SHEARING PROCESS AND AN EXAMPLE OF A BOUNDED SUPPORT FUNCTION IN $S^0(\mathbb{B}^2)$

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Abstract. We introduce a process, that we call *shearing*, which for any given normal Loewner chain produces a normal Loewner chain made of shears automorphisms. As an application, and in stringent contrast to the one-dimensional case, we prove the existence of a starlike bounded function in the class S^0 of the ball $\mathbb{B}^2$ (in fact the restriction of a shear automorphism of $\mathbb{C}^2$) which is a support point for a linear continuous functional.

1. Introduction

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$. Let $S(\mathbb{B}^n)$ denote the family of univalent holomorphic maps from $\mathbb{B}^n$ to $\mathbb{C}^n$ such that $f(0) = 0$ and $df_0 = \mathsf{id}$. Such a class is known to be compact for $n = 1$ (see, *e.g.* [10]) but it is not for $n > 1$.

A *normalized Loewner chain* on $\mathbb{B}^n$ is a family $(f_t)_{t\geq 0}$ of univalent maps from $\mathbb{B}^n$ to $\mathbb{C}^n$ such that $f_s(\mathbb{B}^n) \subset f_t(\mathbb{B}^n)$ for $0 \leq s \leq t$, $f_t(0) = 0$ and $d(f_t)_0 = e^t\mathsf{id}$. A normalized Loewner chain (f_t) is said to be a *normal Loewner chain* if $\{e^{-t}f_t\}_{t\geq 0}$ is normal in $\mathbb{B}^n$. Let $S^0(\mathbb{B}^n)$ be the subset of $S(\mathbb{B}^n)$ of maps which admit *parametric representation*, namely, $f \in S^0(\mathbb{B}^n)$ provided there exists a normal Loewner chain (f_t) in $\mathbb{B}^n$ such that $f_0 = f$. It is known, that in $\mathbb{D} = \mathbb{B}^1$, $S^0(\mathbb{D}) = S(\mathbb{D})$ (see [10]), while $S^0(\mathbb{B}^n)$ is strictly contained in $S(\mathbb{B}^n)$ for $n \geq 2$ (see, *e.g.* [2]). Nonetheless, the class $S^0(\mathbb{B}^n)$ is compact and many similar growth estimates as in the one-dimensional case can be pursued.

For studying extremal problems, it is important to determine the so called *support points*. Endow $\mathsf{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ with the topology of uniform convergence on compacta, which makes it a Fréchet space. A function $f \in S^0(\mathbb{B}^n)$ is called a *support point* if there exists a bounded linear functional $L : \mathsf{Hol}(\mathbb{B}^n, \mathbb{C}^n) \to \mathbb{C}$ such that L is not constant on $S^0(\mathbb{B}^n)$ and $\mathsf{Re}\, L(f) = \max_{h\in S^0(\mathbb{B}^n)} \mathsf{Re}\, L(h)$.

In dimension one it was proved by Schaeffer (see, *e.g.* [13]) that support points are slit functions. In particular they are unbounded.

In higher dimension, much work has been done to study support points in $S^0(\mathbb{B}^n)$ and many evidences that support points in $S^0(\mathbb{B}^n)$ should be unbounded have been obtained, see, *e.g.*, [4], [5], [14], [6], [1], [8], [12].

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However, and very surprisingly, in this paper, we construct an example of a *bounded* starlike map in $S^0(\mathbb{B}^2)$ which is a support point.

Such a map is quite simple and it is in fact the restriction to $\mathbb{B}^2$ of a shear of $\mathbb{C}^2$:

$$\Phi(z_1, z_2) = (z_1 + \frac{3\sqrt{3}}{2} z_2^2, z_2), \quad \forall (z_1, z_2) \in \mathbb{B}^2.$$

Let $L^1_{0,2} : \mathsf{Hol}(\mathbb{B}^2, \mathbb{C}^2) \to \mathbb{C}$ be defined as $L^1_{0,2}(f) := \frac{1}{2}\frac{\partial^2 f_1}{\partial z_2^2}(0)$, where $f = (f_1, f_2) \in \mathsf{Hol}(\mathbb{B}^2, \mathbb{C}^2)$.

