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THE DYNAMICS OF HOLOMORPHIC MAPS NEAR CURVES OF FIXED POINTS.

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ABSTRACT. Let M be a two dimensional complex manifold and $f : M \to M$ a holomorphic map. Let $S \subset M$ be a curve made of fixed points of f, *i.e.* $\mathrm{Fix}(f) = S$. We study the dynamics near S in case f acts as the identity on the normal bundle of the regular part of S. Besides results of local nature, we prove that if S is a globally and locally irreducible compact curve such that $S \cdot S < 0$ then there exists a point $p \in S$ and a holomorphic f-invariant curve with p on the boundary which is attracted by p under the action of f. These results are achieved introducing and studying a family of local holomorphic foliations related to f near S.

INTRODUCTION

Let M be a two-dimensional complex manifold, $\Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $f : M \to M$ be holomorphic and $p \in M$. A *parabolic curve* for f at p is the image of an injective analytic disc $\varphi : \Delta \to M$ such that φ is continuous up to the boundary of Δ, $p = \varphi(1)$, $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$ and for any $q \in \varphi(\Delta)$ it follows that $\lim_{n\to\infty} f^n(q) = p$. Moreover φ is said to be *tangent* to a direction $[v] \in \mathbb{CP}^1$ at p if $[\varphi(\zeta)] \to [v]$ for $\zeta \to 1$ (where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^2 - \{0\}$ onto $\mathbb{CP}^1$).

Theorem 0.1 (Écalle, Hakim, Abate). *If f has an isolated fixed point $p \in M$ and $df_p = Id$ then there exists at least one parabolic curve for f at p.*

This theorem is a complete generalization of the well known one-dimensional Leau-Fatou flower theorem.

The "flower theorem in two-dimensions" has quite an odd story. The first who (partially) proved it in the 1980's was J. Écalle [10] who, using his theory of formal series and resurgence, was able to produce small pieces of f-invariant curves attached to the point p in case f is "generic". In the middle of the nineties, several people felt a need for a complete analytic proof of such a theorem. After some preliminary results of T. Ueda [19] and B. Weickert [20], a major step in this direction has been done by M. Hakim [12] who proved the "flower theorem" for "generic maps" (not only in $\mathbb{C}^2$ but also in $\mathbb{C}^n$). Her idea was to look at $f - Id$ near p. That is to say, if some conditions on the first non-zero homogeneous polynomial in the expansion of

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$f - Id$ are satisfied, then one can apply some Banach spaces techniques to construct parabolic curves. After his results in [1], Abate understood that if Hakim's conditions at p were not satisfied then one could have tried to blow-up the point p to reach a more favorable situation on the exceptional divisor. This is exactly the same strategy exploited by C. Camacho and P. Sad [7] to show that any holomorphic foliation on a two dimensional complex manifold has a separatrix at a singular point. Indeed Abate's proof of the flower theorem follows the same lines of Camacho-Sad argument. Abate defines "singularities" and "reduced singularities" for a holomorphic map and proves a reduction theorem which, roughly speaking, says that after a finite number of blow ups one gets a holomorphic map with only "reduced singularities" on the exceptional divisor. Then he defines an index for holomorphic maps on curves of fixed points and proves an index theorem which allows to localize the characteristic classes of the curve near the singularities of the map. After that he has formally the same ingredients as in the Camacho-Sad theory, and he can argue in the same way to obtain a point where Hakim's theory applies.

Besides giving a complete analytic proof of the flower theorem, Abate's work made evident that, firstly, the dynamics near isolated fixed points can be well understood only once one understands the dynamics near curves of fixed points, secondly, some results on the older theory of holomorphic foliations can be properly translated to give new results in discrete dynamics. An evidence of this second claim is a recent work of the author and F. Tovena [4], in which it is proved a discrete dynamics analogous of a generalization of the Camacho-Sad index Theorem due to T. Suwa [18].

In a sense this similarity with holomorphic foliations has to be expected according to some mathematical folklore: one should always find a *formal* vector field along the curve of fixed points of a holomorphic map (provided this map is tangent to the identity on such a curve) in such a way that the map is the time one flow of this formal vector field, and then somehow use the theory of formal foliations.

One aim of this paper is to provide a new approach which avoids the use of formal foliations but provides an actual link between holomorphic foliations and holomorphic maps. Given a holomorphic self-map f of M which fixes $p \in M$ and such that $df_p = Id$, we associate to f a family of holomorphic 1-forms $\Omega_{f,p}$, see (1.1), which is composed by forms whose flows are "first-order approximations" of f at p. That is to say, the normalizations of such forms all have the same linear part at p (up to scalar multiples). This allows to define a *(reduced) singularity* of f as a (reduced) singularity of the family $\Omega_{f,p}$. With this approach, the *Reduction Theorem 3.3* follows directly from the Seidenberg Reduction Theorem for holomorphic foliations. Then we turn our attention to the case f has a curve of fixed points, say $S \subset M$. In this case, if $p \in S$, we select from the family of 1-forms $\Omega_{f,p}$ those forms which might generate foliations which have S locally as a leaf (see (2.1)). It will turn out that the existence of such forms is dynamically very relevant. Indeed f is said *non-tangential* on S if such forms do not exist, *tangential* if they do. In case f is tangential on S, starting from such forms, we define a flat connection for the normal bundle of (the regular part of) S, outside the singularities of f on S. In particular, in case f has no singularities on S and S is non singular then we have a vanishing theorem (see Theorem 4.6). If S is compact (no matters whether non singular or not) and f has singularities on S then the characteristic classes of S localize around the singularities of f on S producing "residual indices" (see Theorems 5.2, 6.2) and we recover the index theorems of [2] and [4]. All these results can also be regarded as topological obstructions for a curve to be the fixed points locus of a (tangential) holomorphic self-map of M.

The second target of these notes is to study the dynamics near a curve $S \subset M$ of fixed points of f, in case f acts as the identity on the normal bundle N_S of (the regular part of) S. Let p be a non singular point of S. The differential df_p has two eigenvalues (counting multiplicity) at p. One must be 1. The other eigenvalue gives the action of f on $N_{S,p}$. If this eigenvalue has modulo $\neq 1$ then the *center stable/unstable manifold theorem* [21] provides a clear picture of the dynamics of f near S at p (see also [16]). Here we deal with the case where 1 is the only eigenvalue of df on S. This situation is the one we find blowing up a fixed point $q \in M$ where $df_q = Id$, and therefore seems to deserve a special care. After relating the previous work about indices and singularities to blow-ups, we give an algorithm for producing parabolic curves starting with a curve of fixed points whose index is not a positive rational number. Our algorithm is a generalization (and "translation to discrete dynamics") of J. Cano's work [8]. Our argument allows new results also in the holomorphic foliations case, even if we are not going to explicitly state them here. This is the key to several results. For instance, we show that f is non-tangential on a curve S (and identically acting on the normal bundle N_S) if and only if f has a parabolic curve at all but a discrete set of points of S (see Proposition 7.12). Finally we show that if S is compact, globally and locally irreducible and $S \cdot S < 0$ then there exists at least a point of S where f has at least one parabolic curve (see Theorem 7.14). From this we give a new proof of Theorem 0.1.

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1. ONE JET OF FOLIATIONS ATTACHED TO FIXED POINT GERMS

Let M be a two dimensional complex manifold, $p \in M$ and let $f : M \to M$ be holomorphic and such that $f(p) = p$, $df_p = Id$. For $p \in S$ we denote by $\mathcal{O}_p$ the ring of germs of holomorphic functions at p. For $y, x \in \mathcal{O}_p$ let

$$\omega^{y,x,p} := (y \circ f - y)dx - (x \circ f - x)dy.$$

Let us consider the family of germs of holomorphic foliations given by

$$\Omega_{f,p} := \{\omega^{y,x,p} = 0 : y, x \in \mathcal{O}_p, dy_p \wedge dx_p \neq 0\}. \tag{1.1}$$

Note that the vector field

$$X^{y,x,p} := (x \circ f - x)\frac{\partial}{\partial x} + (y \circ f - y)\frac{\partial}{\partial y},$$

is the dual of $\omega^{y,x,p}$. That is to say, $X^{y,x,p}$ generates the foliation defined by $\ker \omega^{y,x,p}$ near p. In other words, one may think of associating to f the foliation generated by the "vector field" $f(p) - p$, except that this is not well defined and depends on the coordinates chosen. However we will show that the first jet of this "vector field" is independent of the coordinates (up to nonzero multiples) and this allows to well-define a first jet of holomorphic foliation associated to f. The dynamics behavior of f is then read by the dynamics of this first jet of holomorphic foliation. We proceed in formalizing this argument.

Let $\hat{\omega}^{y,x,p}$ be the *saturated* of the form $\omega^{y,x,p}$. That is, the form $\hat{\omega}^{y,x,p}$ is obtained from $\omega^{y,x,p}$ dividing its coefficients by their greatest common divisor in $\mathcal{O}_p$. Note that $\hat{\omega}^{y,x,p} = \omega^{y,x,p}$ if and only if p is an isolated fixed point of f.

Let $\varrho := [a_{11}x + a_{12}y + q_1(x,y)]dx - [a_{21}x + a_{22}y + q_2(x,y)]dy$ be a holomorphic one form, with $a_{ij} \in \mathbb{C}$ and $q_j(x,y)$ of order at least two in $(0,0)$. By definition, the *linear part of* ϱ, denoted by $J^1_{(0,0)}\varrho$, is given by the linear transformation $(x,y) \mapsto (a_{21}x + a_{22}y, a_{11}x + a_{12}y)$ and its eigenvalues are called the *eigenvalues of* ϱ at $(0,0)$.

Remark 1.1. Let $U \subset M$ be a coordinate set and $\phi : U \to \mathbb{C}^2$ a local chart. Assume that $\mathrm{Fix}(f) \cap U = \{l = 0\}$ for a suitable $l \in \mathcal{O}(U)$. Then

$$\phi \circ f \circ \phi^{-1} = Id + (l \circ \phi^{-1})^T G, \tag{1.2}$$

for some germ $G = (G_1, G_2)$ of holomorphic self-map of $\mathbb{C}^2$ at $(0,0)$, $G \not\equiv 0$ on $\mathrm{Fix}(f) \cap U$ and $T \geq 1$. As a matter of notations, we will omit to write explicitly the local chart ϕ when not indispensable, *e.g.*, we write simply $f = Id + l^T G$ instead of (1.2). Also, we denote by h' the gradient of $h \in \mathcal{O}_p$ in the given local chart, and by $\langle H, K \rangle$ the scalar product of two germs H, K of holomorphic self-maps of $\mathbb{C}^2$. With these notations, for any $H \in \mathcal{O}_p$ it holds

$$H \circ f - H = \langle H', l^T G \rangle + O(l^{T+1}),$$

where $O(l^{T+1})$ denotes terms divisible by l^{T+1}.

