AN ABSTRACT APPROACH TO LOEWNER CHAINS

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Abstract. We present a new geometric construction of Loewner chains in one and several complex variables which holds on complete hyperbolic complex manifolds and prove that there is essentially a one-to-one correspondence between evolution families of order d and Loewner chains of the same order. As a consequence we obtain an univalent solution $(f_t\colon M \to N)$ for any Loewner-Kufarev PDE. The problem of finding solutions given by univalent mappings $(f_t\colon M \to \mathbb{C}^n)$ is reduced to investigating whether the complex manifold $\cup_{t\geq 0} f_t(M)$ is biholomorphic to a domain in $\mathbb{C}^n$. We apply such results to the study of univalent mappings from the unit ball $\mathbb{B}^n$ to $\mathbb{C}^n$.

1. Introduction

Loewner's partial differential equation

$$\frac{\partial f_s}{\partial s}(z) = -(df_s)_z G(z,s), \quad \text{a.e. } s \geq 0,$$

received much attention from mathematicians since Charles Loewner [23] introduced it in 1923 to study extremal problems in the unit disc $\mathbb{D} \subset \mathbb{C}$ and, later, P.P. Kufarev [20] and C. Pommerenke [27], [28] fully developed the original theory. Such an equation was a cornerstone in the de Branges' proof of the Bieberbach conjecture. In 1999 O. Schramm [33] introduced a stochastic version of the original differential equation, nowadays known as SLE, which, among other things, was a basic tool to prove Mandelbrot's conjecture by himself, G. Lawler and W. Werner.

Loewner's original theory has been extended (see [25], [26], [14], [15], [17], [29]) to higher dimensional balls in $\mathbb{C}^n$ and successfully used to study distortion, star-likeness, spiral-likeness and other geometric properties of univalent mappings in higher dimensions.

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Very recently, the second named author with M. Contreras and S. Díaz-Madrigal [4], [5] and Contreras, Díaz-Madrigal and P. Gumenyuk [7] proposed a general setting for the Loewner theory, which works also on complete hyperbolic complex manifolds. While the classical theory deals with normalized objects, this general theory does not, and encloses the classical theory as a special case.

The aim of this paper is to present a general geometric construction of Loewner chains on complete hyperbolic complex manifolds which does not use any limit process (and thus it is new also for the unit disc case) but relies on the apparently new interpretation of Loewner chains as the direct limit of evolution families, and to give applications of such a theory to geometric properties of univalent mappings on the unit ball. To be more precise, we need some definitions. In the following, M is a complete hyperbolic complex manifold of dimension n, and $d \in [1, +\infty]$. An *L^d-evolution family* on M is a family $(\varphi_{s,t})_{0\leq s\leq t}$ of holomorphic self-mappings of M satisfying the *evolution property*

$$\varphi_{s,s} = \mathsf{id}, \quad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \quad 0 \leq s \leq u \leq t,$$

and $t \mapsto \varphi_{s,t}(z)$ has some L^d_{loc}-type regularity locally uniformly with respect to $z \in M$ (see Definition 2.1).

L^d-evolution families are trajectories of certain time-dependent holomorphic vector fields on M, called *Herglotz vector fields*. An L^d-Herglotz vector field $G(z,t)$ on M is a weak holomorphic vector field in the sense of Carathéodory which satisfies a suitable L^d_{loc}-bound in t uniformly on compacta of M and such that for almost every $t \geq 0$ the vector field $z \mapsto G(z,t)$ is semicomplete (see Definition 2.4).

The main result in [5] states that there is a one-to-one correspondence between evolution families and Herglotz vector fields. The bridge for such a correspondence is given by the following Loewner-Kufarev ODE:

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t), \quad \text{a.e. } t \in [s, \infty). \tag{1.1}$$

Both classical radial and chordal Loewner ODE in the unit disc are just particular cases of such an equation (see [4]).

L^d-Evolution families are strictly related to L^d-Loewner chains. Such chains are defined in the unit disc $\mathbb{D}$ in [7] as families of univalent mappings $(f_t \colon \mathbb{D} \to \mathbb{C})_{t\geq 0}$ such that $f_s(\mathbb{D}) \subseteq f_t(\mathbb{D})$ for all $0 \leq s \leq t$ and satisfying an L^d_{loc} bound in t uniformly on compacta of $\mathbb{D}$. The classical Loewner chains in the unit disc are particular cases of such chains.

The correspondence between evolution families and Loewner chains is provided by a functional equation: an L^d-evolution family and an L^d-Loewner chain are *associated* if

$$f_s = f_t \circ \varphi_{s,t}, \quad 0 \leq s \leq t. \tag{1.2}$$

In [7] it is proved that given an L^d-Loewner chain (f_t) in the unit disc $\mathbb{D}$, the family

$$(\varphi_{s,t} := f_t^{-1} \circ f_s)$$

is an associated L^d-evolution family and, conversely, any L^d-evolution family admits a unique (up to biholomorphisms) associated L^d-Loewner chain. Such a result, as already in the classical theory, is based on a scaling limit process.

Similar results, in the case of L^∞-evolution families in the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ fixing the origin and having a normalized differential at the origin, have been obtained in [14], [17]. In such works Loewner chains are defined as image-increasing sequences of univalent mappings on the ball with image in $\mathbb{C}^n$ fixing the origin and having the differential subjected to some normalization at the origin. Again, Loewner chains are defined starting from normalized evolution families by means of a scaling limit process.

In this paper we propose a definition of L^d-Loewner chains on complete hyperbolic complex manifolds and prove that equation (1.2) provides a one-to-one correspondence (up to biholomorphisms) between L^d-Loewner chains and L^d-evolution families. Since there exist complete hyperbolic complex manifolds (even non-compact ones) which are not biholomorphic to domains in $\mathbb{C}^n$, requiring each f_t to be a univalent mapping from M to $\mathbb{C}^n$ would be unnecessarily restrictive. Hence we give the following definition: let N be a complex manifold of the same dimension of M and let d_N denote the distance induced on N by some Hermitian metric. A family $(f_t : M \to N)_{t\geq 0}$ is an L^d-*Loewner chain* if

LC1. For each $t \geq 0$ fixed, the mapping $f_t : M \to N$ is univalent,
LC2. $f_s(M) \subset f_t(M)$ for all $0 \leq s \leq t < +\infty$,
LC3. For any compact set $K \subset\subset M$ and any $T > 0$ there exists a $k_{K,T} \in L^d([0,T], \mathbb{R}^+)$ such that for all $z \in K$ and for all $0 \leq s \leq t \leq T$

$$d_N(f_s(z), f_t(z)) \leq \int_s^t k_{K,T}(\xi) d\xi.$$

The main results of the present paper can be summarized as follows.

Theorem 1.1. *Let M be a complete hyperbolic complex manifold of dimension n. Let $(\varphi_{s,t})$ be an L^d-evolution family on M. Then there exists an associated L^d-Loewner chain $(f_t\colon M \to N)$. If $(g_t\colon M \to Q)$ is another L^d-Loewner chain associated with $(\varphi_{s,t})$, then there exists a biholomorphism*

$$\Lambda\colon \bigcup_{t\geq 0} f_t(M) \to \bigcup_{t\geq 0} g_t(M)$$

such that

$$g_t = \Lambda \circ f_t, \quad t \geq 0.$$

Conversely, if $(f_t\colon M \to N)$ is an L^d-Loewner chain, then $(\varphi_{s,t} := f_t^{-1} \circ f_s)$ is an associated L^d-evolution family.

The first part of the result holds more generally on taut manifolds (see Theorems 4.7 and 4.10). The second part is proved in Theorem 4.6. In order to prove the result we exploit a *kernel convergence theorem* on complete hyperbolic complex manifolds which we prove in Theorem 3.5.

The L^d-Loewner chain $(f_t\colon M \to N)$ associated with the L^d-evolution family $(\varphi_{s,t})$ is constructed as its direct limit. From a categorical point of view, we consider complex manifolds as objects and holomorphic mappings as morphisms. Define $M_t = M$ for all $t \geq 0$. The pair $((M_t), (\varphi_{s,t}))$ is a direct system indexed by $\mathbb{R}^+$, and the chain (f_t) is defined as the family of canonical morphisms given by the direct limit $(N, (f_t))$ of $((M_t), (\varphi_{s,t}))$. To be more precise, we define an equivalence relation on the product $M \times \mathbb{R}^+$:

$$(x, s) \sim (y, t) \quad \text{iff} \quad \varphi_{s,u}(x) = \varphi_{t,u}(y) \text{ for } u \text{ large enough},$$

and define $N := (M \times \mathbb{R}^+)/_\sim$. Let $\pi\colon M \times \mathbb{R}^+ \to N$ be the projection on the quotient, and let $i_t\colon M \to M \times \mathbb{R}^+$ be the injection $i_t(x) = (x, t)$. The canonical morphisms are the mappings

$$f_t := \pi \circ i_t, \quad t \geq 0,$$

which are injective since the mapping $\varphi_{s,t}$ is injective for all $0 \leq s \leq t$. Equation (1.2) holds and is the universal property of the direct limit $(N, (f_t))$. We endow $N = \bigcup_{t\geq 0} f_t(M)$ with a complex manifold structure which makes the mapping f_t holomorphic for all $t \geq 0$, and we prove the L^d-estimate.

As a consequence of Theorem 1.1, we can define the *Loewner range* $\mathsf{Lr}(\varphi_{s,t})$ of $(\varphi_{s,t})$ as the biholomorphism class of $\bigcup_{t\geq 0} f_t(M)$, where (f_t) is any associated L^d-Loewner chain. The Loewner range can be seen as an analogue of the abstract basin of attraction defined by Fornaess and Stensønes in the setting of discrete holomorphic dynamics with an attractive fixed point [12].

