LOCALIZATIONS, PARTIAL HOLOMORPHIC CONNECTIONS, THE ATIYAH BUNDLE AND THE CAMACHO-SAD INDEX THEOREM

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1. INTRODUCTION

The classical index theorem due to C. Camacho and P. Sad [12] says that if a compact complex curve S in a two dimensional complex manifold is invariant by a holomorphic foliation then its self-intersection number is the sum of Grothendieck residues at the singularities of the restriction of the foliation on the curve.

Such a deep result is nowadays a very important and useful tool in dynamics. For instance, based on such a result Camacho and Sad proved the existence of separatrices for germs of holomorphic vector fields in $\mathbb{C}^2$.

The Camacho-Sad index theorem contains several hypotheses, which we list as follows:

(a) a complex two dimensional manifold M;
(b) a non-singular complex compact curve S;
(c) a holomorphic foliation $\mathcal{F}$ on M;
(d) the curve S is $\mathcal{F}$-invariant.

The Camacho-Sad index theorem has been generalized—and such generalizations profitably used—changing from time to time some of the previous ingredients, by several authors. For instance, A. Lins Neto [15] and T. Suwa [17] replaced (b) by allowing S to be any (possibly singular) curve in M. Then D. Lehmann [13] and T. Suwa and D. Lehmann [14] relaxed (a), (b) by allowing complex manifolds and subvarieties of any (co)dimensions. A rather complete list of other papers related to generalizations of the Camacho-Sad index theorem with respect to (a) and (b) can be found in [18]. More recently, C. Camacho and D. Lehmann [10] and C. Camacho, H. Movasati and P. Sad [11] dropped the hypothesis (d), replacing it by some assumption on the way S sits in M (for instance if S is the zero section of the vector bundle M and the dimension of S equals the dimension of $\mathcal{F}$). Along a different perspective, M. Abate replaced (c) and (d) by the hypothesis of the existence of a holomorphic diffeomorphism pointwise fixing the curve S and "tangent" to it. Later, F. Tovena and the author [9] allowed S to be singular and in [2] those results were extended to any dimension (and used in discrete dynamics). Also, the "tangentiality" hypothesis has been replaced by some assumption on the way S sits in M (see [2], [3]).

In these notes we are going to explain a general strategy, discovered by M. Abate, F. Tovena and the author in [3], to obtain all the previous (and actually any other which will be discovered in future) Camacho-Sad type index theorems.

The strategy is as follows. We first describe a cohomological machinery, essentially developed by D. Lehmann and T. Suwa, which allows to "localize" characteristic classes outside open sets where one has some vanishing representatives. The next step is then to understand what might yield to have vanishing representatives. The Bott vanishing theorem, due to the existence of partial holomorphic connections, is the needed tool. Then we discuss partial holomorphic connections and we see that existence of such objects is equivalent to the existence of particular morphisms into a universal bundle, which we call the *Atiyah bundle*. Such morphisms are then defined naturally from "dynamically objects" such as holomorphic foliations and holomorphic diffeomorphisms, thus closing the circle.

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2. General strategy for localization of characteristic classes

Let X be a n-dimensional complex variety and let $\varphi \in H^\bullet(X)$ be an element of its cohomology. Such a class might represent the obstruction to the existence of a certain global object. For instance—and this is the case we are most interested in here—Chern classes represents the obstruction of existence of global frames for complex vector bundles).

Very roughly speaking, it is interesting to understand which "parts" of the variety really obstruct the existence of the object represented by φ. Such loci, call them S, might not be unique in general but there might be a good choice of them, depending on the problem one is facing. Therefore, once removed the "obstructions", the object exists and thus the cohomology class representing it vanishes on $M \setminus S$. Therefore it "localizes" at S (in a way we will explain later).