Lemma 1.2. *Let $z, w \in \mathcal{O}_p$ be such that $dz_p \wedge dw_p = \delta dx_p \wedge dy_p \neq 0$ for some $\delta \neq 0$. Then*

(1) $\hat{\omega}^{w,z,p}[p] = \delta \hat{\omega}^{y,x,p}[p]$.
(2) *If $\omega^{y,x,p}[p] = 0$ then $J^1_p \hat{\omega}^{w,z,p} = \delta J^1_p \hat{\omega}^{y,x,p}$.*

Proof. We are going to prove the statement in local coordinates $\{U, (x,y)\}$ such that $p = (0,0)$. Write $f = Id + hG$ with $h \in \mathcal{O}_{(0,0)}$ of order ≥ 0 at $(0,0)$ and $G = (G_1, G_2)$ a germ of holomorphic self-map of $\mathbb{C}^2$ at $(0,0)$ with G_1, G_2 relatively prime in $\mathcal{O}_{(0,0)}$. Note that $h = 0$ is the fixed points set of f at $(0,0)$, thus its order at $(0,0)$ is 0 if and only if f has an isolated fixed point at p. For $H \in \mathcal{O}_p$ we denote by $J^j H$ the term of order j in its expansion at $p = (0,0)$.

Indicating with R_1 the terms of order ≥ 1 at $(0,0)$, we have

$$\begin{aligned}\frac{(w\circ f-w)}{h}dz-\frac{(z\circ f-z)}{h}dw\\ &=(z_x\langle w',G\rangle-w_x\langle z',G\rangle)dx+(z_y\langle w',G\rangle-w_y\langle z',G\rangle)dy+R_1\\ &=[J^0z_x\langle J^0w',J^0G\rangle-J^0w_x\langle J^0z',J^0G\rangle]dx\\ &\quad+[J^0z_y\langle J^0w',J^0G\rangle-J^0w_y\langle J^0z',J^0G\rangle]dy+R_1\\ &=J^0(z_xw_y-z_yw_x)(J^0G_2dx-J^0G_1dy)+R_1=\delta(J^0G_2dx-J^0G_1dy)+R_1,\end{aligned}$$

which proves the first statement. Now assume $J^0G=(0,0)$. Then the first jet of $\hat{\omega}^{w,z,p}$ is given by

$$[J^0z_x\langle J^0w',J^1G\rangle-J^0w_x\langle J^0z',J^1G\rangle]dx\\ +[J^0z_y\langle J^0w',J^1G\rangle-J^0w_y\langle J^0z',J^1G\rangle]dy,$$

and a calculation similar to the previous one gives the second claimed result. □

Note that if $\omega^{y,x,p}[p]\neq 0$ then it might happen that $J^1_p\hat{\omega}^{y,x,p}=0$ but $J^1_p\hat{\omega}^{w,z,p}\neq 0$ for some $z,w\in\mathcal{O}_p$.

Definition 1.3. We say that p is a *singularity of* f if $\hat{\omega}^{y,x,p}[p]=0$ for some $y,x\in\mathcal{O}_p$ such that $dx_p\wedge dy_p\neq 0$.

By Lemma 1.2 a point is a singularity of f if and only if $\hat{\omega}^{y,x,p}[p]=0$ for *all* $z,w\in\mathcal{O}_p$.

Remark 1.4. Choose local coordinates (x,y) near p such that $p=(0,0)$ and $f=Id+G$, with G a germ of holomorphic self-map of $\mathbb{C}^2$ at $(0,0)$. Let $G=(G_1,G_2)=h(G_1^\circ,G_2^\circ)$ with h the greatest common divisor of G_1, G_2 and G_1° and G_2° coprime in $\mathcal{O}_p$. In [2] the *pure order* of f at p is defined as the minimum of the order of vanishing of G_j°, $j=1,2$, at p. In [2] a point is a singularity for f if the pure order is at least one. Therefore a point is a singularity of f according to Definition 1.3 if and only if it is a singularity according to [2].

Assume that $p\in M$ is a singularity of f. By Lemma 1.2 all the forms $\hat{\omega}^{y,x,p}$ have the same linear part up to nonzero multiples. Thus all the saturated of the foliations in $\Omega_{f,p}$ coincides at the first order at p. In particular one can define the "reduced singularities" for f according to the type of singularities of the family of saturated of Ω_f.

More precisely, let p be a singularity of f. Let $\lambda_1^{w,z,p},\lambda_1^{w,z,p}\in\mathbb{C}$ denote the eigenvalues of $J^1_p\hat{\omega}^{w,z,p}$. By Lemma 1.2 it follows that $\lambda_1^{w,z,p}=\lambda_1^{w,z,p}=0$ if this is so for all $z,w\in\mathcal{O}_p$, and if $\lambda_1^{w,z,p}\neq 0$ then the ratio $\lambda_2^{w,z,p}/\lambda_1^{w,z,p}$ is independent of z,w.

Definition 1.5. Let $p\in M$ be a singularity for f. We say that p is a *reduced singularity* for f if p is a reduced singularity for $\hat{\omega}^{w,z,p}=0$ for some—and hence any—w,z. That is to say, if $\lambda_1^{w,z,p},\lambda_1^{w,z,p}$ are the eigenvalues of $J^1_p\hat{\omega}^{w,z,p}$ then

($\star_1$) either the eigenvalues $\lambda_1^{w,z,p},\lambda_1^{w,z,p}\neq 0$ and $\lambda_1^{w,z,p}/\lambda_2^{w,z,p}\notin\mathbb{Q}^+$ or

$(\star_2)$ $\lambda_1^{w,z,p} \neq 0$ and $\lambda_2^{w,z,p} = 0$.

2. CURVES OF FIXED POINTS AND SINGULARITIES

Let M be a complex two dimensional manifold, S a (possibly singular) irreducible curve in M, $f : M \to M$ holomorphic such that $f|_S = Id_S$, $f \not\equiv Id_M$. Let $\mathcal{I}(S)_p \subset \mathcal{O}_p$ be the ideal of germs vanishing on S.

If $U \subset M$ is a coordinate set, $l \in \mathcal{O}_p$ a defining function for S at p then $f = Id + l^T G$ for some germ $G = (G_1, G_2)$ of holomorphic self-map of $\mathbb{C}^2$ at $(0,0)$, $G \not\equiv 0$ on S and $T \geq 1$. It is easy to see that T is independent of the chosen chart of the defining function l. We call $T_p(f,S) := T$ the *order of f on S at p.*

Note that if $H \in \mathcal{O}_p$ then

$$\frac{H \circ f - H}{l^T} \equiv \langle H', G \rangle \bmod \mathcal{I}(S)_p$$

Definition 2.1. We say that f is *tangential* on S at p if for a defining function l of S at p

$$\frac{l \circ f - l}{l^T} \equiv 0 \bmod \mathcal{I}(S)_p,$$

i.e., if $\langle l', G \rangle \equiv 0$ on S near p.

Remark 2.2. In [2] and [4], the word *non-degenerate* is used instead of *tangential*. However, as it should be clear after Proposition 2.4, it seems preferable to adopt this terminology.

Note that if f is tangential on S at p for some defining function l then it is so for any defining function. For the proof of this and for a detailed discussion of tangential conditions we refer the reader to [4]. Here we content ourselves to state the following result from [4]:

Proposition 2.3. *If the curve S is globally irreducible then f is non-tangential at $p \in S$ if and only if f is non-tangential at $q \in S$ for every $q \in S$.*

Let Σ_S be the set of singular points of S. Let $U \subset M$ be a coordinate set with coordinates functions (x,y). Let l be a defining function of S on U, *i.e.*, $S \cap U = \{(x,y) : l(x,y) = 0\}$ with $dl_p \neq 0$ for any $p \in U \cap (S - \Sigma_S)$. Up to shrink U, we can assume that $dl_p \neq 0$ for any $p \in U - \Sigma_S$. Let $\tau \in \mathcal{O}(U)$ be such that $d\tau_q \neq 0$ for any $q \in U$ and $d\tau_p \wedge dl_p \neq 0$ for any $p \in (U - \Sigma_S) \cap S$. We call such a τ a *transverse* to S.

Let $T := T_p(f,S)$ be the order of f on S at p for some $p \in U \cap S$. Using the coherence of the ideals sheaf of S it is easy to see that $T_q(f,S) = T$ for any $q \in U \cap S$ (see the proof of *Lemma 2 in* [4]). Therefore the following is a well defined holomorphic one form on U:

$$\omega^{l,\tau} := \frac{\tau \circ f - \tau}{l^T} dl - \frac{l \circ f - l}{l^T} d\tau. \tag{2.1}$$

We have the following proposition which justifies the name "tangential" given in Definition 2.1:

Proposition 2.4. *The map f is tangential on $S \cap U$ if and only if S is a leaf for the family of holomorphic foliations on U given by $\{\omega^{l,\tau} = 0\}$ when varying l among the defining functions of S and τ among the transverses to S.*

Proof. The map f is tangential on S if and only if there exists $\tilde{h} \in \mathcal{O}(U)$ such that

$$l \circ f - l = l^{T+1}\tilde{h}.$$

Thus $\omega^{l,\tau}|_S \equiv 0$ if and only if f is tangential on S. □

Note that, even if f is tangential on S, the foliations $\{\omega^{l,\tau} = 0\}$ on U really depend on l and τ; in particular, apart from $S \cap U$, they generally do not share other leaves.

Assume $p \in \Sigma_S$. Then $dl_p = 0$ for any defining function of S. Therefore in this case $\omega^{l,\tau}[p] = 0$ for any defining function l of S and any transverse τ. In particular p is a singularity for all the family of foliations $\omega^{l,\tau} = 0$.

Now suppose $p \in S \setminus \Sigma_S$. Since $dl_p \wedge d\tau_p \neq 0$ then by Lemma 1.2 it follows that if $\mathrm{Fix}(f) = S$ near p then p is a singularity for f if and only if $\omega^{l,\tau}[p] = 0$, *i.e.*, p is a singularity for f if and only if it is a singularity for all the family of foliations $\omega^{l,\tau} = 0$. Therefore *if f is tangential on S at p and Fix$(f) = S$ near p, then $p \in S \setminus \Sigma_S$ is a singularity for f if and only if there exists one—and hence any—transverse τ to S such that*

$$\frac{\tau \circ f - \tau}{l^T}[p] = 0.$$

On the other hand, in case $\mathrm{Fix}(f)$ is the union of S and another curve S' at $p \in S \setminus \Sigma_S$, the point p might be a singularity for the family of foliations $\omega^{l,\tau} = 0$ but not a singularity for f according to our definition.

It is easy to see that if $\mathrm{Fix}(f)$ at a point p contains two non singular curves S, S' intersecting transversally at p and f is tangential on S and on S' then p is necessarily a singularity for f.

Singularities are the only relevant points for dynamics on tangential curves, indeed we have

Proposition 2.5 (Abate, [2])**.** *Let M be a two dimensional manifold, $S \subset M$ a non singular curve and $p \in S$. Let $f : M \to M$ be holomorphic and such that Fix$(f) = S$ near p. Assume f is tangential on S. If p is not a singularity for f on S then p cannot be an attracting point for f; in particular there are no parabolic curves for f at p.*

On the other hand, a reduced $(\star_1)$ singularity on a tangential curve implies existence of parabolic curves:

Theorem 2.6 (Abate, Hakim)**.** *Let M be a two dimensional complex manifold, $f : M \to M$ holomorphic, $S \subset M$ a curve such that $f|_S = Id|_S$. Let $p \in S$ be a non singular point of S and suppose f is tangential on S at p. If Fix$(f) = S$ near p and p is a reduced $(\star_1)$ singularity of f then there exists at least one parabolic curve for f at p not contained in S.*

For a proof of this result see [2], [12], where a more precise statement about the actual number of parabolic curves is given.