If $(\varphi_{s,t})$ is an L^d-evolution family on the unit disc $\mathbb{D}$ the Loewner range has to be simply connected and cannot be compact, thus by the uniformization theorem it has to be biholomorphic to $\mathbb{D}$ or $\mathbb{C}$, and, as noticed also in [7], the choice depends on the dynamics of $(\varphi_{s,t})$. Generalizing this result we prove the following formula: if (f_t) and $(\varphi_{s,t})$ are associated, then

$$f_s^* \kappa_{\mathsf{Lr}(\varphi_{s,t})} = \lim_{t\to\infty} \varphi_{s,t}^* \kappa_M, \quad s \geq 0,$$

where we recall that, if $g\colon X \to Y$ is an holomorphic mapping between to complex manifolds, the pull-back of the Kobayashi pseudometric κ_Y of Y is given by

$$g^*\kappa_Y(x; v) \doteq \kappa_Y(g(x); (dg)_x(v)), \quad x \in X, v \in T_x X.$$

Using results from Fornaess and Sibony [11] we provide in Theorem 4.18 some conditions on the corank of the Kobayashi pseudometric in order to determine the Loewner range of an L^d-evolution family.

In dimension one, Theorem 1.1 and the uniformization theorem allow to recover both the classical results of Loewner, Kufarev, Pommerenke and the new results by Contreras, Díaz-Madrigal and Gumenyuk. In higher dimensions these results are new.

Let now $G(z, t)$ be an L^d-Herglotz vector field whose flow is given by an L^d-evolution family as in (1.1), and let N be a complex manifold of dimension n. Theorem 1.1 yields that a family of univalent mappings $(f_t\colon M \to N)$ solves the Loewner-Kufarev PDE

$$\frac{\partial f_s}{\partial s}(z) = -(df_s)_z G(z,s), \quad \text{a.e. } s \geq 0, z \in M$$

if and only if it is an L^d-Loewner chain associated with $(\varphi_{s,t})$. A solution given by univalent mappings $(f_t\colon M \to \mathbb{C}^n)$ exists if and only if the Loewner range $\mathsf{Lr}\,(\varphi_{t,s})$ is biholomorphic to a domain in $\mathbb{C}^n$.

In Section 6 we introduce a notion of conjugacy for L^d-evolution families which preserves the Loewner range. In Section 7 we give examples of L^d-Loewner chains in the unit ball generated by the Roper-Suffridge extension operator. In Section 8 we consider spiral-shaped and star-shaped mappings and give a characterization of such mappings.

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2. Evolution families and Herglotz vector fields

In the rest of this paper, unless differently stated, all manifolds are assumed to be connected. Let M be a complex manifold and let d_M denote the distance associated with a given Hermitian metric on M. In the sequel we will also use the Kobayashi pseudodistance k_M on M and the associated Kobayashi pseudometric κ_M on M. For definitions and properties we refer the reader to the books [1], [21].

Definition 2.1. Let M be a taut manifold. A family $(\varphi_{s,t})_{0\leq s\leq t}$ of holomorphic self-mappings of M is an *evolution family of order* $d \in [1,\infty]$ (or L^d-evolution family) if it satisfies the *evolution property*

$$\varphi_{s,s} = \mathsf{id}, \quad \varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}, \quad 0 \leq s \leq u \leq t, \tag{2.1}$$

and if for any $T > 0$ and for any compact set $K \subset\subset M$ there exists a function $c_{T,K} \in L^d([0,T],\mathbb{R}^+)$ such that

$$d_M(\varphi_{s,t}(z), \varphi_{s,u}(z)) \leq \int_u^t c_{T,K}(\xi)d\xi, \quad z \in K,\ 0 \leq s \leq u \leq t \leq T. \tag{2.2}$$

The following lemma is proved in [5, Lemma 2].

Lemma 2.2. *Let $d \in [1,+\infty]$. Let $(\varphi_{s,t})$ be an L^d-evolution family. Let $\Delta := \{(s,t) : 0 \leq s \leq t\}$. Then the mapping*

$$(s,t) \mapsto \varphi_{s,t}$$

from Δ to $\mathsf{hol}(M,M)$ endowed with the topology of uniform convergence on compacta is jointly continuous. Hence the mapping $\Phi(z,s,t) := \varphi_{s,t}(z)$ from $M \times \Delta$ to M is jointly continuous.

Proposition 2.3. *Let $d \in [1,+\infty]$. Let $(\varphi_{s,t})$ be an L^d-evolution family. Then for all $0 \leq s \leq t$ the mapping $(\varphi_{s,t})$ is univalent.*

Proof. We proceed by contradiction. Suppose there exists $0 < s < t$ and $z \neq w$ in M such that $\varphi_{s,t}(z) = \varphi_{s,t}(w)$. Set $r := \inf\{u \in [s,t] : \varphi_{s,u}(z) = \varphi_{s,u}(w)\}$. Since by Lemma 2.2 $\lim_{u\to s+} \varphi_{s,u} = \mathsf{id}$ uniformly on compacta, we have $r > s$. If $u \in (s,r)$,

$$\varphi_{u,r}(\varphi_{s,u}(z)) = \varphi_{u,r}(\varphi_{s,u}(w)),$$

and since $\varphi_{s,u}(z) \neq \varphi_{s,u}(w)$, the mappings $\varphi_{u,r}$, $u \in (s,r)$, are not univalent on a fixed relatively compact subset of M. But by Lemma 2.2 $\lim_{u\to r-} \varphi_{u,r} = \mathsf{id}$ uniformly on compacta, which is a contradiction since the identity mapping is univalent. □

Definition 2.4. A *weak holomorphic vector field of order* $d \in [1,\infty]$ on M is a mapping $G : M \times \mathbb{R}^+ \to TM$ with the following properties:

(i) The mapping $G(z,\cdot)$ is measurable on $\mathbb{R}^+$ for all $z \in M$.
(ii) The mapping $G(\cdot,t)$ is a holomorphic vector field on M for all $t \in \mathbb{R}^+$.
(iii) For any compact set $K \subset\subset M$ and all $T > 0$, there exists a function $C_{K,T} \in L^d([0,T],\mathbb{R}^+)$ such that

$$\|G(z,t)\| \leq C_{K,T}(t), \quad z \in K, \text{ a.e. } t \in [0,T].$$

We recall that a holomorphic vector field G on M is called an *infinitesimal generator* provided the Cauchy problem

$$\begin{cases} \dot{z}(t) = G(z(t)), \\ z(0) = z_0 \end{cases}$$

has a solution $z : \mathbb{R}^+ \to M$ for all $z_0 \in M$.

A *Herglotz vector field of order* $d \in [1,\infty]$ on a complete hyperbolic manifold M is a weak holomorphic vector field of order d such that the holomorphic vector field $z \mapsto G(z,t)$ is an infinitesimal generator for a.e. fixed $t \in \mathbb{R}^+$.

In the sequel we will use the following result which was proved in [5] with some additional assumptions (smoothness of the Kobayashi distance of M and $d = \infty$) and was proved in [3] in its generality.

Theorem 2.5. *Let M be a complete hyperbolic complex manifold. Then for any Herglotz vector field G of order $d \in [1,+\infty]$ there exists a unique L^d-evolution family $(\varphi_{s,t})$ over M such that for all $z \in M$*

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z),t) \quad a.e.\ t \in [s,+\infty). \tag{2.3}$$

Conversely for any L^d-evolution family $(\varphi_{s,t})$ over M there exists a Herglotz vector field G of order d such that (2.3) is satisfied. Moreover, if H is another weak holomorphic vector field which satisfies (2.3) then $G(z,t) = H(z,t)$ for all $z \in M$ and almost every $t \in \mathbb{R}^+$.

3. Kernel convergence on complex manifolds

Let $B(z_0, r) \subset \mathbb{C}^n$ denote the Euclidean open ball of center z_0 and radius $r > 0$ (as customary, we denote by $\mathbb{B}^n := B(0,1)$ the Euclidean open ball centered at the origin and radius 1).

Proposition 3.1. *Let $U \subset \mathbb{C}^n$ be an open set. Let $f_k \colon U \to \mathbb{C}^n$ be a sequence of univalent mappings. Assume that $f_k \to f$ uniformly on compacta and that f is univalent. Then for all $z_0 \in U$ and $0 < s < r$ such that $B(z_0, s) \subset\subset B(z_0, r) \subset\subset U$ there exists $m = m(z_0, s, r)$ such that if $k > m$ then*

$$f(B(z_0, s)) \subset f_k(B(z_0, r)).$$

Proof. Let $K = f\left(\overline{B(z_0, s)}\right)$, $\gamma = \partial B(z_0, r)$ and $\Gamma = f(\gamma)$. Then $K \cap \Gamma = \varnothing$ since f is univalent on U.

Let η be the Euclidean distance between Γ and K. Then $\eta > 0$ and

$$\eta = \min\{\|f(z) - w\| : \ w \in K, \|z - z_0\| = r\}.$$

If $u_0 \in K$ then $\|f(z) - u_0\| \geq \eta$ for $z \in \gamma$, and since $f_k \to f$ uniformly on γ there exists $m > 0$ such that if $k \geq m$ and $z \in \gamma$ then

$$\|f(z) - f_k(z)\| < \|f(z) - u_0\|.$$

Rouché theorem in several complex variables (see [24, Theorem 9.3.4]) yields then that $f_k(z) - u_0$ and $f(z) - u_0$ have the same number of zeros on $B(z_0, r)$ counting multiplicities. But $f(z) - u_0$ has a zero in $B(z_0, r)$ since $u_0 \in K$, and thus $u_0 \in f_k(B(z_0, r))$ for $k \geq m$. The constant $m > 0$ does not depend on $u_0 \in K$, hence we have the result. □

Corollary 3.2. *Let $U \subset \mathbb{C}^n$ be an open set. Let (f_k) be a sequence of univalent mappings $f_k \colon U \to \mathbb{C}^n$ converging uniformly on compacta to a univalent mapping f. Then any compact set $K \subset f(U)$ is eventually contained in $f_k(U)$.*

Proof. All the balls $B(z, s) \subset\subset U$ give an open covering of U. Since K is compact there is a finite number of balls $B(z_i, s_i) \subset\subset U$ such that $K \subset \bigcup_i f(B(z_i, s_i))$, hence Proposition 3.1 yields the result. □

Definition 3.3. Let (Ω_k) be a sequence of open subsets of a manifold M. The *kernel* Ω is the biggest open set such that for all compact sets $K \subset \Omega$ there exists $m = m(K)$ such that if $k \geq m$ then $K \subset \Omega_k$. We say that the sequence (Ω_k) *kernel converges* to Ω (denoted $\Omega_k \to \Omega$) if every subsequence of (Ω_k) has the same kernel Ω.

