Loewner's Theory from the deterministic point of view

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Lecture Notes by Michele Triestino,
with an appendix by Santiago Díaz-Madrigal

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Introduzione

Queste note sono il frutto di un *live-TEXing* durante le lezioni del corso estivo *Loewner's Theory from the deterministic and stochastic point of view*, tenutosi a Cortona dal 15 al 27 agosto 2010. Lo scopo di tale scuola (a mia opinione ben riuscito) è stato di offrire una introduzione alla Teoria di Loewner dal punto di vista dinamico (*catene di Loewner*) e stocastico (*SLE*). I corsi sono stati tenuti da Filippo Bracci e Robert Bauer.

L'idea di avere una revisione delle note da me prese sul momento è sorta verso gli ultimi giorni di corso parlandone con Filippo Bracci. Queste note riportano alla fine un seminario gentilmente tenuto da Santiago Díaz Madrigal a complemento delle lezioni "deterministiche".

Le mie note originali comprendevano anche le lezioni di Robert Bauer, ma la mia (in)competenza probabilistica non è tale da assicurare la correttezza di ció che abbia scritto. Di conseguenza ho preso la decisione di rimuovere questa parte. Gli argomenti trattati in tali lezioni sono in ogni modo altamente ben riportati nel libro, scelto come testo di riferimento, di Lawler *Conformally invariant processes in the plane*.

Tengo a ringraziare caldamente Filippo Bracci e Santiago Díaz-Madrigal per avere avuto la pazienza di rileggere, corregere e completare questo lavoro. Un grazie va anche a tutti gli altri partecipanti della scuola estiva di Cortona, in particolare a Leandro Arosio, Cinzia Bisi, Pasha Gumenyuk, Georgy Ivanov e Caterina Stoppato per aver contribuito con le loro idee nelle risoluzioni degli esercizi proposti e per aver costruito un ottimo clima. Infine grazie all'efficiente e impeccabile segreteria del Palazzone.

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Indice

1 Loewner's chains and related topics (F. Bracci)

1.1 Historic motivation

Karel Löwner was born the 29th of May 1893, at Lány, Bohemia, but was also known as Karl Löwner, from the German version of his first name. He received his Ph.D. from the University of Prague in 1917 under the supervision of Georg Pick. Later on, he spent some years at the universities of Berlin and Cologne. He emigrated to the US in 1939 and changed his name into Charles Loewner. In the United States, he worked at Brown University, Syracuse University and eventually at Stanford University, where he stayed until his death on the 8th of January 1968.

Loewner was motivated in studying the subject bearing nowadays his name, by the following conjecture due to Bieberbach: consider a univalent function $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ on the unit disc $\mathbf{D}$ then $|a_j| \leq j$ for any $j \geq 2$. Using his own method, Loewner solved it for a_3. Bieberbach's conjecture was completely solved in the '80s by L. de Branges, see also [FP85].

The original idea of Charles Loewner was to embed the univalent function into a special flow governed by a nice vector field for which good estimates of the coefficients are available and then recover estimates for the initial function.

1.2 Preliminaries

Theorem 1.2.1 (Riemann Mapping Theorem)**.** *Let D be a simply connected proper domain in $\mathbf{C}$, then there exists a biholomorphic function $f : \mathbf{D} \longrightarrow D$.*

Moreover, if $z_0 \in D$, there exists a unique $f : \mathbf{D} \longrightarrow D$ biholomorphic such that $f(0) = z_0$ and $f'(0) > 0$.

As an exercise, prove the uniqueness.

Exercise 1.2.1 (The Koebe function). Consider $f : z \in \mathbf{D} \longmapsto \frac{z}{(1-z)^2} \in \mathbf{C} - [-\infty, -\frac{1}{4}]$: show that it defines a biholomorphism and expand it into power series at $z = 0$.

Observe that it gives exactly the bounds for the Bieberbach conjecture.

Exercise 1.2.2. Find a biholomorphism from $\mathbf{D}$ onto the upper half-plane $\mathbf{H}$.

Theorem 1.2.2 (Carathéodory)**.** *The Riemann mapping $f : \mathbf{D} \longrightarrow D$ extends continuously up to $\partial\mathbf{D}$ if and only if ∂D is locally connected.*

1.3 The class $\mathcal{S}$

Definition 1.3.1. *We define $\mathcal{S} := \{f : \mathbf{D} \longrightarrow \mathbf{C}$ univalent$, f(0) = 0, f'(0) = 1\}$, where we recall that a* univalent function *is a injective holomorphic function.*

Exercise 1.3.1.

- If $f \in \mathcal{S}$ is such that $|f(z)| \leq 1$ for any $z \in \mathbf{D}$, then $f(z) = z$,
- show that $\nexists f \in \mathcal{S}$ such that $f(\mathbf{D}) = \mathbf{C}$.

Remark 1.3.1.

- In $\mathbf{C}^n$, $n > 1$, there exist simply connected domains $D \subsetneq \mathbf{C}^n$ such that D is biholomorphic to $\mathbf{C}^n$ (Fatou-Bieberbach phenomena).
- In $\mathbf{C}^n$, $n > 1$, there is no Riemann Mapping Theorem. For instance, as already noticed by Poincaré, the unit ball and the polydisc are not biholomorphic in dimension greater than 1 because their automorphisms groups are not isomorphic.

Remark 1.3.2. Given $g : \mathbf{D}_r(z_0) \longrightarrow \mathbf{C}$ univalent,the function $f(z) := \frac{g((z+z_0)r)-g(0)}{rg'(z_0)}$ is in $\mathcal{S}$.

1.4 Estimates on $\mathcal{S}$

Theorem 1.4.1 (Area Theorem). *Let $g(z) = \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \ldots$ be a univalent function in $\mathbf{D} - \{0\}$, with a simple pole at $z = 0$, then $\sum n|b_n|^2 \leq 1$.*

Dimostrazione. Define $D_r := \mathbf{C} - g(\mathbf{D}_r)$ and

$$\mathrm{Area}(D_r) := \iint_{D_r} dx \wedge dy \overset{\text{complex Gauss-Green}}{=} \frac{1}{2i} \int_{\partial D_r} \overline{z} dz = -\frac{1}{2i} \int_{\partial \mathbf{D}_r} \overline{g} d\overline{g};$$

as $\overline{g}(z) = \frac{1}{\overline{z}} + \overline{b_0} + \overline{b_1}\overline{z} + \ldots$ and $dg = g'(z)dz = \left(-\frac{1}{z^2} + b_1 + \ldots\right) dz$ we have

$$\overline{g} dg = \left(\frac{1}{\overline{z}} + \overline{b_0} + \overline{b_1}\overline{z} + \ldots\right)\left(-\frac{1}{z^2} + b_1 + \ldots\right) dz = -\left(\frac{1}{1-z} + \frac{b_1}{\overline{z}} + \ldots\right) dz;$$

we observe that $z \in \partial \mathbf{D}_r$ if and only if $z\overline{z} = r^2$, so that $\overline{z} = \frac{r^2}{z}$ on $\partial \mathbf{D}_r$; moreover, $\int_{\partial \mathbf{D}_r} z^n dz = 0$ for $n \neq -1$ and $\int_{\partial \mathbf{D}_r} z^n \overline{z}^m dz = \begin{cases} 0 & n \neq m-1 \\ 2\pi i r^2 & n = m-1 \end{cases}$.

Thus, $-\frac{1}{2i} \int_{\partial \mathbf{D}_r} \overline{g} dg = \pi \left(\frac{1}{r^2} - \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}\right) \geq 0$. We let r go to 1 and we get the result. □

Corollary 1.4.2. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in \mathcal{S}$, then $|a_2| \leq 2$.*

Dimostrazione. Let $g(z) := \frac{1}{\sqrt{f(z^2)}} \overset{\text{exercise}}{=} \frac{1}{z} - \frac{a_2}{2} z + \sum_j c_j z^{2j+1}$; the claim is that g is univalent in $\mathbf{D}$ with a simple pole at $z = 0$; if so, by the Area Theorem, we have $|a_2| \leq 2$.

We prove the claim. Firstly, g is holomorphic in $\mathbf{D} - \{0\}$, with pole at $z = 0$ (exercise). We show injectivity; suppose $g(z_1) = g(z_2)$, then $f(z_1^2) = f(z_2^2)$, so that $z_1 = \pm z_2$. Since we have observed that g is odd, it can only occur $z_1 = z_2$. □

Theorem 1.4.3 (Koebe's $\frac{1}{4}$-theorem). *If $f \in \mathcal{S}$ then $f(\mathbf{D}) \supset \mathbf{D}_{1/4}$.*

Dimostrazione. Let $w_0 \notin f(\mathbf{D})$; define $g : \mathbf{D} \longrightarrow \mathbf{C}$ as

$$g(z) := \frac{w_0 f(z)}{w_0 - f(z)}.$$

Then (exercise) $g \in \mathcal{S}$. Moreover, g has expansion $g(z) = z + \left(a_2 + \frac{1}{w_0}\right) z^2 + \ldots$. By Corollary 1.4.2,

$$\left|a_2 + \frac{1}{w_0}\right| \leq 2, \quad |a_2| \leq 2,$$

hence $\left|\frac{1}{w_0}\right| \leq \left|a_2 + \frac{1}{w_0}\right| + |a_2| = 4$. □

The Koebe's function realizes the minimum (why?). We need now a technical result.

Theorem 1.4.4 (Growth Theorem). *Given $f \in \mathcal{S}$, then*

(a)

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2},$$

(b)

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3},$$

(c)

$$\frac{1-|z|}{1+|z|} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+|z|}{1-|z|}.$$

Moreover, equalities hold if and only if f is a rotation of the Koebe function.

Dimostrazione. (b) It is true for $z=0$; let $r \in (0,1)$, $|z| = r$, then set

$$g(\zeta) := \frac{f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right) - f(z)}{(1-|z|^2)f'(z)}$$

Exercise 1.4.1. If $a \in \mathbf{D}$, then $T_a(z) := \frac{a-z}{1-\bar{a}z} \in \mathrm{Aut}(\mathbf{D})$ and $T_a(0) = a$.

Exercise 1.4.2. Show that $g \in \mathcal{S}$

Then

$$g(\zeta) = \zeta + a_2\zeta^2 + \dots$$

and $a_2 = a_2(z) \overset{\text{exercise}}{=} \frac{1}{2}\left[(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right]$. By Corollary 1.4.2, we have

$$\left|(1-|z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \le 4$$

and since $|z| = r$,

(*)

$$\left|z\frac{f''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{4r}{1-r^2},$$

(**)

$$\frac{2r^2-4r}{1-r^2} \le \mathfrak{Re}\left[z\frac{f''(z)}{f'(z)}\right] \le \frac{2r^2+4r}{1-r^2}$$

(where (**) is obtained from (*) by means of the triangular inequality).

We note that f is univalent, so that $f'(z) \neq 0$ for any $z \in \mathbf{D}$, $f'(0) = 1$ and we can take a branch of $\log f'(z)$ such that $\log f'(z)|_{z=0} = 0$; writing z in polar coordinates, $z = re^{i\theta}$, we have

$$\begin{aligned}
\frac{\partial}{\partial r}\log|f'(z)| = \frac{\partial}{\partial r}\mathfrak{Re}\left[\log f'(z)\right] &= \\
&= \mathfrak{Re}\left[\frac{\partial}{\partial r}\log f'(z)\right] = \\
&= \mathfrak{Re}\left[\frac{\partial}{\partial z}(\log f'(z))\frac{\partial z}{\partial r}\right] = \\
&= \mathfrak{Re}\left[\frac{f''(z)}{f'(z)}e^{i\theta}\frac{r}{r}\right] = \frac{1}{r}\mathfrak{Re}\left[z\frac{f''(z)}{f'(z)}\right]
\end{aligned}$$

and from (**) we have

$$\frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log\left|f'\left(re^{i\theta}\right)\right| \le \frac{2r+4}{1-r^2};$$

now we integrate in r (since $\log f'(0) = 0$) and we obtain

$$\log\frac{1-r}{(1+r)^3} \le \log\left|f'\left(re^{i\theta}\right)\right| \le \log\frac{1+r}{(1-r)^3}$$

and we get (b) by taking exponentiation.

(a) That's true for $z = 0$, let $r = |z| \in (0, 1)$, then

$$f(z) = \int_{[0,z]} f'(\zeta) d\zeta = \int_0^r f'\left(\rho e^{i\theta}\right) e^{i\theta} d\rho$$

and by the previous estimates

$$|f(z)| \leq \int_0^r \left|f'\left(re^{i\theta}\right)\right| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \frac{r}{(1-r)^2}$$

so that we get the lower bound.

Let $|z_1| = r$ such that $|f(z_1)| = \min\{|f(z)| : |z| = r\} > 0$ and take Γ to be the segment between 0 and $f(z_1)$, $\gamma := f^{-1}(\Gamma)$; then

$$|f(z_1)| = \int_\Gamma |dw| = \int_\gamma |f'(\zeta)|\, |d\zeta| = \int_\gamma \frac{1-|\zeta|}{(1+|\zeta|)^3} d|\zeta| = \int_0^r \frac{1-t}{(1+t)^3} dt = \frac{r}{(1+r)^2},$$

which concludes the proof of (a).

□

1.5 Tutorial - The Poincaré disc

Exercise 1.5.1. 1. Given $a \in \mathbf{D}$, define

$$T_a(z) := \frac{a - z}{1 - \bar{a}z};$$

show that T_a is a conformal transformation of the disc of order 2 and $T_a(0) = a$.

2. Show that $\mathbf{D}$ is homogeneous, i.e. $\mathrm{Aut}(\mathbf{D})$ acts transitively on $\mathbf{D}$.
3. $f \in \mathrm{Aut}(\mathbf{D})$ if and only if f is of the form $f(z) = e^{i\theta} T_a(z)$.

We define the *Poincaré metric* on $\mathbf{D}$ as $d\rho^2 = \frac{dz \otimes d\bar{z}}{(1-|z|^2)^2}$.

1. $d\rho$ is a Hermitian metric on $\mathbf{D}$.
2. If $f : \mathbf{D} \longrightarrow D$ is holomorphic, then $f^* d\rho \leq d\rho$ (and = holds if and only if f is a biholomorphism).

Indeed, $d\rho_z^2(v, w) = \frac{v\bar{w}}{(1-|z|^2)^2}$ and using Schwarz' Lemma one proves that

$$f^*(d\rho_z(v, v)) = d\rho_{f(z)}(df_z(v), df_z(v)) \leq d\rho_z(v, v).$$

More generally, if M is a manifold, with a length function (or a *Finsler metric*) $k : TM \longrightarrow \mathbf{R}^+$ (i.e. $k(v) \geq 0$ and $k(v) = 0$ if and only if $v = 0$, $k(\lambda v) = |\lambda| k(v)$) such that $z \longmapsto k_z$ is upper semi-continuous (that is, given a local frame $v_1, \ldots, v_n$ at a coordinate chart U, then $z \in U \longmapsto k_z(v_j)$ is upper semi-continuous for any $j = 1, \ldots, n$), it is possible to define a distance

$$\mathrm{dist}_k(p, q) = \begin{cases} \infty & \text{if } \nexists \text{ a piecewise smooth curves joining } p \text{ and } q \\ \inf_{\gamma \in S_{p,q}} \int_0^1 k_{\sigma_\gamma}(t)(\sigma'_\gamma(t)) dt & \text{otherwise} \end{cases}$$

where $S_{p,q} :=$ {(absolutely continuous) piecewise smooth curves joining p and q} and σ_γ is a parametrization of γ.

