EVOLUTION FAMILIES AND THE LOEWNER EQUATION II: COMPLEX HYPERBOLIC MANIFOLDS.

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Abstract. We prove that evolution families on complex complete hyperbolic manifolds are in one to one correspondence with certain semicomplete non-autonomous holomorphic vector fields, providing the solution to a very general Loewner type differential equation on manifolds.

1. Introduction

In [11] Loewner developed a tool to embed univalent functions into particular families of univalent functions, nowadays known as *Loewner chains*. Such a tool has been studied and extended by many mathematicians in the past years and has been proved to be very effective in the solution of various problems and conjectures. For definitions, detailed historical discussion and applications, we refer the reader to the books of Pommerenke [13] and Graham and Kohr [7] and to [4].

Loewner chains in the unit disc are related to other two objects: a certain type of non-autonomous holomorphic vector fields (which we call *Herglotz vector fields*) and families of holomorphic self-maps of the unit disc, called *evolution families*. Those three objects are related by means of differential equations, known as *Loewner differential equations*.

In one dimension, most of the work done so far is to relate Loewner chains to Herglotz vector fields and to evolution families when common fixed points inside the disc are present. The main difficulty to treat boundary fixed points and higher dimensional cases is the lack of appropriated distortion theorems. For instance, we present here an apparently unknown example of the lack of a type of Koebe 1/4-theorem in higher dimension (the first named author thanks Francois Berteloot for discussions about it):

Example 1.1. Let $n \geq 2$ and let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic injective map such that $F(\mathbb{C}^n)$ is a proper subset of $\mathbb{C}^n$ (this exists by the well known *Fatou-Bieberbach phenomenon*, see, [14]). Up to translation we can assume that $F(O) = O$. For $\lambda > 0$ let

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$F_\lambda(z) := \lambda F(z)$ and $\Omega_\lambda = F_\lambda(\mathbb{C}^n)$. Then, for all fixed $\epsilon > 0$ there exists $\lambda > 0$ such that $\mathrm{dist}(\partial\Omega_\lambda, O) < \epsilon$. Now, let A be an $n \times n$ invertible matrix such that $d(F_\lambda(Az))_{z=O} = \mathsf{id}$ and let $G(z) := F_\lambda(Az)$. The map $G : \mathbb{B}^n \to \mathbb{C}^n$ is univalent, $G(O) = O$, $dG_O = \mathsf{id}$ and the image of $G(\mathbb{B}^n)$ does not contain the ball of radius ϵ and centered at O.

As a matter of fact, in higher dimension or with boundary fixed points in the unit disc, in order to relate the three objects and to solve the appropriated Loewner differential equations, one is forced to restrict to particular classes of mappings.

Loewner differential equations have been studied in higher and infinite dimension (in the unit ball) especially by Graham, G. Kohr, M. Kohr, H. Hamada, T. Poreda and J.A. Pfaltzgraff. We refer the reader to [7], [8], [9] and references therein.

In this paper, which is a sequel of [4], we study on complex manifolds the Loewner differential equation which relates evolution families and Herglotz vector fields.

To state our main result we need a few definitions. Let M be a complex manifold and let $d(\cdot,\cdot)$ be a distance on M.

Definition 1.2. A family $(\varphi_{s,t})$ is called a *(continuous) evolution family* if

(1) $\varphi_{s,s} = \mathsf{id}_M$.
(2) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \le s \le u \le t < +\infty$.
(3) $\varphi_{s,t} : M \to M$ is holomorphic for all $0 \le s \le t < +\infty$.
(4) For any $0 \le s < +\infty$ and for any compact set $K \subset\subset M$ the function $[s,+\infty) \ni t \mapsto \varphi_{s,t}(z)$ is locally Lipschitz continuous uniformly with respect to K. Namely, fixed $T > 0$, there exists $c_{T,K} > 0$ such that

$$\sup_{z\in K} d(\varphi_{s,t}(z), \varphi_{s,t'}(z)) < c_{T,K}|t - t'|, \tag{1.1}$$

for all $0 \le s \le t, t' \le T$.

A *Herglotz vector field* $G(z,t)$ on M is a weak holomorphic vector field of order ∞ (see Section 2) such that $(dk_M)_{(z,w)}(G(z,t), G(w,t)) \le 0$ for almost every $t \in [0,+\infty)$ and all $z \ne w$. Here k_M is the Kobayashi distance on M and it is assumed to be C^1 outside the diagonal, which is always the case for instance if M is a strongly convex domain with smooth boundary.