We end up this section with some remarks about the case of a non singular curve of fixed points. Suppose thus S is non singular. Then one can choose local coordinates $\{(x,y),U\}$ around p such that $p=(0,0)$, $S\cap U=\{y=0\}$ and, if we write $f=(f_1,f_2)$,

$$\begin{cases} f_1(x,y) &= x+y^\mu g(x,y) \\ f_2(x,y) &= b(x)y+y^\nu h(x,y) \end{cases} \tag{2.2}$$

for some holomorphic functions g,h such that $g(x,0)\not\equiv 0$, $h(x,0)\not\equiv 0$ and natural numbers $\mu\geq 1$, $\nu\geq 2$. Note that if $b(x)=1$ then $T_p(f,S)=\min\{\mu,\nu\}$ and if $b(x)\neq 1$ then $T_p(f,S)=1$. A straightforward computation shows

$$df_{(x,0)}=\begin{pmatrix} 1 & * \\ 0 & b(x) \end{pmatrix}$$

where "$*$" is certainly 0 on $U\cap S$ if $\mu\geq 2$.

Let $N_S:=TM|_S/TS$ be the *normal bundle* of S in M. In the local coordinates $(U,(x,y))$ the projection $[\frac{\partial}{\partial y}]$ of $\frac{\partial}{\partial y}$ under the natural map $TM|_S\to N_S$ is a base frame for N_S over $U\cap S$. Therefore the action of f on N_S over $U\cap S$ is given by

$$[\frac{\partial}{\partial y}]\mapsto [df(\frac{\partial}{\partial y})]=[b(x)\frac{\partial}{\partial y}].$$

Hence $b(x)\equiv 1$ if and only if the action of f over N_S is the identity.

Since N_S has rank one, if S is compact then the action of f on N_S is constant and hence $b(x)\equiv b(f)$ is a constant. Therefore $b(x)\equiv 1$ if and only if f acts on N_S as the identity. That is to say, if S is non singular and compact the spectrum of df_p is $\{1\}$ at some—and hence any—point $p\in S$ if and only if f acts as the identity on N_S.

We remark that if $b(x)\neq 1$ at $p\in S$ then $T_p(f,S)=1$ and f is non-tangential on S at p since $(y\circ f-y)/y=b(x)-1\not\equiv 0$ on $y=0$.

Remark 2.7. Suppose f is given by (2.2) and acts on N_S as the identity (then $b(x)=1$). Up to a linear change of coordinates we can always assume that either $T_p(f,S)=\mu=\nu$ if f is non-tangential on S or $T_p(f,S)=\mu=\nu-1$ if f is tangential on S. In particular if f is non-tangential on S then $\mu\geq 2$ and $df=Id$ along S.

3. REDUCTION OF SINGULARITIES

Let M be a two dimensional complex manifold and let $f:M\to M$ be holomorphic. Let $p\in M$. A *blow-up* or *quadratic transformation* of p is a two dimensional complex manifold $\tilde{M}$ together with a proper holomorphic map $\pi:\tilde{M}\to M$ such that $D:=\pi^{-1}(p)$, called the *exceptional divisor*, is a projective complex line and $\pi:\tilde{M}-D\to M-\{p\}$ is biholomorphic (see, *e.g.*, [13]). Suppose that $p\in M$ is a singularity for f. By definition p is a singularity for $\hat{\omega}^{w,z,p}$ for all $\omega^{w,z,p}\in\Omega_{f,p}$. The 1-forms $\pi^*(\hat{\omega}^{w,z,p})$ are identically zero on D. However one may "saturize" them dividing the coefficients of the forms by their greatest common divisor in order to obtain 1-forms $\tilde{\omega}^{w,z,p}$ with only isolated singularities on D. Once fixed $w,z\in\mathcal{O}_p$, the well known theorem of Seidenberg (see, *e.g.* [6]) assures that after a finite number of blow-ups one obtains a complex two dimensional manifold, still denoted by $\tilde{M}$, together with a proper holomorphic map, still denoted by $\pi : \tilde{M} \to M$ such that

(1) $D := \pi^{-1}(p) = \cup_{\alpha=1}^{N} D_\alpha$ has only *normal crossing singularities*; namely D_α's are complex projective lines intersecting transversally each other and no three of them intersecting at one point;
(2) $\pi : \tilde{M} - D \to M - \{p\}$ is biholomorphic;
(3) The 1-form $\tilde{\omega}^{w,z,p}$ has only isolated reduced singularities on D.

If $\pi : \tilde{M} \to M$ is a quadratic transformation of $p \in M$, the map f induces a holomorphic map $\tilde{f} : \tilde{M} \to \tilde{M}$ such that $\pi \circ f = \tilde{f} \circ \pi$ and $\tilde{f}$ acts on the exceptional divisor D as df_p, if df_p is invertible (see [1]). In particular if p is a singularity for f then $df_p = Id$ and $\tilde{f}|_D = Id|_D$. The map $\tilde{f}$ has isolated singularities on D.

Remark 3.1. Assume $f(p) = p$ and $df_p = Id$. A direct calculation shows that the action of $\tilde{f}$ on the normal bundle N_D of the exceptional divisor $D = \pi^{-1}(p)$ in M is the identity, *i.e.*, $b(\tilde{f}) = 1$.

In [2] Abate proves directly an analogous of Seidenberg's reduction theorem for the map f (see *Theorem 2.3 in* [2]). Here we give another version of such a theorem, with a simpler proof based on Seidenberg's theorem. Before that, we need another definition:

Definition 3.2. Let $p \in M$ be such that $f(p) = p$ and $df_p = Id$. Let $\pi : \tilde{M} \to M$ be the blow-up at p such that $D := \pi^{-1}(p)$ is a complex projective line. We say that p is *dicritical* for f if $\tilde{f} : \tilde{M} \to \tilde{M}$ is non-tangential on D.

The point p is dicritical for f if and only if it is dicritical for any foliation $\hat{\omega}^{w,z,p} = 0$. Indeed $\tilde{f}$ is non-tangential on D if and only if D is not invariant for the saturated of $\pi^*(\hat{\omega}^{w,z,p}) = 0$ for any w, z, which is the definition of *dicritical point* for foliations (see, *e.g.*, [6]).

Theorem 3.3 (Reduction Theorem). *Let M be a two dimensional complex manifold. Let $f : M \to M$ be holomorphic. Let $p \in M$ be a singularity of f. Then there exists a two dimensional complex manifold $\tilde{M}$, a proper holomorphic map $\pi : \tilde{M} \to M$ and a holomorphic map $\tilde{f} : \tilde{M} \to \tilde{M}$ such that*

(1) *$D := \pi^{-1}(p) = \cup_{\alpha=1}^{N} D_\alpha$ has only normal crossing singularities.*
(2) *$\pi : \tilde{M} - D \to M - \{p\}$ is biholomorphic.*
(3) *$\pi \circ \tilde{f} = f \circ \pi$.*
(4) *$\tilde{f}|_D = Id|_D$.*
(5) *$\tilde{f}$ has only isolated reduced or dicritical singularities on D.*

Proof. Let p be a non-dicritical singularity for f. Let $\{U,(x,y)\}$ be local coordinates around p so that $p=(0,0)$ and write $f=(f_1,f_2)$ as

$$\begin{cases} f_1(x,y)= & x+l(x,y)g(x,y) \\ f_2(x,y)= & y+l(x,y)h(x,y) \end{cases} \tag{3.1}$$

where g,h are coprime and l does not divide g and h in $\mathcal{O}_p$. Note that we may assume $l\equiv 1$ if and only if $\mathrm{Fix}(f)=\{p\}$ near p. Moreover, since p is a singularity for f then $g(0,0)=0$ and $h(0,0)=0$. Let $\mu(g)\geq 1$ (*respect.* $\mu(h)\geq 1$) be the order of vanishing of g (*respect.* of h) at $(0,0)$. Write the homogeneous polynomials expansion of g and h as follows

$$g(x,y)=g_{\mu(g)}(x,y)+g_{\mu(g)+1}(x,y)+g^{\bullet}(x,y),$$
$$h(x,y)=h_{\mu(h)}(x,y)+h_{\mu(h)+1}(x,y)+h^{\bullet}(x,y).$$

Note that $g_{\mu(g)}\not\equiv 0$, $h_{\mu(h)}\not\equiv 0$. Let $\mu(l)\geq 0$ be the order of vanishing of l at $(0,0)$. Clearly $\mu(l)+\mu(g)\geq 2$, $\mu(l)+\mu(h)\geq 2$. Let $\pi:\tilde{M}\to M$ be the blow-up at p. Let (u,v) be local coordinates on $\tilde{M}$ such that $\pi(u,v)=(u,uv)$ and $D:=\pi^{-1}(0,0)=\{u=0\}$. Write $\tilde{f}=(\tilde{f}_1,\tilde{f}_2)$ and let $l(u,uv)=u^{\mu(l)}\tilde{l}$, with $\tilde{l}(0,v)\not\equiv 0$. Then

$$\begin{aligned}\tilde{f}_1(u,v)=&u+l(u,uv)g(u,uv)=u+\tilde{l}u^{\mu(l)+\mu(g)}[g_{\mu(g)}(1,v)+O(|u|)],\\ \tilde{f}_2(u,v)=&\frac{uv+l(u,uv)h(u,uv)}{u+l(u,uv)g(u,uv)}=v+\tilde{l}u^{\mu(l)-1}[u^{\mu(h)}h_{\mu(h)}(1,v)\\ &+u^{\mu(h)+1}h_{\mu(h)+1}(1,v)-u^{\mu(g)}vg_{\mu(g)}(1,v)-u^{\mu(g)+1}vg_{\mu(g)+1}(1,v)\\ &-u^{\mu(l)+\mu(g)+\mu(h)-1}\tilde{l}h_{\mu(h)}(1,v)g_{\mu(g)}(1,v)+O(|u|^{\min(\mu(g),\mu(h))+2})]\end{aligned}$$

where, as usual, $O(|u|^m)$ stands for terms of order at least m in u. From this expression follows that p *is dicritical for* f *if and only if* $\mu(g)=\mu(h)$ *and* $yg_{\mu(g)}(x,y)=xh_{\mu(h)}(x,y)$.

Let $\tilde{p}:=(0,v)$. Choose $w=u$, $z=v$.

First suppose $\mu(h)>\mu(g)$. Then $\mu(h)+\mu(l)\geq 3$ and

$$\begin{aligned}\hat{\omega}^{u,v,\tilde{p}}=&[ug_{\mu(g)}(1,v)+O(|u|^2)]dv\\ &-[-vg_{\mu(g)}(1,v)-vug_{\mu(g)+1}(1,v)+u^{\mu(h)-\mu(g)}h_{\mu(h)}(1,v)+O(|u|^2)]du\end{aligned}$$

On the other hand, a straightforward calculation shows that

$$\pi^*(\hat{\omega}^{x,y,p})=\{ug(u,uv)dv-[h(u,uv)-vg(u,uv)]du\},$$

that is, the saturated $[\pi^*(\hat{\omega}^{x,y,p})]^{\upharpoonright}$ is given by

$$\begin{aligned}[\pi^*(\hat{\omega}^{x,y,p})]^{\upharpoonright}=&[ug_{\mu(g)}(1,v)+O(|u|^2)]dv\\ &-[-vg_{\mu(g)}(1,v)-vug_{\mu(g)+1}(1,v)+u^{\mu(h)-\mu(g)}h_{\mu(h)}(1,v)+O(|u|^2)]du.\end{aligned}\tag{3.2}$$

Therefore $\tilde{p}$ is a singularity for $\tilde{f}$ if and only if it is a singularity for $[\pi^*(\hat{\omega}^{x,y,p})]^{\upharpoonright}$ and the part of lowest degree of $\hat{\omega}^{u,v,\tilde{p}}$ is equal to the part of lowest degree of $[\pi^*(\hat{\omega}^{x,y,p})]^{\upharpoonright}$ at $\tilde{p}$.

