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FIXED POINTS OF COMMUTING HOLOMORPHIC MAPS WITHOUT BOUNDARY REGULARITY

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Abstract. We identify a class of domains of $\mathbb{C}^n$ in which any two commuting holomorphic self-maps have a common fixed point.

1. Introduction

In 1964 A.L. Shields [22] proved that a family of continuous functions mapping the closed unit disk into itself, holomorphic in the open disk and commuting under composition, has a common fixed point. Subsequently several authors have generalized the result to different types of domains in $\mathbb{C}^n$ (see [1, 7] and references therein). Generally the proofs require two different approaches depending on the fact that the maps have or not fixed points in the interior of the domain. The case in which the maps have fixed points in the domain is solved by studying the properties of the fixed points set (see [1, 4]), and the hypothesis on the regularity at the boundary of the elements of the family is insignificant. When a map has no fixed points in the domain, then one usually exploits a Wolff-type Lemma, which -roughly speaking- says that the map "squashes" the domain into a unique point of the boundary, producing a boundary fixed point for all the family.

The aim of this paper is to produce existence theorems of common fixed points for commuting holomorphic maps in a wide class of domains in $\mathbb{C}^n$, and without any assumption about the regularity of the maps at the boundary. We deal with two commuting holomorphic self-maps, f and g, of a domain $\Omega \subset \mathbb{C}^n$ -but our method works for a bigger family as well- under the hypothesis that f has no fixed points in Ω and with no assumption on the boundary regularity of f and g. Since f has no fixed points in Ω, if we want to produce a fixed point, it has to belong to $\partial\Omega$. Being f not necessarily continuous up to the boundary we firstly have to specify what "boundary fixed point" means. For the moment it is enough to think "fixed" in the sense of non-tangential limits, since then one applies the results by Čirca [9], Stein [23] and Cima and Krantz [8] to get "admissible" (in some sense) limits available (see section 2). The main part of this work is to find suitable hypotheses on Ω to make f and g have a common boundary fixed point. In sight of the previous results (see [6, 7]) we put on Ω three reasonable hypotheses. The first and the second hypotheses on Ω are substitutes of the Wolff and Wolff-Denjoy Lemmas.

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The third one is a kind of Julia's Lemma involving the Kobayashi pseudodistance k_Ω of Ω (see Def. 2.2). Under these hypotheses we produce a common boundary fixed point for f and g (see Th. 2.7). To make this machinery non-abstract we give some examples and applications, both rediscovering old results and producing some new ones (for instance we show that Theorem 2.7 holds in m-convex domains and strongly pseudoconvex contractible domains, and we give a new proof of a result by Heins [14] generalizing it to the unit ball of $\mathbb{C}^n$).

2. Preliminaries and Statements

Let Δ be the unit open disk of $\mathbb{C}$ and let $\Omega \subset \mathbb{C}^n$ be a domain. The *Kobayashi pseudometric* $\kappa_\Omega : T\Omega \to \mathbb{R}^+$ is defined by:

$$\forall z \in \Omega, \forall v \in T_z\Omega \quad \kappa_\Omega(z; v) = \inf\{|\xi| \mid \exists \varphi \in \mathrm{Hol}(\Delta,\Omega) \ : \varphi(0) = z, d\varphi_0(\xi) = v\}.$$

The *Kobayashi pseudodistance* $k_\Omega : \Omega \times \Omega \to \mathbb{R}^+$ on Ω is the integrated form of κ_Ω. For general properties of the Kobayashi pseudodistance and pseudometric we refer the reader to Jarnicki-Pflug [16]. In particular we recall that k_Ω is contracted by holomorphic maps, *i.e.* if $f \in \mathrm{Hol}(\Omega,\Omega)$ then $k_\Omega(f(z), f(w)) \le k_\Omega(z, w)$ for all $z, w \in \Omega$.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{C}^n$, $z_0 \in \Omega$, $x \in \partial\Omega$ and $R > 0$. The *small horosphere* $E_{z_0}(x, R)$ and the *big horosphere* $F_{z_0}(x, R)$ of center x, pole z_0 and radius R are defined by:

$$E_{z_0}(x, R) := \left\{ z \in \Omega \mid \limsup_{w \to x} [k_\Omega(z, w) - k_\Omega(z_0, w)] < \frac{1}{2} \log R \right\}.$$

$$F_{z_0}(x, R) := \left\{ z \in \Omega \mid \liminf_{w \to x} [k_\Omega(z, w) - k_\Omega(z_0, w)] < \frac{1}{2} \log R \right\}.$$

