BOUNDARY BEHAVIOR OF INFINITESIMAL GENERATORS IN THE UNIT BALL

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Abstract. We prove a Julia-Wolff-Carathéodory type theorem for infinitesimal generators on the unit ball in $\mathbb{C}^n$. Moreover, we study jets expansions at the boundary and give necessary and sufficient conditions on such jets for an infinitesimal generator to generate a group of automorphisms of the ball.

1. Introduction

The classical Julia-Wolff-Carathéodory theorem (see, *e.g.* [2, 13, 16, 18]) is the most powerful tool for studying properties of bounded holomorphic functions of the unit disc $\mathbb{D}$ of $\mathbb{C}$ at a given boundary point. This theorem has been generalized to the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ by W. Rudin (see [17]) and to strongly (pseudo)convex domains and other domains in $\mathbb{C}^n$ by other authors, notably by M. Abate (see [1], [2]. See also [3] for the most recent and complete survey on the subject).

In what follows we are mainly interested in the case of mappings fixing a boundary point. Since the group of automorphisms of $\mathbb{B}^n$ acts bi-transitively on $\partial\mathbb{B}^n$, without loss of generality we restrict our attention to the point $e_1 = (1, 0 \ldots, 0) \in \partial\mathbb{B}^n$.

The maps we are working with are not assumed to be continuous up to the boundary, thus we have to specify the meaning of the term "boundary fixed point". In higher dimensions, in fact, different approaches to boundary limits are possible. We recall them here briefly (see [1], [17] for more information).

Let $R \geq 1$ and let $K(e_1, R) := \{z \in \mathbb{B}^n : |1 - z_1| \leq \frac{R}{2}(1 - \|z\|^2)\}$ be a *Korányi region of vertex e_1 and amplitude R* (see [17, Section 5.4.1], [13]). In [1, Section 2.2.3] a slightly different but essentially equivalent definition is given and used. In order not to excessively burden the notation, since we are only working at e_1, from now on, when we talk about Korányi regions, we will always mean Korányi regions of vertex e_1.

Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a holomorphic map. We say that f has *K-limit L* at e_1 – and we write $K\text{-}\lim_{z \to e_1} f(z) = L$ – if for each sequence $\{z_k\} \subset \mathbb{B}^n$ converging to e_1 such that $\{z_k\}$ belongs eventually to some Korányi region, it follows that $f(z_k) \to L$. We say

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that f has *restricted K-limit* L at e_1 – and we write $\angle_K \lim_{z\to e_1} f(z) = L$ – if for each sequence $\{z_k\} \subset \mathbb{B}^n$ converging to e_1 such that $\|z_k - \langle z_k, e_1\rangle e_1\|^2/(1-|\langle z_k, e_1\rangle|^2) \to 0$ and $\langle z_k, e_1\rangle \to 1$ non-tangentially in $\mathbb{D}$ it follows that $f(z_k) \to L$. Finally, we say that f has *non-tangential limit* L at e_1 and we write $\angle \lim_{z\to e_1} f(z) = L$, if for each sequence $\{z_k\} \subset \mathbb{B}^n$ converging non-tangentially to e_1 – *i.e.*, such that there exists $C > 0$ with $\|z_k - e_1\| \leq C(1-\|z_k\|^2)$ for all $k \geq 1$ – it follows that $f(z_k) \to L$.

One can show that

$$K\text{-}\lim_{z\to e_1} f(z) = L \Longrightarrow \angle_K \lim_{z\to e_1} f(z) = L \Longrightarrow \angle \lim_{z\to e_1} f(z) = L,$$

but the converse to any of these implications is not true in general.

A holomorphic self-map $f : \mathbb{B}^n \to \mathbb{B}^n$ has a *boundary regular fixed point* at e_1 if $\angle \lim_{z\to e_1} f(z) = e_1$ and

$$\alpha_f(e_1) := \liminf_{z\to e_1} \frac{1-\|f(z)\|}{1-\|z\|} < +\infty.$$

Now we can formulate the Julia-Wolff-Carathéodory Theorem for $\mathbb{B}^n$ for boundary regular fixed points in the way we need in this paper. As is customary, we denote by $\{e_1, \ldots, e_n\}$ the standard orthonormal basis in $\mathbb{C}^n$ (the symbol e_1 denotes thus both the point and the direction).

Theorem 1.1 (Rudin). *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be holomorphic. Suppose that e_1 is a boundary regular fixed point for f. Then $K\text{-}\lim_{z\to e_1} f(z) = e_1$. Moreover,*

- ($1'$) *$\langle df_z(e_1), e_1\rangle$ and $\langle df_z(e_h), e_k\rangle$ are bounded in any Korányi region for $h, k = 2, \ldots, n$.*
- ($1''$) *$\langle df_z(e_j), e_1\rangle/(1-z_1)^{1/2}$ is bounded in any Korányi region for $j = 2, \ldots, n$.*
- ($1'''$) *$(1-z_1)^{1/2}\langle df_z(e_1), e_j\rangle$ is bounded in any Korányi region for $j = 2, \ldots, n$.*
- (2) *$\angle_K \lim_{z\to e_1} \frac{1-\langle f(z), e_1\rangle}{1-z_1} = \alpha_f(e_1)$,*
- (3) *$\angle_K \lim_{z\to e_1} \langle df_z(e_1), e_1\rangle = \alpha_f(e_1)$,*
- (4) *$\angle_K \lim_{z\to e_1} \langle df_z(e_j), e_1\rangle = 0$ for $j = 2, \ldots, n$.*
- (5) *$\angle_K \lim_{z\to e_1} \frac{\langle f(z), e_j\rangle}{(1-z_1)^{1/2}} = 0$ for $j = 2, \ldots, n$.*
- (6) *$\angle_K \lim_{z\to e_1} (1-z_1)^{1/2}\langle df_z(e_1), e_j\rangle = 0$ for $j = 2, \ldots, n$.*

One can interpret Julia-Wolff-Carathéodory's theorem as a description of the first jet of a holomorphic self-map of the unit ball at a boundary regular fixed point.

One of the aims of the present paper is to give a corresponding theorem for infinitesimal generator (that is, $\mathbb{R}$-*semicomplete holomorphic vector fields*) on $\mathbb{B}^n$ having a "regular singularity" at e_1.

A holomorphic vector field $G : \mathbb{B}^n \to \mathbb{C}^n$ is said to be an *infinitesimal generator* if the Cauchy problem

$$\begin{cases} \dot{x}(t) = G(x(t)) \\ x(0) = z_0 \end{cases} \tag{1.1}$$

has a solution $x_{z_0} : [0,+\infty) \ni t \mapsto x(t)$ for all $z_0 \in \mathbb{B}^n$. If this is the case, the map $\phi : [0,+\infty) \times \mathbb{B}^n \mapsto \mathbb{B}^n$ given by $\phi_t(z) := x_z(t)$ is real analytic and $z \mapsto \phi_t(z)$ is a univalent holomorphic self-map of $\mathbb{B}^n$ for all fixed $t \in [0,+\infty)$. The family (ϕ_t) is a (continuous) semigroup, namely a continuous morphism of semigroups between $(\mathbb{R}^+,+)$ endowed with the Euclidean topology and $(\mathsf{Hol}(\mathbb{B}^n,\mathbb{B}^n),\circ)$ endowed with the topology of uniform convergence on compacta.

Conversely, any (continuous) semigroup of holomorphic self-maps of $\mathbb{B}^n$ is associated uniquely to an infinitesimal generator. Interior fixed points of the semigroups correspond to singularities of the vector field. At the boundary, the situation is more complicated (see Sections 2 and 3). For the time being, we say that e_1 is a *boundary regular null point* (or BRNP for short) if it is a boundary regular fixed point for the associated semigroup of holomorphic self-maps and we say that $\beta \in \mathbb{R}$ is the *dilation* of G at e_1 if the flow ϕ_1 of G at the time 1 has boundary dilation coefficient $\alpha_{\phi_1}(e_1) = e^\beta$ (see Definition 2.2 for a definition of BRNP which does not involve the associated semigroup).

Now, a version of the Julia-Wolff-Carathéodory Theorem for infinitesimal generators which we are going to prove is the following:

Theorem 1.2. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be an infinitesimal generator. Suppose that*

$$(1.2)\qquad \begin{aligned} &(*) \quad \mathbb{B}^n \ni z \mapsto \frac{|\langle G(z), e_1\rangle|}{|z_1 - 1|} \quad \textit{is bounded in any Korányi region and} \\ &(**) \quad \mathbb{B}^n \ni z \mapsto \frac{|\langle G(z), e_j\rangle|}{|z_1 - 1|^{1/2}} \quad \textit{is bounded in any Korányi region for } j = 2,\dots,n. \end{aligned}$$

Then e_1 is a boundary regular null point for G. Moreover, let $\beta \in \mathbb{R}$ denote the dilation of G at e_1. Then

- $(1')$ *$\langle dG_z(e_1), e_1\rangle$ and $\langle dG_z(e_h), e_k\rangle$ are bounded in any Korányi region for $h,k = 2,\dots,n$,*
- $(1'')$ *$\langle dG_z(e_j), e_1\rangle/(1-z_1)^{1/2}$ is bounded in any Korányi region for $j = 2,\dots,n$,*
- $(1''')$ *$(1-z_1)^{1/2}\langle dG_z(e_1), e_j\rangle$ is bounded in any Korányi region for $j = 2,\dots,n$,*
- (2) *$\angle_K \lim_{z\to e_1} \frac{\langle G(z), e_1\rangle}{z_1 - 1} = \beta$,*
- (3) *$\angle_K \lim_{z\to e_1} \langle dG_z(e_1), e_1\rangle = \beta$,*
- (4) *$\angle_K \lim_{z\to e_1} \langle dG_z(e_j), e_1\rangle = 0$ for $j = 2,\dots,n$.*

In the case where the infinitesimal generator extends smoothly past e_1, Theorem 1.2 is a consequence of Theorem 1.1 applied to the associated semigroup. However, if no regularity is assumed, this way of proceeding does not seem to be possible. Our proof, in fact, does not involve the associated semigroup, but it is based on the properties of infinitesimal generators, and it is contained in Section 4. In particular, we shall prove an intermediate version of the Julia-Wolff-Carathéodory theorem assuming only hypothesis (1.2).$(*)$ (see Proposition 4.1). In Example 4.3, we give an example of an infinitesimal generator which satisfies (1.2).$(*)$ but not (1.2).$(**)$ and for which some implications of Theorem 1.2 do not hold. In Subsection 4.1 we discuss the (dis)similarities between Theorem 1.1 and Theorem 1.2 and some natural open questions raised up from this work.

Next, in Section 5, assuming a C^3 regularity at the BRNP e_1, we describe the jets space of infinitesimal generators, giving complementary results to the ones obtained in [9] for local biholomorphisms of strongly (pseudo)convex domains. In particular, we are interested in finding (minimal, pointwise) necessary and sufficient conditions for an infinitesimal generator to generate a group of automorphisms of $\mathbb{B}^n$. In case of an interior singularity, the condition is rather simple: an infinitesimal generator G with a singularity at $z_0 \in \mathbb{B}^n$ generates a group of automorphisms of $\mathbb{B}^n$ if and only if the spectrum of dG_{z_0} is contained in the imaginary axis $i\mathbb{R}$.