To explain slightly better this point of view, one can think of the complex projective space $X = \mathbb{CP}^N$. Let L be a non-trivial holomorphic line bundle on X. Its first Chern class is not vanishing and measures the obstruction of L from being trivial, namely, the existence of a never zero holomorphic section. If we remove a hyperplane H from X, then we obtain $M := X \setminus H \simeq \mathbb{C}^N$ and the restriction of L to M is trivial. This implies that the first Chern class of L might be localized at H.

Now we describe more in detail the cohomological-homological methods to localize characteristic classes.

Let $P : H^\bullet(X) \to H_{2n-\bullet}(X)$ be the Poincaré homomorphism (isomorphism if X is nonsingular).

Suppose that S is an analytic subset of X and let $U = X \setminus S$. Look at the cohomological exact sequence

$$\ldots \longrightarrow H^\bullet(M,U) \longrightarrow H^\bullet(M) \longrightarrow H^\bullet(U) \longrightarrow \ldots$$

and assume that $H^\bullet(M) \ni \varphi \mapsto 0 \in H^\bullet(U)$. Therefore there exists a lifting $\hat{\varphi} \in H^\bullet(M,U)$ of φ in the relative cohomology. This lifting is not unique in general. Anyhow, by the Alexander homomorphism (isomorphism if S is nonsingular) $A : H^\bullet(M,U) \to H_{2n-\bullet}(S)$ we have the following commuting diagram:

$$
\begin{array}{ccc}
H^\bullet(M,U) & \longrightarrow & H^\bullet(M) \\
{\scriptstyle A}\Big\downarrow & & \Big\downarrow{\scriptstyle P} \\
H_{2n-\bullet}(S) & \xrightarrow{i_*} & H_{2n-\bullet}(M)
\end{array}
\tag{2.1}
$$

An "index theorem" is thus given by the following formula:

$$P(\varphi) = i_*(\hat{\varphi}).$$

In particular if $\bullet = 2n$ and S is a finite set of points, denoting by $\mathrm{Res}(\hat{\varphi}, p) \in \mathbb{C}$ the "residue" at $p \in S$, we have

$$\int_M \varphi = \sum_{p\in S} \mathrm{Res}(\hat{\varphi}, p). \tag{2.2}$$

Typical examples of this situation appears when $\varphi = c_n(TM)$ (the top Chern class) and then the left-hand side of (2.2) is nothing but the Euler characteristic $\chi(M)$ of M. An example is the classical Poincaré-Hopf theorem.

In order to make formula (2.2) really useful one needs to have tools, reasonably good to handle, to compute cohomology. In non-singular spaces such a cohomology is provided by the Čech-de Rham cohomology (see, *e.g.*, [18]).

2.1. **The Čech-de Rham cohomology.** Assume X is non-singular (the case X is singular can be worked out using the so-called "extendable cohomology", see [16]). Let $\mathcal{U}$ be an open covering. For the sake of simplicity, we assume $\mathcal{U} = \{U_0, U_1\}$. Let $\Omega^k(\mathcal{U})$ be the set formed by the triples $(\omega_0, \omega_1, \omega_{01})$ where ω_0 is a k-form on U_0, ω_1 is a k-form on U_1 and ω_{01} is a $(k-1)$-form on the intersection $U_{01} = U_0 \cap U_1$. The operator $D : \Omega^k(\mathcal{U}) \to \Omega^{k+1}(\mathcal{U})$ defined by

$$D(\omega_0, \omega_1, \omega_{01}) = (d\omega_0, d\omega_1, d\omega_{01} + \omega_0|_{U_{01}} - \omega_1|_{U_{01}}),$$

gives rise to a complex $\{\Omega^\bullet(\mathcal{U}), D\}$ whose cohomology $H^\bullet(\mathcal{U})$ is naturally isomorphic (via partition of unity) to the de Rham cohomology of X. In other words, $H^\bullet(\mathcal{U})$ is the cohomology of the double complex Čech-de Rham.

In the Čech-de Rham cohomology setting, "localization" can be interpreted by finding D-closed triples $(\theta_0, \theta_1, \theta_{01})$ whose cohomology class represents the wanted obstruction/object and such that, for example, $\theta_0 \equiv 0$. The class $[(0, \theta_1, \theta_{01})]$ corresponds via the same isomorphism as before to a class which belongs to the relative cohomology $H^\bullet(X, U_0)$.