Note that by the very definition the kernel is an open set, possibly empty. It might be empty as the following example shows:

Example 3.4. Let $M = \mathbb{C}$ and $f_k : \mathbb{D} \to \mathbb{C}$ defined by $f_k(z) = \frac{1}{k}z$. Then (f_k) is a sequence of univalent mappings converging uniformly on compacta to 0, and $f_k(\mathbb{D}) \to \varnothing$.

We have the following result. Another version of the kernel convergence theorem in $\mathbb{C}^n$ may be found in [8].

Theorem 3.5. *[Kernel convergence] Let (f_k) be a sequence of univalent mappings from a complete hyperbolic complex manifold M to a complex manifold N of the same dimension. Suppose that (f_k) converges uniformly on compacta to a univalent mapping f. Then $f(M)$ is a connected component of the kernel Ω of the sequence $(f_k(M))$, and $(f_k^{-1}|_{f(M)})$ converges uniformly on compacta to $f^{-1}|_{f(M)}$. In particular if Ω is connected then $(f_k(M)) \to \Omega$.*

Proof. Let $K \subset f(M)$ be a compact set. We want to prove that eventually $K \subset f_k(M)$. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M such that any U_α is biholomorphic to $\mathbb{B}^n$, and let $\mathcal{H}$ be the open covering of M given by all open subsets H satisfying the following property: there exists $U_\alpha \in \mathcal{U}$ such that $H \subset\subset U_\alpha$ (notice that $f(H)$ is then relatively compact in some coordinate chart of N). Note that f is an open mapping since M and N have the same dimension, thus $f_*\mathcal{U} = \{f(U_\alpha)\}_{U_\alpha \in \mathcal{U}}$ is an open covering of $f(M)$.

Since K is compact there exist a finite number of open subsets $H_i \in \mathcal{H}$ such that $K \subset \bigcup_i f(H_i)$. Note that on H_i the sequence f_k takes eventually values in some $f(U_{\alpha_i})$ thanks to uniform convergence on compacta. By using a partition of unity it is easy to see that there exist a finite number of compact sets K_i such that $K_i \subset f(H_i)$ and $K = \bigcup_i K_i$. Thus we can assume $M \subset \mathbb{C}^n$ and $N = \mathbb{C}^n$, and the claim follows from Corollary 3.2.

Thus $f(M)$ is a subset of the kernel Ω of the sequence $(f_k(M))$. This implies that on any compact set $K \subset f(M)$ the sequence $f_k^{-1} \colon K \to M$ is eventually defined. Let Ω_0 be the connected component of the kernel which contains $f(M)$. We want to prove that $(f_k^{-1}|_{\Omega_0})$ admits a subsequence converging uniformly on compacta. Assume that $(f_k^{-1}|_{\Omega_0})$ is compactly divergent. Since M is complete hyperbolic, this is equivalent to assume that for all fixed $z_0 \in M$ and compact sets $K \subset \Omega_0$ we have

$$\liminf_{k\to\infty} \left(\min_{w \in K} k_M(f_k^{-1}(w), z_0) \right) = +\infty. \tag{3.1}$$

Let $j \geq 0$ and let

$$K(j) := \{f(z_0)\} \cup \bigcup_{k \geq j} \{f_k(z_0)\}.$$

Since $f_k(z_0) \to f(z_0)$, the set $K(j)$ is compact. Since $f(M)$ is open there exists $m > 0$ such that $K(m) \subset f(M) \subset \Omega_0$. But

$$k_M\left(f_k^{-1}(f_k(z_0)), z_0\right) = 0,$$

in contradiction with (3.1).

Let $(f_{k_i}^{-1}|_{\Omega_0})$ be a converging subsequence and let $g : \Omega_0 \to M$ be its limit. Let $w_0 \in \Omega_0$. The sequence $(f_{k_i}^{-1}(w_0))$ is eventually defined and converging to some $z = g(w_0)$. Thus $w_0 = f_{k_i}(f_{k_i}^{-1}(w_0)) \to f(z)$, which implies that $\Omega_0 = f(M)$ and that $g(w_0) = f^{-1}(w_0)$, hence $(f_k^{-1}|_{\Omega_0})$ converges to $f^{-1}|_{\Omega_0}$ uniformly on compacta. $\square$

The condition that the sets are open is important, as the following example shows:

Example 3.6. Let $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Let $f_k : \mathbb{D} \to \mathbb{C}^2$ be defined by $f_k(\zeta) := (\zeta, \frac{1}{k}\zeta)$. Then (f_k) is a sequence of univalent discs which converges uniformly on compacta to the injective disc $\zeta \mapsto (\zeta, 0)$. The only compact set in $\mathbb{C}^2$ which is eventually contained in $f_k(\mathbb{D})$ is $\{0\}$.

4. Loewner chains on complex manifolds

As we will show in what follows, some properties of Loewner chains are related only to the algebraic properties of evolution family and not to L^d regularity. Hence, it is natural to introduce the following:

Definition 4.1. Let M be a complex manifold. An *algebraic evolution family* is a family $(\varphi_{s,t})_{0\leq s\leq t}$ of univalent self-mappings of M satisfying the evolution property (2.1).

Thanks to Proposition 2.3, an L^d-evolution family is an algebraic evolution family (*i.e.*, it is univalent).

Definition 4.2. Let M, N be two complex manifolds of the same dimension. A family of holomorphic mappings $(f_t : M \to N)_{t\geq 0}$ is a *subordination chain* if for each $0 \leq s \leq t$ there exists a holomorphic mapping $v_{s,t} : M \to M$ such that $f_s = f_t \circ v_{s,t}$. A subordination chain (f_t) and an algebraic evolution family $(\varphi_{s,t})$ are *associated* if

$$f_s = f_t \circ \varphi_{s,t}, \quad 0 \leq s \leq t.$$

An *algebraic Loewner chain* is a subordination chain such that each mapping $f_t : M \to N$ is univalent. The *range* of an algebraic Loewner chain is defined as $\mathsf{rg}\,(f_t) := \bigcup_{t\geq 0} f_t(M)$. An algebraic Loewner chain $(f_t\colon M \to N)$ is *surjective* if $\mathsf{rg}\,(f_t) = N$.

Remark 4.3. Equivalently an algebraic Loewner chain can be defined as a family of univalent mappings $(f_t : M \to N)_{t\geq 0}$ such that

$$f_s(M) \subset f_t(M), \quad 0 \leq s \leq t.$$

Definition 4.4. Let $d \in [1, +\infty]$. Let M, N be two complex manifolds of the same dimension. Let d_N be the distance induced by a Hermitian metric on N. An algebraic Loewner chain $(f_t\colon M \to N)$ is a *Loewner chain of order* $d \in [1, +\infty]$ (or L^d-Loewner chain) if for any compact set $K \subset\subset M$ and any $T > 0$ there exists a $k_{K,T} \in L^d([0, T], \mathbb{R}^+)$ such that

$$d_N(f_s(z), f_t(z)) \leq \int_s^t k_{K,T}(\xi)d\xi \tag{4.1}$$

for all $z \in K$ and for all $0 \leq s \leq t \leq T$.

Remark 4.5. By (4.1) the mapping $t \mapsto f_t$ is continuous from $\mathbb{R}^+$ to $\mathsf{hol}(M, N)$. Hence the mapping $\Psi\colon M \times \mathbb{R}^+ \to N$ defined as $\Psi(z, t) = f_t(z)$ is jointly continuous.

Theorem 4.6. *Let $d \in [1, +\infty]$. Let $(f_t \colon M \to N)$ be an L^d-Loewner chain. Assume that M is complete hyperbolic. Let*

$$\varphi_{s,t} := f_t^{-1} \circ f_s, \quad 0 \le s \le t.$$

Then for any Hermitian metric on M, the family $(\varphi_{s,t})$ is an L^d-evolution family on M associated with (f_t).

Proof. It is clear that $(\varphi_{s,t})$ is an algebraic evolution family, so that we only need to prove the L^d-estimate.

Let H be a compact subset of $f_t(M)$. Set

$$L(H,t) := \sup_{\eta,\zeta \in H, \eta \neq \zeta} \frac{d_M(f_t^{-1}(\zeta), f_t^{-1}(\eta))}{d_N(\zeta,\eta)}.$$

Then $L(H,t) < +\infty$, since $w \mapsto f_t^{-1}(w)$ is locally Lipschitz.