In a sense that can be made precise (see section 3 and [1]), every statement on boundary fixed points of $f \in \mathrm{Hol}(\Omega,\Omega)$ is a statement on the behaviour of horospheres under the action of f. For instance if $\Omega = \Delta$, then $E_0(x, R) = F_0(x, R) = \{z \in \Delta \mid (|x - z|^2) \cdot (1 - |z|^2)^{-1}\}$ are the classical horocycles. Moreover $f(E_0(x, R)) \subset E_0(x, R)$ $\forall R > 0$ if and only if f has no fixed points and there exists $\{z_k\} \subset \Delta$ such that $z_k \to x$ and $f(z_k)$ goes to x faster than z_k does. Keeping this in mind, we have the following:

Definition 2.2. Let Ω be a domain in $\mathbb{C}^n$. we say that

(1) Ω has the *property D* if for every $f \in \mathrm{Hol}(\Omega,\Omega)$, the sequence of iterates $\{f^k\}$ is compactly divergent if and only if f has no fixed points in Ω.
(2) Ω has the *property W* if for every $f \in \mathrm{Hol}(\Omega,\Omega)$ such that $\{f^k\}$ is compactly divergent then $\{f^k\}$ converges uniformly on compacta to a constant map $\Omega \ni z \mapsto c \in \partial\Omega$.
(3) Ω has the *property J* if for every $f \in \mathrm{Hol}(\Omega,\Omega)$ such that there exists $\{w_k\} \subset \Omega$ converging to $x \in \partial\Omega$ for which there exists $y \in \partial\Omega$ with $f(w_k) \to y$ and

$$\limsup_{k\to\infty} [k_\Omega(z_0, w_k) - k_\Omega(z_0, f(w_k))] < \infty \tag{2.1}$$

for some $z_0 \in \Omega$, then f has non-tangential limit y at x.

Example 2.3. (1) The unit disk Δ of $\mathbb{C}$ have the properties D and W (Wolff-Denjoy Theorems, see Denjoy [12] and Wolff [24]), and the property J (Julia's Lemma, see Julia [17]). This is the reason we dub D,W and J the properties stated above.

(2) The unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ has the properties D and W (see MacCluer [19]), and the property J (see Rudin [21]).

(3) Any convex domain has the property D (see Abate [1]) but in general a convex domain does not have properties W and J. For instance the bidisk Δ^2 fails to have the property W (check for the map $(z, w) \mapsto (z, (1 + w) \cdot (3 - w)^{-1}))$ and the property J. An example of this last assertion can be built as follows. Let $\{a_n\}$ be the sequence defined by $a_n := (1-1/n^4) \cdot e^{i/n}$. It's easy to see that

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^2} = S < \infty.$$

Starting summing up from $N > 1$ if necessary, we can suppose $S \leq 1$. Then Frostman's Theorem (see Frostman [13]) implies that the Blaschke product B with zero-sequence $\{a_n\}$ has radial limit $\lambda \in \partial\Delta$ at 1 (and we suppose $\lambda = 1$ up to multiply by $\overline{\lambda}$). Moreover B' has radial limit $S \leq 1$ at 1. Hence B has non-tangential limit 1 at 1 (in fact 1 is the Wolff point of B) but $\lim_{n\to\infty} B(a_n) = 0$. Now let γ be a parabolic automorphism of Δ with Wolff point 1. Consider $f : \Delta^2 \to \Delta^2$ defined by $f(z_1, z_2) := (\gamma(z_1), B(z_1))$. Obviously f has non-tangential limit $(1, 1)$ at $(1, 0)$. Now consider $\{z_n\} \subset \Delta^2$ given by $z_n := (a_n, 0)$. Then $\lim_{n\to\infty} f(z_n) = (1, 0)$. Since $\omega(0, B(a_n)) \equiv 0$, a simple computation shows that

$$k_{\Delta^2}(0, z_n) - k_{\Delta^2}(0, f(z_n)) = \omega(0, a_n) - \omega(0, \gamma(a_n)).$$