This is particularly useful with vector bundles. Given a complex vector bundle E of rank r on X, one can compute its Chern classes using the Čech-de Rham cohomology as follows: let ∇_0 be a connection for $E|_{U_0}$; let ∇_1 be a connection for $E|_{U_1}$. Let K_j be the curvature form of ∇_j, $j = 0, 1$. Let $c_i(\nabla_j)$ be the i-th Chern form of ∇_j, defined by $\det(I + K_j) = \sum_{l=0}^{r}(-2\pi i)^l c_l(\nabla_j)$, $j = 0, 1$, $i = 0, \ldots, r$. There exists a $(i-1)$-th form $c_i(\nabla_0, \nabla_1)$ on U_{01}, which we call the *Bott difference form*, and which makes the triple $(c_i(\nabla_0), c_i(\nabla_1), c_i(\nabla_0, \nabla_1))$

D-closed. The class $c_i(E) := [(c_i(\nabla_0), c_i(\nabla_1), c_i(\nabla_0, \nabla_1))] \in H^i(\mathcal{U})$ represents the i-th Chern class of E. The Bott difference form is defined by means of integration along the fibers as follows.

Let $p : U_{01} \times [0,1] \to U_{01}$ be the projection on the first factor and let $\tilde{E} := p^*(E|_{U_{01}})$ be the pull-back bundle. Note that $\tilde{E} \simeq E|_{U_{01}} \times [0,1]$. Then $\tilde{E}$ is a vector bundle over $U_{01} \times [0,1]$ whose fiber at a point (x,t) is $E_x \times \{t\} = E_x$. Thus the sections $\tilde{\sigma} \in C^\infty(\tilde{E}) = C^\infty(E|_{U_{01}} \times [0,1])$ which are "constant along the fibers $[0.1]$", *i.e.*, such that $\tilde{\sigma}(x,t) = \sigma(x)$ for some section σ of $E|_{U_{01}}$, generate $\mathcal{C}^\infty(\tilde{E})$ as a C^∞-module. This means that in order to define a connection $\tilde{\nabla}$ on $\tilde{E}$ it is enough to define it on sections which are constant along $[0,1]$. We let

$$
\begin{aligned}
(2.3) \qquad & (\tilde{\nabla}_v \tilde{\sigma})_{(x,t)} = t((\nabla_0)_v \sigma)_x + (1-t)((\nabla_1)_v \sigma)_x \quad \forall v \in C^\infty(TX|_{U_{01}}), \\
& (\tilde{\nabla}_{\frac{\partial}{\partial t}} \tilde{\sigma})_{(x,t)} = 0.
\end{aligned}
$$

We can now define the *Bott difference form* $c_i(\nabla_0, \nabla_1)$ integrating along the fibers as

$$
c_i(\nabla_0, \nabla_1) := (-1)^{\left[\frac{r}{2}\right]} \int_\Delta c_i(\tilde{\nabla}). \tag{2.4}
$$

Thus, if one is able to find a connection which is trivial with respect to a certain frame on U_0, a connection which is trivial with respect to another frames on U_1, the i-th Chern class is simply obtained as a $(i-1)$ form localized on U_{01}. We also recall how integration works with respect to Čech-de Rham cohomology. Let $\{R_0, R_1\}$ be a honeycomb cell with respect to $\mathcal{U}$. Namely, R_0, R_1 are two smooth compact manifolds with boundary such that $R_j \subset U_j$, $j = 0,1$, the interior of R_0 does not intersect the interior of R_1 and $R_0 \cup R_1 = X$. Therefore, given $[(\theta_0, \theta_1, \theta_{01})] \in H^{2n}(\mathcal{U})$, which corresponds to the class $\theta \in H^{2n}(X)$, we have

$$
\int_X \theta = \int_{R_0} \theta_0 + \int_{R_1} \theta_1 + \int_{\partial R_0} \theta_{01}.
$$