Given a compact subset $K \subset M$ define

$$K_t := \bigcup_{s \in [0,t]} f_s(K).$$

The set K_t is a compact subset of $f_t(M)$ by Remark 4.5 since $K_t = \Psi(K \times [0,t])$. Let $T > 0$ be fixed. We claim that the function $L(K_t, t)$ on $0 \le t \le T$ is bounded by a constant $C(K,T) < +\infty$. Assume that $L(K_t, t)$ is unbounded. Then there exists a sequence $(t_n) \subset [0,T]$, which we might assume converging to some $t \in [0,T]$, such that

$$L(K_{t_n}, t_n) \ge n + 1, \quad \forall n \ge 0.$$

Hence for any $n \ge 0$ there exist $\zeta_n, \eta_n \in K_{t_n}$ such that $\zeta_n \neq \eta_n$ and

$$\frac{d_M(f_{t_n}^{-1}(\zeta_n), f_{t_n}^{-1}(\eta_n))}{d_N(\zeta_n,\eta_n)} \ge n. \tag{4.2}$$

By passing to a subsequence we may assume that $\zeta_n \to \zeta \in K_t$ and $\eta_n \to \eta \in K_t$. By Theorem 3.5, $f_{t_n}^{-1} \to f_t^{-1}$ uniformly on a neighborhood of K_t. By (4.2) we have $\eta = \zeta$, since otherwise

$$\frac{d_M(f_{t_n}^{-1}(\zeta_n), f_{t_n}^{-1}(\eta_n))}{d_N(\zeta_n,\eta_n)} \to \frac{d_M(f_t^{-1}(\zeta), f_t^{-1}(\eta))}{d_N(\zeta,\eta)}.$$

Let U, V be two open subsets of $f_t(M)$, both biholomorphic to $\mathbb{B}^n$ such that $\zeta \in U \subset\subset V \subset\subset f_t(M)$. Since by Theorem 3.5 the sequence $(f_{t_n}^{-1})$ converges to f_t^{-1} uniformly on V, we have that eventually $f_{t_n}^{-1}(U) \subset f_t^{-1}(V)$. The sequence $(f_{t_n}^{-1}|_U)$ is thus equibounded and by Cauchy estimates it is equi-Lipschitz in a neighborhood of ζ, which contradicts (4.2) and thus proves the claim.

Let $\Delta_T := \{(s,t) : 0 \le s \le t \le T\}$. Then the mapping $(s,t) \mapsto f_t^{-1} \circ f_s$ from Δ_T to $\mathsf{hol}(M,M)$ endowed with the topology of uniform convergence on compacta is continuous.

Indeed, let $(s_n, t_n) \to (s,t)$. Let $K \subset M$ be a compact set. By Remark 4.5 the set

$$K(j) := f_s(K) \cup \bigcup_{n\geq j} f_{s_n}(K) = \Psi(K, \{s\} \cup \bigcup_{n\geq j}\{s_n\})$$

is compact. There exists $m > 0$ such that $K(m) \subset f_t(M)$. By Theorem 3.5 the sequence $(f_{t_n}^{-1})$ converges to f_t^{-1} uniformly on $K(m)$. This proves that $(s,t) \mapsto f_t^{-1} \circ f_s$ is continuous.

This implies that the mapping $\Phi\colon M \times \Delta_T \to M$ defined as $\Phi(z,s,t) := \varphi_{s,t}(z)$ is jointly continuous. Hence if $K \subset M$ is a compact set,

$$\hat{K} := \bigcup_{0\leq a\leq b\leq T} \varphi_{a,b}(K) = \bigcup_{0\leq a\leq b\leq T} f_b^{-1}(f_a(K))$$

is compact in M. Therefore, since

$$d_M(\varphi_{s,u}(z), \varphi_{s,t}(z)) = d_M(\varphi_{s,u}(z), \varphi_{u,t}(\varphi_{s,u}(z))),$$

in order to estimate $d_M(\varphi_{s,u}(z), \varphi_{s,t}(z))$ for $z \in K$ and $0 \leq s \leq u \leq t \leq T$ it is enough to estimate $d_M(\zeta, \varphi_{u,t}(\zeta))$ for $\zeta \in \hat{K}$.

Since

$$\begin{aligned} d_M(\zeta, \varphi_{u,t}(\zeta)) &= d_M(f_t^{-1}(f_t(\zeta)), f_t^{-1}(f_u(\zeta))) \leq L(\hat{K}_t, t) d_N(f_t(\zeta), f_u(\zeta)) \\ &\leq C(\hat{K}, T) d_N(f_t(\zeta), f_u(\zeta)) \leq C(\hat{K}, T) \int_u^t k_{\hat{K},T}(\xi) d\xi, \end{aligned}$$

we are done. □

Theorem 4.7. *Any algebraic evolution family $(\varphi_{s,t})$ admits an associated algebraic Loewner chain $(f_t\colon M \to N)$. Moreover if $(g_t\colon M \to Q)$ is a subordination chain associated with $(\varphi_{s,t})$ then there exist a holomorphic mapping $\Lambda\colon \mathsf{rg}\,(f_t) \to Q$ such that*

$$g_t = \Lambda \circ f_t, \quad \forall t \geq 0.$$

The mapping Λ is univalent if and only if (g_t) is an algebraic Loewner chain, and in that case $\mathsf{rg}\,(g_t) = \Lambda(\mathsf{rg}\,(f_t))$.

Proof. Define an equivalence relation on the product $M \times \mathbb{R}^+$:

$$(x,s) \sim (y,t) \quad \text{iff} \quad \varphi_{s,u}(x) = \varphi_{t,u}(y) \text{ for } u \text{ large enough.}$$

and define $N := (M \times \mathbb{R}^+)/_\sim$. Let $\pi\colon M \times \mathbb{R}^+ \to N$ be the projection on the quotient, and let $i_t\colon M \to M \times \mathbb{R}^+$ be the injection $i_t(x) = (x,t)$. Define a family of mappings $(f_t\colon M \to N)$ as

$$f_t := \pi \circ i_t, \quad t \geq 0.$$

Each mapping f_t is injective since $\pi|_{M\times\{t\}}$ is injective, and by construction the family (f_t) satisfies

$$f_s = f_t \circ \varphi_{s,t}, \quad 0 \leq s \leq t.$$

Thus we have $f_s(M) \subset f_t(M)$ for $0 \leq s \leq t$ and $N = \bigcup_{t\geq 0} f_t(M)$.

Endow the product $M \times \mathbb{R}^+$ with the product topology, considering on $\mathbb{R}^+$ the discrete topology. Endow N with the quotient topology. Each mapping f_t is continuous and open, hence it is an homeomorphism onto its image. This shows that N is arcwise-connected and Hausdorff since each $f_t(M)$ is arcwise-connected and Hausdorff. Moreover N is second countable since $N = \bigcup_{k\in\mathbb{N}} f_k(M)$. Now define a complex structure on N by considering the M-valued charts $(f_t^{-1}, f_t(M))$ for all $t \geq 0$. This charts are compatible since $f_t^{-1} \circ f_s = \varphi_{s,t}$ which is holomorphic. Hence the family (f_t) is an algebraic Loewner chain associate with $(\varphi_{s,t})$.

If $(g_t\colon M \to Q)$ is a subordination chain associated with $(\varphi_{s,t})$, then the map $\Psi\colon M \times \mathbb{R}^+ \to Q$

$$(z,t) \mapsto g_t(z)$$

is compatible with the equivalence relation $\sim$. The map Ψ passes thus to the quotient defining a holomorphic mapping $\Lambda\colon N \to Q$ such that

$$g_t = \Lambda \circ f_t, \quad t \geq 0.$$

The last statement is easy to check. □

As a corollary we have the following.

Corollary 4.8. *Let $(\varphi_{s,t})$ be an algebraic evolution family on a complex manifold M. Also let $(f_t\colon M \to N)$ and $(g_t\colon M \to Q)$ be two algebraic Loewner chains associated with $(\varphi_{s,t})$. Then there exists a biholomorphism $\Lambda\colon \mathsf{rg}\,(f_t) \to \mathsf{rg}\,(g_t)$ such that $g_t = \Lambda \circ f_t$ for all $t \geq 0$.*

Thus there exists essentially one algebraic Loewner chain associated with an algebraic evolution family.

Definition 4.9. Let $(\varphi_{s,t})$ be an algebraic evolution family. By Corollary 4.8 the biholomorphism class of the range of an associated algebraic Loewner chain is uniquely determined. We call this class the *Loewner range* of $(\varphi_{s,t})$ and we denote it by $\mathsf{Lr}\,(\varphi_{s,t})$.

Theorem 4.10. *Let $d \in [1, +\infty]$. Let $(\varphi_{s,t})$ be an L^d-evolution family on a taut manifold M, and let $(f_t\colon M \to N)$ be an associated algebraic Loewner chain. Then (f_t) is an L^d-Loewner chain for any Hermitian metric on N.*

Proof. Let $K \subset M$ be a compact set. Let $T > 0$ be fixed. By Lemma 2.2 the subset of M defined as

$$\hat{K} := \bigcup_{0\leq s\leq t\leq T} \varphi_{s,t}(K)$$

is compact. Indeed $\hat{K} = \Psi(K \times \Delta_T)$, where $\Delta_T = \{(s,t) : 0 \leq s \leq t \leq T\}$.

Since f_T is locally Lipschitz there exists $C = C(\hat{K}) > 0$ such that

$$d_N(f_T(z), f_T(w)) \leq C d_M(z,w), \quad z, w \in \hat{K}.$$

The family $(\varphi_{t,T})_{0\le t\le T}$ is equi-Lipschitz on $\hat{K}$, that is there exists $L(\hat{K},T)>0$ such that

$$d_M(\varphi_{t,T}(z),\varphi_{t,T}(w)) \le L(\hat{K},T) d_M(z,w), \quad z,w\in\hat{K},\ t\in[0,T]. \tag{4.3}$$

Indeed assume by contradiction that there exist sequences $(z_n),(w_n)$ in $\hat{K}$, and (t_n) in $[0,T]$ such that

$$\frac{d_M(\varphi_{t_n,T}(z_n),\varphi_{t_n,T}(w_n))}{d_M(z_n,w_n)} \ge n. \tag{4.4}$$

By passing to subsequences we can assume $t_n\to t$, $z_n\to z$ and $w_n\to w$, and by (4.4) it is easy to see that $z=w$.