In the case where the singularity is at the boundary, we prove the following result:

Theorem 1.3. *Let G be an infinitesimal generator on $\mathbb{B}^n$ of class C^3 at e_1. Assume that e_1 is a boundary regular null point with dilation $\beta \in \mathbb{R}$. Then G generates a group of automorphisms if and only if the following conditions are satisfied:*

(1) $\mathsf{Re}\,\langle \frac{\partial G}{\partial z_k}(e_1), e_k\rangle = \frac{\beta}{2}$, *for* $k = 2, \ldots, n$,
(2) $\mathsf{Re}\,\langle \frac{\partial^2 G}{\partial z_1 \partial z_k}(e_1), e_1\rangle = \beta$ *for* $k = 1, \ldots, n$,
(3) $\langle \frac{\partial^2 G}{\partial z_1 \partial z_k}(e_1), e_h\rangle = 0$ *for* $2 \leq k < h \leq n$,
(4) $\mathsf{Re}\,\langle \frac{\partial^3 G}{\partial z_1^3}(e_1), e_1\rangle = 0$.

Moreover, if the previous conditions are satisfied, then $G \equiv 0$ if and only if $\beta = 0$, $\langle \frac{\partial G}{\partial z_k}(e_1), e_k\rangle = 0$ for $k = 2, \ldots, n$ and $\langle \frac{\partial^2 G}{\partial z_1^2}(e_1), e_1\rangle = 0$.

The assumption on the C^3 regularity of G at e_1 can be lowered by assuming the existence of an expansion of G in any Korányi region with vertex e_1, but for the sake of clarity, we will deal only with the C^3 case.

The previous result belongs to the family of so-called "rigidity phenomena", where some minimal conditions on the maps/infinitesimal generators of $\mathbb{B}^n$ at one point imply certain specific forms. For instance, the well known Burns-Krantz rigidity theorem [10] states that a holomorphic self-map of the unit ball which is the identity up to the third order at a boundary point, is the identity *tout court*. Such a result has been extended later to infinitesimal generators (see [14], and also [11]), in the following way: an infinitesimal generator in $\mathbb{B}^n$ which is 0 up to the third order at a boundary point of $\mathbb{B}^n$ is identically zero. In a sense, Theorem 1.3 is a quantitative version of such rigidity phenomena.

The main idea for the proof is to transfer the information on G to a family of infinitesimal generators on $\mathbb{D}$ by means of a method which we call "slice reduction" (see Section 3), first introduced in [7] and implemented here.

Finally, in Section 6 we show with a couple of examples that, contrarily as one might expect, the slice reductions do not preserve the boundary expansion: while in the one dimensional case the quadratic expansion at a BRNP of an infinitesimal generator is always an infinitesimal generator which generates a semigroup of linear fractional maps, in higher dimension this is no longer the case. Moreover, even if the quadratic expansion is an infinitesimal generator, the generated semigroup might not be linear fractional.

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2. Infinitesimal generators on the unit ball and BRNP's

Infinitesimal generators have been characterized in several ways. In the unit disc $\mathbb{D}$, the following powerful characterization is due to Berkson-Porta formula [5]: a holomorphic vector field $g : \mathbb{D} \to \mathbb{C}$ is an infinitesimal generator if and only if there exist $\tau \in \overline{\mathbb{D}}$ and $p : \mathbb{D} \to \{z \in \mathbb{C} : \mathsf{Re}\, z \geq 0\}$ such that

$$g(\zeta) = (\tau - \zeta)(1 - \overline{\tau}\zeta)p(\zeta). \tag{2.1}$$

In the multi-dimensional case, several equivalent characterizations are given both using Euclidean inequalities (see [16]), the Kobayashi metric (see [1]) and pluripotential theory (see [7]). In what follows, we need a characterization only for boundary regular fixed points, which we are going to define.

The function

$$u_{\mathbb{B}^n}(z) := -\frac{1 - \|z\|^2}{|1 - z_1|^2}$$

is the pluricomplex Poisson kernel with a pole at e_1 and its sublevel sets $\{u_{\mathbb{B}^n}(z) < -1/R\}$ for $R > 0$ are called *horospheres* with center e_1 and radius R.

Recall that a function $f : \mathbb{B}^n \to \mathbb{C}^m$ is C^k at e_1 if f and all its partial derivatives up to order k extend continuously to e_1. As the horospheres are smooth ellipsoids, by Whitney's extension theorem, this is equivalent to saying that for each horosphere E with center e_1 there exists a function $\tilde{f}$ (depending on E) of order C^k defined in a open neighborhood of $\overline{E}$ such that $\tilde{f}|_E \equiv f$.

The following characterization of infinitesimal generators in terms of the function $u_{\mathbb{B}^n}$ has been proved in [7, Theorem 3.11]:

Theorem 2.1. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be holomorphic and C^1 at e_1. If $d(u_{\mathbb{B}^n})_z \cdot G(z) \leq 0$ for all $z \in \mathbb{B}^n$ then G is an infinitesimal generator.*

Now we define BRNPs:

Definition 2.2. Let G be an infinitesimal generator in $\mathbb{B}^n$. The point e_1 is a *boundary regular null point*, or BRNP for short, if there exists $b \in \mathbb{R}$ such that

$$(du_{\mathbb{B}^n})_z \cdot G(z) + b u_{\mathbb{B}^n}(z) \leq 0 \quad \forall z \in \mathbb{B}^n.$$

The number

$$\beta := - \inf_{z \in \mathbb{B}^n} \frac{(du_{\mathbb{B}^n})_z \cdot G(z)}{u_{\mathbb{B}^n}(z)}$$

is called the *dilation* of G at e_1.

According to [7, Theorem 0.4] (see also [15]), if G is an infinitesimal generator with the associated semigroup (ϕ_t) then e_1 is a BRNP for G with dilation β if and only if for all $t \geq 0$ it follows

$$u_{\mathbb{B}^n}(\phi_t(z)) \leq e^{-t\beta} u_{\mathbb{B}^n}(z) \quad \forall z \in \mathbb{B}^n.$$

The number $e^{t\beta}$ is the so-called *boundary dilation coefficient* of ϕ_t at e_1. The previous inequality means that a horosphere of center e_1 and radius $R > 0$ is mapped into a horosphere with center e_1 and radius $e^{t\beta} R$.

3. Slice reduction of infinitesimal generators and BRNPs

Let

$$\mathcal{L}_{e_1} := \{v \in \mathbb{C}^n : \|v\| = 1, \langle v, e_1 \rangle = \alpha > 0\}.$$

Also, let

$$\varphi_v(\zeta) := \alpha(\zeta - 1)v + e_1. \tag{3.1}$$

It is easy to see that $\varphi_v : \mathbb{D} \to \mathbb{B}^n$ is holomorphic, and it is a complex geodesic, in the sense that it is an isometry between the Poincaré distance in $\mathbb{D}$ and the Kobayashi distance in $\mathbb{B}^n$. Furthermore, it is well known (see, *e.g.* [1] and [8, Section 1]) that any complex geodesic $\eta : \mathbb{D} \to \mathbb{B}^n$ extends holomorphically through the boundary and moreover, if $e_1 \in \eta(\partial\mathbb{D})$, then there exists an automorphism θ of the unit disc such that $\eta \circ \theta$ is of the form (3.1).

Remark 3.1. A direct computation shows that $u_{\mathbb{B}^n}(\varphi_v(\zeta)) = \frac{1}{\alpha^2} u_{\mathbb{D}}(\zeta)$ for all $\zeta \in \mathbb{D}$.

For a vector $w \in \mathbb{C}^n$ we use the notation $w = (w_1, w'') \in \mathbb{C} \times \mathbb{C}^{n-1}$. The holomorphic map $\rho_v : \mathbb{B}^n \to \mathbb{C}^n$ defined by

$$\rho_v(z_1, z'') := \left(\frac{z_1 + \frac{1}{\alpha}\langle z'', v'' \rangle + \frac{1-\alpha^2}{\alpha^2}(1 - z_1), -\frac{(1-z_1)}{\alpha} v''}{1 + \frac{1}{\alpha}\langle z'', v'' \rangle + \frac{1-\alpha^2}{\alpha^2}(1 - z_1)} \right) \tag{3.2}$$

has the property that $\rho_v(\mathbb{B}^n) = \varphi_v(\mathbb{D})$ and, moreover, $\rho_v \circ \varphi_v(\zeta) = \varphi_v(\zeta)$ for all $\zeta \in \mathbb{D}$.

Finally, we let $\widetilde{\rho}_v := \varphi_v^{-1} \circ \rho_v : \mathbb{B}^n \to \mathbb{D}$, *i.e.*

$$\widetilde{\rho}_v(z_1, z'') = 1 + \frac{\frac{1}{\alpha^2}(z_1 - 1)}{1 + \frac{1}{\alpha}\langle z'', v'' \rangle + \frac{1-\alpha^2}{\alpha^2}(1 - z_1)} = \frac{\alpha \langle z, v \rangle}{1 - z_1 + \alpha \langle z, v \rangle}. \tag{3.3}$$

Note that for every $w=(w_1,w'')\in\mathbb{C}^n$ it follows that

$$d(\widetilde{\rho}_v)_{\varphi_v(\zeta)}(w_1,w'')=\frac{1}{\alpha^2}w_1+\frac{1-\alpha^2}{\alpha^2}(\zeta-1)w_1-\frac{1}{\alpha}(\zeta-1)\langle w'',v''\rangle.$$

Definition 3.2. Let $G:\mathbb{B}^n\to\mathbb{C}^n$ be a holomorphic vector field. Let

$$g_v(\zeta):=d(\widetilde{\rho}_v)_{\varphi_v(\zeta)}(G(\varphi_v(\zeta))).$$

We call the holomorphic vector field $g_v:\mathbb{D}\to\mathbb{C}$ the *slice reduction of G to v.*

More explicitly

$$\begin{aligned}g_v(\zeta)&=\frac{1}{\alpha^2}G_1(\varphi_v(\zeta))+\frac{1-\alpha^2}{\alpha^2}(\zeta-1)G_1(\varphi_v(\zeta))-\frac{1}{\alpha}(\zeta-1)\langle G''(\varphi_v(\zeta)),v''\rangle\\&=\frac{1}{\alpha^2}\zeta G_1(\varphi_v(\zeta))-\frac{\zeta-1}{\alpha}\langle G(\varphi_v(\zeta)),v\rangle.\end{aligned}\tag{3.4}$$

The following version of Julia's lemma for infinitesimal generators was proved in [7, Theorem 0.4] (see also [15, Theorem p.403] and, for the one-dimensional case, see [12, Theorem 1]):

Theorem 3.3. *Let $G:\mathbb{B}^n\to\mathbb{C}^n$ be an infinitesimal generator. Then the following are equivalent:*

(1) *G has BRNP at e_1 and dilation $\beta_0\leq\beta\in\mathbb{R}$,*
(2) *$d(u_{\mathbb{B}^n})_z\cdot G(z)+\beta u_{\mathbb{B}^n}(z)\leq 0$ for all $z\in\mathbb{B}^n$,*
(3) $\dfrac{\mathsf{Re}\,\langle G(z),z\rangle}{1-\|z\|^2}-\mathsf{Re}\,\dfrac{\langle G(z),e_1\rangle}{1-z_1}\leq\dfrac{\beta}{2}$, *for all $z\in\mathbb{B}^n$,*
(4) *for each $v\in\mathcal{L}_{e_1}$ the slice reduction g_v is an infinitesimal generator of the unit disc with BRNP at 1 and dilation $\leq\beta$.*
(5) *there exists $C>0$ such that for all $v\in\mathcal{L}_{e_1}$ it follows*

$$\limsup_{(0,1)\ni r\to1}\frac{|g_v(r)|}{1-r}\leq C.$$

Moreover, if the previous condition is satisfied, then 1 is a BRNP for g_v and the following non-tangential limit exist

$$\angle\lim_{\zeta\to1}g_v'(\zeta)=\angle\lim_{\zeta\to1}\frac{g_v(\zeta)}{\zeta-1}=\beta_v\in\mathbb{R},$$

with $\beta_v\leq\beta$ and

$$\beta_0=\sup_{v\in\mathcal{L}_{e_1}}\beta_v.$$

A sufficient condition for the existence of BRNP, which we will use in the sequel, is contained in the following (see [15]):

Theorem 3.4. *Let $G:\mathbb{B}^n\to\mathbb{C}^n$ be an infinitesimal generator. Assume that*

(1) $\lim_{(0,1)\ni r\to 1} G(re_1) = 0$,

(2) $\liminf_{r\to 1} \mathsf{Re}\, \frac{\langle G(re_1), e_1\rangle}{r-1} < +\infty$.