Our main result is that there is a one-to-one correspondence between evolution families and Herglotz vector fields, namely

Theorem 1.3. *Let M be a complete hyperbolic manifold with Kobayashi distance k_M. Assume that $k_M \in C^1(M \times M \setminus Diag)$. Then for any Herglotz vector field G there exists a unique evolution family $(\varphi_{s,t})$ over M such that for all $z \in M$*

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad a.e.\ t \in [s,+\infty). \tag{1.2}$$

Conversely for any evolution family $(\varphi_{s,t})$ *over* M *there exists a Herglotz vector field* G *such that* (1.2) *is satisfied. Moreover, if* H *is another weak holomorphic vector field which satisfies* (1.2) *then* $G(z,t) = H(z,t)$ *for all* $z \in M$ *and almost every* $t \in [0,+\infty)$.

Such a theorem is not sharp in the sense that some of the hypotheses can be lowered in order to obtain partial results. More precisely we show in Proposition 3.1 that given a Herglotz vector field of order $d \geq 1$ there exists a unique evolution family of the same order d which verifies (1.2) (see Section 2 for definitions). The converse (for L^∞ data) is proved in Proposition 4.1, which holds more generally for taut manifolds. Some remarks about the possibility of dropping the regularity of k_M are also contained at the end of Section 3 and Section 4. We also show that all the elements of an evolution family on taut manifolds must be univalent (see Proposition 5.1).

2. Weak holomorphic vector fields, Herglotz vector fields and evolution families

Let M be a complex manifold of complex dimension n. Let TM denote the complex tangent bundle of M and let $||\cdot||$ be a Hermitian metric along the fibers of TM. We denote by $d(\cdot,\cdot)$ the distance induced on M by $||\cdot||$. Also, we let k_M denote the Kobayashi pseudo-distance on M. For definitions and properties of k_M, of taut manifolds and complete hyperbolic manifolds we refer the reader to, *e.g.*, [1] or [10]. Here we only recall that complete hyperbolic manifolds are taut.

Definition 2.1. A *weak holomorphic vector field of order* $d \geq 1$ on M is a function

$$G : M \times [0,+\infty) \to TM$$

with the following properties:

(1) For all $z \in M$ the function $[0,+\infty) \ni t \mapsto G(z,t)$ is measurable.
(2) For all $t \geq 0$ the function $M \ni z \mapsto G(z,t)$ is holomorphic.
(3) For all compact set $K \subset\subset M$ and all $T > 0$ there exists a function $C_{K,T} \in L^d([0,T],\mathbb{R}^+)$ such that

$$||G(z,t)|| \leq C_{K,T}(t) \tag{2.1}$$

for all $z \in K$ and almost every $t \in [0,T]$.

Lemma 2.2. *Let* $G(z,t)$ *be a weak holomorphic vector field of order* $d \geq 1$ *on* M. *Let* U *be a coordinate open set of* M *such that* $TM|_U \simeq U \times \mathbb{C}^n$. *Let* $P \subset\subset U$ *be a relatively compact polydisc. Let* $T > 0$. *Then there exists* $\widetilde{C}_{P,T} \in L^d([0,T],\mathbb{R}^+)$ *such that*

$$|G(z,t) - G(w,t)| \leq \widetilde{C}_{P,T}(t)|z - w| \tag{2.2}$$

for all $z, w \in P$ *and almost every* $t \in [0,T]$ *(here* $|\cdot,\cdot|$ *denotes the usual Hermitian metric on* $\mathbb{C}^n$*).*

Proof. Let $(r_1, \dots, r_n)$ be the multiradius of P and let $(\tilde{r}_1, \dots, \tilde{r}_n)$ be the multiradius of another polydisc $\tilde{P}$ such that $P \subset\subset \tilde{P} \subset\subset U$. We write

$$\begin{aligned}|G(z,t) - G(w,t)| \leq &|G(z_1, \dots, z_n, t) - G(z_1, \dots, z_{n-1}, w_n, t)| \\ &+ \dots + |G(z_1, w_2 \dots, w_n, t) - G(w_1, \dots, w_n, t)|.\end{aligned}$$