Let U,V be two open subsets of M, both biholomorphic to $\mathbb{B}^n$ such that $z\in V\subset\subset U\subset\subset M$. Since the sequence $(\varphi_{t_n,T})$ converges to $\varphi_{t,T}$ uniformly on U we have that eventually $\varphi_{t_n,T}(V)\subset\varphi_{t,T}(U)$. The sequence $(\varphi_{t_n,T}|_V)$ is thus equibounded and by Cauchy estimates it is equi-Lipschitz in a neighborhood of z, which contradicts (4.4).

Hence, for all $z\in K$ and $0\le s\le t\le T$ we have

$$\begin{aligned} d_N(f_s(z),f_t(z)) &= d_N(f_T(\varphi_{s,T}(z)),f_T(\varphi_{t,T}(z))) \le C d_M(\varphi_{s,T}(z),\varphi_{t,T}(z)) \\ &= C d_M(\varphi_{t,T}(\varphi_{s,t}(z)),\varphi_{t,T}(z)) \le CL(\hat{K},T) d_M(\varphi_{s,t}(z),z) \\ &= CL(\hat{K},T) d_M(\varphi_{s,t}(z),\varphi_{s,s}(z)) \le CL(\hat{K},T)\int_s^t c_{K,T}(\xi)d\xi, \end{aligned}$$

by (2.2). This concludes the proof. □

Corollary 4.11. *Assume that the algebraic evolution family $(\varphi_{s,t})$ on a complete hyperbolic complex manifold M is associated with the algebraic Loewner chain $(f_t\colon M\to N)$. Then $(\varphi_{s,t})$ is an L^d-evolution family if and only if (f_s) is an L^d-Loewner chain.*

Proof. It follows from Theorems 4.10 and 4.6. □

When dealing with evolution families defined on a domain D of a complex manifold N, a natural question is whether there exists an associated Loewner chain whose range is contained in N, or, in other terms, whether the Loewner range is biholomorphic to a domain of N. This question makes particularly sense if $D=\mathbb{B}^n$ and $N=\mathbb{C}^n$. In other words:

Open question: Given an L^d-evolution family on the unit ball $\mathbb{B}^n$ does there exist an associated L^d-Loewner chain with range in $\mathbb{C}^n$?

Remark 4.12. There exists an algebraic evolution family $(\varphi_{s,t})$ on $\mathbb{B}^3$ which does not admit any associated algebraic Loewner chain with range in $\mathbb{C}^3$. This follows from [2, Section 9.4].

There are several works in this direction, answering such a question in some normalized class of evolution families (see [2], [14], [15], [16], [17], [29], [35]) but in its generality the question is still open. Here we give some answers based on the asymptotic behavior of the Kobayashi pseudometric under the corresponding evolution family.

Definition 4.13. Let $(\varphi_{s,t})$ be an algebraic evolution family on a complex manifold M. Let $\kappa_M : TM \to \mathbb{R}^+$ be the Kobayashi pseudometric of M. For $v \in T_zM$ and $s \geq 0$ we define

$$\beta^s_v(z) := \lim_{t\to\infty} \kappa_M(\varphi_{s,t}(z); (d\varphi_{s,t})_z(v)). \tag{4.5}$$

Remark 4.14. Let $0 \leq s \leq u \leq t$. Since the Kobayashi pseudometric is contracted by holomorphic mappings it follows

$$\begin{aligned}\kappa_M(\varphi_{s,t}(z); (d\varphi_{s,t})_z(v)) &= \kappa_M(\varphi_{u,t}(\varphi_{s,u}(z)); (d\varphi_{u,t})_{\varphi_{s,u}(z)}(d\varphi_{s,u})_z(v)) \\ &\leq \kappa_M(\varphi_{s,u}(z); (d\varphi_{s,u})_z(v)),\end{aligned}$$

hence the limit in (4.5) is well defined.

Proposition 4.15. *Let $(\varphi_{s,t})$ be an algebraic evolution family on a complex manifold M. Let $(f_t\colon M \to N)$ be an associated surjective algebraic Loewner chain. Then for all $z \in M$ and $v \in T_zM$ it follows*

$$f_s^*\kappa_N(z; v) = \beta^s_v(z).$$

Proof. Since the chain $(f_t\colon M \to N)$ is surjective, the range N is the union of the growing sequence of complex manifolds $(f_j(M))_{j\in\mathbb{N}}$, thus

$$\kappa_N(f_s(z); (df_s)_z)(v)) = \lim_{j\to\infty} \kappa_{f_j(M)}(f_s(z); (df_s)_z(v)).$$

The result follows from

$$\begin{aligned}\lim_{j\to\infty} \kappa_{f_j(M)}(f_s(z); (df_s)_z)(v)) &= \lim_{j\to\infty} \kappa_M(f_j^{-1}(f_s(z)); (df_j^{-1})_{f_s(z)} \circ (df_s)_z(v)) \\ &= \lim_{j\to\infty} \kappa_M(\varphi_{s,j}(z); (d\varphi_{s,j})_z(v)).\end{aligned} \tag{4.6}$$

□

As corollaries we find (cf. [7, Theorem 1.6])

Corollary 4.16. *Let $(\varphi_{s,t})$ be an algebraic evolution family on the unit disc $\mathbb{D}$. If there exist $z \in \mathbb{D}$, $v \in T_z\mathbb{D}$, $s \geq 0$, such that $\beta^s_v(z) = 0$ then*

i) $\mathsf{Lr}\,(\varphi_{s,t})$ *is biholomorphic to* $\mathbb{C}$,

ii) $\beta^s_v(z) = 0$ *for all* $z \in \mathbb{D}$, $v \in T_z\mathbb{D}$, $s \geq 0$.

If there exist $z \in \mathbb{D}$, $v \in T_z\mathbb{D}$, $s \geq 0$, such that $\beta^s_v(z) \neq 0$ then

i) $\mathsf{Lr}\,(\varphi_{s,t})$ *is biholomorphic to* $\mathbb{D}$,

ii) $\beta^s_v(z) \neq 0$ *for all* $z \in \mathbb{D}$, $v \in T_z\mathbb{D}$, $s \geq 0$.

Proof. Since the Loewner range $\mathsf{Lr}\,(\varphi_{s,t})$ is non-compact and simply connected, by the uniformization theorem it has to be biholomorphic to $\mathbb{D}$ or $\mathbb{C}$. Since

$$\kappa_{\mathbb{C}}(z;v)=0,\quad z\in\mathbb{C}, v\in T_z\mathbb{C},$$
$$\kappa_{\mathbb{D}}(z;v)\neq 0,\quad z\in\mathbb{D}, v\in T_z\mathbb{D},$$

the result follows from Proposition 4.15. □

Corollary 4.17. *Let $d\in[1,+\infty]$. Let $(\varphi_{s,t})$ be an L^d-evolution family on the unit disc $\mathbb{D}$. Then there exists an L^d-Loewner chain (f_t) associated with $(\varphi_{s,t})$ such that $\mathsf{rg}\,(f_t)$ is either the unit disc $\mathbb{D}$ or the complex plane $\mathbb{C}$.*

Proof. It follows from Corollary 4.16 and Theorem 4.10. □

Such a result can be generalized in higher dimension as follows. As customary, let us denote by $\mathsf{aut}(M)$ the group of holomorphic automorphisms of a complex manifold M. Notice that condition M hyperbolic and $M/\mathsf{aut(M)}$ compact implies that M is complete hyperbolic (see [11]).

Theorem 4.18. *Let M be a hyperbolic complex manifold and assume that $M/\mathsf{aut(M)}$ is compact. Let $(\varphi_{s,t})$ be an algebraic evolution family on M. Then*

(1) *If there exists $z\in M$, $s\geq 0$ such that $\beta^s_v(z)\neq 0$ for all $v\in T_zM$ with $v\neq 0$ then $\mathsf{Lr}\,(\varphi_{s,t})$ is biholomorphic to M.*
(2) *If there exists $z\in M$, $s\geq 0$ such that $\dim_{\mathbb{C}}\{v\in T_zM : \beta^s_v(z)=0\}=1$ then $\mathsf{Lr}\,(\varphi_{s,t})$ is a fiber bundle with fiber $\mathbb{C}$ over a closed complex submanifold of M.*

Proof. It follows at once from Proposition 4.15 and [11, Theorem 3.2, Main Theorem]. □

In particular the previous result applies to $M=\mathbb{B}^n$ (or even to the polydiscs in $\mathbb{C}^n$) and we obtain

Corollary 4.19. *Let $(\varphi_{s,t})$ be an algebraic evolution family on the unit ball $\mathbb{B}^n$. If for some $z\in\mathbb{B}^n$, $s\geq 0$ it follows that $\dim_{\mathbb{C}}\{v\in\mathbb{C}^n : \beta^s_v(z)=0\}\leq 1$, then there exists an algebraic Loewner chain $(f_t\colon M\to\mathbb{C}^n)$ associated with $(\varphi_{s,t})$.*

Proof. If the dimension is zero, then by Theorem 4.18 the Loewner range is biholomorphic to $\mathbb{B}^n\subset\mathbb{C}^n$. If the dimension is one, then by Theorem 4.18 the Loewner range is a fiber bundle with fiber $\mathbb{C}$ over a closed complex submanifold of $\mathbb{B}^n$ and by [11, Corollary 4.8] it is actually biholomorphic to $\mathbb{B}^{n-1}\times\mathbb{C}\subset\mathbb{C}^n$. □

If $\dim_{\mathbb{C}}\{v\in\mathbb{C}^n : \beta^s_v(z)=0\}\geq 2$ the complex structure of the Loewner range can be more complicated: the Loewner range of the algebraic evolution family recalled in Remark 4.12 has $\dim_{\mathbb{C}}\{v\in\mathbb{C}^n : \beta^s_v(z)=0\}=2$ and is not biholomorphic to a domain of $\mathbb{C}^3$.

