\tilde{\varphi}(\Delta)) \subseteq \tilde{\varphi}(\Delta)$ and then $\tilde{f}_2(\xi, 0, \ldots, 0) = \ldots = \tilde{f}_n(\xi, 0, \ldots, 0) = 0$. If we set $\Psi(\xi) := \tilde{f}_1(\xi, 0, \ldots, 0)$, then Ψ is a holomorphic self-map of Δ without fixed points and there exists the Wolff point $\eta \in \partial\Delta$ of Ψ. Since $\liminf_{\xi\to\eta} \frac{1-|\Psi(\xi)|}{1-|\xi|} \leq 1$ we get

$$\liminf_{z\to\tilde{\varphi}(\eta)} \frac{1-\|\tilde{f}(z)\|}{1-\|z\|} \leq \liminf_{\substack{z\to\tilde{\varphi}(\eta)\\ z\in\tilde{\varphi}(\Delta)}} \frac{1-\|\tilde{f}(z)\|}{1-\|z\|} = \liminf_{\xi\to\eta} \frac{1-|\Psi(\xi)|}{1-|\xi|} \leq 1.$$

By Theorem (1.8), and since

$$\lim_{r\to 1} \tilde{f}(r\tilde{\varphi}(\eta)) = \lim_{r\to 1} \tilde{f}(r\eta, 0, \dots, 0) = \lim_{r\to 1} \tilde{\varphi}(\tilde{f}_1(r\eta)) = \lim_{r\to 1} \tilde{\varphi}(\Psi(r\eta)) = \tilde{\varphi}(\eta),$$

we have: $\limsup_{r\to 1} |\langle d\tilde{f}_{r\tilde{\varphi}(\eta)}\tilde{\varphi}(\eta), \tilde{\varphi}(\eta)\rangle| \leq 1$; by theorem 2.1 $\tilde{\varphi}(\eta)$ turns out to be the Wolff point of $\tilde{f}$. Then $\Phi(\tau) = \tilde{\varphi}(\eta)$ and hence $\Phi(\tau) \in \overline{\tilde{\varphi}(\Delta)}$ and $\tau \in \overline{\varphi(\Delta)}$. $\square$

Theorem 2.4. *Let $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Let f be without fixed points and let $\tau \in \partial\mathbb{B}^n$ be its Wolff point. If f has a cut complex geodesic and $f\circ g = g\circ f$ then g has restricted K-limit τ at τ.*

Proof. Let $\varphi : \Delta \to \mathbb{B}^n$ be a cut complex geodesic of f. By Theorem 2.3 we get that $\tau \in \overline{\varphi(\Delta)}$. Up to conjugation in $\mathrm{Aut}(\mathbb{B}^n)$, we can suppose $\tau = e_1$ and $\varphi : \xi \to (\xi, 0, \dots, 0)$. As φ is a cut complex geodesic of f we have

$$f_2(\xi, 0, \dots, 0) = \dots = f_n(\xi, 0, \dots, 0) = 0. \tag{2.1}$$

Set $\gamma(t) := f^{k(t)}(2[1-2^{k(t)}(1-t)]f(z_0))$, where $t \in [0,1)$, $z_0 = (0, \dots, 0)$ and $k(t)$ is the greatest integer less than or equal to $-\log_2(1-t)$. Since $\gamma([1-2^{-k}, 1-2^{-k-1}])$ is the image by f^k of the segment S from z_0 to $f(z_0)$, it is easily checked that γ is continuous and $\gamma(t) \to e_1$ as $t \to 1$. Moreover $\lim_{t\to 1} g(\gamma(t)) = \lim_{k\to\infty} g(f^k(S)) = \lim_{k\to\infty} f^k(g(S)) = e_1$ (by the Wolff-Denjoy Theorem for $\mathbb{B}^n$, see [9], [1]). By equation (2.1) it holds that $\gamma(t)$ is a special e_1-curve and by the Lindelöf-Čirka theorem (see [10], [1]) g has restricted K-limit e_1 at e_1. $\square$

Remark 2.5. If g has no fixed points and $\limsup_{r\to 1} |\langle dg_{r\tau}\tau, \tau\rangle| \leq 1$, then τ turns out to be the Wolff point of g. In particular $K\text{-}\lim_{z\to\tau} g(z) = \tau$ (by Theorem 2.1). Again note that we can substitute hypothesis 2) in Theorem 2.1 with an equivalent one: $\limsup_{r\to 1} \left| \frac{1-\langle g(r\tau), \tau\rangle}{1-r} \right| \leq 1$.

3. K-differentiability

We will now state some technical results about Korányi regions and a notion of boundary differentiability.

Lemma 3.1. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $\tau \in \partial\mathbb{B}^n$. If $\liminf_{z\to\tau} \frac{1-\|f(z)\|}{1-\|z\|} = \beta < +\infty$ then there exists $\sigma \in \partial\mathbb{B}^n$ such that f maps Korányi regions with vertex τ into Korányi regions with vertex σ.*

A proof can be found in [10]. In particular every automorphism of $\mathbb{B}^n$ maps Korányi regions onto Korányi regions.