In order to clarify our point of view, we present the following example:

Example 2.1 (Intersection number of the exceptional divisor)**.** Let $\pi : X \to \mathbb{C}^2$ be the two dimensional manifolds obtained by blowing up the point O of $\mathbb{C}^2$. We want to compute the self-intersection number $D \cdot D$ of the exceptional divisor $D := \pi^{-1}(O)$ using the Čech-de Rham cohomology and the localization procedure as described above. Recall that $D \cdot D = \int_D c_1(N_D)$, where $N_D = TX|_D / TD$ is the normal bundle of D in X. Let U_0, U_1 be the two standard charts of X such that $U_0 \cup U_1 = X$, with coordinates (u_0, v_0) on U_0, and (u_1, v_1) on U_1 with $\pi(u_j, v_j) = (u_j v_j, v_j)$, $j = 0,1$ (namely, $D \cap U_j = \{v_j = 0\}$). Then $U_{01} = \{(u_0, v_0) : u_0 \neq 0\}$ and the change of coordinates is given by $u_1 = 1/u_0, v_1 = v_0/u_0$. On the chart U_j we choose the natural frame $[\frac{\partial}{\partial v_j}]$ for $(N_D)|_{U_j}$, where $[V]$ represents the image of a section V of $TX|_D$ into N_D. We define two trivial connections ∇_0 and ∇_1 for $(N_D)|_{U_0}$ and for $(N_D)|_{U_1}$ respectively by imposing:

$$
\nabla_0[\frac{\partial}{\partial v_0}] = \nabla_1[\frac{\partial}{\partial v_1}] = 0.
$$

Thus the 1-form associated to ∇_0 (respectively ∇_1) in such a frame $[\frac{\partial}{\partial v_0}]$ (respectively $[\frac{\partial}{\partial v_1}]$) is zero, hence the curvature in such a frame is zero and it follows obviously that $c_1(\nabla_0) = c_1(\nabla_1) = 0$. Therefore the class $c_1(N_D)$ is represented in the Čech-de Rham cohomology by $[(0, 0, c_1(\nabla_0, \nabla_1))]$. Notice that, although $c_1(\nabla_0) = c_1(\nabla_1) = 0$, this does not mean that $c_1(\nabla_0, \nabla_1) = 0$. Indeed, we consider U_{01} with coordinates (u_0, v_0). Then the connection $\tilde{\nabla}$, defined along the lines described above, has associated 1-form $\tilde{\theta}$ given by $t\theta_0 + (1-t)\theta_1$ where θ_0, θ_1 are the 1-forms of ∇_0, ∇_1 *in the frame* $[\frac{\partial}{\partial v_0}]$. Thus $\theta_0 = 0$, but to compute θ_1 we find

$$0 = (\nabla_1)_{\frac{\partial}{\partial u_1}}[\frac{\partial}{\partial v_1}] = (\nabla_1)_{-u_0^2\frac{\partial}{\partial u_0}}[u_0\frac{\partial}{\partial v_0}] = -u_0^2[\frac{\partial}{\partial v_0}] - u_0^3(\nabla_1)_{\frac{\partial}{\partial u_0}}[\frac{\partial}{\partial v_0}],$$

from which

$$(\nabla_1)_{\frac{\partial}{\partial u_0}}[\frac{\partial}{\partial v_0}] = -\frac{1}{u_0}[\frac{\partial}{\partial v_0}].$$