If $\mu(h) < \mu(g)$ then a similar reasoning leads to the same conclusion.

Suppose finally $\mu(g) = \mu(h)$. This implies that $\mu(l) + \mu(g) \geq 2$. As observed before, since p is non-dicritical for f then $h_{\mu(h)}(1, v) - v g_{\mu(g)}(1, v) \neq 0$ and therefore

$$\begin{aligned}\hat{\omega}^{u,v,\tilde{p}} = [u g_{\mu(g)}(1, v) + O(|u|^2)]dv \\ - [h_{\mu(h)}(1, v) - v g_{\mu(g)}(1, v) + u(h_{\mu(h)+1}(1, v) - v g_{\mu(g)+1}(1, v)) \\ + u^{\mu(l)+\mu(g)-1}\tilde{l} h_{\mu(h)}(1, v) g_{\mu(g)}(1, v) + O(|u|^2)]du\end{aligned}$$

If $\mu(l) + \mu(g) > 2$ we can argue as before to find the same conclusion. So we are left to analyze the case $\mu(l) + \mu(g) = 2$. The singularity for $\tilde{f}$ on the exceptional divisor D are given by $(0, v_0)$ where v_0 is such that

$$h_{\mu(h)}(1, v_0) - v_0 g_{\mu(g)}(1, v_0) = 0.$$

By (3.2), these are exactly the singularities of $[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$. But, unless $h_{\mu(h)}(1, v_0) = 0$, the linear part of $[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$ at $(0, v_0)$ is different to the linear part of $\hat{\omega}^{u,v,\tilde{p}}$ at $(0, v_0)$. However a straightforward calculation shows that the eigenvalues of $\hat{\omega}^{u,v,\tilde{p}}$ at $(0, v_0)$ are given by $g_{\mu(g)}(1, v_0)$ and $\frac{\partial}{\partial v}(-h_{\mu(h)}(1, v) + v g_{\mu(g)}(1, v))_{|v=v_0}$, which are exactly the eigenvalues of $[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$ at $(0, v_0)$.

Summing up, we have shown that the singularities of $\hat{\omega}^{u,v,\tilde{p}}$ on the chart (u, v) of D are exactly the singularities of $[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$ on such a chart and also that the part of lowest degree or, when this degree is one, the eigenvalues of $\hat{\omega}^{u,v,\tilde{p}}$ are equal to the part of lowest degree or to the eigenvalues of $[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$ at such singularities. The same holds for the other chart of D and hence the result follows from the Seidenberg reduction theorem applied to the 1-form $\hat{\omega}^{x,y,p}$. □

Remark 3.4. Let f be given by (3.1). In the proof of Theorem 3.3 we saw that the point p is dicritical for f if and only if $\mu(h) = \mu(g)$ and $y g_{\mu(g)}(x, y) \equiv x h_{\mu(h)}(x, y)$.

Remark 3.5. Note that, with the notations of the proof of Theorem 3.3, in general at a singularity $\tilde{p}$ of $\tilde{f}$ we have $J^1_{\tilde{p}}\hat{\omega}^{u,v,\tilde{p}} \neq J^1_{\tilde{p}}[\pi^*(\hat{\omega}^{x,y,p})]\upharpoonleft$, even if the two linear parts have the same eigenvalues.

4. Connections on non singular curves and the vanishing theorem

Let M be a two dimensional complex manifold, $S \subset M$ a non singular curve. Let l be a defining function of S on U, *i.e.*, $S \cap U = \{(x, y) : l(x, y) = 0\}$ with $dl_p \neq 0$ for any $p \in U$. Let $\tau \in \mathcal{O}(U)$ be a transverse to S, *i.e.*, $dl_p \wedge d\tau_p \neq 0$ for any $p \in U$.

Let L be the line bundle associated to the divisor S. That is to say, if U_j, U_k are coordinate sets such that $U_j \cap U_k \neq \emptyset$ and S is given by $\{l_j = 0\}$ on U_j and by $\{l_k = 0\}$ on U_k, then $l_{jk} := \frac{l_j}{l_k} \in \mathcal{O}^*(U_j \cap U_k)$ are cocycle functions defining L. Then, since $l_{jk} dl_k = dl_j$ on $U_j \cap U_k \cap S$, it follows that $\{dl_j\}$ is a vector bundle homomorphism between $TM_{|S}$ and $L_{|S}$ whose kernel is TS. Therefore $L_{|S} \simeq N_S$, the normal bundle of S. We have the following exact sequence:

$$0 \longrightarrow TS \longrightarrow TM|_S \xrightarrow{dl} N_S \longrightarrow 0. \tag{4.1}$$

Remark 4.1. The morphism dl allows to define a natural holomorphic frame for N_S on $U \cap S$; that is to say, if v is a holomorphic section of TM such that $dl(v|_S) \equiv 1$ then $dl(v|_S)$ can be thought of as a holomorphic frame for N_S on U, which in the sequel we will always denote by E.

Let $f : M \to M$ be holomorphic such that $\mathrm{Fix}(f) = S$. Assume f is tangential on S and moreover suppose that f has no singularities on $S \cap U$. In the local coordinates $\{U, (x,y)\}$ write $f = (f_1, f_2)$ and

$$\begin{cases} f_1(x,y) &= x + l^T g(x,y), \\ f_2(x,y) &= y + l^T h(x,y). \end{cases} \tag{4.2}$$

Note that the map f is, by definition, tangential on S at p if and only if $l_x g + l_y h \equiv 0$ on S.

Let us define the following operator on $TS_p \times N_S|_U$ with value in N_S:

$$\begin{aligned} &\theta^{l,\tau} : T_pS \times N_S|_U \to N_{S,p} \\ &\theta^{l,\tau} : (X, s) \mapsto \theta^{l,\tau}_X(s) := dl_p([\tilde{X}, \tilde{s}]|_S), \end{aligned} \tag{4.3}$$

where $\tilde{s}$ is a section of TM near $S \cap U$ such that $dl(\tilde{s}|_S) = s$ and $\tilde{X}$ is a section of TM on U such that $\tilde{X}(p) = X$ and $\omega^{l,\tau}(\tilde{X}) = 0$ on U.

Lemma 4.2. *For any $X \in T_pS$ and $s = v \cdot E$ section of N_S over $S \cap U$ it follows*

$$\theta^{l,\tau}_X(s) = \left\{ X \cdot v - v \frac{l \circ f - l}{l(\tau \circ f - \tau)}[p] d\tau_p(X) \right\} E. \tag{4.4}$$

In particular $\theta^{l,\tau}_X(s)$ as defined in (4.3) *depends only on X and s and not on $\tilde{s}$, $\tilde{X}$ chosen to define it.*

Proof. Let $\tilde{s} = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ be such that $dl(\tilde{s}|_S) = v$. On U a basis for TS is given by $l_y \frac{\partial}{\partial x} - l_x \frac{\partial}{\partial y}$, therefore $X = \lambda(l_y \frac{\partial}{\partial x} - l_x \frac{\partial}{\partial y})[p]$ for some $\lambda \in \mathbb{C}$. For $T = T_p(f, S)$ the order of f on S at p, let us set

$$\begin{aligned} \tilde{g} =& \frac{\tau \circ f - \tau}{l^T}, \\ \tilde{h} =& \frac{l \circ f - l}{l^{T+1}}. \end{aligned}$$

Since $\omega^{l,\tau} = (-\tilde{h} l \tau_x + \tilde{g} l_x)dx + (-\tilde{h} l \tau_y + \tilde{g} l_y)dy$ it follows that

$$\tilde{X} = wk \left\{ (-\tilde{h} l \tau_y + \tilde{g} l_y)\tfrac{\partial}{\partial x} + (-\tilde{g} l_x + \tilde{h} l \tau_x)\tfrac{\partial}{\partial y} \right\}$$

where $k \in \mathcal{O}^*(U)$ and $w(p) = \frac{\lambda}{k(p)\tilde{g}(p)}$ (note that $\tilde{g} \neq 0$ on S by the previous discussion). Thus

$$\begin{aligned}[\tilde{X}, \tilde{s}] = \{&wk(-\tilde{h}l\tau_y + \tilde{g}l_y)A_x + wk(-\tilde{g}l_x + \tilde{h}l\tau_x)A_y \\ &- A\tfrac{\partial}{\partial x}(wk(-\tilde{h}l\tau_y + \tilde{g}l_y)) - B\tfrac{\partial}{\partial y}(wk(-\tilde{h}l\tau_y + \tilde{g}l_y))\}\tfrac{\partial}{\partial x} \\ &+ \{wk(-\tilde{h}l\tau_y + \tilde{g}l_y)B_x + wk(-\tilde{g}l_x + \tilde{h}l\tau_x)B_y \\ &- A\tfrac{\partial}{\partial x}(wk(-\tilde{g}l_x + \tilde{h}l\tau_x)) - B\tfrac{\partial}{\partial y}(wk(-\tilde{g}l_x + \tilde{h}l\tau_x))\}\tfrac{\partial}{\partial y}\end{aligned}$$

Therefore, after a straightforward calculation we find

$$\begin{aligned}dl_p([\tilde{X}, \tilde{s}]|_{l=0}) &= wkv\tilde{h}(l_x\tau_y - \tau_x l_y)[p] + wk\tilde{g}(-Bl_x l_{yy} + Bl_y l_{xy} \\ &\quad - Al_x l_{xy} + Al_y l_{xx} + l_x l_y A_x - l_x^2 A_y - l_y l_x B_y + l_y^2 B_x)[p] \\ &= v\lambda\frac{\tilde{h}}{\tilde{g}}(l_x\tau_y - \tau_x l_y)[p] - \lambda(l_y\tfrac{\partial}{\partial x} - l_x\tfrac{\partial}{\partial y})[p](dl_p(\tilde{s})) \\ &= -v\frac{\tilde{h}}{\tilde{g}}[p]d\tau_p(X) + X\cdot v,\end{aligned}$$

as wanted. □

Now we want to see how $\theta^{l,\tau}$ varies when varying τ and l.