Definition 3.2. A holomorphic self-map f of $\mathbb{B}^n$ is said to be *K-differentiable* at $\tau \in \partial\mathbb{B}^n$, if $K\text{-}\lim_{z\to\tau} \dfrac{\partial f_j}{\partial z_k}$ exist for $j, k = 1, \dots, n$. In this case we define the *K-differential* of f at τ to be:

$$k\text{-}df_\tau := \lim_{r\to 1} df_{r\tau}.$$

Remark 3.3. If we replace the request for the existence of $K\text{-}\lim_{z\to\tau}\frac{\partial f_j}{\partial z_k}$ with the stronger one concerning the existence of $\lim_{z\to\tau}\frac{\partial f_j}{\partial z_k}$, we get a notion of *differentiability at the boundary.* Of course boundary differentiability implies K-differentiability, and an automorphism of $\mathbb{B}^n$ is differentiable in every boundary point.

As we will soon see, the fact that a holomorphic self-map f is K-differentiable at $\tau\in\partial\mathbb{B}^n$ means that it is C^1 on $K(\tau,M)\bigcup\{\tau\}$, for all $M>1$:

Proposition 3.4. *If $f\in\mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$ is K-differentiable at $\tau\in\partial\mathbb{B}^n$, then*

- *there exists $\sigma\in\overline{\mathbb{B}^n}$ such that $K\text{-}\lim\limits_{z\to\tau}f(z)=\sigma$.*
- *for each $v\in\mathbb{C}^n$ such that $v+\tau\in\mathbb{B}^n$, we get*

$$K\text{-}\lim_{\|v\|\to 0}\frac{f(\tau+v)-\sigma-k\text{-}df_\tau(v)}{\|v\|}=0.$$

Proof. By a straightforward computation we see that a Korányi region is convex. Whence we can apply the mean value Theorem to the real (and to the imaginary) part of each component of f to obtain the assertion. □

The second statement of Proposition 3.4 can be replaced by :
for each differentiable curve $\gamma:[0,1)\to K(\tau,M)$ such that $\lim\limits_{t\to 1}\gamma(t)=\tau$ and $\lim\limits_{t\to 1}\gamma'(t)=v$ we get:

$$\lim_{t\to 1}\frac{f(\gamma(t))-\sigma}{1-t}=k\text{-}df_\tau(v).$$

By using Remark 2.2 and Lemma 3.1 we have the chain rule for K-differentiable maps:

Proposition 3.5. *Let be $f,g\in\mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$. Let g be K-differentiable at $\tau\in\partial\mathbb{B}^n$ and let $\sigma=K\text{-}\lim\limits_{z\to\tau}g(z)$. If f is K-differentiable at σ then $f\circ g$ is K-differentiable at τ and moreover $k\text{-}d(f\circ g)_\tau=k\text{-}df_\sigma\circ k\text{-}dg_\tau$.*

By using Theorem 1.8 for K-differentiable maps we get

Theorem 3.6. *Let $f\in\mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$ be without fixed points, and let $\tau\in\partial\mathbb{B}^n$ be its Wolff point. If f is K-differentiable at τ then*

(1) $\langle k\text{-}df_\tau\tau,\tau\rangle=\alpha(f)\neq 0$,
(2) $\langle k\text{-}df_\tau\tau^\perp,\tau\rangle=0$,

where $\tau^\perp$ is any non-zero vector orthogonal to τ.

Definition 3.7. The *tangent complex plane* $\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ at $\tau\in\partial\mathbb{B}^n$ is given by

$$\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n=\{v\in\mathbb{C}^n:\langle v,\tau\rangle=0\}.$$

Proposition 3.8. *Let $f\in\mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau\in\partial\mathbb{B}^n$. Then $v\in\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ if and only if $k\text{-}df_\tau(v)\in\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$; in particular $\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ is $k\text{-}df_\tau$invariant.*

Proof. It is a straightforward consequence of Theorem 3.6. □

4. Inner Space

Definition 4.1. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be K-differentiable at a point $\tau \in \partial\mathbb{B}^n$. An eigenvector v of k-df_τ is *properly internal* if $v \notin \mathrm{T}^{\mathbb{C}}_\tau \partial\mathbb{B}^n$.

By definition an eigenvector v of k-df_τ is properly internal if and only if there exists $\xi \in \mathbb{C}$ such that $(\xi v + \tau) \in \mathbb{B}^n$.