Exercise 1.5.2. 1. Show that

$$\omega(z_1, z_2) := \mathrm{dist}_{d\rho}(z_1, z_2) = \frac{1}{2} \log \frac{1 + \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|}{1 - \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|} = \frac{1}{2} \log \frac{1 + |T_{z_1}(z_2)|}{1 - |T_{z_1}(z_2)|}.$$

2. ω is a complete distance (balls are relatively compact).

3. Given a holomorphic function $f : \mathbf{D} \longrightarrow \mathbf{D}$ then $\omega(f(z_1), f(z_2)) \leq \omega(z_1, z_2)$ (and = holds if and only if f is a biholomorphism).

4. $\mathrm{Isom}(d\rho) = \mathrm{Aut}(\mathbf{D}) \cup \overline{\mathrm{Aut}(\mathbf{D})}$.

5. The Gauss curvature of $\mathfrak{Re}\,(d\rho)$ is -4.

1.6 Something more about univalent functions

We equip $\mathrm{Hol}(\mathbf{D}, \mathbf{C})$ with the topology of uniform convergence on compact sets.

Proposition 1.6.1. *$\mathcal{S}$ is compact.*

Dimostrazione. Given $\{f_n\} \subset \mathcal{S}$, $\mathbf{K} \Subset \mathbf{D}$, there exists $r \in (0,1)$ such that $K \Subset \mathbf{D}_r$. By the Growth Theorem,

$$|f_n(z)| \leq \frac{|z|}{(1-|z|)^2} \quad \forall\, z \in \mathbf{D}_r,$$

then $\{f_n\}$ is uniformly bounded on compact sets and, by Montel's Theorem, there is a subsequence f_{n_k} converging to $f \in \mathrm{Hol}(\mathbf{D}, \mathbf{C})$. As the f_{n_k}'s are injective, then the Hurwitz' Theorem says that f is either constant or injective. Since $f_{n_k}(0) = 0$ and $f'_{n_k}(0) = 1$ for any n_k, then $f \in \mathcal{S}$. □

Remark 1.6.1. The Hurwitz' Theorem asserts that the limit of a sequence $f_n : \mathbf{D} \longrightarrow \mathbf{C}$ of holomorphic injective functions is either a constant or an injective function.

Definition 1.6.1. *Let $\{G_n\}_{n\in\mathbf{N}}$ be a sequence of open domains in $\mathbf{C}$, with $0 \in G_n$ for every $n \in \mathbf{N}$. We define the* kernel *of $\{G_n\}$ to be the set G as follows:*

1. *if 0 is not an interior point of $\bigcap_{n\in\mathbf{N}} G_n$ then $G := 0$;*

2. *if 0 is an interior point of $\bigcap_{n\in\mathbf{N}} G_n$ then G is the largest open domain containing 0 such that if $K \Subset G$ then $\exists\, n_0 \in \mathbf{N}$ such that $K \Subset G_n$ for any $n > n_0$.*

We say that a sequence of domains $\{G_n\}$ (kernel) *converges to G if each subsequence $\{G_{n_k}\}$ of $\{G_n\}$ has kernel G and, in such a case, we write $G_n \longrightarrow G$.*

Remark 1.6.2. If $G_{n-1} \subset G_n$ for any n, then the kernel is $G = \bigcup_{n\geq 0} G_n$.

Exercise 1.6.1. Compute the kernel of $\{G_n\}$ where $G_n := \mathbf{C} - [1, +\infty) \cup \left(\text{arc } (0 \leq \theta \leq 2\pi - \frac{1}{n}) \text{ of } \partial\mathbf{D}\right)$.

Theorem 1.6.2 (Carathéodory's kernel convergence)**.** *Let $\{G_n\}$ be a sequence of simply connected domains with $0 \in G_n$ and $G_n \neq \mathbf{C}$ for every n. Let $f_n : \mathbf{D} \longrightarrow G_n$ be a biholomorphism such that $f_n(0) = 0$ and $f'_n(0) > 0$. Let G be the kernel of G_n, then f_n converges uniformly to a function f if and only if $G_n \longrightarrow G$ and $G \neq \mathbf{C}$.*

Moreover, in case of convergence:

1. *if $G = \{0\}$ then $f \equiv 0$,*

2. *if $G \neq \{0\}$ then G is simply connected, $f : \mathbf{D} \longrightarrow G$ is biholomorphic and f_n^{-1} converges uniformly on compacta to f^{-1} on G.*

Remark 1.6.3. In the statement 2 of Carathéodory's kernel convergence theorem the sequence $\{f_n^{-1}\}$ is not in general well defined on all of G for all $n \in \mathbf{N}$. However, for every compact subset $K \Subset G$, K is eventually contained in G_n, thus $\{f_n^{-1}\}$ is eventually well defined on K and the statement is that it converges to f^{-1} on K.

Proof of Theorem 1.6.2. Assume that f_n converges to f. We will show that $f(\mathbf{D}) = G$. Once we have proved this, it will follow from Hurwitz' Theorem that f is either constantly zero or univalent; hence $G \neq \mathbf{C}$. Moreover, since f_{n_k} converges to f for any subsequence $\{f_{n_k}\}$, then (exercise) $G_n \longrightarrow G$.

We prove now that $f(\mathbf{D}) = G$. There are two cases:

1. Suppose $f \equiv 0$. If $G \neq \{0\}$, then there exists $\rho > 0$ such that $\mathbf{D}_\rho \subset G_n$ for any $n \in \mathbf{N}$ and we can define $f_n^{-1}|_{\mathbf{D}_\rho} : \mathbf{D}_\rho \longrightarrow \mathbf{D}$ with $f_n^{-1}|_{\mathbf{D}_\rho}(0) = 0$. By Schwarz' Lemma,
$$\left| \left(f_n^{-1}|_{\mathbf{D}_\rho}\right)' (0) \right| \leq \frac{1}{\rho},$$
so that $|f_n'(0)| \geq \rho$ for any n, which contradicts the fact that f_n converges uniformly to 0. Then $G = \{0\}$ and $f(\mathbf{D}) = G$.

2. If $f \not\equiv 0$, then $f : \mathbf{D} \longrightarrow D := f(\mathbf{D})$ is univalent.

   We prove that $D \subset G$: given $K_0 \Subset D$, we take $\Gamma \subset D$ to be a smooth curve whose interior contains K_0 and we set $\gamma := f^{-1}(\Gamma)$. We have to show that there exists n_0 such that $K_0 \subset G_n$ for any $n > n_0$.

   Let $\delta := \operatorname{dist}(K_0, \Gamma) > 0$; then for any $w_0 \in K_0$, $|f(z) - w_0| \geq \delta$ for any $z \in \gamma$. As γ is compact, there exists an $N \in \mathbf{N}$ such that $|f_n(z) - f(z)| < \delta$ for any $z \in \gamma$ and $n \geq N$.

   By Rouché's Theorem, $f_n(z) - w_0$ has the same number of zeroes of $f(z) - w_0$ in the interior of γ, for $|(f_n(z) - w_0) - (f(z) - w_0)| = |f_n(z) - f(z)| < \delta \leq |f(z) - w_0|$. Therefore there exists z_n' such that $f_n(z_n') = w_0$ and so $w_0 \in G_n = f_n(\mathbf{D})$ for any $n \geq N$. Thus $D \subset G$.

   We prove that $G \subset D$. The sequence $f_n^{-1}|_G : G \longrightarrow \mathbf{D}$ is eventually well defined on compact sets. As it is an equi-bounded sequence, the family $\{f_n^{-1}|_G\}$ is normal. Pick $w_0 \in G$, let $\{f_{n_k}^{-1}|_G\}$ be a convergent subsequence and $w_0 \in G_{n_k}$ for any G_{n_k}. Define $g := \lim f_{n_k}^{-1}|_G : G \longrightarrow \overline{\mathbf{D}}$, which is holomorphic and such that $g(0) = 0$. By Hurwitz, either $g \equiv 0$ or $g(G) \subset \mathbf{D}$ and g is univalent. In any case, define the sequence $z_k := f_{n_k}^{-1}(w_0)$ which converges to $g(w_0) =: z_0 \in \mathbf{D}$. As $f_{n_k}(z_0)$ converges to $f(z_0)$, for $\{z_k\} \cup \{z_0\} \Subset \mathbf{D}$, we have that $f(z_0) = w_0$ and $w_0 \in f(\mathbf{D}) = D$.

   Moreover, $id_G = f_{n_k} \circ f_{n_k}^{-1}|_G$ converges uniformly to $f \circ g$, so that $f_{n_k}^{-1}|_G$ converges uniformly to f^{-1} and this holds for the sequence $f_n^{-1}|_G$ as well.

Suppose now that $G_n \longrightarrow G \neq \mathbf{C}$.

1. Suppose $G = \{0\}$. We want to prove that $f_n'(0) \longrightarrow 0$. If not, there is $\delta > 0$ such that $|f_n'(0)| \geq \delta$ for any $n \in \mathbf{N}$. Hence, by Koebe's $\frac{1}{4}$-Theorem, $\mathbf{D}_{\delta/4} \subset G_n$ for any n, contradicting the hypothesis.

   By the Growth Theorem, $|f_n'(z)| \leq f_n'(0)\frac{|z|}{(1-|z|)^2}$ implies that $f_n'(0)$ converges to 0.

2. If $G \neq \{0\}, \mathbf{C}$, we want to prove that $\{f_n'(0)\}$ is bounded. If not, there is $n_k \in \mathbf{N}$ such that $f_{n_k}'(0) > k$ for any k. By Koebe's $\frac{1}{4}$-Theorem, $f_{n_k}(\mathbf{D}) = G_{n_k} \supset \mathbf{D}_{k/4} \longrightarrow \mathbf{C}$ and so $G_{n_k} \longrightarrow \mathbf{C}$ which is a contradiction.

   Thus there is $M > 0$ such that $f_n'(0) \leq M$ for any $n \in \mathbf{N}$. by the Growth Theorem,
$$|f_n(z)| \leq M\frac{|z|}{(1-|z|)^2}$$
and by Montel's Theorem, $\{f_n\}$ is normal. Suppose to have two convergent subsequences $f_{n_k} \longrightarrow g$ and $f_{n_m} \longrightarrow h$. We use the first part of the proof to f_{n_k} and to f_{n_m} in order to get that the kernel of $G_{n_k} := f_{n_k}(\mathbf{D})$ is $G = g(\mathbf{D})$ and that of $G_{n_m} := f_{n_m}(\mathbf{D})$ is $G = h(\mathbf{D})$ (G is the same by hypothesis). By the uniqueness of the Riemann mapping, $h = g$, as $h(0) = g(0)$ and $g'(0), h'(0) > 0$.

□

1.7 Towards Loewner's Theory

Definition 1.7.1. *A* Jordan arc *is a continuous curve* $\gamma : [0,T) \longrightarrow \mathbf{C}$ *which is injective and such that* $\lim_{t\to T} |\gamma(t)| = \infty$ *and* $0 \notin \gamma([0,T))$.

Definition 1.7.2. *A function* $f \in \mathcal{S}$ *is a* single-slit map *if* $f(\mathbf{D}) = \mathbf{C} - \gamma$.

Theorem 1.7.1. *Single-slit maps are dense in* $\mathcal{S}$.

Dimostrazione. Fix $\varepsilon > 0$, $\rho \in (0,1)$, given $f \in \mathcal{S}$, we need to find a single-slit map g such that $|f(z) - g(z)| < \varepsilon$ for all $|z| \leq \rho$.

Define $f_r(z) := \frac{f(rz)}{r}$, $0 < r < 1$ which turns out to be in $\mathcal{S}$ and $f_r \to f$ uniformly. The functions f_r are analytic up to $\partial\mathbf{D}$. Hence it is enough to prove the theorem for $f \in \mathcal{S}$ which extends real analytically and injectively on $\partial\mathbf{D}$.

Define $D := f(\mathbf{D})$, then $\partial D = f(\partial\mathbf{D})$ is a real analytic Jordan curve. Given $w_0 \in \partial D$, we consider a Jordan arc Γ starting at w_0 which does not intersect $\overline{D} - \{w_0\}$. Let $\{w_n\} \subset \partial D$ such that w_n moves counterclockwise on ∂D and such that $w_n \to w_0$.

Let $\gamma_n := \Gamma \cup$"the portion of ∂D from w_0 to w_n", $G_n := \mathbf{C} - \gamma_n$ (which is a simply connected domain).

Exercise 1.7.1. $G_n \to D$.

Let $g_n : \mathbf{D} \longrightarrow G_n$ be the Riemann maps such that $g_n(0) = 0$, $g_n'(0) > 0$; by the previous exercise and Carathéodory Kernel Convergence Theorem, we have that $g_n \to f$ and then $f_n := \frac{g_n}{g_n'(0)}$ is the searched sequence. □

1.8 Parametric representation of single-slit maps

Given a Jordan arc $\psi : [0,T) \longrightarrow \mathbf{C}$ tending to ∞ and omitting 0, we define $D_0 := \mathbf{C} - \gamma$ (γ is the support of the curve ψ); then we consider the single-slit map $f : \mathbf{D} \longrightarrow D_0$.

Set $\gamma_t := \psi([t,T))$, then the domains $D_t := \mathbf{C} - \gamma_t$ are contained one into the other for increasing time t: $D_s \subsetneq D_t$ for $s < t$.

Let $f_t : \mathbf{D} \longrightarrow D_t$ be the Riemann map with $f_t(0) = 0$ and $f'(0) > 0$; then $f_t(z) = \beta(t)[z + b_2(t)z^2 + b_3(t)z^3 + \ldots]$.

Exercise 1.8.1. Show that $t \longmapsto \beta(t)$, $t \longmapsto b_j(t)$ are continuous.

Solution. Let $t \to t_0$, we can take a sequence $\{t_n\}$ converging to t_0; then we want to show that $f_{t_n}'(0) \to f_{t_0}'(0)$. We have that $f_{t_n}(\mathbf{D}) \to f_{t_0}(\mathbf{D})$, so that by Carathéodory Theorem $f_{t_n} \to f_{t_0}$ uniformly on compact sets. □

Exercise 1.8.2. The function $t \longmapsto \beta(t)$ is strictly increasing.

Solution. The composition $f_t^{-1} \circ f_s : \mathbf{D} \longrightarrow \mathbf{D}$, for $s < t$ is such that

$$(f_s^{-1} \circ f_t)'(0) = \frac{f_t'(0)}{f_s'(0)} < 1$$

by Schwarz' Lemma.

We show that $\beta(t) \to \infty$.

Suppose that $\beta(t) < M$ for a certain $M > 0$. Then $f_t'(0) < M$ and so $f_t'(z) \leq M\frac{|z|}{(1-|z|)^2}$; then there exists $f_{t_n} \to g$ for $t_n \to \infty$ and by Carathéodory Theorem $D_{t_n} \longrightarrow \mathbf{C}$. This implies that we have a conformal transformation $g : \mathbf{D} \longrightarrow \mathbf{C}$, which is a contradiction. □

Let $\sigma(s) := \beta^{-1}(e^s)$, $\widetilde{\psi} : s \longmapsto \psi(\sigma(s))$, with $t = \sigma(s)$ we have

$$f_s(z) = e^s\left(z + \sum_{n\geq 2} b_n(s)z^n\right).$$

Exercise 1.8.3. In fact, $\widetilde{\psi} : [0, \infty) \longrightarrow \mathbf{C}$.

f_s is called the *standard parametrization*.