Therefore $\theta_1 = -\frac{du_0}{u_0}$. From this it follows that

$$\tilde{\theta} = t\theta_0 + (1-t)\theta_1 = (t-1)\frac{du_0}{u_0}.$$

Hence the curvature is $\tilde{K} = d\tilde{\theta} = dt \wedge \frac{du_0}{u_0}$ and $c_1(\tilde{\nabla}) = -\frac{1}{2\pi i}dt \wedge \frac{du_0}{u_0}$. Integrating along the fibers we find

$$c_1(\nabla_0, \nabla_1) = -\frac{1}{2\pi i}\frac{du_0}{u_0},$$

and then, setting $R_0 = \{(u_0, 0) : |u_0| \leq 1\}$ and $R_1 = \overline{D \setminus R_0}$, we obtain

$$\begin{aligned} D \cdot D = \int_D c_1(N_D) &= \int_{R_0} c_1(\nabla_0) + \int_{R_1} c_1(\nabla_1) + \int_{\partial R_0} c_1(\nabla_0, \nabla_1) \\ &= 0 + 0 + \int_{|u_0|=1} -\frac{1}{2\pi i}\frac{du_0}{u_0} = -1. \end{aligned}$$

3. Partial holomorphic connections and splittings

As explained in the previous section, in order to localize characteristic classes one is looking for “good reasons” to get certain forms vanished on certain open sets on the ambient manifold. One of these “good reasons” is to have a *partial holomorphic connection* as we define now. As a matter of notation, if E is a complex bundle over a manifold S, we use the letter $\mathcal{E}$ to denote the associated locally free $\mathcal{O}_S$-sheaf of its holomorphic sections.

Definition 3.1. Let F be a sub-bundle of the tangent bundle TS of a complex manifold S. A *partial holomorphic connection along* F on a complex vector bundle E on S is a $\mathbb{C}$-linear morphism $\nabla \colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\nabla(gs) = dg|_{\mathcal{F}} \otimes s + g\nabla s$$

for all $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$. Moreover, if F is involutive, the partial holomorphic connection ∇ is said to be *flat* if $\nabla_u \circ \nabla_v - \nabla_v \circ \nabla_u = \nabla_{[u,v]}$ for all $u, v \in \mathcal{F}$.

Roughly speaking, having a partial holomorphic connection means that one can "differentiate" holomorphically along some directions but not on all of them. The following result, known as the *Bott vanishing theorem* holds:

Theorem 3.2. *Let S be a complex manifold, F a sub-bundle of TS of rank ℓ, and E a complex vector bundle on S. Assume we have a partial holomorphic connection on E along F. Then:*

(i) *every symmetric polynomial in the Chern classes of E of degree larger than $\dim S - \ell + \lfloor \ell/2 \rfloor$ vanishes.*
(ii) *Furthermore, if F is involutive and the partial holomorphic connection is flat then every symmetric polynomial in the Chern classes of E of degree larger than $\dim S - \ell$ vanishes.*

A proof of such a theorem can be found adapting the result in [6], see also [3, Theorem 6.1].

The question now is what guarantees that a partial holomorphic connection exists on a given complex vector bundle. Arguing as in Atiyah [5], one can prove the following: if F is a sub-bundle of TS, there exists an exact sequence

$$O \longrightarrow \mathcal{H}om(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{A}_{E,F} \xrightarrow{\pi_0} \mathcal{F} \longrightarrow O, \tag{3.1}$$

where $\mathcal{A}_{E,F} = \pi_0^{-1}(\mathcal{F})$. It holds

Proposition 3.3. *Let F be a sub-bundle of the tangent bundle TS of a complex manifold S, and let E be a complex vector bundle over S. Then there is a partial holomorphic connection on E along F if and only if the sequence* (3.1) *splits, that is if and only if there is an $\mathcal{O}_S$-morphism $\psi_0 \colon \mathcal{F} \to \mathcal{A}_{E,F}$ such that $\pi_0 \circ \psi_0 = \mathsf{id}$.*

Therefore existence of partial holomorphic connections is related to the existence of certain splittings.