Now from the very definition of ω it follows

$$\lim_{n\to\infty} [\omega(0, a_n) - \omega(0, \gamma(a_n))] = \lim_{n\to\infty} \frac{1 - |\gamma(a_n)|}{1 - |a_n|}.$$

This last limit is finite since by Julia's Lemma $(1-|\gamma(a_n)|^2)\cdot(1-|a_n|^2)^{-1} = |(1-\gamma(a_n))\cdot(1-a_n)^{-1}|^2$ which tends to $[\gamma'(1)]^2 = 1$ as n tends to ∞. Then the property J doesn't hold.

(4) Strongly convex domains have the properties D, W and J (see Abate [1]).

(5) A bounded convex domain Ω in $\mathbb{C}^n$ is called *m-convex* if there exist two constants $C > 0$ and $m \in (0, +\infty)$ such that for every $v \in \mathbb{C}^n$ it holds $r_\Omega(z; v) \leq C[d(\partial\Omega, z)]^{1/m}$ for all $z \in \Omega$, where $r_\Omega(z; v)$ is the radius of the largest analytic disk centered at z, tangent to v and contained in $\overline{\Omega}$, and $d(\partial\Omega, z)$ is the euclidean distance from z to $\partial\Omega$. The m-convex domains satisfy properties D and W (see Mercer [20]) and, as we see in the next section, they also satisfy the property J.

(6) Weakly pseudoconvex contractible domains with real-analytic or smooth boundary in $\mathbb{C}^2$ have the properties D and W (see Zhang-Ren [25]).

(7) Strongly pseudoconvex contractible domains with C^3 boundary in $\mathbb{C}^n$ have the properties D and W (see Ma [18], Huang [15] and Abate [3]); they also have property J as we prove in the next section.

Remark 2.4. (i) It is easy to verify that if the property J holds for a $z_0 \in \Omega$, then it holds for any $z \in \Omega$.

(ii) Abate ([1], Prop. 2.4.15) proved that if Ω is bounded and $f \in \mathrm{Hol}(\Omega,\Omega)$ is such that there exists $\{w_k\} \subset \Omega$ converging to $x \in \partial\Omega$ for which there exists $y \in \partial\Omega$ with $f(w_k) \to y$ and for which equation (2.1) holds, then $f(E_{z_0}(x, R)) \subseteq F_{z_0}(y, \alpha R)$, where

$$1/2 \log \alpha = \liminf_{w \to x} [k_\omega(z_0, w) - k_\Omega(z_0, f(w))],$$

and α is sometimes called "boundary dilatation coefficient" of f at x.

(iii) If Ω is complete hyperbolic and if $f \in \mathrm{Hol}(\Omega,\Omega)$ is such that there exists $\{w_k\} \subset \Omega$ converging to $x \in \partial\Omega$ for which it holds equation (2.1) then there exists $y \in \partial\Omega$ such that $f(w_k) \to y$. Hence if Ω is complete hyperbolic, the hypothesis on the existence of $y \in \partial\Omega$ such that $f(w_k) \to y$ in the statement of the property J is redundant.

(iv) If $\{w'_k\}$ is a second sequence in Ω which satisfies the hypothesis of the property J, then $f(w'_k) \to y$; that is $f(z_k) \to y$ not only for $\{z_k\}$ non-tangential, but also for those $\{z_k\}$ for which

$$\limsup_{k \to \infty} [k_\Omega(z_0, z_k) - k_\Omega(z_0, f(w_k))] < +\infty$$

holds. Then f has actually *J-limit* y at x (see Cor.1.8 of [7]).