Theorem 1.1. *The map $\Phi \in S^0(\mathbb{B}^2)$ is starlike and maximizes $\mathsf{Re}\, L^1_{0,2}$.*

The fact that Φ is starlike and belongs to $S^0(\mathbb{B}^2)$ is known from long (see [15, Example 3], [11, Example 5]), however, we give a proof of it in Lemma 2.3. The fact that Φ maximizes $\mathsf{Re}\, L^1_{0,2}$ follows at once from the following result:

Theorem 1.2. *Let $f(z_1, z_2) = (z_1 + \sum_{\alpha\in\mathbb{N}^2, |\alpha|\geq 2} a^1_\alpha z^\alpha, z_2 + \sum_{\alpha\in\mathbb{N}^2, |\alpha|\geq 2} a^2_\alpha z^\alpha) \in S^0(\mathbb{B}^2)$. Then $|a^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$, and the bound is reached by Φ.*

The proof of Theorem 1.2 relies on a process that we call *shearing process* which has no one-dimensional analogue and it seems to be interesting by itself. Let

$$\mathsf{Hol}_D(\mathbb{B}^2, \mathbb{C}^2) := \{h \in \mathsf{Hol}(\mathbb{B}^2, \mathbb{C}^2) : h(0) = 0 \text{ and } dh_0 \text{ is diagonal and invertible}\}.$$

Definition 1.3. Let $h \in \mathsf{Hol}_D(\mathbb{B}^2, \mathbb{C}^2)$ and write the Taylor expansion at 0 as

$$h(z) = (\lambda z_1 + A z_2^2 + O(|z_1|^2, |z_1 z_2|, \|z\|^3), \mu z_2 + O(\|z\|^2)).$$

Then we define the *shearing of h* to be

$$h^{[c]}(z_1, z_2) := (\lambda z_1 + A z_2^2, \mu z_2) \quad \forall z \in \mathbb{B}^2.$$

The main result of the paper is the following

Theorem 1.4. *Let (f_t) be a normal Loewner chain in $\mathbb{B}^2$. Then $(f_t^{[c]})$ is a normal Loewner chain in $\mathbb{B}^2$. In particular, if $f \in S^0(\mathbb{B}^2)$ then $f^{[c]} \in S^0(\mathbb{B}^2)$.*

Such a theorem is proved in Section 2. Using such a result, in Section 3 we prove Theorem 1.2.

We remark that our construction can be done in any dimension greater than one, but for the sake of clearness, we give the results only in dimension 2. Also, Theorem 1.1 gives an example of a function in $S^0(\mathbb{B}^2)$ which is bounded by a certain $M > 1$ but it is not a "$\log M$-time reachable function" in $S^0(\mathbb{B}^2)$ (see, *e.g.*, [7] for definition), contrary to the one-dimensional case where the set of functions in $S^0(\mathbb{D})$ which are bounded by $M > 1$ coincides with "$\log M$-time reachable functions" in $S^0(\mathbb{D})$.

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2. The shearing process of Loewner chains

2.1. Shearing the class $\mathcal{M}_-$. Let

$$\mathcal{M}_- := \{H \in \mathsf{Hol}(\mathbb{B}^2, \mathbb{C}^2) : H(0) = 0, dH_0 = -\mathsf{id}, \mathsf{Re}\,\langle H(z), z\rangle \leq 0 \quad \forall z \in \mathbb{B}^2\}.$$

We show that the shearing of a holomorphic vector field in the class $\mathcal{M}_-$ is still in the class $\mathcal{M}_-$:

Proposition 2.1. *If $H \in \mathcal{M}_-$, then $H^{[c]} \in \mathcal{M}_-$.*

Proof. It is clear that $H^{[c]}(0) = 0, dH_0^{[c]} = -\mathsf{id}$, so we have to check that $\mathsf{Re}\,\langle H^{[c]}(z), z\rangle \leq 0$ for all $z \in \mathbb{B}^2$. Write the Taylor expansion of H at 0 as

$$H(z) = (-z_1 + \sum_{\alpha\in\mathbb{N}^2:|\alpha|\geq 2} q^1_\alpha z^\alpha, -z_2 + \sum_{\alpha\in\mathbb{N}^2:|\alpha|\geq 2} q^2_\alpha z^\alpha). \tag{2.1}$$