4. The Atiyah bundle

Now we restrict our attention to the case S is a complex submanifold of a complex manifold M, and we study the normal bundle $E = \mathcal{N}_S$ of S into M defined by the natural exact sequence

$$0 \longrightarrow \mathcal{T}_S \longrightarrow \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S \xrightarrow{p} \mathcal{N}_S \to 0,$$

where $\mathcal{I}_S$ is the ideal subsheaf of $\mathcal{O}_M$ of functions identically vanishing on S and $\mathcal{O}_S := \mathcal{O}_M/\mathcal{I}_S$. Let $\mathcal{T}_{M,S(1)} := \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_M/\mathcal{I}_S^2$. Let $\theta : \mathcal{T}_{M,S(1)} \to \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ be the natural $\mathcal{O}_M$-morphism and let $\mathcal{T}^S_{M,S(1)} := \ker(p \circ \theta)$.

In order to give a rough description of the elements of $\mathcal{T}^S_{M,S(1)}$, we use the following notations. Choose a local coordinates patch $(z,w) \in U \times U' \subset \mathbb{C}^m \times \mathbb{C}^s$ such that $S = \{w = 0\}$. Then a vector of type $\sum a_j(z,w)\frac{\partial}{\partial z_j}$ will be denoted with the letter T (as "tangent"), while a vector of the form $\sum b_j(z,w)\frac{\partial}{\partial w_j}$ will be denoted with the letter N (as "normal"). For $q \in \mathbb{N}$, we write T_q if $T_q = \sum a_j(z,w)\frac{\partial}{\partial z_j}$ with $a_j \in \mathcal{I}_S^q$ for all j (namely, if $a_j(z,w) = w^Q \tilde{a}_j(z,w)$ for some multi-indices Q with $|Q| = q$). Similarly we will use the notation N_q for "normal" vectors with all coefficients in $\mathcal{I}_S^q$. Also, we write R_q (all the "rest") for any combinations of vectors of type T_q, N_q.

With such notations, a (local) section of $\mathcal{T}_{M,S(1)}$ is represented by a vector of the form $T_0 + T_1 + N_0 + N_1 + R_2$. And an element in $\mathcal{T}^S_{M,S(1)}$ is represented by a vector of the form $T_0 + T_1 + N_1 + R_2$ (i.e., no terms N_0 appear).

Definition 4.1. The *Atiyah sheaf* of S in M is the quotient sheaf given by

$$\mathcal{A} := \mathcal{T}^S_{M,S(1)}/\mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)}.$$

A (local) section of $\mathcal{A}$ is represented by a vector of the form $T_0 + N_1 + R_2$. The following results are proved in [3]:

Theorem 4.2. *Let $\mathcal{A}$ be the Atiyah sheaf of the complex submanifold S in M. Then:*

(1) *The sheaf $\mathcal{A}$ has a natural structure of locally free $\mathcal{O}_S$-module and the map $\pi : \mathcal{A} \to \mathcal{T}_S$ (locally) defined as $\pi : [T_0 + N_1] \mapsto T_0$ is an $\mathcal{O}_S$-morphism.*
(2) *The Atiyah sheaf $\mathcal{A}$ is isomorphic to the sheaf $A_{\mathcal{N}_S,\mathcal{T}_S}$ defined in* (3.1).
(3) *The Atiyah sheaf $\mathcal{A}$ has a natural structure $\{\cdot,\cdot\}$ of Lie algebroid such that $\pi\{u,v\} = [\pi(u),\pi(v)]$ for all $u, v \in \mathcal{A}$.*
(4) *There exists a natural holomorphic π-connection $\tilde{X} : \mathcal{N}_S \to \mathcal{A}^* \otimes \mathcal{N}_S$ on $\mathcal{N}_S$ given by*

$$\tilde{X}_q(s) = p([v,\tilde{s}])$$

for all $q \in \mathcal{A}$ and $s \in \mathcal{N}_S$, where $v \in \mathcal{T}^S_{M,S(1)}$ and $\tilde{s} \in \mathcal{T}_{M,S(1)}$ are such that $[v]_{\mathcal{A}} = q$ and $p \circ \theta(\tilde{s}) = s$;
(5) *this holomorphic π-connection $\tilde{X}$ is flat.*

Here we recall that a $\mathbb{C}$-bilinear map $\{\cdot,\cdot\}$ is a *Lie algebroid structure* for $\mathcal{A}$ if for all $u, v \in \mathcal{A}$

(a) $\{v,u\} = -\{u,v\}$;
(b) $\{u,\{v,w\}\} + \{v,\{w,u\}\} + \{w,\{u,v\}\} = O$;
(c) $\{g \cdot u, v\} = g \cdot \{u,v\} - \pi(v)(g) \cdot u$ for all $g \in \mathcal{O}_S$ and $u, v \in \mathcal{A}$.