Proposition 4.2. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points, K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. If v is a properly internal eigenvector of k-df_τ, then k-$df_\tau(v) = \alpha(f)v$.*

Proof. Write $v = h\tau + \tau^\perp$ where $\langle \tau, \tau^\perp \rangle = 0$ and $h \neq 0$ by hypothesis. By Theorem 3.6 and since k-$df_\tau(v) = \lambda v$ with $\lambda \in \mathbb{C}$, we have:

$$\lambda h = \langle k\text{-}df_\tau(v), \tau \rangle = h\langle k\text{-}df_\tau(\tau), \tau\rangle + \langle k\text{-}df_\tau(\tau^\perp), \tau \rangle = h\alpha(f).$$

Then $\lambda = \alpha(f)$. □

In force of the above proposition, we can define the *inner space* of a map f:

Definition 4.3. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. We define the *inner space* of f, $\mathbb{A}(f)$, to be the vector space generated by the properly internal eigenvectors of k-df_τ, i.e.

$$\mathbb{A}(f) = \mathrm{span}\{v \in \mathbb{C}^n : k\text{-}df_\tau(v) = \alpha(f)v,\ \langle v, \tau \rangle \neq 0\}.$$

We call *inner vector* each element of $\mathbb{A}(f)$.

Remark 4.4.
1. For each $m \in 0, \dots, n$ there exists $f_m \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ such that $\dim\, \mathbb{A}(f_m) = m$, see §6 for examples with $f_m \in \mathrm{Aut}(\mathbb{B}^n)$.
2. If there exist no properly internal eigenvectors of k-df_τ then $\mathbb{A}(f)= \{0\}$ and we say that $\mathbb{A}(f)$ is *trivial*.
3. By Proposition 4.2 the properly internal eigenvectors have the same eigenvalue, then $\mathbb{A}(f)$ is really a subspace of an eigenspace of k-df_τ. (We will soon see that if $\mathbb{A}(f) \neq \{0\}$ then it is actually the eigenspace which corresponds to the boundary dilatation coefficient of f at τ).
4. $\{\mathbb{A}(f)+\tau\} \bigcap \mathbb{B}^n$ is an open set of $\mathbb{A}(f)+\tau \simeq \mathbb{C}^{\dim\, \mathbb{A}(f)}$.
5. If $\dim\, \mathbb{A}(f)= 1$ then each inner vector is a properly internal eigenvector.
6. If $\dim\, \mathbb{A}(f)> 1$ then there are inner vectors which are not properly internal eigenvectors.

Proposition 4.5. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. If $\mathbb{A}(f)$ is non-trivial then it coincides with the eigenspace corresponding to the eigenvalue $\alpha(f)$ of k-df_τ.*

Proof. Let S be the eigenspace of k-df_τ corresponding to $\alpha(f)$. By Proposition 4.2 we know that $\mathbb{A}(f) \subseteq$ S. Let $v \in$ S. Since $\mathbb{A}(f) \neq \{0\}$ if v is not properly internal then there is a properly internal inner vector w. Then $w - v \in \mathbb{A}(f)$ and therefore $v = w - (w - v)$ belongs to $\mathbb{A}(f)$, too. □

The inner space of a holomorphic self-map of $\mathbb{B}^n$ is an "intrinsic" concept, that is, it is independent under conjugation in $\mathrm{Aut}(\mathbb{B}^n)$:

Proposition 4.6. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \mathbb{B}^n$; if $\chi \in \mathrm{Aut}(\mathbb{B}^n)$, set $\tilde{f} = \chi \circ f \circ \chi^{-1}$. Then the inner space $\mathbb{A}(f)$ is isomorphic to the inner space $\mathbb{A}(\tilde{f})$ with isomorphism $d\chi_\tau$.*

Proof. We recall that χ extends holomorphically in a neighborhood of $\mathbb{B}^n$. Set $\sigma = \chi(\tau)$; we can easily see that $\tilde{f}$ is a holomorphic self-map of $\mathbb{B}^n$ with no fixed points and σ is its Wolff point. Furthermore, by Lemma 3.1, $\tilde{f}$ is K-differentiable at σ. Now if v is a properly internal eigenvector of $k\text{-}df_\tau$ at τ, we get

$$k\text{-}d\tilde{f}_\sigma(d\chi_\tau(v)) = d\chi_\tau \circ k\text{-}df_\tau \circ d\chi_\sigma^{-1} \circ d\chi_\tau(v) = d\chi_\tau \circ k\text{-}df_\tau(v) = \alpha(f) d\chi_\tau(v).$$

Hence $d\chi_\tau(v)$ is an eigenvector of $k\text{-}d\tilde{f}_\sigma$, and since $d\chi_\tau$ is a isomorphism of $T_\tau\mathbb{B}^n$ onto $T_\sigma\mathbb{B}^n$ which maps $\mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$onto $\mathrm{T}^{\mathbb{C}}_\sigma\partial\mathbb{B}^n$, then $d\chi_\tau(v)$ is properly internal and $\mathbb{A}(f) \simeq \mathbb{A}(\tilde{f})$. □

Now we are able to find the relationship between the inner space and the cut complex geodesics of a holomorphic self-map of $\mathbb{B}^n$:

Proposition 4.7. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. Let $\varphi : \Delta \to \mathbb{B}^n$ be a complex geodesic with direction $v \in \mathbb{C}^n$. If $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$, then v is a properly internal eigenvector of $k\text{-}df_\tau$; in particular $\mathbb{A}(f)$ is non-trivial.*