We know that for all $z \in \mathbb{B}^2$

$$0 \geq \mathsf{Re}\,\langle H(z), z\rangle = -|z_1|^2 - |z_2|^2 + \sum_{\alpha\in\mathbb{N}^2,|\alpha|\geq 2} \mathsf{Re}\,\left(q^1_\alpha z^\alpha \overline{z_1}\right) + \sum_{\alpha\in\mathbb{N}^2,|\alpha|\geq 2} \mathsf{Re}\,\left(q^2_\alpha z^\alpha \overline{z_2}\right). \tag{2.2}$$

Let $\eta \in [0, 2\pi)$ be such that $q^1_{0,2} e^{-i\eta} = |q^1_{0,2}|$. Take $z_1 = xe^{i(\theta+\eta)}$ and $z_2 = ye^{i\frac{\theta}{2}}$ for $x, y \geq 0, x^2 + y^2 < 1$ and $\theta \in \mathbb{R}$. Substituting these expressions in (2.2) we obtain

$$\begin{aligned} 0 \geq -x^2 - y^2 + |q^1_{0,2}|xy^2 + &\sum_{\alpha\in\mathbb{N}^2,|\alpha|\geq 2,\alpha\neq(0,2)} x^{\alpha_1+1} y^{\alpha_2} \mathsf{Re}\,\left(q^1_\alpha e^{i(\alpha_1-1)\eta} e^{i[\alpha_1+\frac{\alpha_2}{2}-1]\theta}\right) \\ + \sum_{\alpha\in\mathbb{N}^2,|\alpha|\geq 2} x^{\alpha_1} y^{\alpha_2+1} &\mathsf{Re}\,\left(q^2_\alpha e^{i\alpha_1\eta} e^{i[\alpha_1+\frac{\alpha_2}{2}-\frac{1}{2}]\theta}\right). \end{aligned} \tag{2.3}$$

Now, notice that among all $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 2$ the expression $\alpha_1 + \frac{\alpha_2}{2} - 1 = 0$ has only the solution $\alpha = (0, 2)$, while $\alpha_1 + \frac{\alpha_2}{2} - \frac{1}{2} = 0$ has no solution for $\alpha \in \mathbb{N}^2$ with $|\alpha| \geq 2$. Therefore, all terms in the previous expression except the first three terms are of the form $a\cos(m\theta)$, $a\cos(m\frac{\theta}{2})$ (or with sin instead of cos), for some $a \in \mathbb{R}$ and $m \in \mathbb{N}, m \geq 1$. Thus, if we integrate (2.3) in θ for $\theta \in [0, 4\pi]$ and taking into account that the series converges uniformly on compacta and thus we can exchange the series with the integral, all the terms in (2.3) but the first three, become zero and we obtain:

$$0 \geq -x^2 - y^2 + |q^1_{0,2}|xy^2 \quad \forall x, y \geq 0, x^2 + y^2 < 1. \tag{2.4}$$

From this it follows that

$$\mathsf{Re}\,\langle H^{[c]}(z), z\rangle = -|z_1|^2 - |z_2|^2 - \mathsf{Re}\,\left(q^1_{0,2} z_2^2 \overline{z_1}\right) \leq -|z_1|^2 - |z_2|^2 + |q^1_{0,2}||z_1||z_2|^2 \leq 0,$$

proving that $H^{[c]} \in \mathcal{M}_-$. □

Corollary 2.2. *Let $H \in \mathcal{M}_-$ be given by (2.1). Then $|q^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$.*

Proof. By Proposition 2.2 it is enough to prove the result for $H(z) = (-z_1 + az_2^2, -z_2)$. By (2.4), we have

$$-x^2 - y^2 + |a|xy^2 \le 0 \quad \forall x, y \ge 0, x^2 + y^2 < 1.$$

Studying such a function it is not hard to show that the equation holds only if $|a| \le \frac{3\sqrt{3}}{2}$. $\square$

The previous estimate is sharp, indeed we have the following result:

Lemma 2.3. *Let $H(z) = (-z_1 + \frac{3\sqrt{3}}{2} z_2^2, -z_2)$. Then $H \in \mathcal{M}_-$. Moreover,*

$$f_t(z) = e^t \left(z_1 + \frac{3\sqrt{3}}{2} z_2^2, z_2 \right)$$

is a normal Loewner chain which satisfies the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = d(f_t)_z H(z).$$

In particular, $\Phi = f_0 \in S^0(\mathbb{B}^2)$ and Φ is starlike.