Then

$$\lim_{(0,1)\ni r\to 1} \frac{\langle G(re_1), e_1\rangle}{r-1} = \beta \in \mathbb{R}$$

and e_1 is a BRNP for G with dilation β.

Slice reductions of holomorphic vector fields preserve pluricomplex Green and Poisson functions of strongly convex domains, as shown in [7] (see also [11] for somewhat explicit computations). As a consequence, a holomorphic vector field is an infinitesimal generator if and only if all its slice reductions (with respect to all points $\tau \in \partial\mathbb{B}^n$) are infinitesimal generators in the unit disc. In what follows, we need only a boundary version of this fact, which we prove here explicitly for the unit ball. We start with the following:

Lemma 3.5. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be holomorphic. Then, for all $v = (\alpha, v'') \in \mathcal{L}_{e_1}$, and for all $\zeta \in \mathbb{D}$*

(1) $\dfrac{\mathsf{Re}\,\langle G(\varphi_v(\zeta)), \varphi_v(\zeta)\rangle}{1 - \|\varphi_v(\zeta)\|^2} - \mathsf{Re}\,\dfrac{\langle G(\varphi_v(\zeta)), e_1\rangle}{1 - \langle \varphi_v(\zeta), e_1\rangle} = \dfrac{\mathsf{Re}\,\langle g_v(\zeta), \zeta\rangle}{1 - |\zeta|^2} - \mathsf{Re}\,\dfrac{g_v(\zeta)}{1-\zeta}$

(2) *for all $\delta \in \mathbb{R}$ it follows*

$$d(u_{\mathbb{D}})_\zeta \cdot g_v(\zeta) + \delta u_{\mathbb{D}}(\zeta) = \alpha^2[d(u_{\mathbb{B}^n})_{\varphi_v(\zeta)} \cdot G(\varphi_v(\zeta)) + \delta u_{\mathbb{B}^n}(\varphi_v(\zeta))].$$

Proof. (1) We have $1 - \|\varphi_v(\zeta)\|^2 = \alpha^2(1 - |\zeta|^2)$ and $1 - \langle \varphi_v(\zeta), e_1\rangle = \alpha^2(1-\zeta)$. Write G_1 for $G_1(\varphi_v(\zeta))$ and G_2 for $\langle G(\varphi_v(\zeta)), v\rangle$. Then, taking into account that for all $a \in \mathbb{C}$ it holds $\mathsf{Re}\,(a\zeta) + \mathsf{Re}\,(a\overline{\zeta}) = 2\mathsf{Re}\,\zeta\mathsf{Re}\,a$, and expanding (1), we have

$$\begin{aligned}
&\frac{1}{\alpha}\frac{\mathsf{Re}\,(G_2\overline{\zeta})}{1-|\zeta|^2} - \frac{1}{\alpha}\frac{\mathsf{Re}\,G_2}{1-|\zeta|^2} + \frac{1}{\alpha^2}\frac{\mathsf{Re}\,G_1}{1-|\zeta|^2} - \frac{1}{\alpha^2}\frac{\mathsf{Re}\,G_1}{|1-\zeta|^2} + \frac{1}{\alpha^2}\frac{\mathsf{Re}\,(G_1\overline{\zeta})}{|1-\zeta|^2} - \frac{|\zeta|^2}{\alpha^2}\frac{\mathsf{Re}\,G_1}{1-|\zeta|^2}\\
&+\frac{|\zeta|^2}{\alpha}\frac{\mathsf{Re}\,G_2}{1-|\zeta|^2} - \frac{1}{\alpha}\frac{\mathsf{Re}\,(G_2\overline{\zeta})}{1-|\zeta|^2} + \frac{1}{\alpha^2}\frac{\mathsf{Re}\,(G_1\zeta)}{|1-\zeta|^2} - \frac{|\zeta|^2}{\alpha^2}\frac{\mathsf{Re}\,G_1}{|1-\zeta|^2} - \frac{1}{\alpha}\frac{\mathsf{Re}\,(G_2\zeta)}{|1-\zeta|^2} + \frac{|\zeta|^2}{\alpha}\frac{\mathsf{Re}\,G_2}{|1-\zeta|^2}\\
&+\frac{1}{\alpha}\frac{\mathsf{Re}\,G_2}{|1-\zeta|^2} - \frac{1}{\alpha}\frac{\mathsf{Re}\,(G_2\overline{\zeta})}{|1-\zeta|^2} = \left(\frac{1-|\zeta|^2}{\alpha^2(1-|\zeta|^2)} + \frac{-1+2\mathsf{Re}\,\zeta - |\zeta|^2}{\alpha^2|1-\zeta|^2}\right)\mathsf{Re}\,G_1\\
&+\left(\frac{|\zeta|^2-1}{\alpha(1-|\zeta|^2)} + \frac{|\zeta|^2 - 2\mathsf{Re}\,\zeta + 1}{\alpha|1-\zeta|^2}\right)\mathsf{Re}\,G_2 = 0,
\end{aligned}$$

as we wanted.

(2) A direct computation shows that

$$d(u_{\mathbb{B}^n})_z \cdot G(z) = -2\mathsf{Re}\,\left(\frac{\langle G(z), e_1\rangle}{1-z_1}\right)\frac{1-\|z\|^2}{|1-z_1|^2} + 2\mathsf{Re}\,\frac{\langle G(z), z\rangle}{|1-z_1|^2}.$$

Hence, the result follows from (1) taking into account Remark 3.1. Also, see [7, Eq. (4.7) p.45], where such a formula has been proved for strongly convex domains. $\square$

Also, we need the following lemma which will be useful to move from BRNP with dilation > 0 to BRNP with dilation ≤ 0:

Lemma 3.6. *Let $\beta \in \mathbb{R}$. Define $H_\beta : \mathbb{B}^n \to \mathbb{C}^n$ by*

$$H_\beta(z) = \frac{\beta}{2}(e_1 - z_1 z). \tag{3.5}$$

Then H_β generates a group of (hyperbolic) automorphisms of $\mathbb{B}^n$, with BRNP at e_1 with dilation $-\beta$ and

$$d(u_{\mathbb{B}^n})_z \cdot H_\beta(z) - \beta u_{\mathbb{B}^n}(z) \equiv 0 \quad \forall z \in \mathbb{B}^n. \tag{3.6}$$

Moreover, for all $v \in \mathcal{L}_{e_1}$ the slice reduction is

$$h_v(\zeta) = \frac{\beta}{2}(1 - \zeta^2) = -\beta(\zeta - 1) - \frac{\beta}{2}(\zeta - 1)^2.$$

Proof. It is well known that H_β is a generator of a group of (hyperbolic) automorphisms (see, e.g. [6]) with BRNP at e_1 and dilation $-\beta$. Hence $-H_\beta$ is a generator of a group of automorphisms having BRNP at e_1 with dilation β. Applying Theorem 3.3.(2) at both H_β and $-H_\beta$ we get (3.6).

The form of the slice reductions is a direct computation from the very definition. □

In the paper we will use several times the following trick, whose proof is immediate from Theorem 3.3 and Lemma 3.6, which we state here for the reader convenience:

Corollary 3.7. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be an infinitesimal generator and assume e_1 is a BRNP for G, with dilation δ. Let $\beta \in \mathbb{R}$ and let H_β be given by* (3.5). *Then $G + H_\beta$ is an infinitesimal generator in $\mathbb{B}^n$ with e_1 as BRNP and dilation $\delta - \beta$.*

Now we can prove a boundary characterization of infinitesimal generators at BRNP:

Proposition 3.8. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be holomorphic and C^1 at e_1. Then the following are equivalent:*

1. *G is an infinitesimal generator with BRNP at e_1 and dilation $\leq \beta \in \mathbb{R}$,*
2. *for each $v \in \mathcal{L}_{e_1}$ the slice reduction g_v is an infinitesimal generator of the unit disc with BRNP at 1 and dilation $\leq \beta$.*
3. *$d(u_{\mathbb{B}^n})_z \cdot G(z) + \beta u_{\mathbb{B}^n}(z) \leq 0$ for all $z \in \mathbb{B}^n$.*

Proof. (1) implies (2) and (3) by Theorem 3.3.

If either (2) or (3) holds, the only aim is to show that G is an infinitesimal generator, because then (1) follows from Theorem 3.3.

Let $F := G + H_\beta$, where H_β is given by (3.5).

Assume (2) holds. By Theorem 3.3 it follows that $d(u_{\mathbb{D}})_\zeta \cdot g_v(\zeta) + \beta u_{\mathbb{D}}(\zeta) \leq 0$ for all $\zeta \in \mathbb{D}$ and $v \in \mathcal{L}_{e_1}$. Hence, by Lemma 3.5 and (3.6) it is easy to see that $d(u_{\mathbb{B}^n})_z \cdot F(z) \leq 0$ for all $z \in \mathbb{B}^n$. The same conclusion is obtained directly if (3) holds. By Theorem 2.1 it follows that F is an infinitesimal generator, and so does $G = F - H_\beta$, because infinitesimal generators in the ball form a cone (see [1, Corollary 2.5.29]). □

Finally, we have the following characterization of generators of groups which we will use later.

Proposition 3.9. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be holomorphic and C^1 at e_1. Let $\beta \in \mathbb{R}$. The following are equivalent:*

1. *G generates a group of automorphisms of $\mathbb{B}^n$ with BRNP e_1 and dilation β,*
2. *$d(u_{\mathbb{B}^n})_z \cdot G(z) + \beta u_{\mathbb{B}^n}(z) \equiv 0$,*
3. *for each $v \in \mathcal{L}_{e_1}$ the infinitesimal generator g_v generates a group of automorphisms of $\mathbb{D}$ with BRNP 1 and dilation β,*
4. *for each $v \in \mathcal{L}_{e_1}$ it holds $d(u_{\mathbb{D}})_\zeta \cdot g_v(\zeta) + \beta u_{\mathbb{D}}(\zeta) \equiv 0$.*

Moreover, $G \equiv 0$ (hence the group it generates is the trivial group of automorphisms $\phi_t(z) \equiv z$ for all $t \geq 0$) if and only if for each $v \in \mathcal{L}_{e_1}$ it follows $g_v \equiv 0$. If this is the case then $\beta = 0$.