Proof. By Theorem 2.3 it follows that $\tau \in \overline{\varphi(\Delta)}$ and we can suppose $(\tau + tv) \in K(\tau, M)$ for some $M > 1$ and $t \in \mathbb{R}$, $|t| \ll 1$. By Proposition 3.4 we get $k\text{-}df_\tau(v) = \lim_{t\to 0}(f(\tau + tv) - \tau)/t$. As $f(\tau + tv) \subseteq \varphi(\Delta)$ for every t small, it follows that $k\text{-}df_\tau(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Then v is a properly internal eigenvector of $k\text{-}df_\tau$, and by Proposition 4.2 we get $\lambda = \alpha(f)$. □

If $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ has no fixed points and f is K-differentiable at its Wolff point, then Proposition 4.7 gives a necessary condition for a complex geodesic to be a cut complex geodesic of f; in particular if $\mathbb{A}(f)$= $\{0\}$ then f has no cut complex geodesics. As we see in the following example, the condition fails to be sufficient:

Example 4.8. The holomorphic function defined as $f : \mathbb{B}^2 \to \mathbb{B}^2$,

$$f(z_1, z_2) := \left(\frac{1+z_1}{3-z_1}, \frac{(z_1-1)^2}{z_1-3}\right)$$

has no fixed points, and $(1, 0)$ is its Wolff point (apply Theorem 2.1). The map f is clearly differentiable at $(1, 0)$ and $df_{(1,0)}(1, 0) = (1, 0)$. The vector $(1, 0)$ is a properly internal eigenvector but the complex geodesic $\varphi : z \mapsto (z, 0)$ is not a cut complex geodesic of f.

We really need a sort of "rigidity" on the complex geodesics to get an exact correspondence between the inner vectors and the cut complex geodesics: we say that f has the *rigidity property* if for each complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ there exists a complex geodesic $\eta : \Delta \to \mathbb{B}^n$ such that $f(\varphi(\Delta)) \subseteq \eta(\Delta)$.

Proposition 4.9. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. If f has the rigidity property then a complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ with direction $v \in \mathbb{C}^n$ is a cut complex geodesic if and only if $\tau \in \overline{\varphi(\Delta)}$ and v is a properly internal eigenvector of $k\text{-}df_\tau$.*

Proof. The necessity follows from Theorem 2.3 and Proposition 4.7.
To prove the sufficiency we operate as follows: let $\eta : \Delta \to \mathbb{B}^n$ be such that $f(\varphi(\Delta)) \subseteq \eta(\Delta)$. We have that $(\tau + tv) \in K(\tau, M) \bigcap \varphi(\Delta)$ for some $M > 1$

and $t \in \mathbb{R}$, $|t| \ll 1$. Moreover $f(\tau + tv) \in \eta(\Delta)$. Since $K\text{-}\lim_{z\to\tau} f(z) = \tau$ (see Theorem 1.8) we get $\tau \in \overline{\eta(\Delta)}$. If w is the direction of $\eta(\Delta)$, it follows that

$$\alpha(f)v = k\text{-}df_\tau(v) = \lim_{t\to 0} \frac{f(\tau + tv) - \tau}{t} = \lambda w.$$

Since $\alpha(f) \neq 0$ (see Lemma 1.7), this implies $\lambda \neq 0$ and $w = \alpha(f)\lambda^{-1}v$ and $\varphi(\Delta) = \eta(\Delta)$. □

Since any element in $\mathrm{Aut}(\mathbb{B}^n)$ has the rigidity property (see [10] or [1]), we can apply the above proposition to $\mathrm{Aut}(\mathbb{B}^n)$.

Working with the inner space of a function in the next Proposition 4.10 we will be able to find necessary conditions for maps to commute (better results will be obtained in the next section):

Proposition 4.10. *Let $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be with no fixed points K-differentiable at their common Wolff point*[1] *$\tau \in \partial\mathbb{B}^n$. If $f \circ g = g \circ f$ and if $\mathbb{A}(f)$ has dimension 1 then $k\text{-}dg_\tau$ has one and only one (up to complex multiple) properly internal eigenvector in common with $k\text{-}df_\tau$. In particular $\mathbb{A}(g) \neq \{0\}$.*

Proof. Let v be a generator of $\mathbb{A}(f)$. By Proposition 3.5 $f \circ g$ is K-differentiable at τ and $k\text{-}d(f\circ g)_\tau = k\text{-}df_\tau \circ k\text{-}dg_\tau$ (the same holds for $g \circ f$); hence

$$k\text{-}df_\tau \circ k\text{-}dg_\tau(v) = k\text{-}dg_\tau \circ k\text{-}df_\tau(v) = k\text{-}dg_\tau(\alpha(f)v) = \alpha(f)k\text{-}dg_\tau(v).$$

Since by Proposition 4.5 $\mathbb{A}(f)$ coincides with the eigenspace of $\alpha(f)$ and $\dim \mathbb{A}(f) = 1$, we get $k\text{-}dg_\tau(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. □

5. Generalized Inner Space

Let us start this section with the following result:

Theorem 5.1. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points, K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. The boundary dilatation coefficient of f at τ, $\alpha(f)$, is an eigenvalue for $k\text{-}df_\tau$.*

Proof. Let $a_1, \ldots, a_r$ be the eigenvalues of $k\text{-}df_\tau$, and let

$$A_m = \bigcup_{j=1}^{\infty} \ker(k\text{-}df_\tau - a_m I)^j$$

where $m = 1, \ldots, r$ and I is the identity matrix. The vector spaces A_m are the generalized eigenspaces of $k\text{-}df_\tau$ and therefore $\mathbb{C}^n = \bigoplus_{m=1}^r A_m$. Then there exist $m_0 \in \{1, \ldots, r\}$ and $v \in A_{m_0}$ such that $\langle v, \tau\rangle \neq 0$. For the sake of clarity, we set $a = a_{m_0}$.

By hypothesis we have $v \in \ker(k\text{-}df_\tau - aI)^j$ for some $j \geq 1$. We can suppose $j > 1$, otherwise v is a properly internal eigenvector of $k\text{-}df_\tau$ and by Proposition 4.2 we have $a = \alpha(f)$ and we are done. By Proposition 3.5 (setting $k\text{-}df_\tau{}^0 = I$) the following equality holds:

$$(k\text{-}df_\tau - aI)^j = \sum_{h=0}^{j} \binom{j}{h} (-a)^h k\text{-}df_\tau{}^{(j-h)}. \tag{5.1}$$

[1]The author has recently proved that two commuting holomorphic self-maps of $\mathbb{B}^n$, with no fixed points, have the same Wolff point unless their restrictions to the complex geodesic passing through their different Wolff points are two commuting hyperbolic automorphisms of that geodesic (see [3]).

We claim that $a \neq 0$. Firstly, by Theorem 3.6 and Proposition 3.8 we see that

$$\langle k\text{-}df_\tau{}^k(\tau), \tau\rangle = \alpha(f)^k, \tag{5.2}$$

$$\langle k\text{-}df_\tau{}^k(\tau^\perp), \tau\rangle = 0. \tag{5.3}$$

Now assume that $v = \lambda\tau + \tau^\perp$ with $\lambda \neq 0$. If $a = 0$ then $k\text{-}df_\tau{}^j(v) = 0$, that is

$$0 = \langle k\text{-}df_\tau{}^j(v), \tau\rangle = \langle k\text{-}df_\tau{}^j(\lambda\tau + \tau^\perp), \tau\rangle = \lambda\langle k\text{-}df_\tau{}^j(\tau), \tau\rangle = \lambda\alpha(f)^j \neq 0$$

by Lemma 1.7. This is a contradiction; then $a \neq 0$.
Since $a \neq 0$, from (5.1) we obtain

$$v = -\sum_{h=0}^{j-1} \left(\begin{array}{c} j \\ h \end{array} \right) (-a)^{h-j} k\text{-}df_\tau{}^{(j-h)}(v)$$

and from this we have

$$\langle v, \tau\rangle = -\sum_{h=0}^{j-1} \left(\begin{array}{c} j \\ h \end{array} \right) (-a)^{h-j} \langle k\text{-}df_\tau{}^{(j-h)}(v), \tau\rangle.$$

By equation (5.2) and since $v = \lambda\tau + \tau^\perp$ with $\lambda \neq 0$ we get

$$\langle v, \tau\rangle = -\sum_{h=0}^{j-1} \left(\begin{array}{c} j \\ h \end{array} \right) (-a)^{h-j} \lambda\alpha(f)^{j-h}.$$

Now, as $\langle v, \tau\rangle = \lambda \neq 0$, we have

$$1 = -\sum_{h=0}^{j-1} \left(\begin{array}{c} j \\ h \end{array} \right) (-a)^{h-j} \alpha(f)^{j-h}$$

that is

$$\sum_{h=0}^{j} \left(\begin{array}{c} j \\ h \end{array} \right) (-a)^{h} \alpha(f)^{(j-h)} = 0,$$

hence $(\alpha(f) - a)^j = 0$, and $a = \alpha(f)$. $\square$

Definition 5.2. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. We call *generalized inner space* of f the subspace of $\mathbb{C}^n$ given by:

$$\mathcal{AG}(f) = \bigcup_{j=1}^{\infty} \ker(k\text{-}df_\tau - \alpha(f)I)^j.$$

The space $\mathcal{AG}(f)$ is not reduced to zero by Theorem 5.1 and $k\text{-}df_\tau|_{\mathcal{AG}(f)} : \mathcal{AG}(f) \mapsto \mathcal{AG}(f)$ is a linear operator with only the eigenvalue $\alpha(f)$. By Proposition 4.5 and by Theorem 5.1 it turns out that $\mathbb{A}(f) \subseteq \mathcal{AG}(f)$ and $\mathcal{AG}(f) \neq \{0\}$. Moreover a *Cartan-type decomposition* of $\mathbb{C}^n$ is possible at the Wolff point:

Theorem 5.3. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points, K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$. Then there exists $V(f) \subseteq \mathrm{T}^{\mathbb{C}}_\tau\partial\mathbb{B}^n$ such that $\mathbb{C}^n = \mathcal{AG}(f) \bigoplus V(f)$ is a k-df_τ-invariant decomposition.*

Proof. Let A_m $(m = 1, \dots, k)$ be the generalized eigenspaces of k-df_τ not associated to $\alpha(f)$. Set $V(f) = \bigoplus_{m=1}^{k} A_m$. From the proof of Theorem 5.1 it follows that each generalized eigenvector of k-df_τ that does not belong to $\mathrm{T}_\tau^{\mathbb{C}}\partial\mathbb{B}^n$ is in $\mathcal{AG}(f)$. Then $V(f) \subseteq \mathrm{T}_\tau^{\mathbb{C}}\partial\mathbb{B}^n$ and $\mathbb{C}^n = \mathcal{AG}(f) \bigoplus V(f)$ is a k-df_τ-invariant decomposition since it is the sum of generalized eigenspaces of k-df_τ. □

Definition 5.4. Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \partial\mathbb{B}^n$; we call *generalized inner vectors* the elements of $\mathcal{AG}(f)$ and *generalized properly internal vectors* the elements of $\mathcal{AG}(f)$ that do not belong to $\mathrm{T}_\tau^{\mathbb{C}}\partial\mathbb{B}^n$.

Remark 5.5. If $\mathcal{AG}(f)$ has dimension one, it follows from Theorem 5.3 that $\mathbb{A}(f)=\mathcal{AG}(f)$. In particular if $\mathbb{A}(f)$ is trivial then $\mathcal{AG}(f)$ has dimension greater than or equal to two.

As one can expect, for $\mathcal{AG}(f)$ a property similar to that stated in Proposition 4.6 for $\mathbb{A}(f)$ holds; in fact, since formula (5.1) holds, we can easily generalize the proof of Proposition 4.6 in order to get:

Proposition 5.6. *Let $f \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at its Wolff point $\tau \in \mathbb{B}^n$; if $\chi \in \mathrm{Aut}(\mathbb{B}^n)$, set $\tilde{f} = \chi \circ f \circ \chi^{-1}$. Then the generalized inner space $\mathcal{AG}(f)$ is isomorphic to the generalized inner space $\mathcal{AG}(\tilde{f})$ with isomorphism $d\chi_\tau$. In particular $\alpha(f) = \alpha(\tilde{f})$.*

Before we can state the main result of this section we need another simple preliminary result directly following by formula (5.1):

Lemma 5.7. *Let $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at the common Wolff point $\tau \in \partial\mathbb{B}^n$. If $f \circ g = g \circ f$ then k-dg_τ maps $\mathcal{AG}(f)$ into itself and k-df_τ maps $\mathcal{AG}(g)$ into itself.*

And now we have:

Theorem 5.8. *Let $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points and K-differentiable at the common Wolff point $\tau \in \partial\mathbb{B}^n$. If $f \circ g = g \circ f$ then there exists a common generalized properly internal vector of k-df_τ and k-dg_τ; in particular*

$$\mathcal{AG}(f) \bigcap \mathcal{AG}(g) \neq \{0\}.$$

Proof. Let $v \in \mathcal{AG}(f)$, $v \notin \mathrm{T}_\tau^{\mathbb{C}}\partial\mathbb{B}^n$ (such a v exists by Theorem 5.3). Since Lemma 5.7 holds, then $\mathcal{AG}(f) = A_1 \bigoplus \dots \bigoplus A_k$, where the A_j are the generalized eigenspaces of k-$dg_\tau|_{\mathcal{AG}(f)}$. Then $v = a_1 + \dots + a_k$ $(a_j \in A_j)$, and since $0 \neq \langle v, \tau\rangle = \langle a_1, \tau\rangle + \dots + \langle a_k, \tau\rangle$, there exists $j_o \in \{1, \dots, k\}$ such that $\langle a_{j_0}, \tau\rangle \neq 0$. Then a_{j_0} is a generalized properly internal eigenvector of k-dg_τ, hence $a_{j_0} \in \mathcal{AG}(g)$. □

6. Some applications to automorphisms.

In this last section we apply the results obtained to the case of the automorphisms of $\mathbb{B}^n$ with no fixed points (hyperbolic and parabolic automorphisms). This will give a different (and more geometric) point of view for the study of the structure of $\mathrm{Aut}(\mathbb{B}^n)$. We now recall this statement (see e.g. [5]):

Theorem 6.1. *Let γ be an automorphism of $\mathbb{B}^n$ with no fixed points. If γ has two different fixed points on $\partial\mathbb{B}^n$ (i.e. if it is* hyperbolic*), up to conjugacy, we can suppose*

$$\gamma(z) = \frac{\left(\cosh t_0\, z_1 + \sinh t_0, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n\right)}{\sinh t_0\, z_1 + \cosh t_0} \tag{6.1}$$

where $t_0 \in \mathbb{R} \setminus \{0\}$ and $\theta_j \in \mathbb{R}$ for $j = 2, \dots, n$.