We define a family $(\varphi_t)_{0 \le t < \infty}$ of univalent mappings of $\mathbf{D}$

$$\mathbf{D} \xrightarrow{f=f_0} D_0 \hookrightarrow D_t \xrightarrow{f_t^{-1}} \mathbf{D}\,, \qquad \varphi_t (= \varphi_{0,t}) : \mathbf{D} \longrightarrow \mathbf{D}.$$

We see that $\varphi_t(z) = e^{-t}\left(z + \sum_{n \ge 2} a_n(t) z^n\right)$ for $0 \le t < \infty$.

Exercise 1.8.4. Prove that $a_n(t)$ is a polynomial in $b_2(t), \ldots, b_n(t)$, hence $t \longmapsto a_n(t)$ is continuous.

Theorem 1.8.1 (Loewner)**.** *Let $f \in \mathcal{S}$ be a single-slit map with omitted arc γ. Let $\psi : [0, \infty) \longrightarrow \gamma$ be the standard parametrization and let (f_t) be the associated chain. Then $(\varphi_t) := (f_t^{-1} \circ f_0)$ satisfies:*

$$\frac{\partial \varphi_t}{\partial t} = -\varphi_t(z) \frac{1 + k(t)\varphi_t(z)}{1 - k(t)\varphi_t(z)}, \tag{1.1}$$

where $k : [0, \infty) \longrightarrow \partial \mathbf{D}$ is a continuous function.

Moreover $\lim_{t \to \infty} e^t \varphi_t(z) = f(z)$ uniformly on compact sets.

Equation (1.1) is nowadays called the (classical) *Loewner ODE*.

Definition 1.8.1. *A family of univalent functions $(f_t : \mathbf{D} \longrightarrow \mathbf{C})_{t \in [0,\infty)}$ such that*

- *$f_t(0) = 0$, $f_t'(0) = e^t$,*
- *$f_s(\mathbf{D}) \subset f_t(\mathbf{D})$ for any $0 \le s \le t < \infty$*

is a classical Loewner chain.

We define $\varphi_{s,t} := f_t^{-1} \circ f_s$.

Definition 1.8.2. *A family of holomorphic functions $(\varphi_{s,t} : \mathbf{D} \longrightarrow \mathbf{D})$ such that*

- *$\varphi_{s,s} = id_{\mathbf{D}}$ for any $s \ge 0$,*
- *$\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for any $0 \le s \le u \le t < \infty$,*
- *$\varphi_{s,t}(0) = 0$, $\varphi_{s,t}'(0) = e^{s-t}$*

is a classical evolution family

We will see that $\varphi_{s,t}$ is univalent for any $0 \le s \le t < \infty$.

Set $G(w, t) := -w\, p(w, t)$, where $p(w, t) = \frac{1+k(t)w}{1-k(t)w}$ with $|k(t)| = 1$ for every $t \ge 0$.

Exercise 1.8.5. For any $w \in \mathbf{D}$, $t \ge 0$ we have $\mathfrak{Re}\, p(w, t) \ge 0$.

Definition 1.8.3. *A* classical Herglotz vector field $G(z, t)$ *is a function $G : \mathbf{D} \times [0, \infty) \longrightarrow \mathbf{C}$ (= $T\mathbf{D}$) such that*

1. *$z \longmapsto G(z, t)$ is holomorphic for any $t \ge 0$*
2. *$t \longmapsto G(z, t)$ is measurable for any $z \in \mathbf{D}$*
3. *$G(z, t) = -z\, p(z, t)$ where*
   (a) *$p(0, t) = 1$*
   (b) *$\mathfrak{Re}\, p(z, t) \ge 0$*

We will see that given a classical Loewner chain we get a unique classical evolution family and viceversa. Moreover, given a classical Herglotz vector field we can get a unique classical evolution family and viceversa. The bridge between these last objects is given by the Loewner ODE:

$$\frac{\partial \varphi_{s,t}}{\partial t} = G(\varphi_{s,t}(z), t).$$

1.9 Loewner's Theorem

Given a single-slit map $f \in \mathcal{S}$, denote by $f_t(z) = e^t z + \dots$ the standard parametrization; given $s < t$ we define $\varphi_{s,t} = f_t^{-1} \circ f_s$. We also denote the image of the tip of the slit with $\lambda(t) := f_t^{-1}(\psi(t))$.

Exercise 1.9.1. Show that f_t^{-1} is continuous at $\psi(t)$.

The exercise is not so trivial, use the following result:

Definition 1.9.1. *Let $f : \mathbf{D} \longrightarrow \mathbf{C}$ be a continuous function, $\{C_n\}$ a family of Jordan arcs with $C_n \subset \mathbf{D}$, such that*

1. $\operatorname{diam}(C_n) > \delta > 0$ *for any n,*
2. $f|_{C_n} \to \text{const}$ *as $n \to \infty$.*

Then the family $\{C_n\}$ is called a family of Koebe arcs *for f.*

Theorem 1.9.1 (Koebe)**.** *A univalent function $f : \mathbf{D} \longrightarrow \mathbf{C}$ has no Koebe arcs.*

Proof of Theorem 1.8.1. (1) We show that $\lim_{t\to\infty} e^t \varphi_{s,t}(z) = f_s(z)$.

By the Growth Theorem (applied to $e^{-t} f_t(z) \in \mathcal{S}$), we have

$$\frac{e^t|z|}{(1+|z|)^2} \le |f_t(z)| \le \frac{e^t|z|}{(1-|z|)^2}, \quad |z| < 1; \tag{1.2}$$

for $w \in \mathbf{C}$ given, there is $t \gg 1$, such that $w \in f_t(\mathbf{D})$; let $z = f_t^{-1}(w)$, then by (1.2) we have

$$\left(1 - \left|f_t^{-1}(w)\right|\right)^2 \le e^t \left|\frac{f_t^{-1}(w)}{w}\right| \le \left(1 + \left|f_t^{-1}(w)\right|\right)^2 \le 4, \tag{1.3}$$

which implies that $\left|f_t^{-1}(w)\right| \le 4e^{-t}|w| \to 0$ uniformly as $t \to \infty$.

Then, by (1.3), $\lim_{t\to\infty} e^t \frac{|f_t^{-1}(w)|}{|w|} = 1$ uniformly on compact sets and so $\left\{e^t \frac{f_t(w)}{w}\right\}$ is a normal family converging uniformly on compact sets to the constant function 1, for

$$e^t \frac{f_t^{-1}(w)}{w}\bigg|_{w=0} = e^t (f_t^{-1})'(0) = 1;$$

hence $e^t f_t^{-1}(w) \to w$ uniformly.

Setting $w = f_s(z)$, we get $\lim_{t\to\infty} e^t \varphi_{S,t}(z) = f_s(z)$ uniformly on compact sets.

(2) We have to derive the differential equation (1.1).

We define $\Phi_{s,t}(z) := \log \frac{\varphi_{s,t}(z)}{z}$, choosing the branch of log in such a way that

$$\Phi_{s,t}(0) = \log e^{s-t} = s - t.$$

Exercise 1.9.2. Show that Φ is holomorphic in $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$.

As $\lambda(s)$ belongs to the unit circle and it is mapped to the tip of the slit by $\varphi_{s,t}$, we can take two points $e^{i\alpha}, e^{i\beta}$ on the circle such that the arc from $e^{i\alpha}$ to $e^{i\beta}$ contains $\lambda(s)$ and

$$\varphi_{s,t}(\partial\mathbf{D} - \operatorname{arc}(e^{i\alpha}, e^{i\beta})) \subset \partial\mathbf{D}, \quad \varphi_{s,t}(\operatorname{arc}(e^{i\alpha}, e^{i\beta})) = J_{s,t},$$

where $J_{s,t}$ is the slit ending in $\varphi_{s,t}(\lambda(s))$.

Hence we see that

$$\mathfrak{Re}\,\Phi_{s,t}(z) = \begin{cases} 0 & \text{if } z \in \partial\mathbf{D} - \operatorname{arc}(e^{i\alpha}, e^{i\beta}) \\ < 0 & \text{if } z \in \operatorname{arc}(e^{i\alpha}, e^{i\beta}) \end{cases}$$

Now we use the Poisson formula to get

$$\Phi_{s,t}(z) = \frac{1}{2\pi}\int_\alpha^\beta \mathfrak{Re}\,(\Phi_{s,t}(e^{i\theta}))\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta + \underbrace{i\mathfrak{Im}\,\Phi_{s,t}(0)}_{0}.$$

Remark 1.9.1.

$$s-t = \Phi_{s,t}(0) = \frac{1}{2\pi}\int_\alpha^\beta \mathfrak{Re}\,(\Phi_{s,t}(e^{i\theta}))d\theta.$$

Given now $0 \le m \le s \le t < \infty$,

$$\Phi_{s,t}(\varphi_{m,s}(z)) = \log\frac{\varphi_{s,t}(\varphi_{m,s}(z))}{\varphi_{m,s}(z)} = \log\frac{\varphi_{m,t}(z)}{\varphi_{m,s}(z)}$$

and the LHS is also equal to

$$\frac{1}{2\pi}\int_\alpha^\beta \underbrace{\mathfrak{Re}\,\Phi(e^{i\theta})}_{<0}\frac{e^{i\theta}+\varphi_{m,s}(z)}{e^{i\theta}-\varphi_{m,s}(z)}d\theta \overset{\text{mean value thm}}{=}$$

$$= \frac{1}{2\pi}\left(\int_\alpha^\beta \mathfrak{Re}\,\Phi(e^{i\theta})d\theta\right)\left(\mathfrak{Re}\,\frac{e^{i\eta}+\varphi_{m,s}(z)}{e^{i\eta}-\varphi_{m,s}(z)} + i\mathfrak{Im}\,\frac{e^{i\sigma}+\varphi_{m,s}(z)}{e^{i\sigma}-\varphi_{m,s}(z)}\right),$$

where $e^{i\eta}$ and $e^{i\sigma}$ are in the arc from $e^{i\alpha}$ and $e^{i\beta}$.

When $t \to s$, the arc $(e^{i\alpha}, e^{i\beta})$ goes to $\lambda(s)$ (this fact is true but requires a certain care to be proven correctly). Hence $e^{i\eta}$ and $e^{i\sigma}$ go to $\lambda(s)$ as well, so that

$$\lim_{t\to s}\frac{1}{t-s}\log\frac{\varphi_{m,t}(z)}{\varphi_{m,s}(z)} = \lim_{t\to s}\frac{1}{t-s}\underbrace{\left(\frac{1}{2\pi}\int_\alpha^\beta \mathfrak{Re}\,\Phi_{s,t}(e^{i\theta})d\theta\right)}_{\Phi_{s,t}(0)=s-t}\frac{\lambda(s)+\varphi_{m,s}(z)}{\lambda(s)-\varphi_{m,s}(z)} = -\frac{1+\frac{1}{\lambda(s)}\varphi_{m,s}(z)}{1-\frac{1}{\lambda(s)}\varphi_{m,s}(z)};$$

on the other hand,

$$\frac{1}{t-s}(\log\varphi_{m,t}(z) - \log\varphi_{m,s}(z)) \to \frac{\partial}{\partial s}\log\varphi_{m,s}(z) = \frac{1}{\varphi_{m,s}(z)}\frac{\partial\varphi_{m,s}(z)}{\partial s}.$$

Exercise 1.9.3. Prove the differentiability for $s \to t$, $s \le t$.

(3) We have to show that $t \longmapsto \lambda(t)$ is continuous.

We use Schwarz' reflection principle to extend holomorphically the function

$$\varphi_{s,t} : \mathbf{C} - \operatorname{arc}(e^{i\alpha}, e^{i\beta}) \longrightarrow \mathbf{C} - (J_{s,t} \cup J^*_{s,t}),$$

where $J^*_{s,t}$ is the reflection of $J_{s,t}$.

By Koebe's $\frac{1}{4}$-Theorem (as $e^{t-s}\varphi_{s,t}(z) \in \mathcal{S}$), $\varphi_{s,t}(\mathbf{D}) \supset \mathbf{D}_{\frac{1}{4}e^{s-t}}$, so that $J_{s,t} \cap \mathbf{D}_{\frac{1}{4}e^{s-t}} = \emptyset$.

Hence for any $w \in J_{s,t}$, $|w| > \frac{1}{4}e^{s-t}$ and for any $w \in J^*_{S,t}$, $|w| < 4e^{t-s}$.

Moreover,

$$\lim_{z\to\infty}\frac{\varphi_{s,t}(z)}{z} = \lim_{z\to 0}\frac{z}{\varphi_{s,t}(z)} = \frac{1}{\varphi'_{s,t}(0)} = e^{t-s},$$

as for $|z|>1$, $\varphi_{s,t}(z)=\frac{1}{\varphi_{s,t}(1/z)}$.
By the maximum principle,

$$\left|\frac{\varphi_{s,t}(z)}{z}\right|\leq 4e^{t-s}$$

on $\mathbf{C}-\mathrm{arc}(e^{i\alpha},e^{i\beta})$, so that

$$\lim_{t\longrightarrow s}\left|\frac{\varphi_{s,t}(z)}{z}\right|\leq 4$$

and so $\left\{\frac{\varphi_{s,t}(z)}{z}\right\}$ is a normal family on $\mathbf{C}-\mathrm{arc}(e^{i\alpha},e^{i\beta})$.
Let φ be a limit function of $\left\{\frac{\varphi_{s,t}(z)}{z}\right\}_{t\downarrow s}$; as the arc goes to $\lambda(s)$, it follows that φ is a holomorphic function $\varphi:\mathbf{C}-\{\lambda(s)\}\longrightarrow\mathbf{C}P^1$ with $|\varphi(z)|\leq 4$, so that (exercise) φ is constant and equal to $\varphi(0)=\lim_{t\to s}\varphi'_{s,t}=\lim_{t\to s}e^{s-t}=1$.
Then $\varphi_{s,t}(z)\to z$ uniformly on compact sets in $\mathbf{C}-\{\lambda(s)\}$, when $t\to s$.
Fix $s\geq 0$, $\varepsilon>0$, let $\delta>0$ be such that $0<t-s<\delta$, then the $\mathrm{arc}(e^{i\alpha},e^{i\beta})\subset\mathbf{D}_\varepsilon(\lambda(s))$; set $C:=\partial\mathbf{D}_\varepsilon(\lambda(s)$, $\widetilde{C}:=\varphi_{s,t}(C)$; as C is a compact set, $\varphi_{s,t}|_C\xrightarrow{t\to s}z$, so that for $t\downarrow s$, $\mathrm{diam}(\widetilde{C})\leq 3\varepsilon$.
Then, if $z\in C$,

$$|\lambda(s)-\lambda(t)|\leq|\lambda(s)-z|+|z-\varphi_{s,t}(z)|+|\varphi_{s,t}(z)-\lambda(t)|\leq\varepsilon+\varepsilon+3\varepsilon.$$

Hence $\lim_{t\downarrow s}\lambda(t)=\lambda(s)$.
Example 1.9.1. Get $\lim_{s\uparrow t}\lambda(s)=\lambda(t)$.

□

1.10 Tutorial

Exercise 1.10.1. Let $\Gamma=[-\infty,-1/4]$, $f(z)=\frac{z}{(1-z)^2}$; find the classical Loewner chain for f. Find $k(t)$ in the Loewner equation.

Solution. If we multiply by e^t we get the Loewner chain.
As $\lambda(t)=f_t^{-1}(\psi(t))=f_t^{-1}\left(-\frac{e^t}{4}\right)$, we want to find z such that

$$-\frac{e^t}{4}=e^t\frac{z}{(1-z)^2},$$

whence $z=-1$. □

Exercise 1.10.2. Take $f:\mathbf{D}\longrightarrow\mathbf{D}$ holomorphic such that $f(0)=0$.