By the Cauchy formula and (2.1) (taking into account that the Hermitian metric $||\cdot||$ is equivalent to the metric $|\cdot|$ of $\mathbb{C}^n$ on U) we have

$$\begin{aligned}|G(z_1, \dots, z_n, t) - G(z_1, \dots, z_{n-1}, w_n, t)| &\leq \frac{1}{2\pi} \int_{|\xi| = \tilde{r}_n} \frac{|G(z_1, \dots, z_{n-1}, \xi, t)||z_n - w_n|}{|\xi - z_n||\xi - w_n|} |d\xi| \\ &\leq c_P C_{P,T}(t) |z_n - w_n|,\end{aligned}$$

for some constant $c_P > 0$ which depends only on U and P. Similar estimates hold for the other terms and thus (2.2) follows. $\square$

Remark 2.3. By the Carathéodory theory of ODE's (see, for instance, [5]) it follows that if $G(z,t)$ is a weak holomorphic vector field on M, for any $(s, z) \in [0, +\infty) \times M$ there exist a unique $I(s,z) > s$ and a function $x : [s, I(s,z)) \to M$ such that

1. x is locally absolutely continuous in t.
2. x is the maximal solution to the following problem:

$$\begin{cases} \frac{dx}{dt}(t) := x_*(\frac{\partial}{\partial t}) = G(x(t), t) & \text{for a.e. } t \in [s, I(s,z)), \\ x(s) = z. \end{cases}$$

The number $I(s,z)$ is referred to as the *escaping time* of the couple (s,z).

Remark 2.4. If M is complete hyperbolic with Kobayashi metric k_M, G is a weak holomorphic vector field over M and x is the maximal solution for the couple (s,z) with escaping time $I(s,z) > 0$ as defined in the previous remark, it follows that for any compact set $K \subset\subset M$

$$\limsup_{t \to I(s,z)} k_M(x(t), K) = +\infty.$$

Definition 2.5. Let $d \geq 1$. Let M be a complex manifold. Assume that $k_M \in C^1(M \times M \setminus \mathrm{Diag})$. We let

$$\begin{aligned}\mathcal{H}_d(M) := \{&G(z,t) \text{ weak holomorphic vector field of order } d \\ &: (dk_M)_{(z,w)}(G(z,t), G(w,t)) \leq 0 \quad \forall z, w \in M, z \neq w \text{ and a.e. } t \in [0, +\infty)\}.\end{aligned}$$

We call a *Herglotz vector field of order* d any element $G \in \mathcal{H}_d(M)$. A Herglotz vector field of order ∞ is simply said a *Herglotz vector field.*

Remark 2.6. According to [3] if $M = D$ is a strongly convex domain in $\mathbb{C}^n$ with smooth boundary then G is a Herglotz vector field if and only if it is a weak holomorphic vector field such that for almost every $t \in [0, +\infty)$ the function $D \ni z \mapsto G(z,t)$ is an infinitesimal generator of a semigroup of holomorphic self-maps of D (see, *e.g.* [1] for definitions and properties of semigroups).

Definition 2.7. A family $(\varphi_{s,t})$ is called an *evolution family of order* $d \geq 1$ if

(1) $\varphi_{s,s} = \mathsf{id}_M$.
(2) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$.
(3) $\varphi_{s,t} : M \to M$ is holomorphic for all $0 \leq s \leq t < +\infty$.
(4) For any $T > 0$ and for any compact set $K \subset\subset M$ there exists a function $c_{T,K} \in L^d([0,T], \mathbb{R}^+)$ such that

$$\sup_{z \in K} d(\varphi_{s,t}(z), \varphi_{s,t'}(z)) \leq \int_{t'}^{t} c_{T,K}(\xi) d\xi, \tag{2.3}$$

for all $0 \leq s \leq t' \leq t \leq T$.

An evolution family of order ∞ is simply said an *evolution family*.

Since on a taut manifold M the topology of pointwise convergence on $\mathrm{Hol}(M,M)$ coincides with the topology of uniform convergence on compacta (see, *e.g.*, [1, Corollary 2.1.17]), the proof of the following lemma is similar to that of [4, Proposition 3.4] and we omit it:

Lemma 2.8. *Let M be a taut complex manifold. Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ in M. The map $(s,t) \mapsto \varphi_{s,t} \in \mathrm{Hol}(M,M)$ is jointly continuous. Namely, given a compact set $K \subset M$ and two sequences $\{s_n\}$, $\{t_n\}$ in $[0,+\infty)$, with $0 \leq s_n \leq t_n$, $s_n \to s$, and $t_n \to t$, then $\lim_{n\to\infty} \varphi_{s_n,t_n} = \varphi_{s,t}$ uniformly on K.*

3. From Herglotz vector fields to evolution families

In this section we prove that to each Herglotz vector field of order $d \geq 1$ there corresponds a unique evolution family of order $d \geq 1$.