Lemma 4.3. *Let $\tilde{\tau} \in \mathcal{O}(U)$ be such that $d\tilde{\tau} \wedge dl \neq 0$ on U. Let $\tilde{l} = ul$ for $u \in \mathcal{O}^*(U)$. Then for any $X \in T_pS$*

$$\frac{l\circ f - l}{l(\tau\circ f - \tau)}[p]d\tau_p(X) - \frac{\tilde{l}\circ f - \tilde{l}}{\tilde{l}(\tilde{\tau}\circ f - \tilde{\tau})}[p]d\tilde{\tau}_p(X) = -\{d(\log u)\cdot X\}. \tag{4.5}$$

Proof. Write $f = (f_1, f_2)$ as in (4.2). Let $T = T_p(f, S)$ be the order of f on S at p. Let $\tilde{h} = \frac{l\circ f - l}{l^{T+1}}$. From (4.4) it follows that, if $X = \lambda(l_y\frac{\partial}{\partial x} - l_x\frac{\partial}{\partial y}) \in TS|_U$,

$$\begin{aligned}&\left\{\frac{l\circ f - l}{l(\tau\circ f - \tau)}|_{l=0}d\tau - \frac{l\circ f - l}{l(\tilde{\tau}\circ f - \tilde{\tau})}|_{l=0}d\tilde{\tau}\right\}\cdot X \\ &= \tilde{h}|_{l=0}\frac{\tilde{\tau}_x\tau_y - \tau_x\tilde{\tau}_y}{(\tau_x g + \tau_y h)(\tilde{\tau}_x g + \tilde{\tau}_y h)}(-hdx + gdy)\cdot X \\ &= -\lambda\tilde{h}|_{l=0}\frac{\tilde{\tau}_x\tau_y - \tau_x\tilde{\tau}_y}{(\tau_x g + \tau_y h)(\tilde{\tau}_x g + \tilde{\tau}_y h)}(l_y h + l_x g) = 0,\end{aligned}$$

since $l_y h + l_x g = 0$ on S by definition of non-tangentiality. Now we may assume $\tau = x$. Then

$$\left\{\frac{l\circ f - l}{l^{T+1}g}|_{l=0}dx - \frac{ul\circ f - ul}{ul^{T+1}g}|_{l=0}dx\right\}\cdot X$$
$$= \left\{\frac{\tilde{h}}{g}|_{l=0}dx - l\frac{u\circ f - u}{l^{T+1}ug}|_{l=0}dx - (u\circ f)\frac{l\circ f - l}{ul^{T+1}g}|_{l=0}dx\right\}\cdot X$$
$$= \left\{\frac{\tilde{h}}{g}|_{l=0}dx - \frac{u_x g + u_y h}{ug}|_{l=0}dx - \frac{\tilde{h}}{g}|_{l=0}dx\right\}\cdot X = -\left\{\frac{u_x}{u} + \frac{u_y}{u}\frac{h}{g}\right\}dx\cdot X.$$

Since f is tangential on S, by definition, it follows that $\frac{h}{g} = -\frac{l_x}{l_y}$ on $\{l = 0\}$ and therefore

$$-\left\{\frac{u_x}{u} + \frac{u_y}{u}\frac{h}{g}\right\}dx\cdot X = \left\{\frac{-u_x l_y + u_y l_x}{ul_y}\right\}dx\cdot\lambda(l_y\tfrac{\partial}{\partial x} - l_x\tfrac{\partial}{\partial y})$$
$$= \lambda\frac{-u_x l_y + u_y l_x}{u} = -d(\log u)\cdot X,$$

as wanted. $\square$

Remark 4.4. Let χ be a (local) biholomorphism on U with values in U. Let $V \subseteq U$ a open set such that $f(\chi^{-1}(V)) \subseteq U$. Let $\tilde{\tau} = \tau\circ\chi^{-1}$, $\tilde{l} = l\circ\chi^{-1}$ and $\tilde{f} = \chi\circ f\circ\chi^{-1}_{|V}$. Then, since

$$\frac{\tilde{l}\circ\tilde{f} - \tilde{l}}{\tilde{l}(\tilde{\tau}\circ\tilde{f} - \tilde{\tau})}d\tilde{\tau} = \chi^*\left(\frac{l\circ f - l}{l(\tau\circ f - \tau)}d\tau\right)$$

it follows that $\theta^{\tilde{l},\tilde{\tau}}$ is the expression of $\theta^{l,\tau}$ in the local coordinates $(\chi_1(x,y),\chi_2(x,y))$.

Now let $\{U_j\}$ be a covering of S made of coordinate sets with local coordinates (x_j, y_j). For any j choose a defining function l_j such that $S\cap U_j = \{l_j = 0\}$. By (4.5) we may define on each U_j the operator

$$\theta^j := \theta^{l_j,\tau_j},$$

where τ_j is any transverse to S on U_j. Note that since N_S is a holomorphic line bundle and by (4.4) the operators θ^j can be viewed as $(1,0)$-connections on $N_S|_{U_j}$. We want to show that they indeed glue together to give a $(1,0)$-connection for all of N_S. This is the case if and only if the 1-forms

$$\eta^j := \left\{-\frac{l_j\circ f_j - l_j}{l_j(\tau_j\circ f_j - \tau_j)}|_{l_j=0}\right\}d\tau_j \tag{4.6}$$

defined on the U_j's are such that

$$\eta^j = \eta^k + \frac{du_{kj}}{u_{kj}}, \tag{4.7}$$

whenever $U_k \cap U_j \neq \emptyset$ and $\{u_{jk}\}$ is the system of cocycles defining N_S relative to $\{U_j\}$, and this follows at once from Remark 4.4 and (4.5). We denote by ∇ such a $(1,0)$-connection. Summing up we have proved:

Proposition 4.5. *Suppose M is a two dimensional complex manifold, $S \subset M$ a non singular curve. Let $f : M \to M$ be holomorphic and such that Fix$(f) = S$. Assume that f is tangential on S and that f has no singularities on S. Then there exists a $(1,0)$-connection ∇ for N_S such that, if U is a coordinate open set and (x, y) are local coordinates on U such that $S \cap U = \{y = 0\}$ then the connection 1-form with respect to the frame $[\frac{\partial}{\partial y}]$ of N_S on U is given by*

$$\left\{ -\frac{y \circ f - y}{y(x \circ f - x)}|_{y=0} \right\} dx. \tag{4.8}$$

Proof. We have

$$\nabla_{\frac{\partial}{\partial x}}([\frac{\partial}{\partial y}]) = \theta^{y,x}_{\frac{\partial}{\partial x}}([\frac{\partial}{\partial y}]),$$

and

$$\nabla_{\frac{\partial}{\partial \bar{x}}}([\frac{\partial}{\partial y}]) = 0.$$

From this and (4.4) the statement follows. □

Hence we have

Theorem 4.6 (Vanishing Theorem). *Let M be a two dimensional complex manifold, $S \subset M$ a non singular curve, $f : M \to M$ holomorphic such that Fix$(f) = S$ and f is tangential on S. If f has no singularities on S then there exists a connection ∇ (which we call the* basic connection*) for N_S such that its curvature $K \equiv 0$. In particular the first Chern class $c_1(N_S) = 0$.*

Proof. Let ∇ be the connection for N_S defined in Proposition 4.5. Locally ∇ is given by the 1-form η given by (4.8) in the natural frame E of N_S. Since η has holomorphic coefficients and $K = d\eta - \eta \wedge \eta$, it follows that $K \equiv 0$. Also $c_1(N_S) = -\frac{1}{2\pi i}[c_1(\nabla)] = 0$. □

Remark 4.7. If S is non-compact then $c_1(L) = 0$ for any line bundle L on S (this follows at once from the long exact sequence associated to the exponential sequence). However if S is compact then this is not generally true, and the result is not obvious.

5. Residual index theorem in the non singular case

We use the notations of the previous section. Suppose S is compact, connected and non singular and f is tangential on S. Then f has only a finite number of singularities on S. Let $\Sigma := \{p_1, \dots, p_r\}$ be such singularities and $V := S - \Sigma$. For $\alpha = 1, \dots, r$ let $\{U_\alpha\}$ be a coordinate set of M such that $T_\alpha := U_\alpha \cap S$ is non-empty and simply connected and $\{p_\alpha\} = \Sigma \cap T_\alpha$. Let ∇ be a basic connection for N_V as defined in the previous section. Let W_α be a simply connected open set in S such that $\overline{W_\alpha} \subset T_\alpha$. On each T_α let ∇_α be a connection for $N_S|_{T_\alpha}$. Let ψ be a C^∞ function on S such that ψ has support in $\cup_\alpha T_\alpha$ and $\psi|_{W_\alpha} \equiv 1$ for

$\alpha = 1, \dots, r$. Let $\nabla_1 := \psi \sum \nabla_\alpha + (1-\psi)\nabla$. Then ∇_1 is a connection for N_S and, if K_1 is its curvature, it follows that

$$\frac{-1}{2\pi i}[K_1] = c_1(N_S) \in H^2(S, \mathbb{C}).$$

Note that $K_1 = K$ on $S - \cup_\alpha T_\alpha$, where K is the curvature of ∇. Thus by Theorem 4.6, K_1 has compact support contained in $\cup T_\alpha$.

In particular by the Poincaré duality it follows

$$H^2(S, \mathbb{C}) \simeq H_0(S, \mathbb{C}) \simeq \mathbb{C},$$

where the first isomorphism is given by integration on S. Therefore the first Chern number of N_S, which equals the self-intersection number of S denoted by $S \cdot S$, is given by

$$S \cdot S = \frac{-1}{2\pi i} \int_S K_1 = \frac{-1}{2\pi i} \sum_\alpha \int_{T_\alpha} K_1. \tag{5.1}$$

Let us give the following definition:

Definition 5.1. The *residual index* of f on S at $p \in S$ is given by

$$\mathrm{Ind}(f, S, p) = \frac{1}{2\pi i} \int_{V_p} K_1,$$

where V_p is a simply connected open set in S containing p such that $V_p \cap \cup_\alpha T_\alpha = \emptyset$ if $p \notin \Sigma$ and $V_p = T_\alpha$ if $p = p_\alpha \in \Sigma$.

Note that $\mathrm{Ind}(f, S, p) = 0$ for any $p \in S$ which is not a singularity for f.

The index theorem (due to Abate, who first proved it by other methods, see [2]) follows now directly from (5.1):

Theorem 5.2 (Index Theorem in the non singular case)**.** *Let M be a complex two dimensional manifold, $S \subset M$ a non singular compact connected curve and $f : M \to M$ holomorphic such that Fix$(f) = S$ and f is tangential on S. Then*

$$\sum_{p \in S} \mathrm{Ind}(f, S, p) = S \cdot S.$$

Now we are going to show that $\mathrm{Ind}(f, S, p)$, for p a singularity of f, is in fact independent of the ∇_1 chosen to define it. To do so, we compute $\mathrm{Ind}(f, S, p)$ explicitly.

Let $p \in S$ be a singularity for f, $\{(x, y), U\}$ local coordinates for M such that $p = (0, 0)$ and $S \cap U = \{y = 0\}$. Let $T := T_p(f, S)$ be the order of f on S at p. Suppose $f = (f_1, f_2)$ is given by

$$\begin{cases} f_1(x, y) = & x + y^T g(x, y) \\ f_2(x, y) = & y + y^{T+1} h(x, y) \end{cases} \tag{5.2}$$

where $g(x, 0) \not\equiv 0$. Suppose V, W are simply connected open sets with smooth boundary in S such that $p \in V \subset \overline{V} \subset W \subset \overline{W} \subset U \cap S$ and $\psi \equiv 1$ on V and ψ has compact support in

W. Let η_1 be the connection 1-form of ∇_1 on U, η the connection 1-form of ∇ on $U-\{p\}$ and η_α the connection 1-form of ∇_α (where ∇_α is a connection for N_S on W). Then, in the natural frame $[\frac{\partial}{\partial y}]$, by (4.8) we have

$$\eta_1 = \psi\eta_\alpha + (1-\psi)\eta = \psi\eta_\alpha - (1-\psi)\frac{h(x,0)}{g(x,0)}dx.$$

Since $K_1 = d\eta_1 - \eta_1 \wedge \eta_1 = d\eta_1$ then Stokes' theorem implies (recall that $\psi|_{\partial W} \equiv 0$)

$$\begin{aligned}\mathrm{Ind}(f,S,p) = \frac{-1}{2\pi i}\int_W K_1 = \frac{-1}{2\pi i}\int_W d\eta_1 = \frac{-1}{2\pi i}\int_{\partial W}\eta_1 \\ = \frac{-1}{2\pi i}\int_{\partial W} -\frac{h(x,0)}{g(x,0)}dx = \mathrm{Res}(\frac{h(x,0)}{g(x,0)}dx; x=0).\end{aligned}$$

Thus in the case S is non singular we have

$$\mathrm{Ind}(f,S,p) = \mathrm{Res}(\frac{h(x,0)}{g(x,0)}dx; x=0),$$

which is exactly the index defined in [2].