- If f is not a rotation, then $f^{\circ n}$ converges uniformly on compact sets to the map $(z\longmapsto 0)$.
- If $\lambda=f'(0)\neq 0$, show that

$$\sigma_n(z):=\frac{f^{\circ n}(z)}{\lambda^n}$$

is converging to a unique holomorphic $\sigma:\mathbf{D}\longrightarrow\mathbf{C}$ such that $\sigma(0)=0,\sigma'(0)=1$ and $\sigma\circ f=\lambda\sigma$. Moreover σ is univalent if and only if f is.

Solution. For any $r>0$, there is $c=c(r)$ such that for any $z\in\mathbf{D}_r$,

$$|f(z)|<c|z|.$$

Let C be such that $|f(z)-\lambda z|\leq C|z|^2$, then

$$\left|\frac{f^{n+1}(z)}{\lambda^{n+1}}-\frac{f^n(z)}{\lambda^n}\right|=\frac{1}{|\lambda|^{n+1}}|f^{n+1}(z)-\lambda f^n(z)|\leq\frac{C}{|\lambda|^{n+1}}|f^n(z)|^2.$$

As we can have $c(r)$ such that $c(r)^2<c(r)<|\lambda|$ for some r, we can get to the solution. □

1.11 Background notions and results

Definition 1.11.1. *A* real analytic semigroup of holomorphic self-maps in $\mathbf{D}$ *is a family*

$$\phi_t : [0, \infty) \times \mathbf{D} \longrightarrow \mathbf{D}$$

such that

(1) $\phi_0 = id_{\mathbf{D}}$,

(2) $\phi_{s+t} = \phi_t \circ \phi_s$ *for any* $s, t \in [0, \infty)$,

(3) for any $t \geq 0$, $z \longmapsto \phi_t(z)$ *is holomorphic,*

(4) $\phi_t \xrightarrow{t \to s} \phi_s$ *uniformly on compact sets,*

(5) $t \longmapsto \phi_t(z)$ *is real analytic.*

Properties (1),(2), (3) and (4) say that ϕ is a continuous morphism between the semigroups $(\mathbf{R}^+, +)$ (with the Euclidean topology) and $(\mathrm{Hol}(\mathbf{D}, \mathbf{D}), \circ)$ with the topology of uniform convergence.

Exercise 1.11.1.

1. Assume $\phi_{t_0}(0) = 0$ for some $t_0 > 0$ and, then, if ϕ_{t_0} is not a rotation, prove that $\phi_t(0) = 0$ for any $t \geq 0$.

   If ϕ_{t_0} is a rotation, prove that ϕ_t is a rotation for $t \geq 0$.

2. If ϕ_{t_0} is not a rotation, $t_0 > 0$, $\phi_{t_0}(0) = 0$, prove that if $\lambda_t := \phi_t'(0)$ then $\lambda_t = e^{-at}$ with $\Re\mathrm{e}\, a > 0$ (it can be assumed that ϕ_t is univalent for $t \geq 0$).

3. There is a unique univalent function $h : \mathbf{D} \longrightarrow \mathbf{C}$ such that $h(0) = 0$, $h'(0) = 1$ and such that

   $$h \circ \phi_t = e^{-at} h.$$

   Show that (following Poincaré)

   (a) for fixed $t_0 > 0$ there is a unique h in $\mathbf{C}[[z]]$,

   (b) solve the *homological equation*: suppose $\phi_t(z) = \lambda_t z + a_k(t) z^k + \ldots$, $k \geq 2$, look for $p_k(z) = z + \alpha z^k$ such that

   $$p_k \circ \phi_t = (\lambda_t z + b_{k+1}(t) z^{k+1} + \ldots) \circ p_k(z);$$

   show that there exists $\alpha \in \mathbf{C}$ which does not depend on t;

   (c) set $\widetilde{h}(z) = \lim_{k \to \infty} p_k \circ p_{k-1} \circ \cdots \circ p_1$, show that $\widetilde{h} \circ \phi_t = \lambda_t \widetilde{h}$ with $\widetilde{h} \in \mathbf{C}[[z]]$, hence prove that $\widetilde{h} = h$.

Exercise 1.11.2. Given a semigroup (ϕ_t) with $\phi_t(0) = 0$ and $\phi_t'(0) = e^{-t}$, then $(\phi_{s,t} := \phi_{t-s})$ is a classical evolution family.

It follows that evolution families are a dynamical generalization of semigroups.

1.12 Properties of classical Loewner's chains

Lemma 1.12.1. *Consider* $\mathcal{P} := \{f : \mathbf{D} \longrightarrow \mathbf{H}$ *holomorphic,* $f(0) = 1\}$, *which is called the* Carathéodory class, *then*

1.

$$\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|} \quad \forall\, |z| < 1$$

2.
$$\frac{1-|z|}{1+|z|} \leq \Re\mathrm{e}\, f(z) \leq \frac{1+|z|}{1-|z|}$$

3.
$$|f'(z)| \leq \frac{2\Re\mathrm{e}\, f(z)}{1-|z|^2} \leq \frac{2}{(1-|z|)^2}.$$

Remark 1.12.1. With $\mathbf{H}$ we denote the domain $\{\Re\mathrm{e}\, w > 0\}$.

Exercise 1.12.1. Show that $\mathcal{P}$ is compact.

Lemma 1.12.2. *Let (f_t) be a classical Loewner chain; then*

1.
$$e^t \frac{|z|}{(1+|z|)^2} \leq |f_t(z)| \leq e^t \frac{|z|}{(1-|z|)^2}$$

2.
$$e^t \frac{1-|z|}{(1+|z|)^3} \leq |f_t'(z)| \leq e^t \frac{1+|z|}{(1-|z|)^3}$$

3.
$$|f_s(z) - f_t(z)| \leq \frac{8|z|}{(1-|z|)^4}(e^t - e^s) \quad |z| < 1, t \geq s,$$

it follows that f_t is absolutely continuous uniformly on compact sets.

Dimostrazione. We prove 3. For $s \leq t$, define
$$p_{s,t}(z) = \frac{1+e^{s-t}}{1-e^{s-t}} \frac{z-\phi_{s,t}(z)}{z+\phi_{s,t}(z)}$$
with $\varphi_{s,t} = f_t^{-1} \circ f_s$.

Exercise 1.12.2. $p_{s,t} \in \mathcal{P}$ for any $0 \leq s \leq t < \infty$.

By 1. of Lemma 1.12.1
$$\frac{1+e^{s-t}}{1-e^{s-t}} \left|\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)}\right| \leq \frac{1+|z|}{1-|z|},$$
hence, by Schwarz' Lemma ($|\varphi_{s,t}(z)| < |z|$),
$$|z-\varphi_{s,t}(z)| \leq \frac{1+|z|}{1-|z|}\frac{1-e^{s-t}}{1+e^{s-t}}|z+\varphi_{s,t}(z)| \leq 2|z|(1-e^{s-t})\frac{1+|z|}{1-|z|}$$

Remark 1.12.2. ¿From here we get that $\varphi_{s,t}$ is absolutely continuous in $0 \leq s \leq t < \infty$ uniformly on compact sets.

Then
$$|f_t(z) - f_s(z)| = |f_t(z) - f_t(\varphi_{s,t}(z))| = \left|\int_{\varphi_{s,t}(z)}^{z} f_t'(\zeta)d\zeta\right| \overset{2.\text{ of } 1.12.1}{\leq} e^t \frac{1+|z|}{(1-|z|)^3}|z-\varphi_{s,t}(z)| \leq$$
$$\leq e^t \frac{1+|z|}{(1-|z|)^3} 2|z|(1-e^{s-t})\frac{1+|z|}{1-|z|}$$
□

Theorem 1.12.3. *1. If (f_t) is a classical Loewner chain, then there exists a unique classical evolution family $(\varphi_{s,t})$ such that $f_s = f_t \circ \varphi_{s,t}$ for $0 \leq s \leq t < \infty$.*

2. *If $(\varphi_{s,t})$ is a classical evolution family, there exists a unique classical Loewner chain (f_t) such that $f_s = f_t \circ \varphi_{s,t}$.*

Dimostrazione. The first statement is almost trivial. Let's prove 2.; the idea of Ch. Pommerenke is to show that $f_s(z) = \lim_{t\to\infty} e^t \varphi_{s,t}(z)$. However this method works only for *classical evolution families* (fixing 0). Here we give a different proof based on the extension of the inverse of a univalent mapping, as taken from [LAK10].

Extension of the inverse of a univalent map. Given complex manifolds X and Y of complex dimension n and a univalent function $F : X \longrightarrow Y$, the claim is that there exists a complex manifold Z of complex dimension n and $G : Y \longrightarrow Z$ such that G is biholomorphic and

1. $X \hookrightarrow Z$ is an open set,
2. $G \circ F = id_X$.

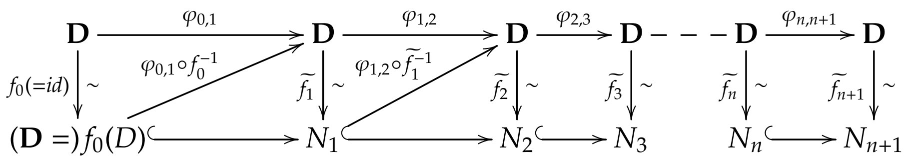
Indeed, define $Z := X \sqcup (Y - F(X))$ as a set and a bijection $H : Z \longrightarrow Y$

$$H(z) = \begin{cases} F(z) & z \in X \\ z & z \in Y - F(X) \end{cases}$$

Exercise 1.12.3. Show that there is a unique structure of complex manifold on Z such that H is a biholomorphism.

Then it is enough to define $G := H^{-1}$.

Start now with any biholomorphism $f_0 : \mathbf{D} \longrightarrow \mathbf{D}$, for example the identity, then by the previous construction we can find Riemann surfaces $N_1, N_2, \ldots$ isomorphic to $\mathbf{D}$ such that the following diagram commutes



Then we can define $N = \bigcup_{n\geq 1} N_n$ which is an abstract Riemann surface, simply connected and non-compact. Then the (abstract) Riemann mapping theorem implies that there is a biholomorphism G from N to either $\mathbf{C}$ or $\mathbf{D}$. We take G such that $\underline{0} = f_0(0) \hookrightarrow N_1 \hookrightarrow N_2 \hookrightarrow \cdots \hookrightarrow N$ is sent to 0.

Define $f_n := G \circ \widetilde{f_n}$ which goes from $\mathbf{D}$ to either $\mathbf{D}$ or $\mathbf{C}$. Anyway, the function $f_n : \mathbf{D} \longrightarrow \mathbf{C}$ is univalent. By construction, $f_{n+1} \circ \varphi_{n,n+1} = f_n$ for any $n \in \mathbf{N}$; let $t \geq 0$, consider $n \in \mathbf{N}$ such that $n \geq t$, define

$$f_t := f_n \circ \varphi_{t,n}$$

and for $t \leq n \leq m$ we have

$$f_m \circ \varphi_{t,m} = f_m \circ (\varphi_{n,m} \circ \varphi_{t,n}) = f_m \circ (\varphi_{m-1,m} \circ \cdots \circ \varphi_{n,n+1}) \circ \varphi_{t,n} = f_{m-1} \circ (\varphi_{m-2,m-1} \circ \cdots \circ \varphi_{n,n+1}) \circ \varphi_{t,n} = f_n \circ \varphi_{t,n}.$$

Hence $G(N) = \bigcup_{t\geq 0} f_t(\mathbf{D})$ and $f_s(\mathbf{D}) \subset f_t(\mathbf{D})$ for any $s \leq t$. Moreover $f_0 = f_t \circ \varphi_{0,t}$ for any $t \geq 0$ and, as $\lambda = f_0'(0) \neq 0$ we differentiate the equality to get

$$\lambda = f_0'(0) = f_t'(0)\varphi_{0,t}'(0) = e^{-t},$$

so that, up to replace f_t with $\frac{f_t}{\lambda}$, we see that $f_t(0) = 0$, $f_t'(0) = e^t$ and so f_t is a classical Loewner chain.

Exercise 1.12.4. Show that $G(N) = \mathbf{C}$.

Open question: Given an evolution family $\varphi_{s,t} : \mathbf{B}^2 \longrightarrow \mathbf{B}^2$, does there exists a Loewner chain associated to $\varphi_{s,t}$ with image in $\mathbf{C}^2$? Leandro Arosio [Aro10] proved that if $\varphi_{s,t}(0) = 0$ with $d(\varphi_{s,t})_0 = e^{(s-t)A}$ with

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Re\mathrm{e}\, \lambda_j < 0,$$

then the question has positive answer.

Uniqueness. Suppose (g_t) is another classical Loewner chain such that $g_t \circ \varphi_{s,t} = g_s$ for $0 \le s \le t$. Define $h(z) := g_t \circ f_t^{-1}(z)$ for $z \in f_t(\mathbf{D})$.

Remark 1.12.3. For any $z \in \mathbf{C}$, there is $t(z) \ge 0$ such that $z \in f_t(\mathbf{D})$ for any $t \ge t(z)$.

Exercise 1.12.5. The function h does not depend on t, i.e. if $s \le t$ and $z \in f_s(\mathbf{D})$, then $g_s \circ f_s^{-1}(z) = g_t \circ f_t^{-1}(z)$.

As a consequence, $h : \mathbf{C} \longrightarrow \mathbf{C}$ is a biholomorphism.

Remember that $\mathrm{Aut}(\mathbf{C}) = \{z \longmapsto az + b, a \neq 0\}$, so that $h(z) = az + b$; as $h \circ f_t = g_t$, then $f_t(0) = g_t(0)$ implies that $b = 0$; moreover, as $f_t'(0) = e^t = g_t'(0)$, we have $a = 1$. This proves uniqueness. □

1.13 The Loewner-Kufarev PDE

Theorem 1.13.1. *Let (f_t^n) be a sequence of classical Loewner chains. Then there is a subsequence $(f_t^{n_k})$ which converges uniformly on compact sets to a classical Loewner chain (f_t).*

Dimostrazione. By Theorem 1.12.3, the sequence (f_t^n) corresponds uniquely to a sequence $(\varphi_{s,t}^n)$. The family $\{\varphi_{s,t}^n\}$ are uniformly bounded on compact sets for any $n \in \mathbf{N}$. Hence $\varphi_{s,t}(z)$ is uniformly continuous on $\{|z| \le 1 - 1/m\} \times \{0 \le s \le t \le m\}$.

Then the Ascoli-Arzelà Theorem implies that there is a converging subsequence $\varphi_{s,t}^{n_k} \to \varphi_{s,t}$ (a diagonal argument is needed).

Keep (f_t) corresponding to $(\varphi_{s,t})$ (which is a classical evolution family). Then $f_t^{n_k} \to f_t$, for $f_t^{n,k}$ is uniformly convergent on $K \times [0, T] \Subset \mathbf{D} \times [0, T]$.

If $f_t^{n_k}$ converges to another function g_t, then $f_t^{n_k} \circ \varphi_{s,t}^{n_k} = f_s^{n_k}$ implies the equality $g_t \circ \varphi_{s,t} = g_s$ and by uniqueness of the associated Loewner chain, we have $g_t \equiv f_t$. □

Corollary 1.13.2. *Let $f \in \mathcal{S}$, then there exists a classical Loewner chain (f_t) such that $f_0 = f$.*

Dimostrazione. As single-slit maps are dense in $\mathcal{S}$, we can find a sequence of single-slit maps f^k converging to f. Then use the parametric representation to construct classical Loewner chains (f_t^k) for f^k.