As is known, in multidimensional spaces, the natural approach regions for limits to the boundary are not cones (*i.e* non-tangential limits) but broader regions called "admissible regions". Generally, once we have radial limit at a point, then a Lindelöf principle assures the existence of limits along complex tangential directions (for a discussion of this matter see Abate [2]). To be more precise, depending on the domain Ω, assuring non-tangential limits gives us "admissible limits" in the sense of Abate's *special and restricted limits* ([1]), Cima and Krantz *hypoadmissible limits* ([8]), Čirca's ([9]) or Stein's ([23]) *admissible limits*. We don't want to enter into details here, but from now on, we say *admissible limits* leaving to the reader the choice of the approach regions he prefers (and that the hypothesis on Ω allow).

Remark 2.5. If Ω is F-convex (see Definition 3.2) then we can actually consider *admissible limits* in the sense of Abate's *K-regions* (see [1, 2]), since condition (2.1) assures existence of J-limits and then one can reason as in the proof of Thm.2.7.14(i) of [1].

First of all we need this lemma:

Lemma 2.6. *Let Ω be a domain in $\mathbb{C}^n$ with the properties D, W and J. Let $f \in \mathrm{Hol}(\Omega,\Omega)$ be a map with no fixed points in Ω. If $x \in \partial\Omega$ is the point defined by the property W, then f has admissible limit x at x.*

Proof. Using the contractive properties of holomorphic maps with respect to the Kobayashi pseudodistance, it is easy to see that the sequence $w_k := f^k(z_0)$ satisfies the hypothesis of property J. □

Theorem 2.7. *Let D be a domain in $\mathbb{C}^n$ having properties D, W and J. Let $f, g \in \mathrm{Hol}(\Omega,\Omega)$ be such that $f \circ g = g \circ f$ and $\mathrm{Fix}(f) = \emptyset$. Then there exists $x \in \partial\Omega$ such that f and g have admissible limit x at x.*

Proof. Since f has no fixed points in Ω, then $\{f^k\}$ is compactly divergent by property D and therefore it converges to a constant map $\Omega \ni z \mapsto x \in \partial\Omega$ by property W. Let $z_0 \in \Omega$ and put $w_k := f^k(z_0)$. Then $w_k \to x$ and

$$\limsup_{k\to\infty}[k_\Omega(z_0, w_k) - k_\Omega(z_0, g(w_k)] = \limsup_{k\to\infty}[k_\Omega(z_0, f^k(z_0)) - k_\Omega(z_0, f^k(g(z_0))] \leq$$
$$\limsup_{k\to\infty}[k_\Omega(f^k(z_0), f^k(g(z_0))] \leq k_\Omega(z_0, g(z_0)),$$

where the above inequalities are motivated by the facts that f commutes with g and that holomorphic maps are contractions for the Kobayashi pseudodistance of Ω.
Furthermore $\lim_{k\to\infty} g(w_k) = \lim_{k\to\infty} f^k(g(z_0)) = x$. Then g has non-tangential limit x at x by property J. Recalling Lemma 2.6 and the above discussion on admissible limits, we complete the proof. □

Remark 2.8. If the domain Ω satisfies only properties W and J, then Theorem 2.7 holds also by replacing the hypothesis on f to have no fixed points with the hypothesis on $\{f^k\}$ to be compactly divergent.

3. The property J

The aim of this section is to make clear the meaning of property J and to prove assertions 5 and 7 of Example 2.3. We start by recalling:

Proposition 3.1 (Abate)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$, let $z_0 \in D$ and let $f \in$ Hol(Ω,Ω) . If $\{w_k\} \subset \Omega$, $w_k \to x \in \partial\Omega$, if $f(w_k) \to y \in \partial\Omega$ and if*

$$\lim_{k\to\infty}[k_\Omega(z_0, w_k) - k_\Omega(z_0, f(w_k))] \leq \frac{1}{2}\log\alpha < \infty,$$

for some $\alpha > 0$, then for each $R > 0$ it holds

$$f(E_{z_0}(x, R)) \subset F_{z_0}(y, \alpha R).$$

Then property J is a stronger request than Proposition 3.1. Indeed -miming Abate [1]- we show that in some complete hyperbolic F-convex domains property J follows from Proposition 3.1. Let us give first the following:

Definition 3.2 (Abate)**.** A domain $\Omega \subset\subset \mathbb{C}^n$ is called *F-convex* if for every $x \in \partial\Omega$

$$\overline{F_{z_0}(x, R)} \bigcap \partial\Omega \subseteq \{x\}$$

holds for every $R > 0$.

Example 3.3.
1. Any strongly pseudoconvex domain with C^2 boundary is F-convex (Abate [3]).
2. Any weakly pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary is F-convex (Zhang-Ren [25]).
3. Any m-convex domain is F-convex (Mercer [20]).

Now we need the following lemma (see [1], [16]):

Lemma 3.4. *Let Ω be a bounded C^2 domain in $\mathbb{C}^n$, $z_0 \in \Omega$ and $x \in \partial\Omega$.*

(i) *There exists a constant $C > 0$ such that for all $z \in \Omega$ it holds*

$$k_\Omega(z_0, z) \leq C - \frac{1}{2}\log d(z, \partial\Omega).$$

(ii) *There exist* $C>0$ *and* $\epsilon>0$ *such that for all* $z_1, z_2 \in \Omega \bigcap B(x,\epsilon)$ *it holds*

$$k_\Omega(z_1,z_2) \leq C - \frac{1}{2}\sum_{j=1,2} \log d(z_j,\partial\Omega) + \frac{1}{2}\sum_{j=1,2}\log\left(d(z_j,\partial\Omega)+\|z_1-z_2\|\right).$$

Proposition 3.5. *Let* Ω *be a bounded* C^2 F*-convex domain in* $\mathbb{C}^n$ *and let* $z_0\in\Omega$. *If for any* $x\in\partial\Omega$ *there exist constants* C *and* $\epsilon>0$ *(depending on* x*) such that*

$$k_\Omega(z_0,z) \geq C - \frac{1}{2}\log d(z,\partial\Omega), \quad \forall z \in B(x,\epsilon)\bigcap\Omega \tag{3.1}$$

then Ω *has the property J.*

Remark 3.6. A domain Ω as in Proposition 3.5 is complete hyperbolic, for Ω is hyperbolic since it is bounded and equation (3.1) implies that the Kobayashi balls are relatively compact.

Proof. Let $f\in \mathrm{Hol}(\Omega,\Omega)$. Suppose that $\{w_k\}\subset\Omega$, $w_k\to x\in\partial\Omega$, $\lim_{k\to\infty} f(w_k) = y\in\partial\Omega$ and

$$\limsup_{k\to\infty}[k_\Omega(z_0,w_k)-k_\Omega(z_0,f(w_k))]<\infty.$$

By Proposition 3.1 it follows that there exists $\alpha>0$ such that for all $R>0$

$$f(E_{z_0}(x,R))\subset F_{z_0}(y,\alpha R). \tag{3.2}$$

Now, let $\{z_k\}\subset\Omega$ be such that $z_k\to x$ non-tangentially, *i.e.* such that there exists $L>0$ such that

$$\frac{\|z_k-x\|}{\mathrm{d}(z_k,\partial\Omega)}\leq L<\infty.$$

By Lemma 3.4(ii) and by (3.1) it follows that there exists $K>0$ such that

$$\limsup_{w\to x}[k_\Omega(z_k,w)-k_\Omega(z_0,w)] \leq \frac{1}{2}\log\left(1+\frac{\|z_k-x\|}{\mathrm{d}(z_k,\partial\Omega)}\right)+\frac{1}{2}\log\|z_k-x\|+K \leq \frac{1}{2}\log(1+L)+\frac{1}{2}\log\|z_k-x\|+K,$$