Proof. A simple computation shows that $H \in \mathcal{M}_-$ and that (f_t) satisfies the Loewner PDE and the hypothesis of [2, Thm. 8.1.6], so it is a normal Loewner chain. Alternatively, one can solve the Loewner ODE $\frac{\partial \varphi_{s,t}(z)}{\partial t} = H(\varphi_{s,t}(z))$, $\varphi_{s,s}(z) = z$ (which is in fact a semigroup equation) and check that $f_s = \lim_{t\to\infty} e^t \varphi_{s,t}$ for $s \in \mathbb{R}^+$. Hence, by [2, Thm 8.1.6] it follows that (f_t) is a normal Loewner chain. It is finally well known (see, *e.g.* [2, Thm. 8.2.1]) that f is starlike if and only if $(e^t f)$ is a normal Loewner chain. $\square$

2.2. **Shearing normal Loewner chains.** We show that the shearing of a normal Loewner chain is still a normal Loewner chain.

The first simple observation is the following lemma, which can be checked by Taylor expansion at 0:

Lemma 2.4. *Let $h, g \in \mathsf{Hol}_D(\mathbb{B}^2, \mathbb{C}^2)$ and suppose $h \circ g$ is well defined. Then for all $z \in \mathbb{B}^2 \cap g^{[c]}(\mathbb{B}^2)$*

$$(h \circ g)^{[c]}(z) = h^{[c]}(g^{[c]}(z)). \tag{2.5}$$

In fact, (2.5) holds for all $z \in \mathbb{C}^2$ if one denotes by $h^{[c]}$ the corresponding shear and not just its restriction to $\mathbb{B}^2$.

Now we can prove Theorem 1.4:

Proof of Theorem 1.4. A *Herglotz vector field $G(z, t)$ associated with the class $\mathcal{M}_-$* is a map $G : \mathbb{R}^+ \times \mathbb{B}^2 \to \mathbb{C}^2$ such that

(i) The mapping $G(z, \cdot)$ is measurable on $\mathbb{R}^+$ for all $z \in \mathbb{B}^2$.
(ii) $G(\cdot, t) \in \mathcal{M}_-$ for a.e. $t \in [0, +\infty)$.

Let (f_t) be a normal Loewner chain and set $\varphi_{s,t} := f_t^{-1} \circ f_s$ for $0 \leq s \leq t$. It is known (see, *e.g.* [2, Chapter 8]) that (f_t) is absolutely continuous in t locally uniformly in z and that there exists a Herglotz vector field $G(z,t)$ associated with the class $\mathcal{M}_-$ such that the following Loewner ODE is satisfied:

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad a.e.\ t \geq 0, \forall z \in \mathbb{B}^2.$$

By Proposition 2.1, $G^{[c]}(z,t)$ is a Herglotz vector field associated with the class $\mathcal{M}_-$. By Lemma 2.4, it follows that

$$\frac{\partial \varphi^{[c]}_{s,t}(z)}{\partial t} = G^{[c]}(\varphi^{[c]}_{s,t}(z), t) \quad a.e.\ t \geq s, \forall z \in \varphi^{[c]}_{s,t}(\mathbb{B}^2) \cap \mathbb{B}^2.$$

In fact, the previous equation holds for all $z \in \mathbb{C}^2$ if considered at level of shears in $\mathbb{C}^2$.