Theorem 1.13.1 implies that $(f_t^{n_k}) \to f_t$ uniformly and $f_0^{n_k} = f^k \to f$ uniformly, so that $f_0 = f$. □

Estimates 1.12.2 show that $t \longmapsto f_t$ is locally absolutely continuous in t, uniformly on compact sets of $\mathbf{D}$ and the same holds for $t \longmapsto \varphi_{s,t}$.

If (f_t) is a Loewner chain with f_0 a single-slit map, by differentiating the equality

$$f_s = f_t \circ \varphi_{s,t},$$

we get

$$0 = \dot{f}_t(\varphi_{s,t}(z)) + f_t'(\varphi_{s,t}(z))G(\varphi_{s,t}(z), t) \quad \text{for a.e. } t \ge 0,$$

where $G(z, t) = -z\, p(z, t)$, with $p(z, t) = \frac{1+k(t)z}{1-k(z)t}$. Then we deduce the *Loewner-Kufarev PDE*

$$\dot{f}_t(z) = f_t'(z)z\frac{1 + k(t)z}{1 - k(t)z} \quad \text{a.e. } t \ge 0, \quad \text{for any } z \in \mathbf{D}.$$

Remark 1.13.1. We should be more careful when we claim differentiability. Actually we get it *a posteriori*.

Exercise 1.13.1. Given $z_0 \in \mathbf{D}$, define

$$\mathscr{F}_{z_0} := \left\{ u : \mathbf{D} \longrightarrow \mathbf{R}^- \ : \ u \text{ is subharmonic}, \overline{\lim_{z\to z_0}} |u(z) - \log|z - z_0|| < \infty \right\}.$$

Let $G_{z_0}(z) := \sup_{u \in \mathscr{F}_{z_0}} u(z)$.

- Show that G_{z_0} is harmonic in $\mathbf{D} - \{z_0\}$, and $G_{z_0}(e^{i\theta}) = 0$ for any $\theta \in \mathbf{R}$.

  ($G_{z_0} \in \mathscr{F}_{z_0}$) is called the *Green function* of $\mathbf{D}$.
- Show that if $f : \mathbf{D} \longrightarrow \mathbf{D}$ is holomorphic, $G_{f(z_0)}(f(z)) \le G_{z_0}(z)$ for any $z \in \mathbf{D}$ (and = holds if and only if f is an automorphism).
- Compare G with the Poincaré distance.

Remark 1.13.2. A function $u : \mathbf{D} \longrightarrow \mathbf{R}^-$ which is upper semi-continuous and such that

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta,$$

is said to be *subharmonic*.

w is harmonic in a domain D if and only if w is subharmonic and for any $u \in \text{subh}(D)$, $u|_{\partial \mathbf{D}_\rho(z_0)} \le w|_{\partial \mathbf{D}_\rho(z_0)}$ implies that $u \le w$ in $D_\rho(z_0)$.

1.14 Loewner's Theory and the Bieberbach Conjecture

If $f \in \mathcal{S}$, $f(z) = z + \sum_{j\ge2} a_j z^j$, we can use Loewner's Theorem to show that $|a_2| \le 2$ (and similarly can be done to show $|a_3| \le 3$).

As single-slit maps are dense in $\mathcal{S}$, we can prove it for this class of functions. Given f, we associate the classical Loewner chain (f_t). Then Loewner's Theorem gives us

$$\dot{f}_t(z) = f_t'(z)\, z\, p(z,t) \quad \forall\, t \ge 0, \tag{1.4}$$

where $f_t(z) = e^t z + \sum_{j\ge2} a_j(t) z^j$, $p(z,t) = \frac{1+k(t)z}{1-k(t)z}$ and $t \longmapsto k(t) \in \partial\mathbf{D}$ is continuous.

The function p is in the Carathéodory class $\mathcal{P}$ and $p(z,t) = 1 + \sum_{n=1}^\infty c_n(t) z^n$. From (1.4), we get

$$e^t z + \sum_{j\ge2} \dot{a}_j(t) z^j = \left(e^t + \sum_{j\ge2} j a_j(t) z^{j-1}\right) z \left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right) = \left(e^t z + \sum_{j\ge2} j a_j(t) z^j\right)\left(1 + \sum_{n\ge1} c_n(t) z^n\right);$$

looking at the coefficient of z^2, we have

$$\dot{a}_2(t) = e^t c_1(t) + 2a_0(t) \quad t \ge 0,$$

so that

$$e^{-2t}\dot{a}_2(t) - 2e^{-t} a_2(t) = e^{-t} c_1(t)$$

which corresponds to

$$\frac{d}{dt}\left[e^{-2t} a_2(t)\right] = e^{-t} c_1(t)$$

and by integration

$$\int_s^\tau \frac{d}{dt}\left(e^{-2t} a_2(t)\right) dt = \int_s^\tau e^{-t} c_1(t) dt.$$

Now, for any $z \in K \Subset \mathbf{D}$ and any $t \geq 0$, we have $|\underbrace{e^{-t} f_t(z)}_{\in \mathcal{S}}| \leq M$, hence there exists $M_j > 0$ such that

$$e^{-t}|a_j(t)| \leq M_j \quad \text{for any } t \geq 0,$$

so that the limit

$$\lim_{\tau \to \infty} \int_s^\tau \frac{d}{dt}\left(e^{-2t} a_2(t)\right) dt$$

exists and it is equal to

$$\underbrace{e^{-2t} a_2(t)\Big|_{t=+\infty}}_{0} - e^{-2s} a_2(s).$$

Hence

$$a_2(s) = -e^{2s} \int_s^\infty c_1(t) e^{-t} dt. \tag{1.5}$$

We claim that $|c_1(t)| \leq 2$ for any $t \geq 0$. Then equality (1.5) implies that $|a_2| = |a_2(0)| \leq 2$.
The claim follows from the following:

Lemma 1.14.1. *Let $p \in \mathcal{P}$, $p(z) = 1 + \sum_{j \geq 1} c_j z^j$, then $|c_j| \leq 2$ for any j.*

Dimostrazione. Given a holomorphic function $f : \mathbf{D} \longrightarrow \overline{\mathbf{H}} = \{\Re\mathrm{e}\, w \geq 0\}$, we have the *Herglotz' representation formula*: there exists a non-decreasing function $\mu : [0, 2\pi] \longrightarrow \mathbf{R}$ such that $\mu(2\pi) - \mu(0) = \Re\mathrm{e}\, f(0)$ and

$$f(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + i\Im\mathrm{m}\, f(0).$$

The reader may prove this formula as an exercise (Hint: define $\mu(r,t) := \frac{1}{2\pi}\int_0^t \Re\mathrm{e}\, f(re^{i\theta}) d\theta, \quad 0 < r < 1, 0 \leq t \leq 2\pi$, taking into account that $\Re\mathrm{e}\, f(z)$ is harmonic, let r go to 1).

Exercise 1.14.1. Use Herglotz' formula to prove the Growth Theorem for the Carathéodory Class (Lemma 1.12.1).

If $p \in \mathcal{P}$, Herglotz' formula gives

$$p(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \quad \mu(2\pi) - \mu(0) = 1;$$

expanding the integrand we get

$$c_j = 2 \int_0^{2\pi} e^{-ijt} d\mu(t),$$

so that

$$|c_j| \leq 2 \int_0^{2\pi} |e^{-ijt}| d\mu(t) = 2 \int_0^{2\pi} d\mu(t) = 2(\mu(2\pi) - \mu(0)) = 2.$$

□

1.15 Generalized Loewner chains

Definition 1.15.1. *A family $(\varphi_{s,t})_{0 \leq s \leq t < \infty}$ of holomorphic maps $\varphi_{s,t} : \mathbf{D} \longrightarrow \mathbf{D}$ is a L^d-*evolution family *if*

1. $\varphi_{s,s} = id_{\mathbf{D}}$,
2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ *for any* $0 \leq s \leq u \leq t < \infty$,

3. *for any $z \in \mathbf{D}$, $T > 0$, there exists $k_{z,T} \in L^d([0,T], \mathbf{R}^+)$ such that*

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\zeta)d\zeta \quad \forall\, o \leq s \leq T.$$

Example 1.15.1. Given $d \geq 1$, $\lambda : [0,\infty) \longrightarrow \mathbf{R}^+$ absolutely continuous and increasing with $\dot{\lambda} \in L^d_{\ell oc}([0,\infty), \mathbf{R})$ (we can write $\lambda \in AC^d_{\ell oc}$), then $\varphi_{s,t}(z) := e^{\lambda(s)-\lambda(t)}z$ is a L^d-evolution family.

Proposition 1.15.1. *A classical evolution family $(\varphi_{s,t})$ is a L^∞-evolution family.*

Dimostrazione. Define $h_{s,t}(z) = \frac{\varphi_{s,t}(z)}{z}$ for $z \neq 0$, which we can extend holomorphically on $\mathbf{D}$ by $h_{s,t}(0) := \varphi'_{s,t}(0) = e^{s-t}$.

Remark 1.15.1. By Schwarz' Lemma, if $f : \mathbf{D} \longrightarrow \mathbf{D}$ is holomorphic with $f(0) = 0$ and $f'(0) = \lambda \in (0,1]$, then $\Re\mathrm{e}\,(1 - f(z)/z) \geq 0$ (exercise).

Then the function $\frac{1-h_{s,t}(z)}{1-h_{s,t}(0)}$ is in $\mathcal{P}$, so that we can use the Growth Theorem for $\mathcal{P}$ to have

$$|1 - h_{s,t}(z)| \leq \frac{1+|z|}{1-|z|}(1 - h_{s,t}(0) = \frac{1+|z|}{1-|z|}e^s(e^{-s} - e^{-t}) \leq \frac{1+|z|}{1-|z|}e^T(e^{-s} - e^{-t});$$

therefore

$$|\underbrace{\varphi_{s,u}(z)}_{w\in\mathbf{D}} - \underbrace{\varphi_{s,t}(z)}_{\varphi_{u,t}(w)}| \leq \left|1 - \frac{\varphi_{u,t}(w)}{w}\right| \leq \frac{1+|w|}{1-|w|}e^T(e^{-u}e^{-t}).$$

By Schwarz' Lemma, $|\varphi_{s,u}(z)| \leq |z|$ and, as $t \longmapsto \frac{1+t}{1-t}$ is an increasing function for $t \in (0,1)$,

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \frac{1+|z|}{1-|z|}e^T(e^{-u} - e^{-t}) \leq \frac{1+|z|}{1-|z|}e^T(t-s).$$

□

Definition 1.15.2. *Let $d \geq 1$. A* weak holomorphic vector field of order d *is a function*

$$G : \mathbf{D} \times [0,\infty) \longrightarrow \mathbf{C}(\simeq T\mathbf{D})$$

such that

1. *for any $z \in \mathbf{D}$, the function $t \in [0,\infty) \longmapsto G(z,t)$ is measurable,*
2. *for any $t \in [0,\infty)$, the function $z \in \mathbf{D} \longmapsto G(z,t)$ is holomorphic,*
3. *for any $K \Subset \mathbf{D}$, $T > 0$, there exists $k_{K,T} \in L^d([0,T], \mathbf{R}^+)$ such that*

$$|G(z,t)| \leq k_{K,T}(t) \quad \forall\, z \in K,\ a.e.\ t \in [0,T].$$

Exercise 1.15.1. Use the Cauchy Formula to show that if $G(z,t)$ is a L^d-weak holomorphic vector field in $\mathbf{D}$, then for any $K \Subset \mathbf{D}$ and $T > 0$, there exists $\widehat{k_{K,T}} \in L^d([0,T], \mathbf{R}^+)$ such that

$$|G(z,t) - G(w,t)| \leq \widehat{k_{K,T}}(t)|z - w| \quad \forall\, z, w \in K,\ \text{a.e. } t \in [0,T].$$

By the Carathéodory Theory of ODE's, it follows that for any $(z,s) \in \mathbf{D} \times [0,\infty)$ there exist a unique $I(z,s) \in (s,\infty]$ and a unique $x : [s, I(z,s)) \longrightarrow \mathbf{D}$ such that

1. x is locally absolutely continuous in $[s, I(z,s))$,

2. x is the solution to the problem

$$\begin{cases} \dot{x}(t) = G(x(t), t) & \text{for a.e. } t \\ x(s) = z \end{cases}$$

3. the interval $[s, I(z,s))$ is maximal (when $I(z,s) < \infty$ maximality corresponds to $\lim_{t\to I(z,s)} \omega(x(s), x(t)) = \infty$).

Definition 1.15.3. *$I(z,s)$ si called the* escaping time *and $t \longmapsto x(t)$ is the* positive trajectory of *G.*

Definition 1.15.4. *Let $G(z,t)$ be a L^d-weak holomorphic vector field in $\mathbf{D}$. We say that G is a L^d-*Herglotz vector field *if for a.e. $t \in [0,\infty)$, the Cauchy problem*

$$\begin{cases} \dot{y}(\xi) = G(y(\xi), t) \\ y(0) = z \end{cases}$$

has a (necessarily unique) solution $y : [0,\infty) \longrightarrow \mathbf{D}$.

Definition 1.15.5. *A holomorphic vector field $H : \mathbf{D} \longrightarrow \mathbf{C}$ is said to be* semi-complete *(or H is an* infinitesimal generator*) if for every $z \in \mathbf{D}$, its flow $\Phi_z^H(t)$ is defined for any $t \geq 0$.*

Remark 1.15.2. We recall that the flow of a vector field H is defined by

$$\begin{cases} \frac{d\Phi_z^H}{dt} = H(\Phi_z^H(t)) \\ \Phi_z^H(0) = z \end{cases}$$

Exercise 1.15.2. Show that $\Phi_z^H(t+s) = \Phi^H_{\Phi_z^H(t)}(s) = \Phi^H_{\Phi_z^H(s)}(t)$.

The following results are in [FBDM08].

Theorem 1.15.2. 1. *For every L^d-evolution family $(\varphi_{s,t})$ in $\mathbf{D}$, there exists a L^d-Herglotz vector field $G(z,t)$ such that*

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t) \quad \forall\, z \in \mathbf{D},\ a.e.\ t \geq s. \tag{1.6}$$

Moreover, if $H(z,t)$ is another Herglotz vector field satisfying (1.6), then $G(\cdot\,,t) = H(\cdot\,,t)$ for a.e. $t \geq 0$.

2. *Conversely, for any L^d-Herglotz vector field $G(z,t)$ in $\mathbf{D}$, there exists a unique L^d-evolution family $(\varphi_{s,t})$ such that (1.6) is satisfied.*

Definition 1.15.6. *A function $p : \mathbf{D} \times [0,\infty) \longrightarrow \mathbf{C}$ is a L^d-*Herglotz function *if*

1. *for any $z \in \mathbf{D}$, the function $t \in [0,\infty) \longmapsto p(z,t) \in L^d_{\ell oc}([0,\infty), \mathbf{C})$,*
2. *for any $t \in [0,\infty)$, the function $z \in \mathbf{D} \longmapsto p(z,t)$ is holomorphic,*
3. *for any $z \in \mathbf{D}$ and a.e. $t \geq 0$, $\Re\mathrm{e}\, p(z,t) \geq 0$.*

Theorem 1.15.3. *$G(z,t)$ is a L^d-Herglotz vector field if and only if there exists a L^d-Herglotz function $p(z,t)$ and a measurable function $\tau : [0,\infty) \longrightarrow \overline{\mathbf{D}}$ such that*

$$G(z,t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z,t) \quad \forall\, z \in \mathbf{D},\ a.e.\ t \geq 0.$$

Moreover, the L^d-Herglotz function $p(z,t)$ is uniquely determined by $G(z,t)$ up to a zero measure set in t, while τ is uniquely determined up to a zero measure set for those t such that $G(z,t) \neq 0$.