Proposition 3.1. *Let M be a complete hyperbolic manifold with Kobayashi distance k_M. Assume that $k_M \in C^1(M \times M \setminus Diag)$. Let $d \geq 1$ and let $G(z,t)$ be a Herglotz vector field of order $d \geq 1$ on M. Then there exists a unique evolution family $(\varphi_{s,t})$ of order d on M which verifies* (1.2).

In the proof, we will use the well-known Gronwall Lemma. For the reader convenience we state it here as needed for our aims.

Lemma 3.2. *Let $\theta : [a,b] \to \mathbb{R}$ be a continuous function and $k \in L^1([a,b],\mathbb{R})$ non-negative. If there exists $C \geq 0$ such that for all $t \in [a,b]$*

$$\theta(t) \leq C + \int_a^t \theta(\xi)k(\xi)d\xi \qquad \textit{(resp., } \theta(t) \leq C + \int_t^b \theta(\xi)k(\xi)d\xi\textit{)},$$

then

$$\theta(t) \le C \exp\left(\int_a^t k(\xi)d\xi\right) \qquad \textit{(resp., } \theta(t) \le C \exp\left(\int_t^b k(\xi)d\xi\right) \textit{).}$$

Proof of Proposition 3.1. For any $s \in [0,+\infty)$ and $z \in M$, let $\varphi_{s,t}(z)$ be the maximal solution with escaping time $I(s,z) > s$ (see Remark 2.3) which satisfies

$$\begin{cases} \frac{\partial \varphi_{s,t}}{\partial t} = G(\varphi_{s,t}(z), t) & \text{for a.e. } t \in [s, I(s,z)), \\ \varphi_{s,s}(z) = z. \end{cases}$$

For all $s \ge 0$ and $z \in M$ the curve $[s, I(s,z)) \ni t \mapsto \varphi_{s,t}(z)$ is locally absolutely continuous.

1. *For all $z, w \in M$ it holds $I(0,z) = I(0,w)$.* Indeed, fix $z, w \in M$. Then the function $h : [0,+\infty) \ni t \mapsto k_M(\varphi_{0,t}(z), \varphi_{0,t}(w))$ is locally absolutely continuous. Differentiating we get

$$\begin{aligned} \overset{\bullet}{h}(t) &= \frac{\partial}{\partial t} k_M(\varphi_{0,t}(z), \varphi_{0,t}(w)) \\ &= (dk_M)_{(\varphi_{0,t}(z),\varphi_{0,t}(w))}(G(\varphi_{0,t}(z),t), (\varphi_{0,t}(w),t)) \le 0 \quad \text{a.e. } t \in [0,+\infty). \end{aligned}$$

Hence h is decreasing in t and therefore $h(t) \le h(0)$ for all $t \in [0,+\infty)$. If $I(0,z) < I(0,w)$, since k_M is complete hyperbolic and $\{\varphi_{0,t}(w)\}_{t \in [0, I(0,z)]} \subset\subset M$, by Remark 2.4, we have

$$+\infty = \limsup_{t \to I(0,z)} k_M(\varphi_{0,t}(z), \varphi_{0,t}(w)) \le k_M(z,w) < +\infty,$$

a contradiction. Then $I(0,z) \ge I(0,w)$. A similar argument shows that $I(0,z) \le I(0,w)$, thus Step 1 follows.

We let $I := I(0,z)$ for some $z \in M$.

2. *For all $s < I$ and all $z \in M$ it follows $I(s,z) = I$.* Indeed, repeating the argument in Step 1 with s substituting 0, we obtain that for all $z, w \in M$ and $0 \le s \le t < \min\{I(s,z), I(s,w)\}$ it follows

$$k_M(\varphi_{s,t}(z), \varphi_{s,t}(w)) \le k_M(z,w). \tag{3.1}$$

Now, let $w = \varphi_{0,s}(z)$. Notice that w is well defined if $s < I$. By uniqueness of solutions of ODE's it follows that $\varphi_{s,t}(w) = \varphi_{s,t}(\varphi_{0,s}(z)) = \varphi_{0,t}(z)$. Therefore, from (3.1) it follows

$$k_M(\varphi_{s,t}(z), \varphi_{0,t}(z)) \le k_M(z, \varphi_{0,s}(z)).$$

Arguing as in Step 1 we obtain that $I(s,z) = I$ for all $s < I$.