Example 4.20. Let $(\varphi_{s,t})$ be an algebraic evolution family of $\mathbb{B}^2$ such that $\varphi_{s,t}(0)=0$ for all $0\leq s\leq t$ and $(d\varphi_{s,t})_0=e^{A(t-s)}$ where A is a diagonal matrix with eigenvalues $i\theta$, $\theta\in\mathbb{R}$ and $\lambda\in\mathbb{C}$ for some $\mathsf{Re}\,\lambda\leq 0$. Then $\dim_{\mathbb{C}}\ker\beta^s_v(0)\leq 1$ (it is either 1 if $\mathsf{Re}\,\lambda<0$ or

0 if $\operatorname{Re}\lambda = 0$ which is the case if and only if $\varphi_{s,t}$ are automorphisms). Therefore in such a case there exists an algebraic Loewner chain with range in $\mathbb{C}^2$.

The previous example can be generalized as follows:

Example 4.21. Let $G(z,t)$ be an L^∞-Herglotz vector field in $\mathbb{B}^n$ such that $G(0,t) \equiv 0$ and $(d_z G)_{z=0}(\cdot, t) = A(t)$ where $A(t)$ is a diagonal $n \times n$ matrix with eigenvalues $\lambda_1(t), \dots, \lambda_n(t)$ where $\lambda_j : \mathbb{R}^+ \to \mathbb{C}$ are functions of class L^∞ such that $\operatorname{Re}\lambda_j(t) \leq 0$ for almost every $t \geq 0$ and $j = 1, \dots, n$. Assume that there exists $C > 0$ such that

$$\int_0^t \operatorname{Re}\lambda_j(\tau) d\tau \geq -C, \quad t \geq 0,\ j = 1, \dots, n-1.$$

Let $(\varphi_{s,t})$ be the associated L^∞-evolution family in $\mathbb{B}^n$. Then $\varphi_{s,t}(0) = 0$ and $(d\varphi_{s,t})_0$ is the diagonal matrix with eigenvalues $\exp\left(\int_s^t \lambda_j(\tau) d\tau\right)$ for $j = 1, \dots, n$. Hence $\dim_{\mathbb{C}} \ker \beta_v^s(0) \leq 1$ and there exists an associated L^∞-Loewner chain with range in $\mathbb{C}^n$.

5. Loewner-Kufarev PDE

In this section we prove that L^d-Loewner chains on complete hyperbolic complex manifolds are the univalent solutions of the Loewner-Kufarev partial differential equation, as in the classical theory of Loewner chains on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ (see [14], [17]). Other results related to the solutions of the Loewner-Kufarev PDE on $\mathbb{B}^n$ may be found in [8].

Proposition 5.1. *Let M be a complete hyperbolic complex manifold, and let $(f_t \colon M \to N)$ be a Loewner chain of order $d \in [1, +\infty]$ on M. Then there exists a set $E \subset \mathbb{R}^+$ (independent of z) of zero measure such that for every $s \in (0, +\infty) \setminus E$, the mapping*

$$M \ni z \mapsto \frac{\partial f_s}{\partial s}(z) \in T_{f_s(z)} N$$

is well-defined and holomorphic on M.

Proof. The manifold $M \times (0, +\infty)$ has a countable basis $\mathfrak{B}$ given by products of the type $B \times I$, where $B \subset M$ is an open subset biholomorphic to a ball, and $I \subset (0, +\infty)$ is a bounded open interval. Let $\mathcal{V}$ be a countable covering of N by charts. By Remark 4.5 the mapping $\Psi \colon M \times (0, +\infty) \to N$ is continuous, hence there exists a covering $\mathcal{U} \subset \mathfrak{B}$ of $M \times (0, +\infty)$ such that for any $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subset \Psi^{-1}(V)$.

We will prove that for each $U = B \times I \in \mathcal{U}$ there exists a set of full measure $I' \subseteq I$ such that $B \ni z \mapsto \frac{\partial f_s}{\partial s}(z)$ is well defined and holomorphic for all $s \in I'$. Hence $M \ni z \mapsto \frac{\partial f_s}{\partial s}(z)$ will be well defined and holomorphic for $s \in \mathbb{R}^+ \setminus \bigcup_{\mathcal{U}} (I \setminus I')$ which is a set of full measure in $\mathbb{R}^+$.

We can assume that $M = \mathbb{B}^n$, $N = \mathbb{C}^n$, and that the distance d_N is the Euclidean distance. Since $t \mapsto f_t(z)$ is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in \mathbb{B}^n$, we deduce that for each $z \in \mathbb{B}^n$, there is a null set $E_1(z) \subset I$ such that for each $t \in I \setminus E_1(z)$, there exists the limit

$$\frac{\partial f_t}{\partial t}(z) = \lim_{h \to 0} \frac{f_{t+h}(z) - f_t(z)}{h}.$$

By definition there exists a function $p_k \in L^d(I, \mathbb{R}^+)$ such that

$$\|f_s(z) - f_t(z)\| \leq \int_s^t p_k(\xi) d\xi, \quad z \in \mathbb{B}^n_{1-1/k}, \quad s \leq t \in I. \tag{5.1}$$

Also, since $p_k \in L^d(I, \mathbb{R}^+)$, we may find a null set $E_2(k) \subset I$ such that for each $t \in I \setminus E_2(k)$, there exists the limit

$$p_k(t) = \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} p_k(\xi) d\xi, \quad k \in \mathbb{N}. \tag{5.2}$$

Next, let Q be a countable set of uniqueness for the holomorphic functions on $\mathbb{B}^n$ and let

$$E = \left(\bigcup_{q \in Q} E_1\,(q) \right) \bigcup \left(\bigcup_{k=1}^{\infty} E_2(k) \right).$$

Then E is a null subset of $\mathbb{R}^+$, which does not depend on $z \in \mathbb{B}^n$. Arguing as in the proof of [7, Theorem 4.1(1)(a)], it is not difficult to see that (5.1) and (5.2) imply that for each $s \in I \setminus E$, the family

$$\{(f_{s+h}(z) - f_s(z))/h, \quad 0 < |h| < \mathsf{dist}(s, \partial I)\}$$

is relatively compact and has a unique accumulation point for $|h| \to 0$ by Vitali Theorem in several complex variables, proving the result. $\square$

Theorem 5.2. *Let M be a complete hyperbolic complex manifold and let N be a complex manifold of the same dimension. Let $G : M \times \mathbb{R}^+ \to TM$ be a Herglotz vector field of order $d \in [1, +\infty]$ associated with the L^d-evolution family $(\varphi_{s,t})$. Then a family of univalent mappings $(f_t\colon M \to N)$ is an L^d-Loewner chain associated with $(\varphi_{s,t})$ if and only if it is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in M$ and solves the Loewner-Kufarev PDE*

$$\frac{\partial f_s}{\partial s}(z) = -(df_s)_z G(z, s), \quad a.e.\ s \geq 0, z \in M. \tag{5.3}$$

Proof. Since $G(z,t)$ and $(\varphi_{s,t})$ are associated there exists a null set $E_1 \subset \mathbb{R}^+$ such that for all $s \geq 0$, for all $t \in [s, +\infty) \setminus E_1$ and for all $z \in M$,

$$\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t).$$

Let now (f_t) be an L^d-Loewner chain associated with $(\varphi_{s,t})$. By Proposition 5.1, there is a null set $E_2 \subset \mathbb{R}^+$ such that $z \mapsto \frac{\partial f_s}{\partial s}(z)$ is well defined and holomorphic for all

$s \in (0,+\infty) \setminus E_2$. The set $E = E_1 \cup E_2$ has also zero measure. It is clear that the mapping

$$t \mapsto L_t(z) := f_t(\varphi_{0,t}(z))$$

is locally absolutely continuous on $\mathbb{R}^+$ locally uniformly with respect to $z \in M$, in view of the conditions (4.1) and (2.2). Also $L_t(z) = f_0(z)$ for $z \in M$. Differentiating the last equality with respect to $t \in (0,+\infty) \setminus E$ we obtain

$$\begin{aligned} 0 &= (df_t)_{\varphi_{0,t}(z)} \frac{\partial \varphi_{0,t}}{\partial t}(z) + \frac{\partial f_t}{\partial t}(\varphi_{0,t}(z)) \\ &= (df_t)_{\varphi_{0,t}(z)} G(\varphi_{0,t}(z), t) + \frac{\partial f_t}{\partial t}(\varphi_{0,t}(z)), \end{aligned}$$

for all $t \in (0,+\infty) \setminus E$ and for all $z \in M$. Hence

$$\frac{\partial f_t}{\partial t}(w) = -(df_t)_w G(w,t),$$

for all w in the open set $\varphi_{0,t}(M)$ and for all $t \in (0,+\infty) \setminus E$. The identity theorem for holomorphic mappings provides the result.

To prove the converse, fix $s \geq 0$ and let

$$L_t(z) := f_t(\varphi_{s,t}(z))$$

for $t \in [s,+\infty)$ and $z \in M$. In view of the hypothesis, it is not difficult to deduce that

$$\frac{\partial L_t}{\partial t}(z) = 0, \quad \text{a.e. } t \geq 0, \quad \forall z \in M.$$

Hence $L_t(z) \equiv L_s(z)$, i.e. $f_t(\varphi_{s,t}(z)) = f_s(z)$ for all $z \in M$ and $0 \leq s \leq t$, which means that (f_t) is an algebraic Loewner chain associated with $(\varphi_{s,t})$. Hence (f_t) is an L^d-Loewner chain by Theorem 4.10. □

The following result has been proved with different method also in [19].