When we deal with the classical Loewner vector field, we have

$$G(z,t) = -z\frac{1+zk(t)}{1-zk(t)}$$

and $\tau(t) \equiv 0$.

Definition 1.15.7. *A* L^d-Loewner chain *is a family* (f_t) *of functions* $f_t : \mathbf{D} \times [0,\infty) \longrightarrow \mathbf{C}$ *such that*

1. *for every* $t \geq 0$, $z \longmapsto f_t(z)$ *is univalent,*
2. *for every* $0 \leq s \leq t < \infty$, $f_s(\mathbf{D}) \subset f_t(\mathbf{D})$,
3. *for every* $K \Subset \mathbf{D}$ *and* $T > 0$, *there exists* $k_{K,T} \in L^d([0,T],\mathbf{R}^+)$ *such that*

$$|f_t(z) - f_s(z)| \leq \int_s^t k_{K,T}(\xi)\,d\xi \quad \forall\, z \in K, 0 \leq s \leq t \leq T.$$

The following result is in [MDCG10] (for a different proof see also [LAK10].)

Theorem 1.15.4. 1. *Let* $(\varphi_{s,t})$ *be a* L^d*-evolution family, then there exists a* L^d*-Loewner chain* (f_t) *such that*

$$f_s = f_t \circ \varphi_{s,t} \quad \forall\, 0 \leq s \leq t < \infty. \tag{1.7}$$

If (g_t) *is another Loewner chain satisfying* (1.7) *then, setting* $\Omega := \bigcup_{t\geq 0} f_t(\mathbf{D})$ *and* $\Omega' := \bigcup_{t\geq 0} g_t(\mathbf{D})$, *there exists a biholomorphic map* $\Phi : \Omega \longrightarrow \Omega'$ *such that* $g_t = \Phi \circ f_t$.

2. *Given a* L^d*-Loewner chain* (f_t)*, the family* $\varphi_{s,t} := f_t^{-1} \circ f_s$ *is a* L^d*-evolution family and* (1.7) *holds.*

Corollary 1.15.5. *We have the Loewner-Kufarev PDE*

$$\frac{\partial f_s}{\partial s}(z) = -\frac{\partial f_s}{\partial z}(z)G(z,s) \quad \text{for a.e. } s \geq 0. \tag{1.8}$$

Exercise 1.15.3. Give an example of L^d-Loewner chain (f_t) such that $\bigcup_{t\geq 0} f_t(\mathbf{D})$ is not biholomoprhic to $\mathbf{C}$.

Definition 1.15.8. *A* (continuous) one-parameter semigroup

$$\Phi : (\mathbf{R}_0^+, +) \longrightarrow (\mathrm{Hol}(\mathbf{D},\mathbf{D}), \circ)$$

is a semigroup morphism which is continuous with respect to the Euclidean topology on $\mathbf{R}_0^+$ *and the topology of uniform convergence on compact sets on* $\mathrm{Hol}(\mathbf{D},\mathbf{D})$.

Explicitly,

1. $\Phi_0 = id_{\mathbf{D}}$,
2. $\Phi_{t+s} = \Phi_t \circ \Phi_s$,
3. $\Phi_t \xrightarrow{t \to t_0} \Phi_{t_0}$ *uniformly.*

Theorem 1.15.6 (Berkson-Porta)**.** *Let* (Φ_t) *a one-parameter semigroup of holomorphic self-maps of* $\mathbf{D}$. *Then*

1. $t \longmapsto \Phi_t(z)$ *is real analytic for any* $z \in \mathbf{D}$,
2. *there exists a unique holomorphic vector field* $G : \mathbf{D} \longrightarrow \mathbf{C}(= T\mathbf{D})$ *such that*

$$\frac{\partial \Phi_t(z)}{\partial t}(z) = G(\Phi_t(z)).$$

Remark 1.15.3. The same problem can be studied for complex time, but monodromy problems may (often) arise.

Theorem 1.15.7 (Flow-box Theorem)**.** *Consider a domain $\Omega \subset \mathbf{C}^n$ and a holomorphic map $F : \Omega \longrightarrow \mathbf{C}^n$. Let $K \Subset \Omega$, then there exist $\delta > 0$ and an open neighbourhood U of K and a map $X : (-\delta, \delta) \times U \longrightarrow \Omega$ such that X is real analytic in (t, z), it is holomorphic in z for fixed t and it verifies*

$$\begin{cases} \frac{\partial X}{\partial t}(t,z) = F(X(t,z)) \\ X(0,z) = z \end{cases}$$

Moreover X is unique.

As a consequence of the Flow-box Theorem, we get that if G is a semicomplete holomorphic vector field then it generates a one-parameter semigroup.

Proof of Theorem 1.15.6. Take $K \Subset \mathbf{D}$. Let $\alpha \in (0,1)$ and let $\widehat{K}$ be the convex hull of $\Phi([0,\alpha] \times K)$. Note that $\widehat{K}$ is compactly contained in $\mathbf{D}$.

Let $0 < \mu \ll 1$, let $\delta \in (0, \alpha]$ such that

$$\sup_{z \in \widehat{K}} |\Phi_t'(z) - 1| < \mu \quad \forall\, t < \delta.$$

Exercise 1.15.4. Prove that such a δ exists.

Then, for any $z \in \mathbf{K}$, $t \in [0, \delta]$, we have

$$|\Phi_{2t}(z) - 2\Phi_t(z) + z| = \left| \int_{[z,\Phi_t(z)]} \frac{d}{ds}[\Phi_t(\zeta) - \zeta] d\zeta \right| \leq \mu \, |\Phi_t(z) - z| \,.$$

Hence,

$$\begin{aligned} |\Phi_{2t}(z) - z| = |2\,(z - \Phi_t(z)) - (\Phi_{2t}(z) - 2\Phi_t(z) + z)| \geq 2\,|z - \Phi_t(z)| - |\Phi_{2t}(z) - 2\Phi_t(z) + z| \geq \\ \geq 2\,|\Phi_t(z) - z| - \mu\,|\Phi_t(z) - z| = (2 - \mu)\,|\Phi_t(z) - z| \,. \end{aligned} \quad (1.9)$$

We take $(2-\mu)^{-1} \leq 2^{-2/3}$; moreover, for $k \in \mathbf{N}$ such that $2^{k\delta} \geq 1$, we set

$$M := 2^{2k/3} \sup \left\{ |\Phi_t(z) - z| \, : \, z \in K, t \in [2^{-k}, 1] \right\}.$$

From (1.9), we get

$$|\Phi_t(z) - z| \leq \frac{1}{2-\mu} \, |\Phi_{2t}(z) - z| \leq 2^{-2/3} \, |\Phi_{2t}(z) - z| \,. \quad (1.10)$$

We claim that

$$|\Phi_t(z) - z| \leq M\, t^{2/3} \quad \forall\, t \in [0,1],\; z \in K. \quad (1.11)$$

Indeed,

a) for $t \in [2^{-k}, 1]$,
$$|\Phi_t(z) - z| \leq 2^{-2k/3} M = (2^{-k})^{2/3} M \leq t^{2/3} M;$$

b) for $t \in [0, 2^{-k})$, there is $m \in \mathbf{N}$ such that $1 > 2^m t \geq 2^{-k}$, so that
$$|\Phi_t(z) - z| \leq 2^{-2/3}\, |\Phi_{2t}(z) - z| \leq (2^{-2/3})^m (2^{-k})^{2/3} M \leq t^{2/3} M.$$

We argue in the same way for a compact convex set K_1 such that $K_1 \supset \mathrm{int}(\widehat{K})$ so that we find $M_1 > 0$ such that

$$|\Phi_t(z) - z| \le M_1 t^{2/3} \quad \forall\, t \in [0,1], z \in K_1.$$

Using Cauchy inequalities, we find $\widetilde{M} > 0$ such that

$$|\Phi_t'(z) - 1| \le \widetilde{M} t^{2/3} \quad \forall\, z \in \widehat{K}, t \in [0,1]. \tag{1.12}$$

Then

$$|\Phi_{2t}(z) - 2\Phi_t(z) + z| \le \int_{[z,\Phi_t(z)]} \left| \frac{d}{ds}\Phi_t(\zeta) - \zeta \right| |d\zeta| \overset{(1.12)}{\le} \widetilde{M} t^{2/3} |\Phi_t(z) - z| \overset{(1.11)}{\le} M\widetilde{M} t^{4/3}.$$

For any $z \in K$, $t \in [0,1)$ we obtain

$$\left| \frac{\Phi_{2t}(z) - z}{2t} - \frac{\Phi_t(z) - z}{t} \right| = \frac{1}{2t} |\Phi_{2t}(z) - 2\Phi_t(z) + z| \le \frac{\widetilde{M} M t^{4/3}}{2t} = \frac{\widetilde{M} M t^{1/3}}{2},$$

so that

$$\lim_{n\to\infty} \frac{\Phi_{2^{-n}}(z) - z}{2^{-n}} =: G(z)$$

exists uniformly on compact sets.

$G : \mathbf{D} \longrightarrow \mathbf{C}$ is holomorphic, since for $t_0 > 0$, $\Phi([0,t_0) \times \{z_0\})$ is compact in $\mathbf{D}$,

$$\frac{\Phi_{2^{-n}}(\Phi_t(z_0)) - \Phi_t(z_0)}{2^{-n}} \xrightarrow{\text{unif}} G(\Phi_t(z_0)) \quad \forall\, t \in [0,t_0].$$

Let now $F(t) := \int_0^t \Phi_s(z) ds$, then $\int_0^t \Phi_{s+2^{-n}}(z) ds = F(t+2^{-n}) - F(2^{-n})$ and

$$\lim_{n\to\infty} \frac{F(t+2^{-n}) - F(t)}{2^{-n}} = \int_0^t G(\Phi_s(z)) ds$$

or

$$\Phi_t(z) - z = \int_0^t G(\Phi_s(z)) ds,$$

as desired. □

We have established an explicit correspondence between semigroups and infinitesimal generators G. It easy to check that $G(0) = 0$ if and only if $\Phi_t(0) = 0$. In this situation, discs centered at 0 are mapped by any Φ_t into smaller discs with the same center.

Given the hyperbolic distance, we can look at it as a function $\omega : \mathbf{D} \times \mathbf{D} \longrightarrow \mathbf{R}^+$ that is C^∞ on $\mathbf{D} \times \mathbf{D} - \mathrm{diag}(\mathbf{D} \times \mathbf{D})$. Then, if Φ_t maps discs centered at 0 into smaller discs centered at 0 for all $t \ge 0$, it follows that the differential of the Poincaré distance is contracted by the infinitesimal generator. A similar argument holds even if the semigroup does not fix any point. Namely,

Theorem 1.15.8. *A holomorphic map $G : \mathbf{D} \longrightarrow \mathbf{C}$ is an infinitesimal generator if and only if*

$$(d\omega)_{(z,w)}(G(z), G(w)) \le 0 \quad \text{for any } z, w \in \mathbf{D},\ z \ne w.$$

Exercise 1.15.5 (**). Using Theorem 1.15.8 prove directly the *Berkson-Porta Formula*: a holomorphic vector field $G : \mathbf{D} \longrightarrow \mathbf{C}$ is an infinitesimal generator if and only if there exist $\tau \in \overline{D}$ and a holomorphic function $p : \mathbf{D} \longrightarrow \overline{\mathbf{H}}$ such that

$$G(z) = (z - \tau)(\overline{\tau} z - 1) p(z).$$

The Berkson-Porta Formula is unique [Hint: let G be an infinitesimal generator given by $G(z) = (z-\tau)(\overline{\tau}z - 1)p(z)$ with $\tau \in \mathbf{D}$ and $p : \mathbf{D} \longrightarrow \overline{\mathbf{H}}$ holomorphic. If $G(z) = (z-\sigma)(\overline{\sigma}z - 1)q(z)$, with $\sigma \in \mathbf{D}$ and $q : \mathbf{D} \longrightarrow \overline{\mathbf{H}}$ holomorphic then

$$p(z) = \frac{(z-\sigma)(\overline{\sigma}z - 1)}{(z-\tau)(\overline{\tau}z - 1)} q(z) : \mathbf{D} \longrightarrow \overline{H}$$

If $\tau \in \mathbf{D}$, uniqueness follows at once. If $\tau \in \partial\mathbf{D}$ and $\sigma \ne \tau$, multiply both sides by $1 - |z|$ and apply the Growth Theorem to get a contradiction.]

Open question: Characterize semi-complete holomorphic vector field in higher dimension.

In higher dimension, one can replace the Poincaré distance with the Kobayashi distance (a natural distance contracted by holomorphic mappings), see [BDM10], from where Theorem 1.15.8 is also taken.

Proof of Theorem 1.15.8. If G is an infinitesimal generator, then the Flow-box Theorem gives a semigroup (ϕ_t). For $z \neq w$ fixed, define $h(t) := \omega(\phi_t(z), \phi_t(w))$. We will prove later that one-parameter semigroups are injective: for any $t \geq 0$, $\phi_t : \mathbf{D} \longrightarrow \mathbf{D}$ is univalent. Then

$$h(t+s) = \omega(\phi_t(\phi_s(z)), \phi_t(\phi_s(w))) \leq \omega(\phi_s(z), \phi_s(w)) = h(s) \quad \forall\, t, s \geq 0,$$

so that $t \longmapsto h(t)$ is an increasing function which is C^∞ by the Berkson-Porta Theorem.

Then

$$0 \geq \dot{h}(0) = \frac{d}{dt}\omega(\phi_t(z), \phi_t(w))|_{t=0} = d\omega_{(\phi_0(z),\phi_0(w))}(\dot{\phi}_0(z), \dot{\phi}_0(w)) = d\omega_{(z,w)}(G(z), G(w)).$$

Conversely, define

$$\begin{cases} \frac{dx}{dt} = G(x(t)) \\ x(0) = w \end{cases} \tag{1.13}$$

and let $x_w : [0, \delta_w) \longrightarrow \mathbf{D}$ be the maximal solution to (1.13). We show that $\delta_w = \infty$ for any $w \in \mathbf{D}$.

Given $z \neq w$ in $\mathbf{D}$, set $g(t) := \omega(x_z(t), x_w(t))$ which is a well-defined function, real analytic on $[0, \delta_z \wedge \delta_w)$ and $x_z(t) \neq x_w(t)$ for any t, by the uniqueness of the solution of the Cauchy problem. We have

$$g'(t) = d\omega_{(x_z(t),x_w(t))}(\dot{x}_z, \dot{x}_w) = d\omega_{(x_z(t),x_w(t))}(G(x_z), G(x_w)) \leq 0$$

by the hypothesis. It follows that g is non-increasing, whence $\delta_z = \delta_w$, for if $\delta_z > \delta_w$, then

$$\lim_{t \to \delta_w} \omega(x_z(t), x_w(t)) = \infty$$

(as $x_w(\delta_w)$ reaches the unit circle, while $x_z(\delta_w)$ does not) but the distance between $x_z(t)$ and $x_w(t)$ does not increase in time. Contradiction.

Hence, we can set $\delta := \delta_z = \delta_w > 0$. By the Flow-box Theorem, $\phi : \mathbf{D} \times [0, \delta) \longrightarrow \mathbf{D}$ is real-analytic in t and holomorphic in z.