3. $I = +\infty$ (and thus $I(s,z) = +\infty$ for all couple $(s,z) \in [0,+\infty) \times M$ by Step 2).

Fix $z_0 \in M$. We are going to show that there exists $\delta > 0$ such that for all $s \in [0, I)$ it follows $I(s,z) \ge s + \delta$. If this is true and $I < +\infty$, then letting $I - \delta < s < I$ it follows that $I(s,z) > I$ contradicting Step 2, and thus $I = +\infty$.

To prove the existence of $\delta > 0$ as before, let U be a local chart of M which trivializes TM and such that z_0 has coordinates O. With no loss of generality we can assume that U contains a closed polydisc P which contains the ball $\mathbb{B}$ of radius 1 and center O in $\mathbb{C}^n$.

Let $r < 1$. Let $\mathbb{B}_r := \{z \in \mathbb{C}^n : |z| \leq r\}$. By the very definition of weak holomorphic vector field of order d there exists $C := C_{\mathbb{B}_r,[0,I+2]} \in L^d([0, I+2], \mathbb{R}^+)$ such that

$$|G(z,t)| \leq C(t) \tag{3.2}$$

for all $z \in \mathbb{B}_r$ and almost every $t \in [0, I+2]$. Moreover, by Lemma 2.2, there exists $\widetilde{C} := \widetilde{C}_{P,[0,I+2]} \in L^d([0, I+2], \mathbb{R}^+)$ such that

$$|G(z,t) - G(w,t)| \leq \widetilde{C}(t)|z - w| \tag{3.3}$$

for all $z, w \in \mathbb{B}_r$ and almost every $t \in [0, I+2]$.

The functions $[0, I+2] \ni u \mapsto \int_0^u C(\tau)d\tau$, $[0, I+2] \ni u \mapsto \int_0^u \widetilde{C}(\tau)d\tau$ are absolutely continuous and therefore there exists $\delta > 0$ (which we can suppose strictly less than 1) such that for all $s \in [0, I+1]$ it holds

$$\int_s^{s+\delta} C(\tau)d\tau \leq r, \quad \int_s^{s+\delta} \widetilde{C}(\tau)d\tau \leq r. \tag{3.4}$$

For $s \in [0, I+1]$ let us define by induction

$$\begin{cases} x_0^s(t) := O \\ x_n^s(t) = \int_s^t G(x_{n-1}^s(\tau), \tau)d\tau \quad t \in [s, s+\delta]. \end{cases}$$

We notice that $|x_0^s(t)| = 0 < r$. Assuming that $|x_{n-1}^s(t)| \leq r$ for all $t \in [s, s+\delta]$, by (3.2) and (3.4) we have

$$|x_n^s(t)| \leq \int_s^t |G(x_{n-1}^s(\tau), \tau)|d\tau \leq \int_s^t C(\tau)d\tau \leq r, \quad t \in [s, s+\delta]$$

which, by induction, implies that $x_n^s(t)$ is well defined for all $n \in \mathbb{N}$ and $t \in [s, s+\delta]$ and $|x_n^s(t)| \leq r$.

Now, by (3.3) and (3.4) we have

$$\begin{aligned} |x_n^s(t) - x_{n-1}^s(t)| &\leq \int_s^t |G(x_{n-1}^s(\tau), \tau) - G(x_{n-2}^s(\tau), \tau)|d\tau \\ &\leq \int_s^t \widetilde{C}(\tau)|x_{n-1}^s(\tau) - x_{n-2}^s(\tau)|d\tau \\ &\leq \max_{\tau \in [s,s+\delta]} |x_{n-1}^s(\tau) - x_{n-2}^s(\tau)| \int_s^t \widetilde{C}(\tau)d\tau, \end{aligned}$$

for $n \in \mathbb{N}$ and $t \in [s, s+\delta]$. From this follows that $\{x_n^s\}$ is a Cauchy sequence in the Banach space $C^0([s, s+\delta], \mathbb{C}^n)$. Therefore it converges uniformly on $[s, s+\delta]$ to a function $x^s \in C^0([s, s+\delta], \mathbb{B}_r)$. By (3.2) and the Lebesgue dominated converge theorem it follows that

$$x^s(t) = \int_s^t G(x^s(\tau), \tau)d\tau \quad \forall t \in [s, s+\delta],$$

or, in other words,

$$\varphi_{s,t}(z_0) = x^s(t) \quad \forall t \in [s, s+\delta]$$

which proves that $I(z_0, s) \geq s + \delta$ as needed.