and the last term tends to $-\infty$ as $k\to\infty$. Therefore, for each $R>0$, it follows that $\{z_k\}\subset E_{z_0}(x,R)$ eventually, and hence, by (3.2), that $\{f(z_k)\}\subset F_{z_0}(y,\alpha R)$ eventually. If we prove that the limit points of $\{f(z_k)\}$ must belong to $\partial\Omega$, then since Ω is F-convex, it will result

$$\lim_{k\to\infty} f(z_k)\in\partial\Omega\bigcap\overline{F_{z_0}(y,R)}\subseteq\{y\},$$

and hence $\lim_{k\to\infty} f(z_k)=y$.
So, let $\overline{y}$ be a limit point of $\{f(z_k)\}$, that is, suppose $\lim_{m\to\infty} f(z_{k_m})=\overline{y}$. By Lemma 3.4(i) it follows that

$$k_\Omega(z_0,f(z_{k_m}))\leq C+\frac{1}{2}\log\frac{1}{\mathrm{d}(f(z_{k_m}),\partial\Omega)},$$

and then $\overline{y}\in\Omega$ if and only if, for every m,

$$k_\Omega(z_0,f(z_{k_m}))\leq K<\infty. \tag{3.3}$$

The domain Ω being hyperbolic means that the topology of Ω coincides with the one induced by k_Ω. So if $\epsilon>0$, if $B(\overline{y},\epsilon)$ is the Kobayashi ball of center $\overline{y}$ and radius ϵ and if $\{f(z_{k_m})\}\subset B(\overline{y},\epsilon)$ eventually, then by (3.3) it follows that $\{f(z_{k_m})\}\subset B(z_0,\epsilon+K)$, eventually. Then we have

(1) $\{f(z_{k_m})\} \subset F_{z_0}(y, R)$ for all $R > 0$, eventually,
(2) $\{f(z_{k_m})\} \subset B(z_0, C)$ for some $C > 1$, eventually.

But if we choose $R = \exp(2C)^{-1}$ it follows that

$$F_{z_0}(y, R) \bigcap B(z_0, C) = \emptyset.$$

And this is a contradiction. □

Corollary 3.7. *Let $\Omega \subset\subset \mathbb{C}^n$ be a F-convex domains with C^2 boundary which satisfies properties D and W. Let $f, g \in \mathrm{Hol}(\Omega,\Omega)$ be such that $f \circ g = g \circ f$ and f has no fixed points in Ω. If the point $x \in \partial\Omega$ defined by the property W for f is a strongly pseudoconvex point then f and g have admissible limit x at x.*

Proof. The point x being strongly pseudoconvex implies that equation (3.1) holds (see, *e.g.* [25]), then Proposition 3.5 assures that Ω has property J, and then Theorem 2.7 applies. □

More specifically we have the two following examples:

Corollary 3.8. *Let $\Omega \subset\subset \mathbb{C}^n$ be a m-convex C^2 domain. Let $f, g \in \mathrm{Hol}(\Omega,\Omega)$ be such that $f \circ g = g \circ f$ and f has no fixed point in Ω. Then there exists $x \in \partial\Omega$ such that f and g have admissible limit x at x.*

Proof. The only thing to prove is that Ω satisfies equation (3.1). But this follows from Mercer [20]. □

Corollary 3.9. *Let $\Omega \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex contractile C^2 domain. Let $f, g \in \mathrm{Hol}(\Omega,\Omega)$ be such that $f \circ g = g \circ f$ and f has no fixed point in Ω. Then there exists $x \in \partial\Omega$ such that f and g have admissible limit x at x.*

Proof. The result follows from Abate [1] (in which equation (3.1) is proved for strongly pseudoconvex domains) and from the above remarks. □

Remark 3.10. If Ω is a weakly pseudoconvex domain in $\mathbb{C}^2$ with real-analytic or smooth boundary, it is known that Ω satisfies equation (3.1) for almost all points of $\partial\Omega$ (since the strongly pseudoconvex points are dense in $\partial\Omega$, see [25]). However, we don't know if Ω has the property J.