If $\delta < \infty$, then we can take $t \in (\delta, 2\delta)$ and $\delta > s > 0$ such that $t - s < \delta$. Define

$$\widetilde{\phi}_t(z) := \begin{cases} \phi_t(z) & \text{if } t < s \\ \phi_{t-s}(\phi_s(z)) & \text{if } t \geq s \end{cases}$$

which is well defined for any $t \in [0, \delta')$ with $\delta' > \delta$, contradicting maximality of the solution. □

Exercise 1.15.6. Let Φ_t be a one-parameter semigroup (in $\mathbf{D}$), define $\varphi_{s,t} := \Phi_{t-s}$ for $0 \leq s \leq t < \infty$. Show that $(\varphi_{s,t})$ is a L^∞-evolution family (Hint: use the Berkson-Porta Formula).

Proposition 1.15.9. *Let $(\varphi_{s,t})$ be a L^d-evolution family. Then for any $0 \leq s \leq t < \infty$, $\varphi_{s,t}$ is injective.*

Dimostrazione. Suppose that for some $0 < s_0 < t_0$ and $z_0 \neq w_0$ in $\mathbf{D}$, we have $\xi = \varphi_{s_0,t_0}(z_0) = \varphi_{s_0,t_0}(w_0)$. Let

$$t_1 := \inf\{t \in [s_0, t_0] \,:\, \varphi_{s_0,t}(z_0) = \varphi_{s_0,t}(w_0)\}.$$

Then, as it can be easily proved that $\lim_{t \downarrow s_0} \varphi_{s_0,t} = \varphi_{s_0,s_0} = id$ uniformly on compact sets (exercise), we must have $t_1 > s_0$. For $t \in (s_0, t_0)$,

$$\varphi_{t,t_1}(\varphi_{s_0,t}(z_0)) = \varphi_{t,t_1}(\varphi_{s_0,t}(w_0)),$$

so that φ_{t,t_1} is not injective.

We leave as a further exercise to prove that $\lim_{t \uparrow t_1} \varphi_{t,t_1} = \varphi_{t_1,t_1}$ uniformly on compact sets, contradicting the injectivity of the identity function. □

Exercise 1.15.7. If (ϕ_t) is a one-parameter semigroup, then ϕ_t is injective for any $t \geq 0$.

Theorem 1.15.10. *Let $G(z,t)$ be a L^d-Herglotz vector field. Then $G(z,t)$ is semi-complete, namely $\forall\,(z,s) \in \mathbf{D} \times [0,\infty)$, the escaping time $I(z,s) = \infty$.*

Dimostrazione. Consider

$$\begin{cases} \dot{x}(t) = G(x(t),t) \\ x(s) = z \end{cases}$$

Call $\phi_{s,z} : [s, I(z,s)) \longrightarrow \mathbf{D}$ the maximal positive trajectory.

Let $w \in \mathbf{D}$; assume $I(z,s) \leq I(w,s)$ and define the absolutely continuous function

$$h(t) := \omega\left(\phi_{s,z}(t), \phi_{s,w}(t)\right).$$

As $G(\cdot\, t)$ is an infinitesimal generator for a.e. $t \geq 0$, Theorem 1.15.8 implies that

$$(d\omega)_{(\phi_{s,z}(t),\phi_{s,w}(t))}\left(G\left(\phi_{s,z}(t)\right), G\left(\phi_{s,w}(t)\right)\right) \leq 0 \quad \text{for a.e. } t \geq 0.$$

Therefore $\dot{h}(t) \leq 0$ for a.e. $t \geq 0$ and so $h(t) \leq h(s)$ for any $t \in [s, I(z,s))$.

Suppose to have a strict inequality $I(z,s) < I(w,s)$; when $t \to I(z,s)$, we have $\phi_{s,z}(t) \to \partial\mathbf{D}$, while $\phi_{s,z}(t) \to \phi_{s,w}(I(z,s)) \in \mathbf{D}$.

Thus, $\phi_{s,w}(t) \in K$ for any $t \in [s, I(z,s)]$ and so $h(t) \xrightarrow{t \to I(z,s)} \infty$, contradicting the fact that $h(t) \leq h(s) < \infty$. Then $I(z,s) = I(w,s)$.

Let $I := I(0,0) > 0$, let $s < I$ and set $z := \phi_{0,0}(s)$. Consider $s < t < I(0,s) \wedge I$, then

$$\phi_{s,z}(t) = \phi_{s,\phi_{0,0}(s)}(t) \overset{\text{uniqueness of ODE}}{=} \phi_{0,0}(t)$$

and

$$\omega\left(\phi_{s,0}(t), \phi_{0,0}(t)\right) = \omega\left(\phi_{s,0}(t), \phi_{s,z}(t)\right) = h(t) \leq h(s) = \omega(0,z) = \omega\left(0, \phi_{0,0}(s)\right).$$

Arguing as before, we get $I(0,s) = I$ for any $s < I$

We prove that there exists $\delta > 0$ such that $I(0,s) \geq s + \delta$ for any $s \in [0, I)$, whence $I = \infty$.

Let $0 < r < 1$ and let $k_{r,I+2}, \widehat{k}_{r,I+2} \in L^d([0, I+2], \mathbf{R}^+)$ be such that

$$|G(z,t)| \leq k_{r,I+2}(t) \text{ and } |G(z,t) - G(w,t)| \leq \widehat{h}_{r,I+2}(t)|z - w| \text{ for any } z, w \in \partial\overline{\mathbf{D}_r} \text{ and for a.e. } t \in [0, I+2]. \quad (1.14)$$

The integral functions

$$u \longmapsto \int_0^u k_{r,I+2}(\xi)\, d\xi, \quad u \longmapsto \int_0^u \widehat{k}_{r,I+2}(\xi)\, d\xi$$

are absolutely continuous, so that there is $\delta \in (0,1)$ such that for any $s \in [0, I+1]$

$$\int_s^{s+\delta} k_{r,I+2}(\xi)\, d\xi \leq r \quad \text{and} \quad \int_s^{s+\delta} \widehat{k}_{r,I+2}(\xi)\, d\xi \leq r$$

hold.

Thus, if $f : [s, s+\delta] \longrightarrow \overline{\mathbf{D}}_r$ is a measurable function, then

$$\left|\int_s^t G(f(\xi),\xi)\, d\xi\right| \leq \int_s^t \left|G(f(\xi),\xi)\right|\, d\xi \leq \int_s^t k_{r,I+2}(\xi)\, d\xi \leq r \quad \forall\, t \in [s, s+\delta]. \quad (1.15)$$

For $s \in [0, I+1]$, we define by recursion, for $t \in [s, s+\delta]$,

$$\begin{cases} x_{1,0}(t) := 0 \\ x_{s,n}(t) := \int_s^t G(x_{s,n-1}(\xi), \xi)\, d\xi \end{cases}$$

By (1.15), we have $|x_{s,n}(t)| \leq r$, whence

$$|x_{s,n}(t) - x_{s,n-1}(t)| \leq \int_s^t |G(x_{s,n-1}(\xi), \xi) - G(x_{s,n_2}(\xi), \xi)|\, d\xi \overset{(1.14)}{\leq} \int_s^t \widehat{k}_{r,I+2}(\xi) |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)|\, d\xi \leq$$
$$\leq \max_{\xi \in [s,s+\delta]} |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)| \int_s^t \widehat{k}_{r,I+2}(\xi)\, d\xi \leq r \max_{\xi \in [s,s+\delta]} |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)|.$$

Thus, the sequence $\{x_{s,n}\}$ is a Cauchy sequence in the Banach space $C^0([s, s+\delta], \mathbf{C})$ and converges uniformly to a continuous function $x : [s, s+\delta] \longrightarrow \mathbf{C}$.

Since $|G(x_{s,n}(t), t)| \leq k_{r,I+2}(t)$, the Lebesgue's Dominated Convergence Theorem gives

$$x(t) = \int_s^t G(x(\xi), \xi)\, d\xi \quad \forall\, t \in [s, s+\delta].$$

By the uniqueness of the ODE solution, $\phi_{s,0} = x$ on $[s, s+\delta]$. Then $I(0, s) \geq s + \delta$ and $I = \infty$, as desired. □

2 Dynamics of univalent functions of the disc (S. Díaz-Madrigal)

Given $\varphi \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$, we denote by φ_n the n-th iteration of φ. For $z \in \mathbf{D}$, we want to study

$$\lim_{n \to \infty} \varphi_n(z).$$

2.1 Two particular examples: The automorphic case

Consider $\varphi \in \mathrm{Aut}(\mathbf{D})$ which is not trivial (i.e. different from the identity); then we know $\varphi = \lambda \alpha_a$ with $|\lambda| = 1$ and $\alpha_a(z) = \frac{a-z}{1-\bar{a}z}$. Searching for the fixed points of φ, some computations show that $\lambda \alpha_a(z) = z$ has two only possibilities:

- elliptic case: there exists $\tau \in \mathbf{D}$ such that $\varphi(\tau) = \tau$,
- non-elliptic case: there is τ_1 (or $\tau_1 \neq \tau_2$) $\in \partial \mathbf{D}$ such that $\varphi(\tau_i) = \tau_i$.

To go further, we need to introduce a fundamental notion in dynamics. We present it in the abstract setting of two arbitrary sets Ω and $\widehat{\Omega}$.

Definition 2.1.1 (Conjugation)**.** *Given $f : \Omega \longrightarrow \Omega$ and a bijective map g from $\widehat{\Omega}$ onto Ω, we define $\widehat{f} = g^{-1} \circ f \circ g$, which trivially verifies $\widehat{f_n} = g^{-1} \circ f_n \circ g$, and say that f and $\widehat{f}$ are conjugated by the intertwinning map g. From the dynamical point of view, the study of the sequence (f_n) is indistinguishable from the study of $(\widehat{f_n})$.*

We talk of *semi-conjugation* when we only have g (not necessarily injective) such that $g \circ \widehat{f} = f \circ g$. For instance, we met semi-conjugation in 1.5.1 where we showed that given $\phi \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$ with $\phi(0) = 0$, $|\phi'(0)| = \lambda \in (0, 1)$, then there exists $\sigma \in \mathrm{Hol}(\mathbf{D}, \mathbf{C})$ such that $\sigma \circ \phi = \lambda \sigma$. In other words, ϕ (on $\mathbf{D}$) was conjugated with the function λz (on $\sigma(\mathbf{D})$).

Coming back to our problem, we notice that, in the *elliptic case*, it was showed in 1.5.1 that φ is conjugated to a rotation. Namely, using α_τ and applying Schwarz' Lemma, we obtain that φ is conjugated with $\phi(z) = \lambda z$, where $|\lambda| = 1$ and different from one. Clearly, $\phi_n(z) = \lambda^n z$ does not converge to a single point (depends on the value of λ).

Hence, we are forced to introduce another basic notion in dynamics and, as before, we present it in an abstract setting.

Definition 2.1.2. *Given a metric space X and a sequence $(x_n) \subset X$, we define the* ω-limit *of this sequence as*

$$\omega(x_n) := \left\{ x \in X \, : \, \exists\, n_k \uparrow \infty \text{ such that } \lim_{k\longrightarrow\infty} x_{n_k} = x \right\}.$$

A sequence has a limit if and only if its ω-limit reduces to a single point.

If $\lambda = e^{2\pi i q}$ with $q \in \mathbf{Q} - \mathbf{Z}$ (as $\varphi \neq id$), then $\omega(\phi_n(z))$ consists of a finite number of points on the circle $C(0, |z|)$ (for $z \neq 0$) and its cardinal does not depend on z. When q is irrational, then it is well-known that $\omega(\phi_n(z)) = C(0, |z|)$.

This certainly closes the analysis of $(\phi_n(z))$. The following step is to conjugate back the ω-limit of $\phi_n(z)$ to that of $\varphi_n(z)$. This leads to the Hyperbolic Geometry of the unit disc; recall that we can define the hyperbolic pseudo-distance as $\widetilde{\omega}(z_1, z_2) = |\alpha_{z_1}(z_2)|$. Then,

$$\omega(\varphi_n(z)) = C_H(\tau, \omega(\tau, z)) = \{w \in \mathbf{D} \, : \, \widehat{\omega}(w, \tau) = \widehat{\omega}(z, \tau)\}.$$

Let us pass to the *non-elliptic case*. Conjugating with adequate Cayley maps, we can move our study from automorphisms of the unit disc to automorphisms of the right half-plane fixing ∞.

Remark 2.1.1. The Cayley map (associated with $\tau \in \partial\mathbf{D}$) is the map $\sigma_\tau(z) = \frac{\tau+z}{\tau-z}$, $z \in \mathbf{D}$. It is a biholomorphism from $\mathbf{D}$ onto $\mathbf{H}$ which is also a bicontinuous function from $\overline{\mathbf{D}}$ to $\overline{\mathbf{H}}$. Notice that σ maps τ to ∞.

The conjugated map is of the form $\phi(w) = \alpha w + ib$ with $\alpha > 0$ and $b \in \mathbf{R}$. We distinguish two cases.

- Suppose $\alpha = 1$. Then $b \neq 0$ and $\phi_n(w) = w + inb \to \infty$ so $\omega(\phi_n(z)) = \{\infty\}$;
- When $\alpha \neq 1$, we may even assume that $\phi(w) = \beta w$, with $\beta > 1$. It follows also that $\omega(\phi_n(z)) = \{\infty\}$.

The difference between the above two possibilities is *how* the sequence enters into the ω-limit: in the first case we have tangency, in the second one we have transversality.

It is straightforward to pass the above study in the right half-plane to the unit disc. Namely, the following theorem holds.

Theorem 2.1.1. *Let φ be a holomorphic automorphism of the unit disc without fixed points (in the unit disc). Then, there exists $\tau \in \partial\mathbf{D}$ such that $\varphi_n(z) \to \tau$, for every $z \in \mathbf{D}$.*

2.2 The (general) discrete case

Consider $\varphi \in \mathrm{Hol}(\mathbf{D}, \mathbf{D})$ which is non-trivial. Then we classify similarly *elliptic* and *non-elliptic* maps regarding the fixed points of φ.

Suppose that φ has an inner fixed point (elliptic case); up to conjugation, we can suppose that the fixed point is 0. Then $|\phi'(0)| \leq 1$. In the case of equality, we are in the case of a rotation and this was studied in the previos section. When $|\phi'(0)| < 1$, then Exercise 1.10.2 gives $\phi_n(z) \to 0$ for any $z \in \mathbf{D}$.

Suppose now that φ has no inner fixed points.

Theorem 2.2.1 (Denjoy-Wolff (Boundary case))**.** *There exists $\tau \in \partial\mathbf{D}$ such that $\omega(\varphi_n(z)) = \{\tau\}$.*

Dimostrazione. Bearing in mind the former section (indeed, the last theorem of that section) we may and do assume that φ is not an automorphism.

Step I: Construction of τ.

Given $r \in (0,1)$, define $\varphi_r := r\varphi$; we have $|r\varphi(z)| \leq r$ and $\varphi_r : \overline{D(0,r)} \longrightarrow \overline{D(0,r)}$. By Brower's Fixed Point Theorem, there exists $\tau_r \in \overline{D(0,r)}$ such that $\varphi_r(\tau_r) = \tau_r$. Then we can take $r_n \uparrow 1$ so that $\tau_{r_n} \uparrow \tau \in \overline{\mathbf{D}}$ which lies actually on the circle (otherwise φ would have a fixed point in $\mathbf{D}$).

Step II: $\mathrm{Aut}(\mathbf{D}) \cap \omega_{\mathrm{Hol}(\mathbf{D},\mathbf{C})}(\varphi_n) = \emptyset$.