Corollary 5.3. *Let $d \in [1,\infty]$. Let M be a complete hyperbolic complex manifold and let N be a complex manifold of the same dimension. Every L^d-Loewner chain $(f_t\colon M \to N)$ solves a Loewner-Kufarev PDE.*

Proof. By Theorem 4.6 there exists an L^d-evolution family $(\varphi_{s,t})$ associated with (f_t). By Theorem 2.5 there exists a Herglotz vector field $G(z,t)$ of order d associated with $(\varphi_{s,t})$. Theorem 5.2 yields then that the family $(f_t\colon M \to N)$ satisfies

$$\frac{\partial f_s}{\partial s}(z) = -(df_s)_z G(z,s), \quad \text{a.e. } s \geq 0, z \in M.$$

□

From Theorems 4.7 and 4.10 we easily obtain the following corollary. Let $G(z,t)$ be an L^d-Herglotz vector field associated with the L^d-evolution family $(\varphi_{s,t})$.

Corollary 5.4. *Let M be a complete hyperbolic complex manifold of dimension n. The Loewner-Kufarev PDE (5.3) admits a solution given by univalent mappings $(f_t\colon M \to N)$ where N is a complex manifold N of dimension n. Any other solution with values in a complex manifold Q is of the form $(\Lambda \circ f_t)$ where $\Lambda\colon \mathsf{rg}\,(f_t) \to Q$ is holomorphic. Hence a solution given by univalent mappings $(h_t\colon M \to \mathbb{C}^n)$ exists if and only if the Loewner range $\mathsf{Lr}\,(\varphi_{s,t})$ is biholomorphic to a domain in $\mathbb{C}^n$.*

6. Conjugacy

We introduce a notion of conjugacy for L^d-evolution families which preserves the Loewner range. This can be used to put an L^d-evolution family in some normal form without changing its Loewner range (cf. [7, Proposition 2.9]).

Definition 6.1. Let $d \in [1, +\infty]$. Let $(h_t\colon M \to Q)$ be an L^d-Loewner chain such that each $h_t\colon M \to Q$ is a biholomorphism. We call $(h_t\colon M \to Q)$ a family of *intertwining mappings of order d*. If $(\varphi_{s,t}), (\psi_{s,t})$ are L^d-evolution families on M, Q respectively and

$$\psi_{s,t} \circ h_s = h_t \circ \varphi_{s,t}, \quad 0 \le s \le t,$$

then we say that $(\varphi_{s,t})$ and $(\psi_{s,t})$ are *conjugate*. It is easy to see that conjugacy is an equivalence relation.

Lemma 6.2. *Let $(h_t\colon M \to Q)$ be a family of intertwining mappings of order d and let $(f_t\colon Q \to N)$ an L^d-Loewner chain. Then $(f_t \circ h_t\colon M \to N)$ is an L^d-Loewner chain.*

Proof. It is clear that $(f_t \circ h_t\colon M \to N)$ is an algebraic Loewner chain. Let $T > 0$ and let $K \subset M$ be a compact set. The set $\hat{K} := \bigcup_{0 \le t \le T} h_t(K) \subset Q$ is compact by Remark 4.5, and the family $(f_t)_{0 \le t \le T}$ is equi-Lipschitz on $\hat{K}$ (see (4.3)). Thus if $0 \le s \le t \le T$ and $z \in K$,

$$\begin{aligned} d_N(f_t(h_t(z)), f_s(h_s(z))) &\le d_N(f_t(h_t(z)), f_t(h_s(z))) + d_N(f_t(h_s(z)), f_s(h_s(z))) \\ &\le L(\hat{K}, T) d_Q(h_t(z), h_s(z)) + \int_s^t k_{\hat{K},T}(\xi) d\xi \\ &\le L(\hat{K}, T) \int_s^t h_{K,T}(\xi) d\xi + \int_s^t k_{\hat{K},T}(\xi) d\xi. \end{aligned} \tag{6.1}$$

□

Remark 6.3. Two conjugated L^d-evolution families have essentially the same associated L^d-Loewner chains. Namely, if $(g_t\colon Q \to N)$ is an L^d-Loewner chain associated with $(\psi_{s,t})$ which is conjugate to $(\varphi_{s,t})$ on M via $(h_t\colon M \to Q)$, then $(g_t \circ h_t\colon M \to N)$ is an L^d-Loewner chain associated with $(\varphi_{s,t})$. In particular, $\mathsf{Lr}\,(\varphi_{s,t})$ is biholomorphic to $\mathsf{Lr}\,(\psi_{s,t})$.

Proposition 6.4. *Let $(\psi_{s,t})$ be an L^d-evolution family on a complete hyperbolic complex manifold Q and let $(h_t\colon M \to Q)$ be a family of intertwining mappings of order d. Then the family $(h_t^{-1} \circ \psi_{s,t} \circ h_s)$ is an L^d-evolution family on M.*

Proof. By Theorem 4.10, there exists an L^d-Loewner chain $(f_t\colon Q \to N)$ associated with $(\psi_{s,t})$. By Lemma 6.2 the family $(f_t \circ h_t)$ defines an L^d-Loewner chain $(g_t\colon M \to N)$. Then

$$h_t^{-1} \circ \psi_{s,t} \circ h_s = h_t^{-1} \circ f_t^{-1} \circ f_s \circ h_s = g_t^{-1} \circ g_s,$$

which by Theorem 4.6 is an L^d-evolution family on M. □

Let now M be the unit ball $\mathbb{B}^n$.

Definition 6.5. Take $a \in \mathbb{B}^n$. Let $P_a(z) := \frac{\langle z,a\rangle}{\|a\|^2}a$ for $a \neq 0$, $P_0 = 0$, $Q_a(z) := z - P_a(z)$ and $s_a := (1 - \|a\|^2)^{1/2}$. Then

$$\varphi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}$$

is an automorphism of the ball $\mathbb{B}^n$ (see, *e.g.*, [1] or [32]).

We can now show that in order to study the Loewner range of an L^d-evolution family on $\mathbb{B}^n$ one can assume that it fixes the origin.

Corollary 6.6. *Let $(\psi_{s,t})$ be an L^d-evolution family on $\mathbb{B}^n$. There exists a conjugate L^d-evolution family $(\varphi_{s,t})$ such that*

$$\varphi_{s,t}(0) = 0, \quad 0 \leq s \leq t.$$

Proof. Set $a(t) := \psi_{0,t}(0)$. Since

$$\|\varphi_{a(t)}(w) - \varphi_{a(s)}(w)\| \leq C(K,T)\|a(t) - a(s)\|, \quad w \in K,\ 0 \leq s \leq t \leq T,$$

the family $(\varphi_{a(t)})$ is a family of intertwining mappings of order d. Define

$$\varphi_{s,t} := \varphi_{a(t)}^{-1} \circ \psi_{s,t} \circ \varphi_{a(s)},$$

which is an L^d-evolution family by Proposition 6.4. Since $\varphi_{a(t)}(0) = a(t)$, we have $\varphi_{0,t}(0) = 0$ for all $t \geq 0$, and by the evolution property $\varphi_{s,t}(0) = 0$ for all $0 \leq s \leq t$. □

7. Extension of Loewner chains from lower dimensional balls

The following result provides examples of L^d-Loewner chains on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, which are generated by the Roper-Suffridge extension operator [31]. This operator preserves convexity (see [31]), starlikeness and the notion of parametric representation (see e.g. [17] and the references therein).

Theorem 7.1. *Let $d \in [1,+\infty]$ and $(f_t\colon \mathbb{D} \to \mathbb{C})$ be an L^d-Loewner chain on the unit disc $\mathbb{D}$ such that $|\arg f_t'(0)| < \pi/2$ and $|\arg(f_s'(0)/f_t'(f_t^{-1} \circ f_s(0)))| < \pi/2$ for $t \geq s \geq 0$. Also let $(F_t\colon \mathbb{B}^n \to \mathbb{C}^n)$ be given by*

$$F_t(z) = \Big(f_t(z_1), \tilde{z}e^{t/2}\sqrt{f_t'(z_1)}\Big), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n, \quad t \geq 0. \tag{7.1}$$

Then (F_t) is an L^d-Loewner chain.

Proof. It is easy to see that F_t is univalent on $\mathbb{B}^n$ for $t \geq 0$. Let $(\varphi_{s,t})$ be the L^d-evolution family associated with (f_t) (see Theorem 4.6 or [4]). Also let $\Phi_{s,t} : \mathbb{B}^n \to \mathbb{C}^n$ be given by

$$\Phi_{s,t}(z) = \Big(\varphi_{s,t}(z_1), \tilde{z}e^{(s-t)/2}\sqrt{\varphi_{s,t}'(z_1)}\Big), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,\ t \geq s \geq 0.$$

Then $\Phi_{s,t}$ is a univalent mapping on $\mathbb{B}^n$ and in view of the Schwarz-Pick lemma, we have

$$\|\Phi_{s,t}(z)\|^2 = |\varphi_{s,t}(z_1)|^2 + \|\tilde{z}\|^2 e^{s-t}|\varphi_{s,t}'(z_1)|$$

$$< |\varphi_{s,t}(z_1)|^2 + (1-|z_1|^2)e^{s-t} \cdot \frac{1-|\varphi_{s,t}(z_1)|^2}{1-|z_1|^2} \leq 1,\ z \in \mathbb{B}^n,\ t \geq s \geq 0.$$

Hence $\Phi_{s,t}(\mathbb{B}^n) \subseteq \mathbb{B}^n$, and since $F_s(z) = F_t(\Phi_{s,t}(z))$ for $z \in \mathbb{B}^n$ and $t \geq s \geq 0$, we obtain that $F_s(\mathbb{B}^n) \subseteq F_t(\mathbb{B}^n)$ for $s \leq t$. In view of the above relations, we deduce that (F_t) is an algebraic Loewner chain and $(\Phi_{s,t})$ is the associated algebraic evolution family.