Remark 5.3. 1. The index $\mathrm{Ind}(f,S,p)$ is exactly the Camacho-Sad index at the point p for the family of holomorphic foliations defined by $\omega^{l,\tau}=0$, where $\omega^{l,\tau}$ is given by (2.1).

2. In the hypotheses of Theorem 5.2 we assume that $\mathrm{Fix}(f)=S$. This hypothesis can be easily relaxed by simply asking for $f|_S = Id|_S$. As explained in section 2, if there is another curve S' of fixed points of f, intersecting S at a point p, then the foliation $\omega^{l,\tau}=0$ defined by (2.1) has a singularity at p, no matter whether p is a singularity for f or not. Therefore one cannot define the basic connection at p (see section 4). However one can add the point p to the set Σ and argue as before. Note that, even in this case, if p is not a singularity for f it follows from the formula above that $\mathrm{Ind}(f,S,p)=0$.

6. RESIDUAL INDEX THEOREM IN THE SINGULAR CASE

In this section we let S have some isolated singularity. Then the setting is the following: M is a two dimensional complex manifold, $S \subset M$ is a compact (globally) irreducible connected compact curve and $f: M \to M$ holomorphic such that $\mathrm{Fix}(f)=S$ and f is tangential on S.

Let $\Sigma := \Sigma_f \cup \Sigma_S$, where Σ_f is the set of singularities of f on S and Σ_S is the set of singularities of S. Let $\{U_j,(x_j,y_j)\}$ be local coordinates such that $S\cap U_j = \{l_j=0\}$. Let L be the line bundle associated to the divisor S, whose cocycles are given by $l_j/l_k \in \mathcal{O}^*(U_j\cap U_k)$. Let V be a open neighborhood of Σ, such that $V=\cup V_\alpha$ and the V_α's are pairwise disjoint, simply connected and so that each one contains only one point of Σ, say p_α. Moreover, if $T:=V\cap S$ and $T_\alpha := V_\alpha \cap S$ we require that ∂T_α is a smooth regular curve for any α. We are going to define a C^∞ connection $\tilde{\nabla}$ for L such that $\tilde{\nabla}$ restricted to $S-V$ coincides with the basic connection for N_{S-V} defined by the l_j's. To do this, let W be a tubular neighborhood of $S-\Sigma$ and $\rho: W\to S-\Sigma$ be the projection. Since $\rho^*(L|_{S-\Sigma})$ equals L once restricted to

$S-\Sigma$, then $\rho^*(L|_{S-\Sigma})$ is C^∞ isomorphic to L over W. On $L|_W$ we put the pull back connection $\tilde\nabla_0 := \rho^*(\nabla_0)$ where ∇_0 is the basic connection for $L|_{S-\Sigma}$ defined as in section 4. On each V_α let $\tilde\nabla_\alpha$ be any connection for $L|_{V_\alpha}$. For any α let $\tilde V_\alpha \subset V_\alpha$ be a simply connected open set such that $p_\alpha \in \tilde V_\alpha$. Let ψ be a C^∞ function on M such that $\psi \equiv 1$ on $\tilde V_\alpha$ for any α and $\mathrm{supp}(\psi) \subset V$. Finally let $\tilde\nabla := \psi \sum \tilde\nabla_\alpha + (1-\psi)\tilde\nabla_0$. Then $\tilde\nabla$ is a connection for L over a neighborhood of S. Therefore, if $\tilde K$ is the curvature of $\tilde\nabla$, we have

$$S\cdot S = \int_{[S]} c_1(L) = \frac{-1}{2\pi i}\int_{[S]} \tilde K \tag{6.1}$$

where $[S] \in H_2(M,\mathbb{C})$ is the homology class of S. Since $\tilde K = K = 0$ on $S - T$, where K is the curvature of ∇_0 on $S-\Sigma$, then it follows that

$$\frac{-1}{2\pi i}\int_{[S]} \tilde K = \frac{1}{2\pi i}\sum_\alpha \int_{T_\alpha} \tilde K. \tag{6.2}$$

We are now in the position to define the residual index at singular points of S:

Definition 6.1. The *residual index* of f on S at $p_\alpha \in \Sigma_S$ is given by

$$\mathrm{Ind}(f,S,p_\alpha) = \frac{-1}{2\pi i}\int_{T_\alpha} \tilde K.$$

From (6.1) and (6.2) it follows the index theorem in the singular case, already proved in [4] with a different technique:

Theorem 6.2 (Index Theorem in the singular case)**.** *Let M be a two dimensional complex manifold, $S \subset M$ a globally irreducible compact connected curve (possibly with singularities) and $f : M \to M$ holomorphic such that Fix(f) $= S$ and f is tangential on S. Then*

$$\sum_{p\in S} \mathit{Ind}(f,S,p) = S\cdot S.$$

Now we want to calculate the index in the case $p \in \Sigma_S$. Therefore we have to evaluate

$$\frac{-1}{2\pi i}\int_{T_\alpha} \tilde K.$$

We may suppose that on V_α the curve S is given by $l = 0$. By (4.4) the 1-form of the connection $\tilde\nabla$ on $S \cap V_\alpha$ with respect to the natural frame E associated to l is given by

$$\tilde\eta = \psi\eta_\alpha + (1-\psi)\left\{-\frac{l\circ f - l}{l(\tau\circ f-\tau)}|_{l=0}\right\} d\tau,$$

where τ is any transverse to S outside p and η_α is the connection 1-form of ∇_α (where ∇_α is a connection for L on V_α). Then $\tilde K = d\tilde\eta - \tilde\eta\wedge\tilde\eta = d\tilde\eta$ on T_α and by Stokes' theorem we have

$$\mathrm{Ind}(f,S,p) = \frac{-1}{2\pi i}\int_{T_\alpha}\tilde K = \frac{-1}{2\pi i}\int_{T_\alpha} d\tilde\eta = \frac{-1}{2\pi i}\int_{\partial T_\alpha}\tilde\eta = \frac{1}{2\pi i}\int_{\partial T_\alpha}\left\{\frac{l\circ f - l}{l(\tau\circ f-\tau)}\right\} d\tau,$$

as in [4].

Remark 6.3. Note that if ∂T_α is not connected, the integral is a sum of integrals each of which gives the index of f at p of the irreducible component of S intersecting such component of ∂T_α.

Remark 6.4. As in the non singular case, the index at a point $p \in S$ is exactly the Suwa-Camacho-Sad index for the family of holomorphic foliations $\omega^{l,\tau} = 0$ given by (2.1). Also, Theorem 6.2 holds with the hypothesis that $f|_S = Id|_S$ instead of $\mathrm{Fix}(f) = S$, see Remark 5.3.

7. DYNAMICS NEAR CURVES OF FIXED POINTS

In his paper, Abate [2] proves that if f is a (germ of) holomorphic diffeomorphism of $\mathbb{C}^2$ with an isolated fixed point $p \in \mathbb{C}^2$ and such that $df_p = Id$ then there exists at least one parabolic curve for f at p. In this section we are going to investigate what happens in the case $f : M \to M$ has a curve of fixed points and f acts as the identity on the normal bundle of the regular part of S.

In what follows we need these results about indices and blow-ups (see [2] and [4]):

Lemma 7.1. *Let M be a two dimensional complex manifold. Let $f : M \to M$ be holomorphic and $S \subset M$ a curve such that $f|_S = Id_S$ and f is tangential on S. Let $p \in S$ be such that S is irreducible at p. Let $\pi : \tilde{M} \to M$ be the blow-up of M at p, and $\tilde{f} : \tilde{M} \to \tilde{M}$ be the map induced by f. Let $D := \pi^{-1}(p)$ and let $\tilde{S} := \overline{\pi^{-1}(S - \{p\})}$ be the strict transform of S. Then*

1. *The map $\tilde{f}$ is tangential on $\tilde{S}$.*
2. *Let $\{\tilde{p}\} = D \cap \tilde{S}$. Then $\mathrm{Ind}(\tilde{f}, \tilde{S}, \tilde{p}) = \mathrm{Ind}(f, S, p) - m^2$, where $m \geq 1$ is the multiplicity of S at p (in particular $m = 1$ if and only if S is non singular at p).*

Now we can start our study of dynamics. First, we need some calculations for the indices:

Lemma 7.2. *Let M be a two dimensional complex manifold, $S \subset M$ a curve. Let $f : M \to M$ be holomorphic such that $f|_S = Id_S$ and f is tangential on each component of S.*

1. *If $p \in S$ is a non singular point of S dicritical for f then $\mathrm{Ind}(f, S, p) = 1$.*
2. *If $p \in S$ belongs to two non singular irreducible branches S_1, S_2 of S intersecting transversally at p and p is a non-dicritical reduced singularity $(\star_2)$ for f on S then either $\mathrm{Ind}(f, S_1, p) = 0$ or $\mathrm{Ind}(f, S_2, p) = 0$.*
3. *If $p \in S$ belongs to two non singular irreducible branches of S intersecting transversally at p and p is a non-dicritical reduced singularity $(\star_1)$ for f on S, then*
$$\mathrm{Ind}(f, S_1, p) \cdot \mathrm{Ind}(f, S_2, p) = 1.$$
4. *If $p \in S$ is a non singular point of S which is a non-dicritical reduced singularity $(\star_2)$ of f on S*
   - *either $\mathrm{Ind}(f, S, p) = 0$,*
   - *or, after one blow-up, the induced map $\tilde{f}$ has a reduced singularity $(\star_2)$ at the intersection of the strict transform of S with the exceptional divisor and a reduced singularity $(\star_1)$ on a non singular point of the exceptional divisor.*

The statement (2), (3) and (4) of Lemma 7.2 are in [2]. Note also that by Remarks 5.3 and 6.4 the previous Lemmas 7.1 and 7.2 follow from the same properties for foliations. We give here the proof of Lemma 7.2.(1) for it seems not to be written anywhere else.