Suppose that it is not true and take $h \in \mathrm{Aut}(\mathbf{D}) \cap \omega_{\mathrm{Hol}(\mathbf{D},\mathbf{C})}(\varphi_n)$, then we can take a sequence $\varphi_{n_k} \xrightarrow{k} h$ uniformly on compact sets and we set, for $n_k > 1$, $\phi_k := \varphi_{n_k - 1}$.

Montel's Theorem gives that $\phi_{k_\ell} \xrightarrow{\ell} g$ uniformly, where $g \in \mathrm{Hol}(\mathbf{D},\mathbf{C})$; as $|\phi_{k_\ell}| \leq 1$, we have $|g| \leq 1$ in $\mathbf{D}$, so that either g is constant or $g(\mathbf{D}) \subset \mathbf{D}$.

We write

$$\phi_{k_\ell} \circ \varphi = \varphi_{n_{k_\ell}-1} \circ \varphi = \varphi_{n_{k_\ell}}$$

and taking limits we have $g \circ \varphi = h$ which gives that $g(\mathbf{D}) \subset \mathbf{D}$ and φ is injective.

Writing in the other sense $\varphi \circ \phi_{k_\ell} = \varphi_{n_{k_\ell}}$ and taking limits again, we have $\varphi \circ g = h$ so that φ is surjective and $\varphi \in \mathrm{Aut}(\mathbf{D})$, giving a contradiction.

Step III: For any $K > 0$, $\varphi(\mathrm{Hor}(\tau,K)) \subset \mathrm{Hor}(\tau,K)$, where $\mathrm{Hor}(\tau,K)$ denotes the *horocycle*

$$\left\{ z \in \mathbf{D} \;:\; \frac{|z-\tau|^2}{1-|z|^2} < K \right\} = D\left(\frac{1}{K+1}\tau, \frac{K}{K+1}\right).$$

Horocycles are deep related to the open discs associated with the hyperbolic metric (so also to the pseudo-hyperbolic metric).

Namely, for $a \in \mathbf{D}$ and $r \in (0,1)$, set $\widetilde{D}_H(a,r) := \{z \in \mathbf{D} \;:\; \widetilde{\omega}(a,z) < r\}$; then it can be proved (exercise) that

$$\widetilde{D}_H(a,r) = \left\{ z \in \mathbf{D} \;:\; \frac{|1-\bar{a}z|^2}{1-|z|^2} < \frac{1-|a|^2}{1-r^2} \right\}.$$

Note the similarity with the definition of horocycles.

By Schwarz' Lemma, if $\phi \in \mathrm{Hol}(\mathbf{D},\mathbf{D})$ with $\phi(0) = 0$, then $|\phi(z)| \leq |z|$ for any $z \in \mathbf{D}$. This statement is equivalent to the fact that $\phi(D(0,r)) \subset D(0,r)$ and henceforth $\phi_n(D(0,r)) \subset D(0,r)$.

If $\phi \in \mathrm{Hol}(\mathbf{D},\mathbf{D})$ is such that $\phi(\tau) = \tau \in \mathbf{D}$, the same proves that $\phi(\widetilde{D}_H(\tau,r)) \subset \widetilde{D}_H(\tau,r)$.

In fact, this holds for any $h \in \mathrm{Hol}(\mathbf{D},\mathbf{D})$ and $a \in \mathbf{D}$:

$$h(D_H(a,r)) \subset D_H(h(a),r).$$

We pass to prove Step III. We are going to use the above statement. We have $\varphi_{r_n}(\tau_{r_n}) = \tau_{r_n}$ for any $n \in \mathbf{N}$. Fix $z_0 \in \mathbf{D}$ and set $t_n = \widetilde{\omega}(z_0, \tau_{r_n})$. Then,

$$\varphi_{r_n}\left(\overline{\widetilde{D}_H(\tau_{r_n}, t_n)}\right) \subset \overline{\widetilde{D}_H(\tau_{r_n}, t_n)},$$

which gives

$$\frac{|1-\varphi_{r_n}(z_0)\overline{\tau_{r_n}}|^2}{1-|\varphi_{r_n}(z_0)|^2} \leq \frac{1-|\tau_{r_n}|^2}{1-t_n^2}$$

and, taking the limits on both sides,

$$\frac{|1-\varphi(z_0)\overline{\tau}|}{1-|\varphi(z_0)|^2} \leq \frac{|1-z_0\overline{\tau}|^2}{1-|z_0|^2}.$$

Step IV: We have (it is enough) to prove that $\omega_{\mathrm{Hol}(\mathbf{D},\mathbf{C})}(\varphi_n) = \{\tau\}$, i.e. we have to show that for any $(n_k) \uparrow \infty$ there exists $(m_k) \subset (n_k)$ such that $\varphi_{m_k} \xrightarrow{k} \tau$.

Take an arbitrary subsequence (φ_{n_k}), then by Montel's Theorem, $(m_k) \subset (n_k)$ and $\varphi_{m_k} \xrightarrow{k} g \in \mathrm{Hol}(\mathbf{D}, \mathbf{C})$. As $|\varphi_n| \le 1$, we have $|g| \le 1$ in $\mathbf{D}$, so that, by the Open Mapping Theorem, either g is constant or $g(\mathbf{D}) \subset \mathbf{D}$. This gives the following alternatives:

(i) $g = p \in \partial\mathbf{D}$,

(ii) $g \equiv p \in \mathbf{D}$,

(iii) $g(\mathbf{D}) \subset \mathbf{D}$ and $g(\mathbf{D})$ is an open set.

Using Step II, it is possible to prove that case (iii) is impossible (exercise). Case (ii) cannot occur neither: consider $K > 0$ with $\overline{\mathrm{Hor}(\tau, K)} \not\ni p$. Then, by Step III, necessarily any limit of $(\varphi_{n_k}(z))$ must belong to $\overline{\mathrm{Hor}(\tau, K)}$.

It remains (i); then, using again Step II, $\varphi_n(\mathrm{Hor}(\tau, K)) \subset \mathrm{Hor}(\tau, K)$ so the limit of $\varphi_{n_k}(z)$ must belong to $\overline{\mathrm{Hor}(\tau, K)}$. Therefore, $p \in \overline{\mathrm{Hor}(\tau, K)} \cap \partial\mathbf{D}$. This implies $p = \tau$. □

2.3 The rational (also continuous or autonomous) case

In the previous section we have been considering φ_n, with n a natural number. This is usually called in dynamics, the discrete setting. A question, going back at least to Abel's times is if it is possible to give a reasonable/coherent meaning to φ_n, for n rational or even real.

The fundamental fact is to look at the semigroup property $\varphi_{n+m} = \varphi_n \circ \varphi_m$ of discrete iteration. If we want this property to hold for continuous time too we must impose that

$$\begin{cases} \varphi_{s+t} = \varphi_s \circ \varphi_t & s, t \ge 0 \\ \varphi_0 = id_{\mathbf{D}} \end{cases}.$$

However, this is not enough to create a fruitful theory as the following example shows.

Example 2.3.1. Consider any non-measurable function $\theta : [0 + \infty) \longrightarrow \partial\mathbf{D}$ verifying

$$\begin{cases} \theta(s+t) = \theta(s)\theta(t) & s, t \ge 0 \\ \theta(0) = 1 \end{cases}.$$

and consider $\varphi_t(z) = e^{i\theta(t)}z$. Clearly, the above properties are satisfied but the properties of the φ_t with respect to t are really bad.

Therefore we also ask for some kind of continuity in t. Namely, we add the condition $\lim_{t\to 0} \varphi_t(z) = z$ for any $z \in \mathbf{D}$, which is equivalent to the request that $\varphi_s \xrightarrow{s \to t} \varphi_t$ in $\mathrm{Hol}(\mathbf{D}, \mathbf{C})$. In this way, we arrive to the concept of semigroups of holomorphic functions in the unit disc (see again Definition 1.15.8). In this case, we wonder about the behaviour of $\varphi_t(z)$ for $t \to \infty$.

It is known $\varphi_t(z)$ can be differentiated (in fact, they are real-analytic) in t and can be seen as solutions to problems of the form

$$\begin{cases} \dot{w} = F(w) \\ w(0) = z \end{cases}$$

with F holomorphic in $\mathbf{D}$, which is an autonomous dynamical system in $\mathbf{D}$ and the flow is semi-complete on the right: $T_z = \infty$ for any $z \in \mathbf{D}$.

Conversely, given any autonomous dynamical system on $\mathbf{D}$ of such a form, the flow defines (in a natural way) a semigroup of holomorphic functions in the unit disc.

As it was shown in Exercise 1.15.5, those semi-complete vector fields in the unit disc are those of the form $G(z) = (z - \tau)(\overline{\tau}z - 1)p(z)$, with $\tau \in \overline{\mathbf{D}}$ and $p \in \mathrm{Hol}(\mathbf{D}, \mathbf{C})$ with $\Re\mathrm{e}\, p \ge 0$.

What about the limit $\varphi_t(z)$ for $t \to \infty$? We have either

(i) $\exists\, t_0 > 0$ such that φ_{t_0} is a (non-trivial) elliptic automorphism, or

(ii) $\forall\, t > 0$, φ_t is not an elliptic automorphism.

For (i), there exists $\tau \in \mathbf{D}$ with $\varphi_t(\tau) = \tau$ and $\omega(\varphi_t(z)) = C_H(\tau, \omega(z, \tau))$ so that we can think of rotations (in the hyperbolic sense).

For (ii), we have a continuous version of the Denjoy-Wolff Theorem.

Theorem 2.3.1. *There exists $\tau \in \overline{D}$ such that $\lim_{t\to\infty} \varphi_t(z) = \tau$ for any $z \in \mathbf{D}$.*

Dimostrazione. Set $\varphi = \varphi_1$. Then, by the Denjoy-Wolff Theorem, there exists $\tau \in \overline{D}$ such that $\varphi_n(z) \to \tau$. Indeed, this happens for any sequence (q_n) of natural numbers converging to ∞.

Now take (t_n) an arbitrary sequence of real numbers converging to ∞. Then, we write $t_n = [t_n] + p_n$, $p_n \in [0, 1]$ and consider a subsequence (n_k) such that $p_{n_k} \to p_0 \in [0, 1]$.

Then by semigroup properties, $\varphi_{t_{n_k}}(z) = \varphi_{[t_{n_k}]}(\varphi_{p_{n_k}}(z)) \to \tau$. That is, there is always a subsequence of (φ_{t_n}) converging to τ so the proof is done. □

2.4 The non-autonomous case

First of all, we recall the basic question about asymptotic behaviour in the non-autonomous setting. Consider now non-autonomous dynamical systems

$$\begin{cases} \dot{w} = F(w, t) & t \in I \text{ an interval} \\ w(s) = z \end{cases} \qquad (\mathbf{P}_{z,s})$$

with $F(\,\cdot\,, t) \in \mathrm{Hol}(\mathbf{D}, \mathbf{C})$ and $F(z, \cdot\,)$ good (at least measurable on t). The solutions $\Phi_{z,s}(t)$ are defined for $t \in [s, T_{z,s})$, where $s < T_{z,s} \le +\infty$ ($T_{z,s}$ is called the escaping time).

We wonder which is the behaviour of $\Phi_{z,s}(t)$ when $t \to T_{z,s}$.

In the theory of generalized Loewner chains, we have found semi-complete vector fields of the form $G(z, t) = (z - \tau_t)(\overline{\tau_t} z - 1)p(z, t)$. Moreover, we know that, as a matter of fact, solutions to the corresponding problems $(\mathbf{P}_{z,s})$ are the evolution families $\varphi_{s,t}(z)$. We recall that they verify the properties

1. $\varphi_{s,s} = id_{\mathbf{D}}$,
2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for $0 \le s \le u \le t$,
3. for any $z \in \mathbf{D}$ and $T > 0$, there is a $k \in L^p$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k(\xi) d\xi \quad \text{for } 0 \le s \le u \le t \le T.$$

So, what about $\varphi_{s,t}(z)$ for $t \to \infty$? We fix $z \in \mathbf{D}$, $s \ge 0$; we restrict to the case $\tau_t \equiv$ const.

Let's suppose $\tau \in \mathbf{D}$ ($\tau = 0$); then we have three situations:

(i) for any $z \in \mathbf{D}$ and $s \ge 0$, $\varphi_{s,t} \to 0$

(ii) for any $s \ge 0$, $\varphi_{s,t} \to h_s$ with h_s a univalent map from $\mathbf{D}$ to $\mathbf{D}$

(iii) for any $s \ge 0$ and $z \in \mathbf{D}$, $\omega(\varphi_{s,t}(z)) \subset C(0, r_{s,z})$ and it is not a point.

This latter case splits again:

(a) for any $z \in \mathbf{D}$ and $s \ge 0$, $\omega(\varphi_{s,t}(z)) = C(0, r_{s,z})$ and $r_{s,z} \in (0, 1)$

(b) for any $z \in \mathbf{D}$ and $s \geq 0$, $\omega(\varphi_{s,t}(z))$ is a closed proper arc of $C(0, r_{s,z})$ with the same *(hyperbolic) angular extend*

Definition 2.4.1. *The* angular extend *is defined to be*

$$\theta := \frac{\textit{length arc}}{\textit{radius}}$$

and its hyperbolic analogous is

$$\theta := \frac{\textit{(hyperbolic) length arc}}{\sinh r}.$$

Case (i) happens if and only if for every (resp. for some) $z \in \mathbf{D}$, $\Re\mathrm{e}\, p(z, \cdot) \notin L^1([0, \infty))$. This fact clarifies some results in Loewner theory and rational iteration. Indeed, in classical Loewner's Theory, since

$$f_s(z) = \lim_{t \to \infty} \frac{\varphi_{s,t}(z)}{e^{t-s}}$$

converges, we have (i). Now, note $p(0, t) = 1$ and $\int_0^t \Re\mathrm{e}\, p(0, s)\, ds = t \to \infty$.

In the semigroup case ($\tau = 0$) and if (φ_t) is not elliptic-automorphic (there is $t_0 > 0$ such that φ_{t_0} is elliptic and an automorphism), also clearly we have (i). Now, note $\Re\mathrm{e}\, p(z) = p_0 \neq 0$ and

$$\int_0^t \Re\mathrm{e}\, p(z, s) ds = \int_0^t p_0\, ds = p_0\, t \to \infty.$$

For the boundary case $\tau \in \partial\mathbf{D}$, the alternative turns out to be the following:

(i) for any $z \in \mathbf{D}$ and $s \geq 0$, $\varphi_{s,t}(z) \to \tau$ for $t \to \infty$,

(ii) for any $s \geq 0$, $\varphi_{t,s} \to h_s$ univalent with $h_s(\mathbf{D}) \subset \mathbf{D}$,

(iii) for any $s \geq 0$ and $z \in \mathbf{D}$, $\omega(\varphi_{s,t}(z)) \subset \partial\overline{\mathrm{Hor}(\tau, K_{s,z})}$, with $K_{s,z} > 0$ and it is not a point.

This last case splits again:

(a) for any $z \in \mathbf{D}$ and $s \geq 0$, $\omega(\varphi_{s,t}(z)) = \partial\overline{\mathrm{Hor}(\tau, K_{r,s})}$,

(b) for any $z \in \mathbf{D}$ and $s \geq 0$, $\omega(\varphi_{s,t}(z))$ is a proper arc with τ as one of the end-points,

(c) for any $z \in \mathbf{D}$ and $s \geq 0$, $\omega(\varphi_{s,t}(z))$ is a proper arc contained in $\mathbf{D}$ with the same *horocycle-angular extend*.

By *horocycle-angular extend*, we mean the angular extend from the horocycle point of view; that is

$$\frac{\text{hyperbolic length of the arc}}{K_{r,s}}$$

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