Proof. (2) is equivalent to (4) by Lemma 3.5.

If (1) holds then $-G$ is an infinitesimal generator on $\mathbb{B}^n$ with BRNP at e_1 and dilation $-\beta$. Hence Theorem 3.3 applied to G and $-G$ implies (2). If (2) holds, then (1) follows from Proposition 3.8 applied to G and $-G$. Similarly, (3) is equivalent to (4).

Finally, by (3.4) it is easy to see that $G \equiv 0$ if and only if $g_v \equiv 0$ for all $v \in \mathcal{L}_{e_1}$. $\square$

4. The Julia-Wolff-Carathéodory Theorem for infinitesimal generators

As a matter of notation, we write $\angle \lim$ for non-tangential limits, $\angle_K \lim$ for restricted K-limits and $K - \lim$ for K-limits.

Proposition 4.1. *Let G be an infinitesimal generator on $\mathbb{B}^n$. Suppose $\lim_{(0,1)\ni r\to 1} G(re_1) = 0$ and*

$$\mathbb{B}^n \ni z \mapsto \frac{|\langle G(z), e_1\rangle|}{|z_1 - 1|} \quad \text{is bounded in any Korányi region.} \tag{4.1}$$

Then e_1 is a BRNP for G. Moreover, if $\beta \in \mathbb{R}$ is the dilation of G at e_1, then

- ($1'$) *$\mathbb{B}^n \ni z \mapsto \langle dG_z(e_1), e_1\rangle$ is bounded in any Korányi region,*
- ($1''$) *$\mathbb{B}^n \ni z \mapsto \frac{\langle dG_z(e_j), e_1\rangle}{|z_1-1|^{1/2}}$ is bounded in any Korányi region for $j = 2, \ldots, n$,*
- (2) *$\angle_K \lim_{z\to e_1} \frac{\langle G(z), e_1\rangle}{z_1 - 1} = \beta$,*
- (3) *$\angle_K \lim_{z\to e_1} \langle dG_z(e_1), e_1\rangle = \beta$.*

Proof. By hypotheses of the theorem clearly guarantee that the hypotheses of Theorem 3.4 are satisfied so that e_1 is a BRNP for G.

($1'$) The proof is based on an application of the Cauchy formula and it is similar to the one given by Rudin for the case of holomorphic self-maps of the unit ball (see [17, p.180]). For the sake of completeness, we sketch it here.

Let $R \geq 1$ and let $K(e_1, R) = \{z \in \mathbb{B}^n : |1 - z_1| \leq \frac{R}{2}(1 - \|z\|^2)\}$ be a Korányi region. Let $R' > R$ and $\delta := \frac{1}{3}(\frac{1}{R} - \frac{1}{R'})$. By [17, Lemma 8.5.5] if $z \in K(e_1, R)$ and

$\lambda \in \mathbb{C}$ is such that $|\lambda| \leq \delta|1-z_1|$ and $u'' \in \mathbb{C}^{n-1}$ is such that $\|u''\| \leq \delta|1-z_1|^{1/2}$ then $(z_1+\lambda, z''+u'') \in K(e_1, R')$.

Now, fix $z \in K(e_1, R)$ and let $r = r(z) := \delta|1-z_1|$. By the Cauchy formula

$$\begin{aligned}\langle dG_z(e_1), e_1\rangle &= \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\langle G(z_1+\zeta, z''), e_1\rangle}{\zeta^2} d\zeta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle G(z_1+re^{i\theta}, z''), e_1\rangle}{z_1+re^{i\theta}-1}\left(1-\frac{1-z_1}{re^{i\theta}}\right) d\theta.\end{aligned}$$

Now, by the choice of r, the points $(z_1+re^{i\theta}, z'') \in K(e_1, R')$, hence by (4.1) there exists a constant $C>0$ (which depends only on R, R') such that

$$\frac{|\langle G(z_1+re^{i\theta}, z''), e_1\rangle|}{|z_1+re^{i\theta}-1|} \leq C.$$

Also, $|1-\frac{1-z_1}{re^{i\theta}}| \leq 1+1/\delta$. Hence, the function $K(e_1, R) \ni z \mapsto \langle dG_z(e_1), e_1\rangle$ is bounded.

(1$''$) We argue as before, but, fixed $z \in K(e_1, R)$, we take $r = r(z) := \delta|1-z_1|^{1/2}$. Hence, for $j = 2, \ldots, n$, we have

$$\begin{aligned}\frac{\langle dG_z(e_j), e_1\rangle}{|1-z_1|^{1/2}} &= \frac{1}{2\pi i|1-z_1|^{1/2}}\int_{|\zeta|=r} \frac{\langle G(z+\zeta e_j), e_1\rangle}{\zeta^2} d\zeta \\ &= \frac{1}{2\pi\delta}\int_0^{2\pi} \frac{\langle G(z+re^{i\theta}e_j), e_1\rangle}{|1-z_1|} e^{-i\theta} d\theta.\end{aligned}$$

By the choice of r, the points $z+re_j \in K(e_1, R')$, $j = 2, \ldots, n$, and we can conclude as before.

(2) Let us consider the slice reduction $g_{e_1}(\zeta) = \langle G(\zeta e_1), e_1\rangle$. By Theorem 3.4 it follows that $\lim_{(0,1)\ni r\to 1} g_{e_1}(r)/(r-1) = \beta$. Since the function $\mathbb{B}^n \ni z \mapsto \langle G(z), e_1\rangle/(z_1-1)$ is bounded in any Korányi region by (4.1), Čirca's theorem [17, Theorem 8.4.8] implies (2).

(3) By Theorem 3.3, we have $\lim_{(0,1)\ni r\to 1} g'_{e_1}(r) = \beta$, that is, $\lim_{(0,1)\ni r\to 1}\langle dG_{re_1}(e_1), e_1\rangle = \beta$. By (1) the map $\mathbb{B}^n \ni z \mapsto \langle dG_z(e_1), e_1\rangle$ is bounded in any Korányi region, and once again (3) follows by Čirca's theorem [17, Theorem 8.4.8]. $\square$

Assuming slightly more regularity at e_1 we can prove the following intermediate result:

Proposition 4.2. *Let G be an infinitesimal generator on $\mathbb{B}^n$. Suppose $\angle\lim_{z\to e_1} G(z) = 0$ and*

$$\mathbb{B}^n \ni z \mapsto \frac{|\langle G(z), e_1\rangle|}{|z_1-1|} \quad \text{is bounded in any Korányi region.} \tag{4.2}$$

Then e_1 is a BRNP for G and 1 is a BRNP for g_v for all $v \in \mathcal{L}_{e_1}$. Moreover, if $\beta \in \mathbb{R}$ denotes the dilation of G at e_1 and β_v denotes the dilation of g_v at 1, then for all $v \in \mathcal{L}_{e_1}$ it follows $\beta_v = \beta$.

Proof. Let $v \in \mathcal{L}_{e_1}$. Let g_v be the slice reduction to v of G. Write $G = (G_1, G'')$. Taking into account that for all $v \in \mathcal{L}_{e_1}$ the curve $(0,1) \ni r \mapsto \varphi_v(r)$ tends to e_1 non-tangentially it follows that $\lim_{(0,1)\ni r\to 1} G(\varphi_v(r)) \to 0$. By Theorem 3.3 and Proposition 4.1

$$\begin{aligned}\beta_v &= \lim_{(0,1)\ni r\to 1} \frac{g_v(r)}{r-1} \\ &= \lim_{(0,1)\ni r\to 1} \frac{\frac{1}{\alpha^2}G_1(\varphi_v(r)) + \frac{1-\alpha^2}{\alpha^2}(r-1)G_1(\varphi_v(r)) - \frac{1}{\alpha}(r-1)\langle G''(\varphi_v(r)), v''\rangle}{r-1} \\ &= \frac{1}{\alpha^2}\lim_{(0,1)\ni r\to 1}\frac{G_1(\varphi_v(r))}{r-1} = \frac{1}{\alpha^2}\lim_{(0,1)\ni r\to 1}\frac{G_1(\varphi_v(r))}{\langle \varphi_v(r), e_1\rangle - 1}\frac{\langle \varphi_v(r), e_1\rangle - 1}{r-1} \\ &= \frac{1}{\alpha^2}\beta\alpha^2 = \beta,\end{aligned}$$

and we are done. □

Proof of Theorem 1.2. The hypothesis (1.2) implies that $\angle\lim_{z\to e_1} G(z) = 0$ and (4.1). Thus, Proposition 4.1 applies and ($1''$), (2) and (3) follow.

($1'$) The boundness of $\langle dG_z(e_1), e_1\rangle$ in any Korányi region follows again from Proposition 4.1. The proof that $\mathbb{B}^n \ni z \mapsto \langle dG_z(e_h), e_k\rangle$ is bounded in any Korányi region for $h, k = 2, \ldots, n$ is similar to the proof of ($1''$) in Proposition 4.1. Thus, we just sketch it here. Let R, R', δ as in the proof of Proposition 4.1. Fix $z \in K(e_1, R)$ and let $r = r(z) := \delta|1-z_1|^{1/2}$. Then for $h, k = 2, \ldots, n$,

$$\begin{aligned}\langle dG_z(e_h), e_k\rangle &= \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{\langle G(z+\zeta e_h), e_k\rangle}{\zeta^2}d\zeta = \frac{1}{2\pi}\int_0^{2\pi}\frac{\langle G(z+re^{i\theta}e_h), e_k\rangle}{r}e^{-i\theta}d\theta \\ &= \frac{1}{2\pi\delta}\int_0^{2\pi}\frac{\langle G(z+re^{i\theta}e_h), e_k\rangle}{|1-z_1|^{1/2}}e^{-i\theta}d\theta.\end{aligned}$$

By the choice of r, the points $z + re_h \in K(e_1, R')$, $h = 2, \ldots, n$. Hence (1.2).(∗∗) guarantees that $z \mapsto \langle dG_z(e_h), e_k\rangle$ is bounded in $K(e_1, R)$.

($1'''$) We retain the notations introduced in the proof of Proposition 4.1. Fix $z \in K(e_1, R)$ and let $r = r(z) := \delta|1 - z_1|$. Then, for $j = 2, \ldots, n$

$$\begin{aligned}|1-z_1|^{1/2}\langle dG_z(e_1), e_j\rangle &= \frac{|1-z_1|^{1/2}}{2\pi i}\int_{|\zeta|=r}\frac{\langle G(z_1+\zeta, z''), e_j\rangle}{\zeta^2}d\zeta \\ &= \frac{1}{2\pi\delta}\int_0^{2\pi}\frac{\langle G(z_1+re^{i\theta}, z''), e_j\rangle}{|1-(z_1+re^{i\theta})|^{1/2}}e^{-i\theta}\left|\frac{1-(z_1+re^{i\theta})}{1-z_1}\right|^{1/2}d\theta.\end{aligned}\tag{4.3}$$

Again, by the choice of r, the points $(z_1 + re^{i\theta}, z'') \in K(e_1, R')$. Since $|1-(z_1+re^{i\theta})|/|1-z_1| \leq 1+\delta$, hypothesis (1.2).(∗∗) guarantees that $z \mapsto |1-z_1|^{1/2}\langle dG_z(e_1), e_j\rangle$ is bounded in $K(e_1, R)$.