Thus, since by [2, Theorem 8.1.3] the solution to the Loewner ODE is unique, in particular, for all $s,t \in \mathbb{R}$ such that $0 \leq s \leq t$, $\varphi^{[c]}_{s,t}(\mathbb{B}^2) \subset \mathbb{B}^2$. Hence, since again by Lemma 2.4 we have $f_s^{[c]} = f_t^{[c]} \circ \varphi^{[c]}_{s,t}$ for all s,t such that $0 \leq s \leq t$, it follows that $f_s^{[c]}(\mathbb{B}^2) \subset f_t^{[c]}(\mathbb{B}^2)$ for all s,t such that $0 \leq s \leq t$. It is now easy to check that $(f_t^{[c]})$ is a normal Loewner chain. □

3. Sharp bound for the coefficient $a^1_{0,2}$

Theorem 3.1. *Let* $f = (z_1 + \sum_{\alpha \in \mathbb{N}^2 : |\alpha| \geq 2} a^1_\alpha z^\alpha, z_2 + \sum_{\alpha \in \mathbb{N}^2 : |\alpha| \geq 2} a^2_\alpha z^\alpha) \in S^0(\mathbb{B}^2)$. *Then* $|a^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$. *Such an estimate is sharp and it is reached by* Φ.

Proof. Let (f_t) be a normal Loewner chain such that $f_0 = f$. By Theorem 1.4, $f^{[c]} \in S^0(\mathbb{B}^2)$ and it has a parametric representation given by $f_t(z) = (e^t z_1 + a(t) z_2^2, e^t z_2)$, where $a : \mathbb{R}^+ \to \mathbb{R}$ is a bounded absolutely continuous function and $a(0) = a^1_{0,2}$. Let $\varphi_{s,t} := f_t^{-1} \circ f_s$. As in the proof of Theorem 1.4, there exists a Herglotz vector field $G(z,t)$ associated with the class $\mathcal{M}_-$ such that

$$\frac{\partial \varphi^{[c]}_{s,t}(z)}{\partial t} = G^{[c]}(\varphi^{[c]}_{s,t}(z), t) \quad a.e.\ t \geq s, \forall z \in \mathbb{B}^2. \tag{3.1}$$

Write $G^{[c]}(z,t) = (-z_1 + q(t) z_2^2, -z_2)$ and $\varphi^{[c]}_{s,t} = (e^{s-t} z_1 + a(s,t) z_2^2, e^{s-t} z_2)$ for $0 \leq s \leq t$. Writing down explicitly (3.1), we obtain

$$\begin{cases} \frac{\partial a(s,t)}{\partial t} & = -a(s,t) + q(t) e^{2(s-t)} \quad a.e.\ t \geq s, \\ a(s,s) & = 0 \end{cases}$$

from which it follows that for $0 \leq s \leq t$

$$a(s,t) = e^{s-t} \int_s^t q(\tau) e^{s-\tau} d\tau.$$

By Corollary 2.2 we get $|a(s,t)| \leq \frac{3\sqrt{3}}{2} e^{s-t}(1-e^{s-t})$. By [2, Thm. 8.1.5] $\lim_{t\to\infty} e^t \varphi_{s,t}^{[c]} = f_s^{[c]}$ (uniformly on compacta) for all $s \in \mathbb{R}^+$. Hence

$$|a(s)| = \lim_{t\to\infty} |e^t a(s,t)| \leq \lim_{t\to\infty} e^t \frac{3\sqrt{3}}{2} e^{s-t}(1-e^{s-t}) = \frac{3\sqrt{3}}{2} e^s,$$

from which $|a^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$. □

Remark 3.2. Let $\Phi_a(z_1, z_2) := (z_1 + az_2^2, z_2)$ for $a \in \mathbb{C}$. It was known that $\Phi_a \in S^0(\mathbb{B}^2)$ for $|a| \leq \frac{3\sqrt{3}}{2}$ since such a map is starlike (see [15, Example 3]), while it was known that $\Phi_a \notin S^0(\mathbb{B}^2)$ for $|a| \geq 2\sqrt{15}$ (see [3, Remark 3.5]) since it does not satisfy the growth estimates for the class $S^0(\mathbb{B}^2)$. In fact, due to Theorem 3.1, $\Phi_a \notin S^0(\mathbb{B}^2)$ for $|a| > \frac{3\sqrt{3}}{2}$.

Remark 3.3. The map $\Phi_a(z_1, z_2) := (z_1 + az_2^2, z_2)$ for $a = \frac{1}{2}$ is a bounded support point for the set $\mathcal{K}$ of normalized univalent maps from $\mathbb{B}^n$ to $\mathbb{C}^n$ whose image is convex (see [9, Thm. 2.7]).

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