Proof of Lemma 7.2.(1). By Remark 2.7 we may choose local coordinates (x, y) around p in such a way that $p = (0, 0)$, $S = \{y = 0\}$ and f is given by (5.2), where $T \geq 1$ and $g(x, 0) \not\equiv 0$, $h(x, 0) \not\equiv 0$. Since p is dicritical it follows from Remark 3.4 that $g_{\mu(g)}(x, y) = xh_{\mu(h)}(x, y)$, where $g_{\mu(g)}$ and $h_{\mu(h)}$ are the first non-zero terms in the homogeneous expansion of g and h respectively, and $\mu(g) = \mu(h) + 1$. Then, setting $\Gamma = (e^{2\pi i\theta}, 0)$ for $\theta \in (0, 1)$, $g(x, y) = g_{\mu(g)}(x, y) + g^\bullet(x, y)$ and $h(x, y) = h_{\mu(h)}(x, y) + h^\bullet(x, y)$ we have

$$
\begin{aligned}
2\pi i \text{ Ind}(f, S, p) &= \int_\Gamma \frac{y \circ f - y}{y(x \circ f - x)} dx = \int_\Gamma \frac{h_{\mu(h)}(x, y) + h^\bullet(x, y)}{g_{\mu(g)}(x, y) + g^\bullet(x, y)} dx \\
&= \int_\Gamma \frac{h_{\mu(h)}(x, 0)}{g_{\mu(g)}(x, 0) + g^\bullet(x, 0)} dx + \int_\Gamma \frac{h^\bullet(x, 0)}{g_{\mu(g)}(x, 0) + g^\bullet(x, 0)} dx.
\end{aligned}
$$

Now the second integral is the residue at $x = 0$ of the ratio of two holomorphic functions with numerator of order $\geq \mu(h) + 1$ at 0 and denominator of order $\mu(g) = \mu(h) + 1$ at 0, therefore the ratio is holomorphic at $x = 0$ and the integral is zero. As for the first integral

$$
\begin{aligned}
\int_\Gamma \frac{h_{\mu(h)}(x, 0)}{g_{\mu(g)}(x, 0) + g^\bullet(x, 0)} dx &= \int_\Gamma \frac{h_{\mu(h)}(x, 0)}{h_{\mu(h)}(x, 0)x + g^\bullet(x, 0)} dx \\
&= \int_\Gamma \frac{1}{x + \frac{g^\bullet(x,0)}{h_{\mu(h)}(x,0)}} dx = 2\pi i,
\end{aligned}
$$

since $g^\bullet(x, 0)/h_{\mu(h)}(x, 0)$ has order $\geq \mu(g) + 1 - \mu(h) = \mu(g) + 1 - \mu(g) + 1 = 2$ at $x = 0$. Therefore $\text{Ind}(f, S, p) = 1$ as wanted. □

Now we are going to somehow mimic the work of Cano [8] for the continuous dynamics in order to find out an algorithm for producing parabolic curves. However, contrarily to the continuous case, we have to worry about dicritical points and non-tangential curves, since it is by no means obvious that there must exist infinitely many parabolic curves in such situations. This is actually true as we show later, but the proof is based on the algorithm itself.

Definition 7.3. Let M be a two dimensional complex manifold, $f : M \to M$ holomorphic. Let $\text{Fix}(f) = S$, for S a curve in M. Assume that f is tangential on each component of S.

- We say that a point $p \in S$ is *of type* (C_1) if S is nonsingular at p and

$$\text{Ind}(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}.$$

- We say that a point $p \in S$ is *of type* (C_2) if S has two nonsingular branches S_0, S_1 at p, intersecting transversally at p and there exists a real number $r > 0$ such that

$$\mathrm{Ind}(f, S_0, p) \notin \mathbb{Q}_{(\geq -1/r)} = \{a \in \mathbb{Q} : a \geq -1/r\}$$
$$\mathrm{Ind}(f, S_1, p) \in \mathbb{Q}_{(\leq -r)} = \{a \in \mathbb{Q} : a \leq -r\}.$$

Remark 7.4. If $q \in S$ is of type (C_2) then by Lemma 7.2.(2) and (3) the point q cannot be a reduced singularity for f on the branches of S at q. Note also that by Lemma 7.2 a point of type (C_1) or (C_2) cannot be dicritical.

We start with the following lemma whose proof goes exactly as in [8] (we only note that blowing up a (C_1) or (C_2) point produces a divisor on which the blow up map is tangential by the previous note):

Lemma 7.5. *Let M be a two dimensional complex manifold, $f : M \to M$ be holomorphic and such that Fix$(f) = S$, for a curve S. Let $q \in S$ be a point of type $(C_1), (C_2)$. Let $\pi : \tilde{M} \to M$ be the blow-up at q and $E := \pi^{-1}(S)$ the total transform of S. Let $\tilde{f}$ be the holomorphic map induced by f on $\tilde{M}$. Then there exists a point $\tilde{q} \in E$ of type (C_1) or (C_2).*

Definition 7.6. Let S be a curve in a complex two dimensional manifold M. Let $f : M \to M$ be holomorphic and assume Fix$(f) = S$. Let $p \in S$. We say that p is an *appropriate singularity* for f if after a finite number of blow ups there exists a point of type (C_1) or (C_2) on the total transform of S.

The importance of appropriate singularities comes from the following result:

Proposition 7.7. *Let M be a two dimensional complex manifold, $S \subset M$ a curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$. Let $p \in S$ be an appropriate singularity for f. Then there exists at least one parabolic curve for f at p.*

Proof. By definition, after a finite number of blow ups we obtain a (C_1) or (C_2) point. By the theorem of resolution of singularities for curves, (see, *e.g.*, [13]) we may assume the total transform of S has only normal crossing singularities. By Lemma 7.5 and by Theorem 3.3, after a finite number of blow-ups at (C_1) or (C_2) points, we find either a nonsingular point of the total transform or a corner which is a reduced singularity for the induced map of type, respectively, (C_1) or (C_2). However by Remark 7.4 it cannot be of type (C_2). Thus there must be a reduced singularity of type (C_1) which is a non singular point of the total transform, say q. By Lemma 7.2.(4) we may assume (up to blow up once more if necessary) that the point q is a reduced singularity of type $(\star_1)$. Thus Theorem 2.6 applies producing (at least) a parabolic curve not contained in the total transform of S, which projects down to a parabolic curve for f at p. □

By definition, a nonsingular point $p \in S$ of a curve S such that Fix$(f) = S$ near p, f is tangential on S and $\mathrm{Ind}(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$, is an appropriate singularity for f. However there are some more interesting examples, as the following results show:

Proposition 7.8. *Let M be a two dimensional complex manifold, $S \subset M$ be a (possibly singular and non-compact) curve. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$.*

(1) *If f acts on the normal bundle of the regular part of S as the identity and f is non-tangential on S then every $p \in S$, except at most a discrete subset of S, is an appropriate singularity for f.*
(2) *Let $p \in S$ be such that S_p is a generalized irreducible cusp, i.e., there exist local coordinates (x, y) such that $p = (0,0)$, $S = \{y^m = x^n\}$, $m < n$ and S is irreducible at $(0,0)$. If f is tangential on S and Ind$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ then p is an appropriate singularity.*

Proof. 1. Let

$$\mathcal{S}_f := \{p \in S : \frac{l \circ f - l}{l^{T_p(f,S)}}[p] = 0, \forall\, l \in \mathcal{O}_p : (l)_p = \mathcal{I}(S)_p\}.$$

Note that $\mathcal{S}_f$ is a discrete set which contains the set of singularities of f on S. Let $\mathcal{S}_S$ be the (discrete) set of singular points of S. Let $\mathcal{S} := \mathcal{S}_f \cup \mathcal{S}_S$. Let $p \in S \setminus \mathcal{S}$. We can choose local coordinates (x, y) around p in such a way that $p = (0,0)$, $S = \{y = 0\}$ and $f = (f_1, f_2)$ is given by

$$\begin{cases} f_1(x,y) &= x + y^T g(x,y) \\ f_2(x,y) &= y + y^T h(x,y) \end{cases}$$

where $T \geq 2$ (by Remark 2.7) and $h(0,0) \neq 0$. Write $g(x,y) = a_0 + g_1(x,y)$ where $a_0 \in \mathbb{C}$, $g_1(0,0) = 0$ and $h(x,y) = b_0 + h_1(0,0)$ with $b_0 \in \mathbb{C}$, $b_0 \neq 0$ and $h_1(0,0) = 0$. Now let $\pi : \tilde{M} \to M$ be the blow-up of M at p and $\tilde{f}$ the map induced on $\tilde{M}$ by f. If (u, v) are local coordinates around $\pi^{-1}(p)$ such that $\pi(u, v) = (u, uv)$, then the exceptional divisor $D := \pi^{-1}(p)$ is given by $\{u = 0\}$ and the strict transform $\tilde{S}$ of S is given by $\{v = 0\}$. The point $\tilde{p} = (u = 0, v = 0)$ is the only intersection between $\tilde{S}$ and D. Writing $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ we have

$$\begin{cases} \tilde{f}_1(u,v) &= u + u^T v^T a_0 + v^T o(|u|^T) \\ \tilde{f}_2(u,v) &= v + u^{T-1} v^T [b_0 - v a_0] + v^T o(|u^{T-1}|) \end{cases}$$

Therefore the order of $\tilde{f}$ on D at $(0,0)$ is $T - 1$, and $\tilde{f}$ is tangential on D. Now we calculate the index of $\tilde{f}$ at $(0,0)$ on D. Let Γ be the cycle given by $u = 0$, $v = re^{2\pi i\theta}$ for $\theta \in [0,1]$, $0 < r << 1$. Then

$$2\pi i \ \mathrm{Ind}(\tilde{f}, D, (0,0)) = \int_\Gamma \frac{u \circ \tilde{f} - u}{u(v \circ \tilde{f} - v)} dv = \int_\Gamma \frac{a_0 + O(u)}{b_0 - v a_0 + O(u)} dv = 0.$$

Since $D \cdot D = -1$ then there exists $q \in D - \tilde{S}$ such that $\Re e \mathrm{Ind}(\tilde{f}, D, q) < 0$, and then a (C_1) point as wanted.

2. If p is a dicritical point for f then we blow it up and find that the blow up map $\tilde{f}$ is non-tangential on the exceptional divisor and acting as the identity on its normal bundle. Then p is an appropriate singularity by point 1.