4. $(\varphi_{s,t})$ *is an "algebraic" evolution family*, namely, Properties (1) and (2) of Definition 2.7 hold. This follows at once by the uniqueness of solutions of ODE's.

5. *For all fixed* $0 \leq s \leq t < +\infty$ *the map* $M \ni z \mapsto \varphi_{s,t}(z) \in M$ *is holomorphic*. Let $z_0 \in M$ and let $0 \leq s \leq t < +\infty$. The absolutely continuous (compact) curve $[s,t] \ni \eta \mapsto \varphi_{\eta,t}(z_0)$ is covered by a finite number of coordinates charts, so that we can find a partition $s = t_0 < t_1 < \ldots < t_m = t$ such that each curve $[t_j, t_{j+1}] \ni \eta \mapsto \varphi_{\eta, t_{j+1}}(z_0)$ is contained in a coordinates chart. By Property (2) of Definition 2.7 (which holds for $(\varphi_{s,t})$ by Step 4), holomorphicity of $\varphi_{s,t}$ at z_0 will follow as soon as we can show holomorphicity of φ_{t_{j-1},t_j} at $\varphi_{s,t_j}(z_0)$ for $j = 1, \ldots, m$. Therefore, we can suppose that the curve $[s,t] \ni \eta \mapsto \varphi_{\eta,t}(z_0)$ is contained in a local chart.

By (3.1), for any open neighborhood V of z_0, relatively compact in U, there exists a open neighborhood $W \subset U$ of z_0 such that for any $w \in W$ and all $\eta \in [s,t]$ it follows that $\varphi_{s,\eta}(w) \in V$.

Since holomorphy is a local property, we can work on the local chart U (centered at z_0), which we may assume is the ball $\mathbb{B} := \{z \in \mathbb{C}^n : |z| < 1\}$ of center O and radius 1. In such local coordinates we can write $G = (G_1, \ldots, G_n)$. For $\eta \in [s,t]$ let

$$A(\eta) := \begin{pmatrix} \frac{\partial G_1(\varphi_{s,\eta}(O),\eta)}{\partial z_1} & \cdots & \frac{\partial G_1(\varphi_{s,\eta}(O),\eta)}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial G_n(\varphi_{s,\eta}(O),\eta)}{\partial z_1} & \cdots & \frac{\partial G_n(\varphi_{s,\eta}(O),\eta)}{\partial z_n} \end{pmatrix}.$$

By Cauchy formula and (2.1) (arguing similarly to the proof of Lemma 2.2), each entry of $A(\eta)$ is a L^d-measurable function in η. It is well known that the following system of ODE's

$$\begin{cases} \frac{dH}{d\eta}(\eta) = -H(\eta) \cdot A(\eta) \\ H(s) = \mathsf{Id} \end{cases}$$

has a unique continuous solution $H(\eta)$ which is an invertible $n \times n$ matrix for all η.

Let $v \in \mathbb{C}^n$ be such that $|v| = 1$. We will prove that

$$\lim_{h \in \mathbb{C}, |h| \to 0} \frac{\varphi_{s,t}(hv) - \varphi_{s,t}(0)}{h} = H^{-1}(t) \cdot v, \tag{3.5}$$

showing that $z \mapsto \varphi_{s,t}(z)$ is holomorphic.

Since the topology induced by k_M coincides with the one of M, the previous argument based on (3.1) shows that there exists $\delta > 0$ such that

$$\sup\{|\varphi_{s,\eta}(z)| : |z| < \delta, \eta \in [s,t]\} < \frac{1}{5n}. \tag{3.6}$$