4. Using property J in old contexts

In this section we give an example of possible uses of property J in the form stated in Definition 2.2 in contexts where property J was known in different suites. In particular we give a new proof of a well-known Heins' Theorem, that is

Theorem 4.1 (Heins). *Let $\gamma \in \mathrm{Aut}(\Delta)$ be hyperbolic, and $f \in \mathrm{Hol}(\Delta,\Delta)$ be such that*

$$f \circ \gamma = \gamma \circ f.$$

Then either f is a hyperbolic automorphism of Δ with the same fixed points as γ, or $f = id_\Delta$.

Our proof of Heins' Theorem is not simpler than the original one (see [14]), but it has the value that it doesn't require any explicit knowledge of the automorphisms group of Δ. For this reason our proof can be translated to prove an analogous result in the unit ball $\mathbb{B}^n$ (see also [11] and [10]).

Proof. Suppose $f \neq \mathrm{id}_\Delta$. Up to conjugation we can assume that γ has fixed points 1 and -1 and $\gamma^k(z) \to 1$ for any $z \in \Delta$ (*i.e.* 1 is the Wolff point of γ). Notice that γ^{-1} commutes with f:

$$\gamma^{-1} \circ f = \gamma^{-1} \circ f \circ \gamma \circ \gamma^{-1} = \gamma^{-1} \circ \gamma \circ f \circ \gamma^{-1} = f \circ \gamma^{-1}.$$

Now, since Δ satisfies properties D, W and J (see Example 2.3), we can apply Theorem 2.7 (with g equals, respectively, to γ and to γ^{-1}) in order to obtain that f has non-tangential limits 1 at 1 and -1 at -1. By the classical Schwarz Lemma it follows that f has no fixed points in Δ and furthermore it is easy to verify that either 1 or -1 is the Wolff point of f (otherwise γ would have three different boundary fixed points). Suppose w.l.o.g. f has Wolff point 1. Equation (2.1) assures that f has finite boundary dilatation coefficients at 1 and -1, call them $d_f(-1)$ and $d_f(1)$. Recall that if $x \in \partial\Delta$ the "boundary dilatation coefficient" of f at x is defined by

$$\frac{1}{2}\log d_f(x) := \liminf_{w \to x}[k_\Delta(0,w) - k_\Delta(0,f(w))].$$

Notice that this is not the classical definition, but it agrees with the classical one. Arguing as in the proof of Theorem 2.7 -letting $w_k = (\gamma^{-1})^k(z)$- we see that $1/2 \log d_f(-1) \leq k_\Delta(z, f(z))$ for any $z \in \Delta$. From the definition of k_Δ -the Poincaré distance of Δ- and a straightforward calculation we have

$$d_f(-1) \leq \frac{1 + |\Phi_z(f(z))|}{1 - |\Phi_z(f(z))|} \tag{4.1}$$

where $\Phi_z(w)$ is the Möebius transformation of Δ which brings z to 0. In particular we can choose to evaluate equation (4.1) for $z = r$, $r \to 1^-$. From the expression of $\Phi_r(f(r))$ and from the classical Julia-Wolff-Carathéodory Theorem (see *e.g.* [1]) it follows that $d_f(-1) \leq 1/d_f(1)$ (for details see Lemma 3.6 of [7]). Now, the Behan's Lemma (a version of Julia's Lemma, see Lemma 8 of Behan [5]) implies that f is a hyperbolic automorphism. □

A word-by-word translation of the previous proof allows the following theorem (first discovered by de Fabritiis and Gentili [11]):

Theorem 4.2. *Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$. Let γ be an hyperbolic automorphism of $\mathbb{B}^n$ and let f be a holomorphic self-map of $\mathbb{B}^n$ which commutes to γ. Then, up to conjugation with automorphisms, it holds*

(a) $z_1 \mapsto f_1(z_1, 0, \ldots, 0)$ *is a hyperbolic automorphism of* Δ.
(b) $f_2(z_1, 0, \ldots, 0) = \ldots = f_n(z_1, 0, \ldots, 0) = 0$ *for any* $z_1 \in \Delta$.

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