(4) Let $v \in \mathcal{L}_{e_1}$. Let g_v be the slice reduction to v of G. Since $\varphi_v'(\zeta) = \alpha v$,

$$\begin{aligned} g_v'(\zeta) = {}& \frac{1}{\alpha}\langle dG_{\varphi_v(\zeta)}(v), e_1\rangle + (\zeta-1)\frac{1-\alpha^2}{\alpha}\langle dG_{\varphi_v(\zeta)}(v), e_1\rangle - (\zeta-1)\langle dG''_{\varphi_v(\zeta)}(v), v''\rangle \\ & + \frac{1-\alpha^2}{\alpha^2}\langle G(\varphi_v(\zeta)), e_1\rangle - \frac{1}{\alpha}\langle G''(\varphi_v(\zeta)), v''\rangle. \end{aligned}$$

By Proposition 4.2 and Theorem 3.3 we have $\lim_{(0,1)\ni r\to 1} g_v'(r) = \beta$.

Taking into account that $(0,1) \ni r \mapsto \varphi_v(r)$ tends to e_1 non-tangentially, we have $\lim_{r\to 1} G(\varphi_v(r)) = 0$ and $r \mapsto \langle dG_{\varphi_v(r)}(v), e_1\rangle$ is bounded by $(1^{'})$ and $(1'')$. Moreover, by $(1^{'})$ and $(1^{'''})$ it follows that for $v = \alpha e_1 + \sum_{j=2}^n v_j e_j \in \mathcal{L}_{e_1}$

$$\begin{aligned} \lim_{r\to 1}(r-1)\langle dG''_{\varphi_v(r)}(v), v''\rangle &= \lim_{r\to 1}(r-1)\left(\sum_{j=2}^n \alpha\overline{v_j}\langle dG_{\varphi_v(r)}(e_1), e_j\rangle\right. \\ &\qquad \left. + \sum_{h,k=2}^n v_h\overline{v_k}\langle dG_{\varphi_v(r)}(e_h), e_k\rangle\right) = \lim_{r\to 1}\sum_{j=2}^n \alpha\overline{v_j}(r-1)\langle dG_{\varphi_v(r)}(e_1), e_j\rangle \\ &= \sum_{j=2}^n \alpha\overline{v_j}\lim_{r\to 1}\frac{(r-1)}{(1-\langle\varphi_v(r), e_1\rangle)^{1/2}}(1-\langle\varphi_v(r), e_1\rangle)^{1/2}\langle dG_{\varphi_v(r)}(e_1), e_j\rangle \\ &= -\sum_{j=2}^n \frac{\overline{v_j}}{\alpha}\lim_{r\to 1}(r-1)^{1/2}(1-\langle\varphi_v(r), e_1\rangle)^{1/2}\langle dG_{\varphi_v(r)}(e_1), e_j\rangle = 0. \end{aligned}$$

Therefore,

$$\beta = \lim_{(0,1)\ni r\to 1} g_v'(r) = \frac{1}{\alpha}\lim_{(0,1)\ni r\to 1}\langle dG_{\varphi_v(r)}(v), e_1\rangle. \tag{4.4}$$

Expanding (4.4), and taking into account (3), we have

$$\begin{aligned} \beta &= \frac{1}{\alpha}\lim_{(0,1)\ni r\to 1}\left(\langle dG_{\varphi_v(r)}(\alpha e_1), e_1\rangle + \sum_{j=2}^n v_j\langle dG_{\varphi_v(r)}(e_j), e_1\rangle\right) \\ &= \beta + \frac{1}{\alpha}\lim_{(0,1)\ni r\to 1}\left(\sum_{j=2}^n v_j\langle dG_{\varphi_v(r)}(e_j), e_1\rangle\right), \end{aligned}$$

from which it follows that, for all choices of v

$$\lim_{(0,1)\ni r\to 1}\left(\sum_{j=2}^n v_j\langle dG_{\varphi_v(r)}(e_j), e_1\rangle\right) = 0.$$

For the arbitrariness of v we have

$$\lim_{(0,1)\ni r\to 1}\langle dG_{\varphi_v(r)}(e_j), e_1\rangle = 0 \quad j = 2, \ldots, n.$$

Since the function $\mathbb{B}^n \ni z \mapsto \langle dG_z(e_j), e_1\rangle$ is bounded in every Korányi region and has limit 0 along a non-tangential curve, by Čirca's theorem [17, Theorem 8.4.8], it has restricted K-limit 0, and this proves (4). □

Example 4.3. (cfr. [7, Example 4.2]). Let $G(z_1, z_2) := (0, -z_2/(1-z_1))$. Then G is an infinitesimal generator in $\mathbb{B}^n$ with BRNP e_1 and dilation $\beta = 0$. Note that $|\langle G(z), e_1\rangle| \equiv 0$, and Proposition 4.1 applies. However, $\angle \lim_{z\to e_1} G(z)$ does not exist. In fact, dG_z is not bounded in any Korányi region: a direct computation shows that

$$dG_z = \begin{pmatrix} 0 & 0 \\ -\frac{z_2}{(1-z_1)^2} & -\frac{1}{1-z_1} \end{pmatrix}.$$

Moreover, given $v = (\alpha, v_2) \in \mathcal{L}_{e_1}$ it is easy to see that

$$g_v(\zeta) = (1 - \frac{1}{\alpha^2})(\zeta - 1),$$

hence, the dilation β_v of g_v at 1 is $1 - 1/\alpha^2$. Thus, $\beta_v < 0 = \beta$ for all $v \in \mathcal{L}_{e_1} \setminus \{e_1\}$, and $\beta_{e_1} = 0$.

4.1. **(Dis)similarities between the Julia-Wolff-Carathéodory theorems for maps and for infinitesimal generators and open questions.** Hypothesis (1.2) is stronger than the corresponding starting hypothesis in Rudin's theorem, which involves only finiteness of the liminf defining $\alpha_f(e_1)$. In fact, part of the work in proving Rudin's theorem is devoted to show that such a condition, via Julia's lemma, implies boundness of suitable functions in any Korányi region. Julia's lemmas for infinitesimal generators (see Theorem 3.3) are however – and, in a certain sense, very naturally – weaker than those for self-mappings and this forced us to use such a stronger hypothesis. We do not know whether there exists any weaker condition in terms of liminf of some function of G which assures (and it is equivalent to) hypothesis (1.2).

It would be also interesting to find an example (if any) of an infinitesimal generator satisfying the hypothesis of Proposition 4.2 but not hypothesis (1.2).

Moreover, with our techniques, we are not able to prove (or disprove) for infinitesimal generators the statements corresponding to (5) and (6) of Theorem 1.1. Namely, under the hypothesis of Theorem 1.2, we do not know whether for $j = 2, \ldots, n$ it holds

$$\angle_K \lim_{z\to e_1} \frac{\langle G(z), e_j\rangle}{(1-z_1)^{1/2}} = 0, \quad \angle_K \lim_{z\to e_1} (1-z_1)^{1/2}\langle dG_z(e_1), e_j\rangle = 0. \tag{4.5}$$

By Čirca's theorem [17, Theorem 8.4.8] and Theorem 1.2.($1'''$) and using (4.3), these results hold if one can prove that for $j = 2, \ldots, n$

$$\lim_{(0,1)\ni r\to 1} \frac{\langle G(re_1), e_j\rangle}{(1-r)^{1/2}} = 0 \tag{4.6}$$

In the case of Rudin's theorem, the corresponding radial limit is proven using Julia's lemma and the strong constrain of sending the ball into itself.

Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a holomorphic self-map having a BRFP at e_1. Then $G(z) := f(z) - z$ is an infinitesimal generator (see [18, Corollary 3.3.1] and [16]) and, using Theorem 1.1, it is not hard to see that G satisfies (1.2) at e_1. Again by Theorem 1.1 it is easy to see that G satisfies (4.6), and hence (4.5). Therefore, for the dense subclass of infinitesimal generators of the form $f(z) - z$ with $f : \mathbb{B}^n \to \mathbb{B}^n$ holomorphic, the full analogue of Rudin's theorem holds.

Thus, it is reasonable to believe that even in the general case (4.6) holds and should follow from Julia's lemma for infinitesimal generators and the condition of being an infinitesimal generators. However, we are not able to prove the result.

5. Higher order jets of generators at BRNPs

Let G be an infinitesimal generator on $\mathbb{B}^n$ having a boundary regular null point (BRNP) at e_1 and assume G is C^3 at e_1. We can expand G in the form

$$G(z) = T(z - e_1) + Q_2(z - e_1) + Q_3(z - e_1) + o(|z - e_1|^3), \tag{5.1}$$

where Q_j is a n-tuple of homogeneous polynomial of degree j for $j = 2, 3$. Then by Theorem 1.2 we can write,

$$T = \begin{pmatrix} \beta & 0 & \dots & 0 \\ t_2 & s_{22} & \dots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ t_n & s_{n2} & \dots & s_{nn} \end{pmatrix} \tag{5.2}$$

where $\beta \in \mathbb{R}$ is the dilation of G at e_1 and $t_j, s_{jk} \in \mathbb{C}$. We set $S = (s_{jk})_{j,k=2,\dots,n}$.

Also, we write $(x, y) \in \mathbb{C} \times \mathbb{C}^{n-1}$ with $y = (y_2, \dots, y_n)$ and use multi-indices notations. Namely, $y^J = y_2^{j_2} \cdots y_n^{j_n}$, if $J = (j_2, \dots, j_n)$; for a multi-index $I = (i_1, \dots, i_n)$ we let $|I| = \sum_{j=1}^n i_j$.

We let

$$Q_2(x, y) = \left(\sum_{I=(i_1,J)\in\mathbb{N}^n, |I|=2} q^1_{i_1,J} x^{i_1} y^J, \dots, \sum_{I=(i_1,J)\in\mathbb{N}^n, |I|=2} q^n_{i_1,J} x^{i_1} y^J \right), \tag{5.3}$$

for some $q^k_{i_1,J} \in \mathbb{C}$.

Now we characterize boundary jets of infinitesimal generators:

Proposition 5.1. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be an infinitesimal generator of class C^3 at e_1. Assume that e_1 is a BRNP with dilation $\beta = 0$. Let* (5.1) *be the expansion of G at e_1, with T given by* (5.2) *and Q_2 given by* (5.3). *Then $\mathsf{Re}\, q^1_{2,0} \geq 0$ and $\mathsf{Re}\, s_{kk} \leq -|q^1_{0,e_{2k}}|$ for all $k = 2, \dots, n$.*

Moreover, $\mathsf{Re}\, g''_v(1) = 0$ for all $v \in \mathcal{L}_{e_1}$ if and only if

$$\mathsf{Re}\, q^1_{2,0} = \mathsf{Re}\, s_{kk} = 0, \quad k = 2, \dots, n, \tag{5.4}$$

and, $g''_v(1) = 0$ for all $v \in \mathcal{L}_{e_1}$ if and only if $q^1_{2,0} = s_{kk} = 0, \quad k = 2, \dots, n.$

If (5.4) *holds, then*

- S *is anti-Hermitian,*
- $q^1_{0,J} = 0$ *for all* $J \in \mathbb{N}^n$, $|J| = 2$,
- $q^1_{1,e_k} = q^h_{0,e_k+e_h}$ *for* $h, k = 2, \ldots, n$,
- $q^h_{0,e_k+e_l} = 0$ *for all* $h, k, l = 2, \ldots, n$ *with* $h \neq k, l$,
- $\mathsf{Im}\, q^k_{1,e_k} = \mathsf{Im}\, q^1_{2,0} = 0$ *and* $\mathsf{Re}\, q^k_{1,e_k} \geq 0$ *for* $k = 2, \ldots, n$,
- *the matrix* $\tilde{Q} := (q^k_{1,e_h})_{h,k=2,\ldots,n}$ *is Hermitian and positive semi-definite.*

Moreover, there exists $\delta \leq 0$ *such that*

$$Q^1_3(x, y) = \delta x^3 + \sum_{j=2}^{n} (q^1_{1,e_j} + \overline{q}^j_{2,0}) x^2 y_j. \tag{5.5}$$

Finally, if $\mathsf{Re}\, g''_v(1) = 0$ *for all* $v \in \mathcal{L}_{e_1}$ *then* $g'''_v(1) = 0$ *for all* $v \in \mathcal{L}_{e_1}$ *if and only if* $\mathsf{Re}\, q^k_{1,e_k} = 0$, $q^h_{1,e_k} = 0$ *for* $2 \leq k < h \leq n$ *and* $\delta = 0$.