Moreover, a *holomorphic π-connection* is a $\mathbb{C}$-linear map $\tilde{X} : \mathcal{N}_S \to \mathcal{A}^* \otimes \mathcal{N}_S$ such that

$$\tilde{X}(gs) = \pi^*(dg) \otimes s + g\tilde{X}(s)$$

for all $g \in \mathcal{O}_S$ and $s \in \mathcal{N}_S$, where $\pi^* : \mathcal{T}_S^* \to \mathcal{A}^*$ is the dual map of π. We write $\tilde{X}_q(s) := \tilde{X}(s)(v)$ for $s \in \mathcal{N}_S$ and $q \in \mathcal{A}$. It is *flat* in the sense that for all $u, v \in \mathcal{A}$

$$\tilde{X}_u \circ \tilde{X}_v - \tilde{X}_v \circ \tilde{X}_u = \tilde{X}_{\{u,v\}}.$$

The Atiyah sheaf $\mathcal{A}$ and its canonical connection determine all possible partial holomorphic connections on $\mathcal{N}_S$. Indeed, [3, Theorem 5.9]:

Theorem 4.3. *Let S be a submanifold of a complex manifold M, and let $\mathcal{F}$ be a locally $\mathcal{O}_S$-free subsheaf of the tangent bundle $\mathcal{T}_S$. Then:*

(i) *if $\psi\colon \mathcal{F} \to \mathcal{A}$ is an $\mathcal{O}_S$-morphism such that $\pi\circ\psi = \mathsf{id}$ then the map $\nabla^\psi\colon \mathcal{N}_S \to \mathcal{F}^*\otimes\mathcal{N}_S$ given by*
$$\nabla^\psi_v(s) = \tilde{X}_{\psi(v)}(s)$$
for all $v \in \mathcal{F}$ and $s \in \mathcal{N}_S$ is a partial holomorphic connection on $\mathcal{N}_S$ along F;

(ii) *there exists a partial holomorphic connection on $\mathcal{N}_S$ along $\mathcal{F}$ if and only if there exists an $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F} \to \mathcal{A}$ such that $\pi \circ \psi = \mathsf{id}$;*

(iii) *if $\mathcal{F}$ is involutive, then the partial holomorphic connection ∇^ψ is flat if and only if $\psi\colon \mathcal{F} \to \mathcal{A}$ is a Lie algebroid morphism.*

Therefore, in order to find partial holomorphic connection for $\mathcal{N}_S$ along $\mathcal{F}$ one has to determine all possible $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F} \to \mathcal{A}$ such that $\pi \circ \psi = \mathsf{id}$.

5. THE CAMACHO-SAD INDEX THEOREM REVISED

Let S be a complex compact reduced irreducible subvariety of a complex manifold M. Assume that the normal bundle of S in M, a priori defined only outside the singularities of S, "extends" over the singularities of S, namely, there exists a coherent $\mathcal{C}^\infty_M$-modules $\mathcal{N}$ (with a finite syzygy of locally free modules) such that $\mathcal{N} \otimes \mathcal{O}_S = \mathcal{N}_S$ on the regular locus of S. For instance this is always the case if S is a hypersurface or if S is the zero section of a vector bundle M. We are interested in localizing the characteristic classes of $\mathcal{N}$ on S, with respect to some dynamical object which acts on S (such as holomorphic foliations for which S is or not invariant, holomorphic diffeomorphisms leaving S fixed). Using the Čech-de Rham cohomology as explained before, one can always put the singular locus of S into some open set of a suitable covering of S so that we can try to use Theorem 4.3 only on the regular part of S.