Let $P := \{z \in \mathbb{B}^n : \max_j |z_j| < 1/(4n)\}$ be the polydisc of center O and multi-radius $1/(4n)$. Let $\widetilde{C}_{P,t} \in L^d([0,t])$ be as in (2.2) (with $T = t$). We let

$$\theta(\eta) := |\varphi_{s,\eta}(hv) - \varphi_{s,\eta}(0)|,$$

where $h \in \mathbb{C}$ is such that $|h| < \delta$ and $\eta \in [s,t]$. Then $\theta \in C^0([s,t], \mathbb{R}^+)$. Since $\varphi_{s,\eta}(z)$ solves (1.2), and $\varphi_{s,\eta}(hv) \in P$ for all $\eta \in [s,t]$ and $|h| < \delta$ by (3.6), it follows that

$$\begin{aligned}\theta(\eta) &= \left| hv + \int_s^\eta G(\varphi_{s,\xi}(hv), \xi) d\xi - \int_s^\eta G(\varphi_{s,\xi}(O), \xi) d\xi \right| \\ &\leq |h| + \int_s^\eta |G(\varphi_{s,\xi}(hv), \xi) - G(\varphi_{s,\xi}(O), \xi)| \ d\xi \\ &\leq \theta(s) + \int_s^\eta \widetilde{C}_{P,t}(\xi) d\xi.\end{aligned}$$

Gronwall's Lemma 3.2 implies then

$$\theta(\eta) \leq \theta(s) \exp\left(\int_s^\eta \widetilde{C}_{P,t}(\xi) d\xi \right) \quad \forall \eta \in [s,t].$$

Therefore, setting $C(s,t,P) := \exp(\int_s^t \widetilde{C}_{P,t}(\xi) d\xi)$, we have

$$|\varphi_{s,\eta}(hv) - \varphi_{s,\eta}(0)| \leq |h| C(s,t,P) \quad \forall \eta \in [s,t]. \tag{3.7}$$

Fix h and let $f_h(\eta) := (\varphi_{s,\eta}(hv) - \varphi_{s,\eta}(O))/h$. Let us denote $\varphi_{s,\eta}(z) = (\varphi^1_{s,\eta}(z), \ldots, \varphi^n_{s,\eta}(z))$. In what follows we assume that $\varphi^j_{s,\eta}(hv) \neq \varphi^j_{s,\eta}(O)$ for all $0 < h << 1$, $j = 1, \ldots, n$, and leave to the reader the (obvious) changes in case $\varphi^j_{s,\eta}(hv) \equiv \varphi^j_{s,\eta}(O)$. Then, for almost every $\eta \in [s,t]$, it follows

$$\begin{aligned}\frac{df_h}{d\eta}(\eta) &= \frac{\frac{\partial \varphi_{s,\eta}(hv)}{\partial \eta} - \frac{\partial \varphi_{s,\eta}(O)}{\partial \eta}}{h} = \frac{G(\varphi_{s,\eta}(hv), \eta) - G(\varphi_{s,\eta}(O), \eta)}{h} \\ &= \frac{(\varphi^1_{s,\eta}(hv) - \varphi^1_{s,\eta}(O))}{h} \frac{G(\varphi_{s,\eta}(hv), \eta) - G((\varphi^1_{s,\eta}(O), \varphi^2_{s,\eta}(hv), \ldots, \varphi^n_{s,\eta}(O)), \eta)}{\varphi^1_{s,\eta}(hv) - \varphi^1_{s,\eta}(O)} \\ &+ \ldots + \frac{(\varphi^n_{s,\eta}(hv) - \varphi^n_{s,\eta}(O))}{h} \frac{G((\varphi^1_{s,\eta}(O), \ldots, \varphi^{n-1}_{s,\eta}(O), \varphi^n_{s,\eta}(hv)), \eta) - G(\varphi_{s,\eta}(O), \eta)}{\varphi^n_{s,\eta}(hv) - \varphi^n_{s,\eta}(O)} \\ &= A(\eta) f_h(\eta) + \sum_{j=1}^n L^h_j(\eta),\end{aligned}$$

where

$$\begin{aligned}L^h_1(\eta) = \frac{(\varphi^1_{s,\eta}(hv) - \varphi^1_{s,\eta}(O))}{h} \Bigg[&\frac{G(\varphi_{s,\eta}(hv), \eta) - G((\varphi^1_{s,\eta}(O), \varphi^2_{s,\eta}(hv), \ldots, \varphi^n_{s,\eta}(O)), \eta)}{\varphi^1_{s,\eta}(hv) - \varphi^1_{s,\eta}(O)} \\ &- \frac{\partial G}{\partial z_1}(\varphi_{s,\eta}(O), \eta) \Bigg],\end{aligned}$$

and $L_2^h(\eta), \ldots, L_n^h(\eta)$ are defined similarly. Thus, multiplying by $H(\eta)$,

$$H(\eta)\frac{df_h}{d\eta}(\eta) - H(\eta)A(\eta)f_h(\eta) = H(\eta)\sum_{j=1}^{n} L_j^h(\eta) \quad \text{a.e. } \eta \in [s,t],$$

and, by the very definition of H, it holds

$$\frac{d}{d\eta}(H(\eta)f_h(\eta)) = H(\eta)\sum_{j=1}^{n} L_j^h(\eta) \quad \text{a.e. } \eta \in [s,t].$$