Proof. Let $v \in \mathcal{L}_{e_1}$, and let $g_v : \mathbb{D} \to \mathbb{C}$ be the slice reduction of G with respect to v. Let G be given by (5.1), with $Tv = (T^1 v, T'' v) \in \mathbb{C} \times \mathbb{C}^{n-1}$, $Q_2(v) = (Q^1_2(v), Q''_2(v)) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and $Q_3(v) = (Q^1_3(v), Q''_3(v)) \in \mathbb{C} \times \mathbb{C}^{n-1}$. By Theorem 1.2,

$$T^1 v = \langle Tv, e_1 \rangle = \langle dG_{e_1}(v), e_1 \rangle = \alpha\beta = 0.$$

Thus, a direct computation from (3.4) shows that

$$g_v(\zeta) = a_v(\zeta - 1)^2 + b_v(\zeta - 1)^3 + o(|\zeta - 1|^3),$$

with

$$\begin{aligned} a_v &= Q^1_2(v) - \langle T'' v, v'' \rangle, \\ b_v &= (1 - \alpha^2) Q^1_2(v) + \alpha Q^1_3(v) - \alpha \langle Q''_2(v), v'' \rangle. \end{aligned} \tag{5.6}$$

Now, $g_v(\zeta) = a_v(\zeta - 1)^2 + b_v(\zeta - 1)^3 + o(|\zeta - 1|^3)$ is an infinitesimal generator in the unit disc and thus, by Berkson-Porta formula, it has to hold $\mathsf{Re}\,(a_v + b_v(\zeta - 1) + o(|\zeta - 1|)) \geq 0$. Therefore, in particular, $\mathsf{Re}\, a_v \geq 0$ for all $v \in \mathcal{L}_{e_1}$ (see also, [19]).

By writing down explicitly the condition $\mathsf{Re}\, a_v \geq 0$, we find that for all $v = (\alpha, v_2, \ldots, v_n)$ with $\alpha \in (0, 1]$, $\sum_{j=2}^n |v_j|^2 = 1 - \alpha^2$,

$$\begin{aligned} &\sum_{2 \leq j \leq k \leq n} \mathsf{Re}\,(q^1_{0,e_j+e_k} v_j v_k) - \sum_{j,k=2}^{n} \mathsf{Re}\,(s_{kj} v_j \overline{v}_k) \\ &+ \alpha \left[\sum_{j=2}^{n} \mathsf{Re}\,(q^1_{1,e_j} v_j) - \sum_{k=2}^{n} \mathsf{Re}\,(t_k \overline{v}_k) \right] + \alpha^2 \mathsf{Re}\, q^1_{2,0} \geq 0. \end{aligned} \tag{5.7}$$

For $\alpha = 1, v'' = 0$, we find $\mathsf{Re}\, q^1_{2,0} \geq 0$.

When $\alpha \to 0$, the previous inequality implies that the term of degree 0 in α has to have real part ≥ 0, namely

$$\sum_{2\leq j\leq k\leq n} \mathsf{Re}\,(q^1_{0,e_j+e_k} v_j v_k) - \sum_{j,k=2}^{n} \mathsf{Re}\,(s_{kj} v_j \overline{v}_k) \geq 0 \tag{5.8}$$

for $\|v''\| = 1$. Now, fix $k \in \{2, \ldots, n\}$ and substitute v'' with $e^{i\theta_k} e_k$ for $\theta_k \in [0, 2\pi]$. We obtain $\mathsf{Re}\,(q^1_{0,2e_k} e^{2i\theta_k}) - \mathsf{Re}\, s_{kk} \geq 0$, which, for the arbitrariness of θ_k, implies $\mathsf{Re}\, s_{kk} \leq -|q^1_{0,2e_k}|$ for $k = 2, \ldots, n$.

Now, $\mathsf{Re}\, g''_v(1) = 0$ if and only if $\mathsf{Re}\, a_v = 0$ for all $v \in \mathcal{L}_{e_1}$. Therefore, it is clear from the previous considerations that if $\mathsf{Re}\, g''_v(1) = 0$ for all $v \in \mathcal{L}_{e_1}$, then necessarily $\mathsf{Re}\, q^1_{2,0} = 0$. Moreover, the left hand side of (5.8) is equal to 0. Therefore, fixing $k \in \{2, \ldots, n\}$ and substituting v'' with $e^{i\theta_k} e_k$ for $\theta_k \in [0, 2\pi]$ we obtain $\mathsf{Re}\,(q^1_{0,2e_k} e^{2i\theta_k}) - \mathsf{Re}\, s_{kk} = 0$. Integrating with respect to θ_k in $[0, 2\pi]$ the harmonic term vanishes and we obtain $\mathsf{Re}\, s_{kk} = 0$ for $k = 2, \ldots, n$.

Assume that $\mathsf{Re}\, s_{kk} = 0$ for $k = 2, \ldots, n$. Therefore the non-harmonic part in (5.8) is zero, and we claim that this implies that (5.8) is, in fact, identically 0. Indeed, we rewrite (5.8) as

$$\sum_{2\leq j\leq k\leq n} \mathsf{Re}\,(q^1_{0,e_j+e_k} v_j v_k) - \sum_{2\leq j<k\leq n} \mathsf{Re}\,[(s_{kj} + \overline{s}_{jk}) v_j \overline{v}_k] \geq 0.$$

Taking $|v_k| = 1, v_j = 0$ for $j \neq k$, we find immediately $q^1_{0,2e_k} = 0$, $k = 2, \ldots, n$. Next, we take $v_2 = \zeta$, $v_3 = \pm\zeta$ with $|\zeta| = 1/\sqrt{2}$ and $v_4 = \ldots = v_n = 0$ and we obtain

$$\pm\left(\mathsf{Re}\,(q^1_{0,e_2+e_3}\zeta^2) - \mathsf{Re}\,(s_{32} + \overline{s}_{23})|\zeta|^2\right) \geq 0.$$

Decoupling the harmonic and non-harmonic terms by integrating as before, we obtain

$$\mathsf{Re}\,(s_{32} + \overline{s}_{23}) = 0, \quad q^1_{0,e_2+e_3} = 0.$$

Finally, taking $v_2 = \zeta$, $v_3 = e^{i\theta}\zeta$ with $|\zeta| = 1/\sqrt{2}$, $\theta \in [0, 2\pi]$ and $v_4 = \ldots = v_n = 0$ we obtain

$$-|\zeta|^2 \mathsf{Re}\,[(s_{32} + \overline{s}_{23})e^{-i\theta}] \geq 0$$

which implies $s_{32} + \overline{s}_{23} = 0$. A similar argument works for the other indices. This proves that S is anti-Hermitian and $q^1_{0,J} = 0$ for all $|J| = 2$. Moreover, this proves that the terms of degree 0 in α in (5.7) are identically zero. Therefore, the condition $\mathsf{Re}\, s_{kk} = 0$ for all k is sufficient for (5.7) in degree zero in α to be equal to zero for all v when $\alpha \to 0$.

Now, since the terms of degree 0 in α in (5.7) are vanishing identically, the terms of degree 1 in α has to have non negative real part as $\alpha \to 0$, that is

$$\sum_{k=2}^{n} \mathsf{Re}\,[(q^1_{1,e_k} - \overline{t}_k) v_k] \geq 0, \tag{5.9}$$

which clearly implies $q^1_{1,e_k} = \bar{t}_k$ for $k = 2, \ldots, n$. Hence, if $\mathsf{Re}\, s_{kk} = \mathsf{Re}\, q^1_{2,0} = 0$ then $\mathsf{Re}\, g''_v(1) = 0$ for all $v \in \mathcal{L}_{e_1}$.

From the previous considerations it follows easily that $g''_v(1) = 0$ for all $v \in \mathcal{L}_{e_1}$ if and only if $q^1_{2,0} = s_{kk} = 0$ for $k = 2, \ldots, n$.

Now, assume (5.4). Then for all $v \in \mathcal{L}_{e_1}$ we have $\mathsf{Re}\, a_v = 0$ namely, $g_v(\zeta) = (\zeta - 1)^2[i\tilde{a}_v + b_v(\zeta - 1) + o(|\zeta - 1|)]$ for all $v \in \mathcal{L}_{e_1}$, where $\tilde{a}_v \in \mathbb{R}$. Berkson-Porta's formula (see [19]) implies then $b_v \in \mathbb{R}$ and $b_v \leq 0$.

Taking into account what we have already proved, writing $Q^1_3(v) = \sum_{|(i_1,J)|=3} p^1_{i_1,J}\alpha^{i_1}v^J$, where we used the multi-indices notation $v^J = v_2^{j_2}\cdots v_n^{j_n}$, from (5.6), the condition $b_v \leq 0$ becomes

$$
\begin{aligned}
&(1-\alpha^2)\left(q^1_{2,0}\alpha^2 + \alpha\sum_{k=2}^n q^1_{1,e_k}v_k\right) + \alpha\sum_{|(i_1,J)|=3} p^1_{i_1,J}\alpha^{i_1}v^J \\
&- \alpha\sum_{k=2}^n\sum_{|(i_1,J)|=2} q^k_{i_1,J}\alpha^{i_1}v^J\overline{v}_k \leq 0.
\end{aligned}
\tag{5.10}
$$

For $\alpha = 1, v'' = 0$, we obtain $p^1_{3,0} \leq 0$. And, if $g'''_v(1) = 0$ for all v, that is $b_v = 0$, then $p^1_{3,0} = 0$.