If S is invariant by a holomorphic foliation $\mathcal{F}$ of M then one can construct a Lie algebroid morphism $\psi : \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \mathcal{A}$ outside the singularity of the restriction $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ of the foliation $\mathcal{F}$ on S, and the classical Camacho-Sad theorem [12] follows then from Theorem 4.3 and the previous constructions. The same argument works for the various generalizations of the Camacho-Sad index theorem due, with direct and different methods, to T. Suwa [17], A. Lins-Neto [15] and D. Lehmann-T.Suwa [14].

If S is not invariant by the foliation $\mathcal{F}$ but it is "well embedded" into M (for instance if S is the zero section of a vector bundle M; 2-linearizable is enough, see [3], [4]) and the dimension of S equals the dimension of $\mathcal{F}$ then again it is possible to define a morphism $\psi : \mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S \to \mathcal{A}$ outside the singularity of the restriction $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ of the foliation $\mathcal{F}$ on S. The same is true if S is "well embedded" into M and $\mathcal{F} \otimes \mathcal{O}_M/\mathcal{I}_S^2$ is trivial (for instance if $\mathcal{F}$ is generated by a certain number of globally defined vector fields). In all those cases we obtain a generalization of the Camacho-Sad index theorem arguing as above. Such an index theorem for the case S is the zero section of a vector bundle and the dimension of S equals the dimension of the foliation is due to C. Camacho and D. Lehmann [10], while for M of dimension 2, S non singular curve 2-linearizable is due to C. Camacho, H. Movasati and P. Sad [11]. The other generalizations, plus another one with cohomological conditions, are contained in [3, Theorem 7.21].

Another "dynamical condition" with guarantees the existence of a (Lie algebroid) morphism $\psi : \mathcal{F} \to \mathcal{A}$ (outside some closed analytic subset of S) from some subsheaf $\mathcal{F}$ of $\mathcal{T}_S$ is given in case S is a hypersurface pointwise fixed by a biholomorphism f of (a neighborhood of S in) M which is "tangential" to S. Morally speaking, given the diffeomorphism f pointwise fixing S, one can look at its "logarithm", namely, the formal vector field v whose time one flow is f. Such a vector field turns out to be semi-formal, namely, it is a section of $\mathcal{T}_S \otimes \lim_{\leftarrow} \mathcal{O}_M/\mathcal{I}_S^m$, and it is singular along S (namely, its restriction to $\mathcal{O}_S$ is identically zero). The map f is then 'tangential" to S if and only if the desingularization of v is tangent to S. The "singularities" of f along S are exactly the singularities of the desingularization of v on S. The Camacho-Sad type index theorem was obtained first by M. Abate [1] for the case S is a non singular curve in a two dimensional complex manifold, then by F. Tovena and the author [9] for the case S is singular. The general case is in [2] (see also [7] and [8] for the case where M is also allowed to have some simple singularity and [16] for general cases with M singular).

In case the diffeomorphism f is non-tangential to S (or S has codimension greater than one), one can also construct morphism $\psi : \mathcal{F} \to \mathcal{A}$ (outside some closed analytic subset of S) in case S is "well embedded" in M or some cohomological condition is satisfied (see [3, Theorem 8.10]).

Then, collecting the previous mentioned constructions with foliations and diffeomorphisms under the sentence S is "dynamically subjected to a holomorphic object", the very abstract form of the Camacho-Sad index theorem is:

Theorem 5.1. *Let S be a complex compact reduced irreducible subvariety of a complex manifold M with extended normal bundle $\mathcal{N}$. Assume that S is "dynamically subjected to a holomorphic object" outside some closed analytic set Σ. Then every symmetric polynomial in the Chern classes of $\mathcal{N}$ of suitable degree localizes around $\Sigma \cup$ Sing(S).*

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