Suppose p is not dicritical and $m > 2$ (otherwise p is (C_1)). We assume that $S = \{(x,y) : y^m - x^n = 0\}$ near $p = (0,0)$. Blowing up the point p, in the coordinates $x = u, y = uv$ it follows that the strict transform $\tilde{S}$ of S is given by $\{v^m - u^{n-m} = 0\}$ and the exceptional divisor $D = \{u = 0\}$. Thus $\tilde{p} = \tilde{S} \cap D$ is $\tilde{p} = (0,0)$. By Lemma 7.1 it follows $\mathrm{Ind}(\tilde{f}, \tilde{S}, \tilde{p}) \notin \mathbb{Q}_{(\geq -m^2)}$, where $\tilde{f}$ is the blow up of f. If there are no (C_1) points on the exceptional divisor $D \setminus \{\tilde{p}\}$ then by Theorem 5.2 it follows that $\mathrm{Ind}(\tilde{f}, D, \tilde{p}) \in \mathbb{Q}_{(\leq -1)}$ (and in particular $\tilde{p}$ is not dicritical). Since S is irreducible then $m \neq n - m$. Suppose first $n - m > m$. Blow up the point $\tilde{p}$ and denote by S_0 the strict transform of $\tilde{S}$, by D_0 the strict transform of D and by D_1 the new exceptional divisor. Also denote by f_1 the blow up of $\tilde{f}$. It follows that if $D_0 \cap D_1 = \{q_1\}$ and $S_0 \cap D_1 = \{q_0\}$ then $q_0 \neq q_1$. By Lemma 7.1 we have $\mathrm{Ind}(f_1, S_0, q_0) \notin \mathbb{Q}_{(\geq -2m^2)}$ and $\mathrm{Ind}(f_1, D_0, q_1) \in \mathbb{Q}_{(\leq -2)}$. Thus, if there are no (C_1) points on $D_1 \setminus \{q_0, q_1\}$ and q_1 is not (C_2) then again by Theorem 5.2 we have $\mathrm{Ind}(f_1, D_1, q_0) \in \mathbb{Q}_{(\leq -1/2)}$. Note that S_0 is given by $\{y^m - x^{n-2m} = 0\}$. Again, if $n - 2m > m$ we blow up the point and argue as before. We continue this way until finding $k \in \mathbb{N}$ such that $n - km < m$. At this point, if there are no (C_1) or (C_2) points on the total transform, arguing as before, denoting by q the intersection between the exceptional divisor D and the strict transform $\tilde{S}$ of S, and by $\tilde{f}$ the blow up of f, we have $\mathrm{Ind}(\tilde{f}, \tilde{S}, q) \notin \mathbb{Q}_{(\geq -km^2)}$ and $\mathrm{Ind}(\tilde{f}, D, q) \in \mathbb{Q}_{(\leq -1/k)}$. Now, blowing up at q we obtain a triple intersection at one point, say q_0. Arguing as before, if there are no (C_1) points on the exceptional divisor D_1 and f_0 denote the blow up of $\tilde{f}$, D_0 the strict transform of D, S_0 the strict transform of $\tilde{S}$, we have $\mathrm{Ind}(f_0, S_0, q_0) \notin \mathbb{Q}_{(\geq -km^2-(n-km)^2)}$, $\mathrm{Ind}(f_0, D_0, q_0) \in \mathbb{Q}_{(\leq -(k+1)/k)}$ and $\mathrm{Ind}(f_0, D_1, q_0) \in \mathbb{Q}_{(\leq -1)}$. From this point on, each time we blow up the singular point of the strict transform of S we obtain a triple intersection until the strict transform is nonsingular and intersecting the other two lines transversally. Also, if at each step there are no $(C_1), (C_2)$ points, we end up with a triple S_0, D_0, D_1 of nonsingular curves intersecting transversally at one point q and such that $\mathrm{Ind}(f, S_0, q) \notin \mathbb{Q}_{(\geq -M)}$, $\mathrm{Ind}(f, D_0, q) \in \mathbb{Q}_{(\leq -(1+1/r))}$ and $\mathrm{Ind}(f, D_1, q) \in \mathbb{Q}_{(\leq -1)}$, with $r > 0$ and $M > 2r + 1$ (this can be seen by an induction argument). Then, blowing up once more and arguing in the usual way, if there are no (C_1) points, we certainly obtain a (C_2) point. $\square$

More generally one might ask whether all the locally irreducible singularities which are the tangential fixed points set of a holomorphic map are appropriate if the index is not a positive rational number nor zero. Refining a bit the previous argument one can show that this is the case if the index is not rational nor zero. However, while this paper was under reviewing, the question has been affirmatively solved by F. degli Innocenti [9] by means of a careful study of the variation of indices with respect to the resolution process of the singularity. We state here the following theorem, referring the reader to [9] or to a forthcoming paper by the same author, for the proof.

Theorem 7.9 (degli Innocenti). *Let M be a two dimensional complex manifold, $S \subset M$ be a (possibly singular and non-compact) curve. Let $f : M \to M$ be holomorphic such that*

Fix$(f) = S$. *Let* $p \in S$ *be such that* S_p *is irreducible. If* f *is tangential on* S *and Ind*$(f, S, p) \notin \mathbb{Q}^+ \cup \{0\}$ *then* p *is an appropriate singularity for* f.

In particular by Propositions 7.7 and 7.8 we have

Theorem 7.10. *Let* M *be a two dimensional complex manifold,* $S \subset M$ *be a (possibly singular and non-compact) curve. Let* $f : M \to M$ *be holomorphic such that Fix*$(f) = S$. *Suppose that* f *acts on the normal bundle of the regular part of* S *as the identity and that* f *is non-tangential on* S. *Then for every* $p \in S$, *except at most a discrete subset of* S, *there exists at least one parabolic curve for* f *at* p. *In particular if* S *is compact then* f *has parabolic curves at every point of* S *but at most a finite set.*

Remark 7.11. In the situation of Theorem 7.10 it would be interesting to know whether the parabolic curves fill an open set around S. Some results in this directions are obtained in *Theorem 5.3 in* [5] for the case of the blow up of a dicritical point.

Note that also the converse is true:

Proposition 7.12. *Let* M *be a two dimensional complex manifold, let* $S \subset M$ *be a (possibly non compact and singular) curve. Let* $f : M \to M$ *be holomorphic such that Fix*$(f) = S$. *Suppose that* f *acts on the normal bundle of the regular part of* S *as the identity. If there exists a non-discrete subset* $A \subset S$ *such that* f *has parabolic curves at every* $p \in A$ *then* f *is non-tangential on* S.

Proof. If f were tangential on S then the union of the singularities of f on S and the singular points of S would form a discrete set B. By Proposition 2.5 then $A \subseteq B$, which contradicts the hypothesis. $\square$

Another application is to dicritical points (cfr. *Theorem 3.1.(ii) in* [2], where part of the following result is achieved by direct methods):

Proposition 7.13. *Let* M *be a two dimensional complex manifold,* $p \in M$. *Let* $f : M \to M$ *be holomorphic and such that* $f(p) = p$, $df_p = Id$. *Then* p *is dicritical for* f *if and only if for every direction but a finite number, there exists a parabolic curve for* f *at* p *tangent to such a direction at* p.

Proof. Let $\pi : \tilde{M} \to M$ be the blow-up of M at p, $D := \pi^{-1}(p)$ be the exceptional divisor and $\tilde{f} : \tilde{M} \to \tilde{M}$ be the holomorphic blow-up of f. Recall that by Remark 3.1 the map $\tilde{f}$ acts as the identity on the normal bundle N_D of D in $\tilde{M}$. Suppose p is dicritical for f. Thus by Theorem 7.10 every point but a finite number of D has parabolic curves for $\tilde{f}$ which blow down to parabolic curves for f tangent to all directions but a finite number. Viceversa if each direction but a finite number is a tangent for a parabolic curve for f at p then $\tilde{f}$ has parabolic curves at every point of D but a finite number. Since $\tilde{f}$ acts as the identity on the normal bundle of D then it must be non-tangential on D by Proposition 7.12 and hence p is dicritical. $\square$

We also have the following global result:

Theorem 7.14. *Let M be a two dimensional complex manifold, $S \subset M$ a compact, globally and locally irreducible curve with $S \cdot S < 0$. Let $f : M \to M$ be holomorphic such that Fix$(f) = S$ and f is tangential on S. Then there exists a point $p \in S$ such that f has at least one parabolic curve at p.*

Proof. By Theorem 6.2 there exists $p \in S$ such that $\Re e\mathrm{Ind}(f, S, p) < 0$. If such a point is nonsingular for S then it is (C_1) and we are done. Same if it is a singularity satisfying one of the hypothesis of Proposition 7.8. In general however one can argue following the same lines of [17] (see also *p.40 in* [6]) for the holomorphic foliations case to prove the existence of an appropriate singularity for f on S and then apply Proposition 7.7. □

Note that Theorem 7.14 and Theorem 7.10 imply that: *if M is a two dimensional complex manifold, $f : M \to M$ holomorphic, $S \subset M$ a compact, globally and locally irreducible curve such that $S \cdot S < 0$, Fix$(f) = S$ and f acts as the identity on the normal bundle of the regular part of S then there exists at least one point $p \in S$ such that f has parabolic curves at p.*

Also we recover *Abate's flowers theorem,* [2]:

Corollary 7.15 (Abate)**.** *Let f be a (germ of) biholomorphism of $\mathbb{C}^2$ such that 0 is a isolated fixed point for f and $df_0 = Id$. Then there exists a parabolic curve for f at 0.*

Proof. If 0 is dicritical then apply Proposition 7.13. If 0 is non-dicritical then blowing up 0 we find a holomorphic map $\tilde{f}$ which has the exceptional divisor D as fixed points set and is tangential on it. Since $D \cdot D = -1$ then we can apply Theorem 7.14. □

REFERENCES

1. M. Abate, *Diagonalization of non-diagonalizable discrete holomorphic dynamical systems*. Amer. J. Math. 122 (2000), 757-781.
2. M. Abate, *The residual index and the dynamics of holomorphic maps tangent to the identity*. Duke Math. J. 107, 1, (2001), 173-207.
3. P. Baum and R. Bott, *Singularities of holomorphic foliations*. J. Diff. Geom. 7 (1972), 279-342.
4. F. Bracci and F. Tovena, *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*. Math. Z. 242, 3, (2002), 481-490.
5. F. E. Brochero-Martínez, *Groups of germs of analytic diffeomorphisms in* $(\mathbb{C}^2, 0)$. J. Dynamic. and Contr. Sys., vol. 9, (2003).
6. M. Brunella, *Birational geometry of foliations*. First Latin American Congress of Math., IMPA, Rio de Janeiro, Brazil 2000.
7. C. Camacho and P. Sad, *Invariant varieties through singularities of holomorphic vector fields*. Ann. of Math. (2) 115 (1982), 579-595.
8. J. Cano, *Construction of invariant curves for singular holomorphic vector fields*. Proc. Amer. Math. Soc. 125 (9) 1997, 2649-2650.
9. F. degli Innocenti, *Dinamica di germi di foliazioni e diffeomorfismi olomorfi vicino a curve singolari*. Tesi di Laurea, Firenze, 2003.
10. J. Écalle, *Les fonctions résurgentes, Tome III: L'équation du pont et la classification analytiques des objects locaux*. Publ. Math. Orsay, 85-5, Université de Paris-Sud, Orsay, 1985.
11. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*. Prentice-Hall 1965.

12. M. Hakim, *Analytic transformations of* $(\mathbb{C}^p, 0)$ *tangent to the identity*, Duke Math. J. 92 (1998), 403-428.
13. H. B. Laufer, *Normal two dimensional singularities*. Ann. Math. Stud. 71 (1971).
14. D. Lehmann, *Résidus des sous-variétés invariants d'un feuilletage singulier*. Ann. Inst. Fourier, Grenoble, 41, 1 (1991), 211-258.
15. D. Lehmann and T. Suwa, *Residues of holomorphic vector fields relative to singular invariant subvarieties*. J. Diff. Geom. 42, 1, (1995), 165-192.
16. Y. Nishimura, *Automorphisms analytiques admettant des sous-variétés de points fixés attractives dans la direction transversale* J. Math. Kyoto Univ. 23-2 (1983), 289-299.
17. M. Sebastiani, *Sur l'existence de séparatrices locales des feuilletages des surfaces*. An. Acad. Bras. Ci., (1997) 69 (2), 159-162.
18. T. Suwa *Indices of holomorphic vector fields relative to invariant curves on surfaces*. Proc. Amer. Math. Soc. 123 (1995), 2989-2997.
19. T. Ueda, *Analytic transformations of two complex variables with parabolic fixed points*. Preprint 1997.
20. B. J. Weickert, *Attracting basins for automorphisms of* $\mathbb{C}^2$. Invent. Math. 132 (1998), 581-605.
21. H. Wu, *Complex stable manifolds of holomorphic diffeomorphisms*. Indiana Univ. J. Math. 42 (1993), 1349-1358.

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