Now, as before, we start looking at terms of smallest degree in α when $\alpha \to 0$. Since there are no terms of degree 0 in α, the smallest degree is 1, and we get

$$
\sum_{k=2}^n q^1_{1,e_k}v_k + \sum_{|I|=3} p^1_{0,I}v^I - \sum_{k=2}^n\sum_{|J|=2} q^k_{0,J}v^J\overline{v}_k \leq 0, \tag{5.11}
$$

for all $v \in \mathbb{C}^{n-1}$ with $\|v\| = 1$. Replacing v by $-v$ the left-hand side of (5.11) changes sign. Therefore we deduce that

$$
\sum_{k=2}^n q^1_{1,e_k}v_k + \sum_{|I|=3} p^1_{0,I}v^I - \sum_{k=2}^n\sum_{|J|=2} q^k_{0,J}v^J\overline{v}_k = 0, \tag{5.12}
$$

for all $v \in \mathbb{C}^{n-1}$ with $\|v\| = 1$. Replacing v by $e^{i\theta}v$ for $\theta \in [0, 2\pi]$ in (5.12), dividing the equation by $e^{i\theta}$ and integrating with respect to θ in $[0, 2\pi]$ the harmonic terms vanish and we obtain

$$
\sum_{k=2}^n q^1_{1,e_k}v_k - \sum_{k=2}^n\sum_{|J|=2} q^k_{0,J}v^J\overline{v}_k = 0. \tag{5.13}
$$

Equation (5.12) implies then $\sum_{|I|=3} p^1_{0,I}v^I = 0$, which is possible only if $p^1_{0,I} = 0$ for all $|I| = 3$. Taking $v_k = e_k$ for $k = 2, \ldots, n$ in (5.13) we obtain $q^1_{1,e_k} = q^k_{0,2e_k}$ for $k = 2, \ldots, n$.

Now, let $2 \leq k_1 < k_2 \leq n$ and let $v = \frac{1}{\sqrt{2}}(e^{i\theta_1}e_{k_1} + e^{i\theta_2}e_{k_2})$ for $\theta_1, \theta_2 \in [0, 2\pi]$. Expanding (5.13) with such a choice of v, multiplying by $2\sqrt{2}$ and taking into account that $q^1_{1,e_k} =$

$q^k_{0,2e_k}$ for $k = 2, \dots, n$, we obtain

$$\begin{aligned}0 =& e^{i\theta_1}(2q^1_{1,e_{k_1}} - q^{k_1}_{0,2e_{k_1}} - q^{k_2}_{0,e_{k_1}+e_{k_2}}) + e^{i\theta_2}(2q^1_{1,e_{k_2}} - q^{k_2}_{0,2e_{k_2}} - q^{k_1}_{0,e_{k_1}+e_{k_2}})\\ &- e^{i(2\theta_2-\theta_1)}q^{k_1}_{0,2e_{k_2}} - e^{i(2\theta_1-\theta_2)}q^{k_2}_{0,2e_{k_1}}\\ &= e^{i\theta_1}(q^1_{1,e_{k_1}} - q^{k_2}_{0,e_{k_1}+e_{k_2}}) + e^{i\theta_2}(q^1_{1,e_{k_2}} - q^{k_1}_{0,e_{k_1}+e_{k_2}}) - e^{i(2\theta_2-\theta_1)}q^{k_1}_{0,2e_{k_2}} - e^{i(2\theta_1-\theta_2)}q^{k_2}_{0,2e_{k_1}},\end{aligned}$$

from which we deduce that $q^1_{1,e_k} = q^h_{0,e_k+e_h}$ and $q^k_{0,2e_h} = 0$ for $k \neq h \in \{2, \dots, n\}$.

Finally, let $2 \leq k_1 < k_2 < k_3 \leq n$ and consider $v = \frac{1}{\sqrt{3}}(e^{i\theta_1}e_{k_1} + e^{i\theta_2}e_{k_2} + e^{i\theta_3}e_{k_3}$ for $\theta_1, \theta_2, \theta_3 \in [0, 2\pi]$. Expanding (5.13) with such a choice of v, multiplying by $3\sqrt{3}$, we obtain

$$\begin{aligned}0 =& e^{i\theta_1}(3q^1_{1,e_{k_1}} - q^{k_1}_{0,2e_{k_1}} - q^{k_2}_{0,e_{k_1}+e_{k_2}} - q^{k_3}_{0,e_{k_1}+e_{k_3}}) + e^{i\theta_2}(3q^1_{1,e_{k_2}} - q^{k_2}_{0,2e_{k_2}} - q^{k_1}_{0,e_{k_1}+e_{k_2}} - q^{k_3}_{0,e_{k_2}+e_{k_3}})\\ &+ e^{i\theta_3}(3q^1_{1,e_{k_3}} - q^{k_3}_{0,2e_{k_3}} - q^{k_1}_{0,e_{k_1}+e_{k_3}} - q^{k_2}_{0,e_{k_2}+e_{k_3}}) - e^{i(2\theta_2-\theta_1)}q^{k_1}_{0,2e_{k_2}} - e^{i(2\theta_3-\theta_1)}q^{k_1}_{0,2e_{k_3}}\\ &- e^{i(2\theta_1-\theta_2)}q^{k_2}_{0,2e_{k_1}} - e^{i(2\theta_3-\theta_2)}q^{k_2}_{0,2e_{k_3}} - e^{i(2\theta_1-\theta_3)}q^{k_3}_{0,2e_{k_1}} - e^{i(2\theta_2-\theta_3)}q^{k_3}_{0,2e_{k_2}}\\ &- e^{i(\theta_1+\theta_3-\theta_2)}q^{k_2}_{0,e_{k_1}+e_{k_3}} - e^{i(\theta_2+\theta_3-\theta_1)}q^{k_1}_{0,e_{k_2}+e_{k_3}} - e^{i(\theta_1+\theta_2-\theta_3)}q^{k_3}_{0,e_{k_1}+e_{k_2}} = 0.\end{aligned}$$

The first three lines of the previous equation do not give any new information, but the last one implies that $q^k_{0,e_h+e_l} = 0$ for $k, h, l = 2, \dots, n$ and $k \neq h, l$.

Therefore, the term of degree 1 in α, for $\alpha \to 0$ in (5.10) identically vanishes. So we look at terms of degree 2 in α as $\alpha \to 0$. We have

$$q^1_{2,0} + \sum_{|I|=2} p^1_{1,I} v^I - \sum_{j,k=2}^{n} q^k_{1,e_j} v_j \overline{v}_k \leq 0, \tag{5.14}$$

for all $v \in \mathbb{C}^{n-1}$ with $\|v\| = 1$. Replacing v_j with $e^{i\theta_j}v_j$ for $\theta_j \in [0, 2\pi]$ and integrating, we get rid of the harmonic terms and we find

$$q^1_{2,0} - \sum_{k=2}^{n} q^k_{1,e_k} |v_k|^2 \leq 0.$$

Taking $v = e_k$ we obtain $q^1_{2,0} - q^k_{1,e_k} \leq 0$. Since $\mathsf{Re}\, q^1_{2,0} = 0$, this implies $\mathsf{Im}\, q^k_{1,e_k} = \mathsf{Im}\, q^1_{2,0}$ and $\mathsf{Re}\, q^k_{1,e_k} \geq 0$ for $k = 2, \dots, n$.

Now, the harmonic part in (5.14) must be real, that is

$$\sum_{2\leq j\leq l\leq n} p^1_{1,e_j+e_l} v_j v_l - \sum_{k=2}^{n} \sum_{j=2, j\neq k}^{n} q^k_{1,e_j} v_j \overline{v}_k \in \mathbb{R}.$$

Taking $v = e_k$, this immediately implies $p^1_{1,2e_k} = 0$. Taking $v = \frac{1}{\sqrt{2}}e^{i\theta}(e_{k_1} + e_{k_2})$ with $2 \leq k_1 < k_2 \leq n$ and $\theta \in [0, 2\pi]$, we obtain

$$e^{2i\theta} p^1_{1,e_{k_1}+e_{k_2}} - (q^{k_1}_{1,e_{k_2}} + q^{k_2}_{1,e_{k_1}}) \in \mathbb{R},$$

which implies $p^1_{1,e_j+e_k}=0$ for $j,k=2,\ldots,n$, $j\neq k$. Next, setting $v=\frac{1}{\sqrt{2}}(e_{k_1}+e^{i\theta}e_{k_2})$ with $\theta\in[0,2\pi]$, $2\leq k_1<k_2\leq n$, we obtain $(e^{-i\theta}q^{k_1}_{1,e_{k_2}}+e^{i\theta}q^{k_2}_{1,e_{k_1}})\in\mathbb{R}$, that is

$$\mathsf{Im}\,[e^{-i\theta}(q^{k_1}_{1,e_{k_2}}-\overline{q}^{k_1}_{1,e_{k_2}})]=0,$$

hence $q^{k_1}_{1,e_{k_2}}-\overline{q}^{k_2}_{1,e_{k_1}}=0$. This, together with (5.14) implies that the matrix $\tilde{Q}=(q^h_{1,e_k})_{h,k=2,\ldots,n}$ is Hermitian and positive semi-definite.

From the previous considerations, we also note that $b_v=0$ implies $\mathsf{Re}\,q^k_{1,e_k}=0$ and $q^h_{1,e_k}=0$ for $2\leq k<h\leq n$, while, this latter condition implies that the term of order 2 in α as $\alpha\to 0$ in (5.10) vanishes identically.

Now, we are left to impose the condition that the remaining terms give a non-positive real number. Looking at terms of degree 3 in α in (5.10), we have

$$0=\mathsf{Im}\left[\sum_{j=2}^n(-q^1_{1,e_j}v_j+p^1_{2,e_j}v_j-q^j_{2,0}\overline{v}_j)\right]=\sum_{j=2}^n\mathsf{Im}\,[(-q^1_{1,e_j}+p^1_{2,e_j}-\overline{q}^j_{2,0})v_j],$$

for all $v\in\mathbb{C}^{n-1}$ with $\|v\|=1$, which clearly implies $-q^1_{1,e_j}+p^1_{2,e_j}-\overline{q}^j_{2,0}=0$ for all $j=2,\ldots,n$. In particular, the term of degree 3 in α always vanishes identically.

Finally, we impose the condition that the terms of degree 4 in α in (5.10) are real. This means

$$\mathsf{Im}\,\left[-q^1_{2,0}+p^1_{3,0}\right]=0.$$

Taking into account that we already proved that $p^1_{3,0}\leq 0$, this implies that $\mathsf{Im}\,q^1_{2,0}=0$. Note also that if the terms of degree $0,1,2,3$ in (5.10) vanish, which implies in particular that $q^1_{2,0}=0$, then the terms of order 4 are vanishing if and only if $p^1_{3,0}=0$.

Summing up, if $\mathsf{Re}\,g''_v(1)=0$ all $v\in\mathcal{L}_{e_1}$ then $g'''_v(1)=0$ for all $v\in\mathcal{L}_{e_1}$ if and only if the terms of degree 2 and 4 in α in (5.10) are identically vanishing (because those of degree 1 and 3 always do). This, in turn, is equivalent to $\mathsf{Re}\,q^k_{1,e_k}=0$, $q^h_{1,e_k}=0$ for $2\leq k<h\leq n$ and $p^1_{3,0}=0$. And this ends the proof. □

Proposition 5.2. *Let G be an infinitesimal generator on $\mathbb{B}^n$, C^3 at e_1. Assume that e_1 is a BRNP with dilation 0. Then G generates a group of automorphisms if and only if the following conditions are satisfied:*

(1) $\mathsf{Re}\,\langle\frac{\partial G}{\partial z_k}(e_1),e_k\rangle=0$, *for $k=2,\ldots,n$,*
(2) $\mathsf{Re}\,\langle\frac{\partial^2 G}{\partial z_1\partial z_k}(e_1),e_1\rangle=0$, *for $k=1,\ldots,n$,*
(3) $\langle\frac{\partial^2 G}{\partial z_1\partial z_k}(e_1),e_h\rangle=0$ *for $2\leq k<h\leq n$,*
(4) $\mathsf{Re}\,\langle\frac{\partial^3 G}{\partial z_1^3}(e_1),e_1\rangle=0$.