It remains to prove that (F_t) is of order d. Since $(\varphi_{s,t})$ is an evolution family of order d, we deduce in view of [4, Theorem 6.2] that there exists a Herglotz vector field $g(z_1,t)$ of order d such that

$$\frac{\partial \varphi_{s,t}}{\partial t}(z_1) = g(\varphi_{s,t}(z_1), t), \quad \text{a.e.} \quad t \in [s,+\infty), \quad \forall z_1 \in \mathbb{D}.$$

Now, let $G = G(z,t) : \mathbb{B}^n \times \mathbb{R}^+ \to \mathbb{C}^n$ be given by

$$G(z,t) = \Big(g(z_1,t), \frac{\tilde{z}}{2}(-1 + g'(z_1,t))\Big), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,\ t \geq 0.$$

Then $G(z,t)$ is a weak holomorphic vector field of order d on $\mathbb{B}^n$. Indeed, the measurability and holomorphicity conditions from the definition of a weak holomorphic vector field are satisfied. We next prove that for each $r \in (0,1)$ and $T > 0$, there exists $C_{r,T} \in L^d([0,T], \mathbb{R})$ such that

$$\|G(z,t)\| \leq C_{r,T}(t), \quad \|z\| \leq r, \quad t \in [0,T].$$

But the above condition can be easily deduced by using the fact that $g(z_1,t)$ is a Herglotz vector field of order d on $\mathbb{D}$ and by the Cauchy integral formula.

On the other hand, since $\varphi_{s,t}$ is locally absolutely continuous on $[s,+\infty)$ locally uniformly with respect to $z_1 \in \mathbb{D}$, it follows in view of Vitali's theorem (see e.g. [28, Chapter 6]) that

$$\frac{\partial}{\partial t}\Big(\frac{\partial \varphi_{s,t}(z_1)}{\partial z_1}\Big) = \frac{\partial}{\partial z_1}\Big(\frac{\partial \varphi_{s,t}(z_1)}{\partial t}\Big), \quad \text{a.e.} \quad t \geq s, \quad \forall z_1 \in \mathbb{D}.$$

Using the above equality, we obtain by elementary computations that

$$\frac{\partial \Phi_{s,t}(z)}{\partial t} = G(\Phi_{s,t}(z), t), \qquad \text{a.e.} \qquad t \geq s, \quad \forall z \in \mathbb{B}^n.$$

Therefore, as in the proof of [5, Proposition 2], we deduce that

$$(dk_{\mathbb{B}^n})_{(z,w)}(G(z,t), G(w,t)) \leq 0$$

for a.e. $t \in \mathbb{R}^+$, $z \neq w$. Hence $G(z,t)$ is a Herglotz vector field of order d on $\mathbb{B}^n$. Also, as in the proof of [5, Proposition 1], we deduce that $(\Phi_{s,t})$ is an evolution family of order d. Finally, we conclude that the associated algebraic Loewner chain (F_t) is of order d on $\mathbb{B}^n$ by Theorem 4.10. This completes the proof. □

Corollary 7.2. *Let $f : \mathbb{D} \to \mathbb{C}$ be a univalent function such that $|\arg f'(0)| < \pi/2$. Assume that (f_t) is an L^d-Loewner chain on $\mathbb{D}$ such that $f_0 = f$, $|\arg f_t'(0)| < \pi/2$ for $t \geq 0$, and $|\arg(f_s'(0)/f_t'(f_t^{-1} \circ f_s(0)))| < \pi/2$ for $t \geq s \geq 0$. Then $F = \Phi_n(f)$ can be imbedded in a L^d-Loewner chain on $\mathbb{B}^n$, where Φ_n is the Roper-Suffridge extension operator,*

$$\Phi_n(f)(z) = (f(z_1), \tilde{z}\sqrt{f'(z_1)}), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n.$$

Proof. The desired L^d-Loewner chain is given by (7.1). □

8. Spiral-shapedness and Star-shapedness

Definition 8.1. Let $\Omega \subset \mathbb{C}^n$ and let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$, where

$$m(A) = \min\{\mathsf{Re}\,\langle A(z), z\rangle : \|z\| = 1\}.$$

We say that Ω is *spiral-shaped with respect to* A if $e^{-tA}(w) \in \Omega$ whenever $w \in \Omega$ and $t \in \mathbb{R}^+$. If $A = \mathsf{id}$ and Ω is spiral-shaped with respect to id, we say that Ω is *star-shaped*.

If f is a univalent mapping on $\mathbb{B}^n$, then f is called *spiral-shaped with respect to* A if the image domain $\Omega = f(\mathbb{B}^n)$ is spiral-shaped with respect to A. In addition, if $A = \mathsf{id}$ and f is spiral-shaped with respect to id, we say that f is *star-shaped* (see [10]).

Remark 8.2. It is clear that if f is spiral-shaped with respect to A, then $0 \in \overline{f(\mathbb{B}^n)}$. Moreover, if $0 \in f(\mathbb{B}^n)$, then the above definition reduces to the usual definition of spiral-likeness (respectively star-likeness) (see [18] and [34]).

We next present some applications of Theorem 5.2 to the case $M = \mathbb{B}^n$. The first result provides necessary and sufficient conditions for a locally univalent mapping on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ to be spiral-shaped, and thus univalent on $\mathbb{B}^n$.

Remark 8.3. We remark that the equivalence between the conditions (i) and (iii) in Theorem 8.4 below was first obtained by Elin, Reich and Shoikhet (see the proof of [10, Proposition 3.5.2]; cf. [10, Proposition 3.7.2]; [30]) by different arguments (compare [9]). In the case $f(0) = 0$, the analytic characterization of spiral-likeness is due to Gurganus [18] and Suffridge [34].

Theorem 8.4. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally univalent mapping such that $0 \in \overline{f(\mathbb{B}^n)}$. Also let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$. Then the following conditions are equivalent:*
(i) f is spiral-shaped with respect to A;
(ii) The family $(f_t := e^{tA} f(z))_{t \geq 0}$ is an L^∞-Loewner chain.
(iii) f is univalent on $\mathbb{B}^n$ and

$$\mathsf{Re}\,\langle (df_z)^{-1} Af(z), z\rangle \geq (1 - \|z\|^2)\mathsf{Re}\,\langle (df_0)^{-1} Af(0), z\rangle, \quad z \in \mathbb{B}^n. \tag{8.1}$$

Proof. The equivalence between the conditions (i) and (ii) is immediate. Now, we assume that the condition (ii) holds. Then f is univalent on $\mathbb{B}^n$. Let $G(z,t)$ be the Herglotz vector field of order ∞ given by Corollary 5.3. A direct computation from (5.3) implies

$$G(z,t) = -(df_z)^{-1} Af(z), \quad t \geq 0, \quad z \in \mathbb{B}^n. \tag{8.2}$$

Since by the very definition a Herglotz vector field is a semicomplete vector field for almost every $t \geq 0$, it follows that $-(df_z)^{-1} Af(z)$ is semicomplete. Hence, by [10, Proposition 3.5.2] (where the sign convention is different from the one adopted here), we deduce the relation (8.1), as claimed.

Conversely, assume that the condition (iii) holds. Clearly (f_t) is a family of univalent mappings on $\mathbb{B}^n$ such that the mapping $t \mapsto f_t(z)$ is of class C^∞ on $\mathbb{R}^+$ for all $z \in \mathbb{B}^n$. Also (f_t) satisfies the differential equation

$$\frac{\partial f_t}{\partial t}(z) = -(df_t)_z G(z,t), \qquad \text{a.e.} \quad t \geq 0, \quad \forall\, z \in \mathbb{B}^n, \tag{8.3}$$

where $G(z,t)$ is given by (8.2). In view of the condition (8.1) and [10, Lemma 3.3.2], we deduce that the mapping $G(z,t)$ is a semicomplete vector field for all $t \geq 0$, and thus it is a Herglotz vector field of order ∞ by [6, Theorem 0.2]. Hence (f_t) is an L^∞-Loewner chain by Theorem 5.2. This completes the proof. $\square$

We next give the following analytic characterization of star-shapedness on the unit ball $\mathbb{B}^n$ (cf. [10]). In the case $f(0) = 0$, the inequality in the third statement becomes the well known analytic characterization of star-likeness for locally univalent mappings on $\mathbb{B}^n$ (see [13], [17], [34] and the references therein). Necessary and sufficient conditions for star-likeness with respect to a boundary point are given in [22].

Corollary 8.5. *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally univalent mapping such that $0 \in \overline{f(\mathbb{B}^n)}$. Then the following conditions are equivalent:*
(i) f is star-shaped;
(ii) The family $(f_t := e^t f(z))_{t \geq 0}$ is an L^∞-Loewner chain.
(iii) f is univalent on $\mathbb{B}^n$ and

$$\mathsf{Re}\,\langle (df_z)^{-1} f(z), z\rangle \geq (1 - \|z\|^2)\mathsf{Re}\,\langle (df_0)^{-1} f(0), z\rangle, \quad z \in \mathbb{B}^n.$$

Corollary 8.6. *Let $f : \mathbb{D} \to \mathbb{C}$ be a star-shaped function on $\mathbb{D}$ such that $|\arg f'(0)| < \pi/2$ and $|\arg(f'(0)/f'(f^{-1}(\lambda f(0))))| < \pi/2$ for $\lambda \in (0,1]$. Also let $F = \Phi_n(f)$. Then F is also star-shaped on $\mathbb{B}^n$.*

Proof. Since f is star-shaped, it follows that $f_t(z_1) = e^t f(z_1)$ is an L^∞-Loewner chain by Corollary 8.5. Let (F_t) be the chain given by (7.1). In view of Theorem 7.1, (F_t) is an L^∞-Loewner chain on $\mathbb{B}^n$. Moreover, since $0 \in \overline{F(\mathbb{B}^n)}$ and $F_t(z) = e^t F(z)$, we deduce that the mapping $F = F_0$ is star-shaped on $\mathbb{B}^n$, by Corollary 8.5. This completes the proof. □

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