Suppose $t_0 > 0$. An automorphism as (6.1) fixes the two points e_1 and $-e_1$ and, by a straightforward calculation, we can see that e_1 is its Wolff point. Another direct calculation about $d\gamma_{e_1}$ implies that $\alpha(\gamma) = e^{-2t_0}$ and $\mathbb{A}(\gamma) = \mathcal{AG}(\gamma) = \text{span}\{e_1\}$. By Proposition 4.9 the automorphism γ has one, and only one, cut complex geodesic, that is the complex geodesic which passes through e_1 with direction e_1. Note that the second fixed point of γ (i.e. $-e_1$) belongs to the cut complex geodesic. By Proposition 4.6 we can state:

Proposition 6.2. *Let γ be a hyperbolic automorphism of $\mathbb{B}^n$. Then there is one, and only one, cut complex geodesic passing through the two fixed points of γ and with direction the unique (up to $\mathbb{C}$-multiples) inner vector of $d\gamma$ at its Wolff point.*

That is, given a hyperbolic automorphism γ of $\mathbb{B}^n$, it is equivalent to know the fixed points of γ, its cut complex geodesic or its inner space. Now we pass to the case of parabolic automorphisms (see [6]):

Theorem 6.3. *Let η be an automorphism of $\mathbb{B}^n$ with no fixed points. If η has only one fixed point on $\partial\mathbb{B}^n$ (i.e. if it is* parabolic*), up to conjugacy, we can suppose*

$$\eta(z) = \frac{\left(e^{i\theta_1} z_1, \dots, e^{i\theta_{n-1}} z_{n-1}, (1-it)z_n + it\right)}{-itz_n + 1 + it} \tag{6.2}$$

with $t \in \mathbb{R} \setminus \{0\}$ and $\theta_j \in \mathbb{R}$ for $j = 1, \dots, n-1$; or

$$\eta(z) = \frac{\left(e^{i\theta_1} z_1, \dots, e^{i\theta_{n-2}} z_{n-2}, z_{n-1} + sz_n - s, -sz_{n-1} + (1-\beta)z_n + \beta\right)}{-sz_{n-1} - \beta z_n + 1 + \beta} \tag{6.3}$$

with $Re\beta > 0$, $s = \sqrt{2Re\beta}$ and $\theta_j \in \mathbb{R}$ for $j = 1, \dots, n-2$.

An automorphism of type (6.2) or (6.3) has only one fixed point e_n at the boundary, that is its Wolff point. A straightforward calculation shows that $\alpha(\eta) = 1$. If η is of type (6.2), then $\mathbb{A}(\eta)$ is non-trivial, since e_n is an eigenvector of $d\eta_{e_n}$ and in particular we can easily see that $\mathcal{AG}(\eta) = \mathbb{A}(\eta)$ and $\dim \mathcal{AG}(\eta) = 1 + \#\{j = 1, \dots, n-1 : e^{i\theta_j} = 1\}$. If η is of type (6.3), then $\mathbb{A}(\eta) = \{0\}$; the eigenspace corresponding to $\alpha(\eta)$ is a m-dimensional space generated by $\{e_{n-1}\} \bigcup \{e_j : e^{i\theta_j} = 1\}$ where m is the number of j such that $e^{i\theta_j} = 1$, and $\mathcal{AG}(\eta)$ is generated by $\mathbb{A}(\eta) \bigcup \{e_n\}$. This implies that e_n is a generalized properly internal eigenvector which is not a inner vector. Hence $\mathcal{AG}(\eta)$ has dimension equal to $2 + \#\{j = 1, \dots, n-2 : e^{i\theta_j} = 1\}$. It turns out that a parabolic automorphism of type (6.2) has at least one cut complex geodesics, and a parabolic automorphism of type (6.3) has no cut complex geodesic. Since the inner and the generalized inner spaces are "intrinsic", we can easily generalize these results:

Proposition 6.4. *Let $\eta \in \text{Aut}(\mathbb{B}^n)$ be a parabolic automorphism. Then*

(1) *η is conjugated to a parabolic automorphism of type (6.2) if and only if $\mathbb{A}(\eta)$ is non-trivial, and in this case η has a cut complex geodesic for each properly internal vector.*

(2) *η is conjugated to a parabolic automorphism of type (6.3) if and only if $\mathbb{A}(\eta)$ is trivial, and in this case η has no cut complex geodesics.*

If $K \subset \mathbb{C}^n$ is a subset biholomorphic to the unit ball of $\mathbb{C}^m$ (for some m) through a biholomorphism Λ then a holomorphic self-map f of K can be regarded as a holomorphic self-map of $\mathbb{B}^m$ (namely $\Lambda \circ f \circ \Lambda^{-1}$). Keeping this in mind we can state:

Theorem 6.5. *Let γ be an automorphism of $\mathbb{B}^n$ with no fixed points and let $\tau \in \partial\mathbb{B}^n$ be its Wolff point. Let T be a m-dimensional subspace of $\mathbb{C}^n$. Set $L = \{T+y\} \bigcap \mathbb{B}^n$, for some $y \in \partial\mathbb{B}^n$, and suppose $L \neq \emptyset$. If $\gamma|_L$ is a well-defined self-map of L, then $\tau \in \overline{L}$ and $\gamma|_L$ is a hyperbolic (parabolic) automorphism of L if and only if γ is hyperbolic (parabolic) of $\mathbb{B}^n$.*

Proof. It is clear that $\gamma|_L$ is an automorphism of L with no fixed points, and since L is biholomorphic to the unit ball of $\mathbb{C}^m$, $\gamma|_L$ has at least one and at most two boundary fixed points. Therefore if γ is a parabolic automorphism of $\mathbb{B}^n$ then $\gamma|_L$ has only one boundary fixed point which has to be τ.
Now we can assume γ to be a hyperbolic automorphism of $\mathbb{B}^n$. We have to prove that both fixed points of γ belong to $\overline{L}$. If $\gamma|_L$ has only one boundary fixed point, say $\sigma \in \partial\mathbb{B}^n$, then σ has to be the Wolff point of $\gamma|_L$. By Theorem 1.4 and by iterating $\gamma|_L$ we have $(\gamma|_L)^k(z) \to \sigma$ as $k \to \infty$ for all $z \in L$; since $\gamma = \gamma|_L$ on L then $\sigma = \tau$. As L is the image under a translation and a dilatation of $\mathbb{B}^m$, then there is at least one generalized properly internal vector (see Theorem 5.3) for $d(\gamma|_L)_\tau$, say $w \in T$. But then $w \in \mathcal{AG}(\gamma) = \mathbb{A}(\gamma)$, and hence $\mathbb{A}(\gamma) \subseteq T$. By Proposition 6.2, the only cut complex geodesic of γ belongs to L, in particular both the boundary fixed points of γ belong to $\overline{L}$. □

The above theorem states that if γ is an automorphism of $\mathbb{B}^n$ with Wolff point $\tau \in \partial\mathbb{B}^n$ and L is an affine subset of $\mathbb{B}^n$ then $\gamma(L) \subseteq L$ implies that there is a m-dimensional subspace T of $\mathbb{C}^n$ such that $T \bigcap \mathcal{AG}(\gamma) \neq \{0\}$ and $L = \{T+\tau\} \bigcap \mathbb{B}^n$. In particular if γ is a hyperbolic automorphism of $\mathbb{B}^n$ and $m = 1$ then L is the only cut complex geodesic of γ.
If γ is an automorphism of $\mathbb{B}^n$ with Wolff point $\tau \in \partial\mathbb{B}^n$, and $L = \{T+\tau\} \bigcap \mathbb{B}^n$ (T as in Theorem 6.5), a simple condition which guarantees γ to be a well-defined self-map of L is that $d\gamma_\tau(T) \subseteq T$.

Theorem 6.6. *Let f, g be parabolic automorphisms of $\mathbb{B}^n$. Suppose $f \circ g = g \circ f$ and $\mathbb{A}(g) \neq \{0\}$.*

(1) *If $\mathbb{A}(f) \neq \{0\}$ then there is at least one common cut complex geodesic, or equivalently $\mathbb{A}(f) \bigcap \mathbb{A}(g) \neq \{0\}$.*
(2) *If $\mathbb{A}(f) = \{0\}$ then $dim(\mathcal{AG}(f) \bigcap \mathcal{AG}(g)) \geq 2$.*

Proof. It is well known that f and g have the same Wolff point (see e.g. [1]). If $\mathbb{A}(f) \neq \{0\}$ then $\mathbb{A}(f) = \mathcal{AG}(f)$ and $\mathbb{A}(g) = \mathcal{AG}(g)$. From Theorem 5.8 it follows that there is at least one common properly internal eigenvector; by Proposition 6.4 part 1 is proved.
Now we prove the second statement: note that $\mathcal{AG}(g)$ has dimension ≥ 2, by Proposition 4.10. Moreover $\mathcal{AG}(f)$ has dimension greater than or equal to 2 by Remark 5.5. By Lemma 5.7, $df_\tau(\mathcal{AG}(g)) \subseteq \mathcal{AG}(g)$ and then f is a well-defined self-map of $W := (\mathcal{AG}(g) + \tau) \bigcap \mathbb{B}^n$. Since $\mathbb{A}(f|_W) \subseteq \mathbb{A}(f)$ (up to isomorphic images) and $\mathcal{AG}(f|_W) \subseteq \mathcal{AG}(f)$ then $\dim \mathcal{AG}(f|_W) \geq 2$, otherwise $\mathbb{A}(f) \neq \{0\}$

by Remark 5.5. So there are at least two linearly indipendent generalized inner eigenvectors v, w for $d(f|_W)_\tau$, with $v, w \in \mathcal{AG}(f|_W)$. Therefore $v, w \in T_\tau W = \mathcal{AG}(g)$ and we are done. $\square$

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