Moreover, if the previous conditions are satisfied, then $G\equiv 0$ if and only if $\langle\frac{\partial G}{\partial z_k}(e_1),e_k\rangle=0$ for $k=2,\ldots,n$ and $\langle\frac{\partial^2 G}{\partial z_1^2}(e_1),e_1\rangle=0$.

Proof. By Proposition 5.1, the hypotheses are equivalent to the fact that for all $v \in \mathcal{L}_{e_1}$ the slice retraction g_v has the property that $g_v'(1) = \mathsf{Re}\, g_v''(1) = g_v'''(1) = 0$. By [19, Corollary 4] this is equivalent to the fact that g_v is a generator of a group of automorphisms of $\mathbb{D}$ for all $v \in \mathcal{L}_{e_1}$ (and $g_v \equiv 0$ if and only if $g_v''(1) = 0$). Then the statement follows from Proposition 3.9. □

Lemma 5.3. *Let $G : \mathbb{B}^n \to \mathbb{C}^n$ be an infinitesimal generator, C^3 at e_1 and with BRNP at e_1 and dilation $\beta \in \mathbb{R} \setminus \{0\}$. Let H_β be given by* (3.5) *and let $\tilde{G} := G + H_\beta$. Then $\tilde{G}$ has a BRNP at e_1 with dilation* 0. *Moreover, denoting by $\tilde{s}_{jk}$, $\tilde{q}^j_{i_1,J}$ the elements in the expansion of $\tilde{G}$ at e_1, we have $\tilde{s}_{kk} = s_{kk} - \frac{\beta}{2}$, $\tilde{q}^1_{2,0} = q^1_{2,0} - \frac{\beta}{2}$, $\tilde{q}^k_{1,e_k} = q^k_{1,e_k} - \frac{\beta}{2}$, $k = 2, \dots, n$ and $\tilde{q}^h_{1,e_k} = q^h_{1,e_k}$ for $2 \le k < h \le n$.*

Proof. By Corollary 3.7, the vector field $G + H_\beta$ is an infinitesimal generator with BRNP at 1 and dilation $= 0$. Now,

$$
\begin{aligned}
H_\beta(z) = &\left(-\beta(z_1 - 1), -\frac{\beta}{2}z_2, \dots, -\frac{\beta}{2}z_n\right) \\
&+ \left(-\frac{\beta}{2}(z_1 - 1)^2, -\frac{\beta}{2}(z_1 - 1)z_2, \dots, -\frac{\beta}{2}(z_1 - 1)z_n\right).
\end{aligned}
$$

From this the statements follow easily. □

Proof of Theorem 1.3. Let $\tilde{G} := G + H_\beta$. Thanks to Lemma 5.3, the hypotheses on G implies that $\tilde{G}$ satisfies the hypothesis of Proposition 5.2, and the result follows. □

6. On the quadratic expansion at BRNPs

In [19] it is shown that if $g : \mathbb{D} \to \mathbb{C}$ is an infinitesimal generator in $\mathbb{D}$ which is $C^3(1)$ with expansion $g(z) = z - 1 + a(z-1)^2 + o(|z-1|^2)$ then the quadratic part $z \mapsto z - 1 + a(z-1)^2$ is always an infinitesimal generator in $\mathbb{D}$ which generates a semigroup of linear fractional self-maps of the unit disc.

In higher dimension the same result is false, and, even when the quadratic part is an infinitesimal generator, it might not generate a semigroup of linear fractional maps. The underlying reason is that slice reductions at a BRNP of an infinitesimal generator do not preserve the degree of expansion at the boundary (cfr. (5.6)), so that the quadratic part of the infinitesimal generator in $\mathbb{B}^n$ might generate a cubic term on some slice reduction. We present the following examples.

Example 6.1. Let $F : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$
F(z_1, z_2) = -\left(z_1 - 1, \frac{5z_2}{4(2 - z_2)}\right).
$$

We claim that F is an infinitesimal generator. Indeed, for each $z \in \partial\mathbb{B}^2$ we have

$$\langle -F(z), z\rangle = |z_1|^2 - \overline{z_1} + \frac{5|z_2|^2}{4(2-z_2)} = (1-\overline{z_1}) + |z_2|^2\left(\frac{5}{4(2-z_2)} - 1\right).$$

Hence

$$\begin{aligned}\mathsf{Re}\,\langle -F(z), z\rangle &\geq 1 - \sqrt{1-|z_2|^2} + |z_2|^2\mathsf{Re}\left(\frac{5}{4(2-z_2)} - 1\right)\\ &\geq |z_2|^2\left(\frac{1}{1+\sqrt{1-|z_2|^2}} + \frac{5}{4(2+|z_2|)} - 1\right)\\ &= r^2\,\frac{5-3\sqrt{1-r^2}-4r\sqrt{1-r^2}}{4(2+r)(1+\sqrt{1-r^2})},\end{aligned} \tag{6.1}$$

where $r = |z_2|$.

A standard computation shows that the expression $5-3\sqrt{1-r^2}-4r\sqrt{1-r^2}$ is positive on the segment $[0,1]$. Therefore, one concludes that $\mathsf{Re}\,\langle F(z), z\rangle < 0$ for all $z \in \partial\mathbb{B}^2$. Taking into account that F is holomorphic past the boundary of $\mathbb{B}^2$, one can apply Theorem 3.3.(3) with $\beta = 0$, and Proposition 3.8 to see that F is an infinitesimal generator on $\mathbb{B}^2$ (see also [16, Corollary 7.1]).

On the other hand, denote by $\tilde{F}$ the quadratic expansion of F at e_1, namely,

$$\tilde{F}(z) = -\left(z_1 - 1, \frac{5z_2}{8}\left(1+\frac{z_2}{2}\right)\right).$$

For this mapping

$$\langle -\tilde{F}(z), z\rangle = |z_1|^2 - \overline{z_1} + \frac{5|z_2|^2}{8}\left(1+\frac{z_2}{2}\right).$$

In particular, at the point $z_1 = \frac{1}{\sqrt{2}}$, $z_2 = -\frac{1}{\sqrt{2}}$ we have

$$\langle \tilde{F}(z), z\rangle = -\frac{13}{16} + \frac{37}{32\sqrt{2}} \approx 0.005 > 0.$$

So, $\tilde{F}$ is not a semigroup generator on the ball $\mathbb{B}^2$.

Example 6.2. Let $F : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$F(z_1, z_2) = -\left(z_1 - 1, \frac{3z_2}{(2-z_2)}\right).$$

We claim that F is an infinitesimal generator on $\mathbb{B}^2$. Indeed, for each $z \in \partial\mathbb{B}$ we have

$$\langle -F(z), z\rangle = |z_1|^2 - \overline{z_1} + \frac{3|z_2|^2}{2-z_2} = (1-\overline{z_1}) + |z_2|^2\left(\frac{3}{2-z_2} - 1\right).$$

Since the inequality $\mathsf{Re}\,\frac{1}{2-z_2} > \frac{1}{3}$ holds for all z_2, $|z_2| < 1$, we conclude that $\mathsf{Re}\, -\langle F(z), z\rangle > 0$ for all $z \in \partial\mathbb{B}$. As in the previous example taking into account that F

is holomorphic past the boundary of $\mathbb{B}^2$, one can apply Theorem 3.3.(3) with $\beta = 0$, and Proposition 3.8 to see that F is an infinitesimal generator on $\mathbb{B}^2$ (see also [16, Corollary 7.1]).

On the other hand, denote by $\tilde{F}$ the the quadratic expansion of F at e_1, namely,

$$\tilde{F}(z) = -\left(z_1 - 1, \frac{3z_2}{2}\left(1 + \frac{z_2}{2}\right)\right).$$

One can easily see that $\tilde{F}$ generates a semigroup of holomorphic self-maps of $\mathbb{B}^2$ which does not consist of linear fractional maps (cfr [6]).

References

[1] M. Abate, *Iteration theory of holomorphic maps on taut manifolds*, Mediterranean Press, Rende, 1989.

[2] M. Abate, R. Tauraso, *The Julia-Wolff-Caratheodory theorem(s).* Contemp. Math. 222 (1999), 161-172.

[3] M. Abate, *Angular derivatives in several complex variables.* Real methods in complex and CR geometry. Lectures given at the C.I.M.E. Summer School held in Martina Franca, June 30-July 6, 2002. Edited by D. Zaitsev and G. Zampieri. Lecture Notes in Mathematics, 1848. Springer-Verlag, Berlin, (2004), 1–47.

[4] D. Aharonov, S. Reich, D. Shoikhet, *Flow invariance conditions for holomorphic mappings in Banach spaces.* Math. Proc. R. Ir. Acad. 99A (1999), no. 1, 93-104.

[5] E. Berkson, H. Porta, *Semigroups of analytic functions and composition operators.* Michigan Math. J. 25 (1978), no. 1, 101-115.

[6] F. Bracci, M. D. Contreras, S. Díaz-Madrigal *Infinitesimal generators associated with semigroups of linear fractional maps.* J. Anal. Math., 102, (2007) 119-142.

[7] F. Bracci, M. D. Contreras, S. Díaz-Madrigal, *Pluripotential theory, semigroups and boundary behavior of infinitesimal generators in strongly convex domains.* J. Eur. Math. Soc. 12 (2010), 23-53.

[8] F. Bracci, G. Patrizio, S. Trapani, *The pluricomplex Poisson kernel for strongly convex domains.* Trans. Amer. Math. Soc., 361, 2, (2009), 979-1005.

[9] F. Bracci, D. Zaitsev, *Boundary jets of holomorphic maps between strongly pseudoconvex domains.* J. Funct. Anal., 254, (2008), 1449-1466.

[10] D. M. Burns, S. G. Krantz, *Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary.* J. Amer. Math. Soc. 7 (1994), no. 3, 661-676.

[11] T. Casavecchia, *A rigidity condition for generators in strongly convex domain.* Complex Var. Elliptic Equ. 55 (2010), no. 12, 1131-1142.

[12] M. D. Contreras, S. Díaz-Madrigal, Ch. Pommerenke, *On boundary critical points for semigroups of analytic functions.* Math. Scand. 98 (2006), no. 1, 125-142.

[13] C. C. Cowen, B. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[14] M. Elin, M. Levenshtein, S. Reich, D. Shoikhet, *A rigidity theorem for holomorphic generators on the Hilbert ball.* Proc. Amer. Math. Soc. 136 (2008), no. 12, 4313-4320.

[15] M. Elin, S. Reich, D. Shoikhet, *A Julia-Carathéodory theorem for hyperbolically monotone mappings in the Hilbert ball.* Israel J. Math. 164 (2008), 397-411.

[16] S. Reich, D. Shoikhet, David *Nonlinear semigroups, fixed points, and geometry of domains in Banach spaces*. Imperial College Press, London, 2005.

[17] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], 241. Springer-Verlag, New York-Berlin, 1980

[18] D. Shoikhet, *Semigroups in geometrical function theory*. Kluwer Academic Publishers, Dordrecht, 2001.

[19] D. Shoikhet, *Another look at the Burns-Krantz theorem*. J. Anal. Math. 105 (2008), 19-42.

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