Integrating we obtain

$$H(t)f_h(t) - f_h(s) = \sum_{j=1}^{n}\int_s^t H(\eta)L_j^h(\eta)d\eta.$$

But $f_h(s) = v$ for all h. Moreover (2.2) and (3.7) imply that there exists a constant $a > 0$ (independent of h) such that $|H(\eta)L_j^h(\eta)| \leq a\widetilde{C}_{P,t}(\eta)$ for all h. The Lebesgue dominated convergence theorem implies then $\lim_{h\to 0}\int_s^t H(\eta)L_j^h(\eta)d\eta = 0$. Hence

$$\lim_{h\in\mathbb{C}, h\to 0} H(t)f_h(t) - v = 0,$$

proving (3.5).

6. *for all fixed* $T > 0$ *and* $K \subset\subset M$ *compact set, there exists a function* $c_{T,K} \in L^d([0,T],\mathbb{R}^+)$ *such that* (2.3) *is satisfied for all* $0 \leq s \leq t' \leq t \leq T$. Fix $s \geq 0$ and $K \subset\subset M$. Let $T > 0$ and assume $s \leq t' \leq t \leq T$. Let V be a compact set in M which contains $\{\varphi_{s,\eta}(z)\}$ for all $z \in K$ and $\eta \in [s,T]$. Since the curve $\eta \mapsto \varphi_{s,\eta}(z)$ is absolutely continuous, and the distance between two points can be computed as the infimum of absolutely continuous curves (see, *e.g.*, [6]), by (2.1)

$$\begin{aligned} d(\varphi_{s,t'}(z), \varphi_{s,t}(z)) &\leq \int_{t'}^{t} \|\frac{\partial\varphi_{s,\eta}(z)}{\partial t}\|d\eta = \int_{t'}^{t}\|G(\varphi_{s,\eta}(z),\eta)\|d\eta \\ &\leq \int_{t'}^{t} C_{T,V}(\tau)d\tau, \end{aligned}$$

and we are done.

Thus $\{\varphi_{s,t}(z)\}$ is an evolution family of order $d \geq 1$ over M. Uniqueness follows at once from the uniqueness of solutions of ODE's. □

Remark 3.3. One can drop the hypothesis of regularity of k_M from the statement of Proposition 3.1 by giving a meaning in the sense of currents and distributions to the inequality $dk_M(G,G) \leq 0$, similarly to what has been done by M. Abate in [2] for infinitesimal generators of semigroups using the Kobayashi infinitesimal metric. We leave the details to the interested reader.

4. From evolution families to Herglotz vector fields

Proposition 4.1. *Let M be a taut manifold with Kobayashi distance k_M. Assume that $k_M \in C^1(M \times M \setminus Diag)$. Then for any evolution family $(\varphi_{s,t})$ over M there exists a Herglotz vector field $G \in \mathcal{H}_\infty(M)$ which verifies* (1.2)*. Moreover, if H is another weak holomorphic vector field which satisfies* (1.2) *then $G(z,t) = H(z,t)$ for all $z \in M$ and almost every $t \in [0,+\infty)$.*

Proof. We show that the Herglotz vector field $G(z,t)$ is "morally" the vector field tangent to the (locally absolutely) continuous curve $[0,+\infty) \ni h \mapsto \varphi_{t,t+h}(z)$ at $h = 0$.

First of all, we fix a local coordinates chart U. Thus, there exists a biholomorphism $\Phi : U \to \Phi(U) \subset \mathbb{C}^n$. We are going to define a holomorphic vector field which solves (1.2) on $\Phi(U)$ and then pull it back via Φ. With a customary abuse of notation, in the sequel we avoid writing Φ, for instance, we write $\varphi_{s,t}(z)$ instead $\Phi \circ \varphi_{s,t}(z)$ and so on.

1'. *Local definition of an approximation family.* Let $U' \subset\subset U$ be a relatively compact open subset of U and let $T > 0$. By Lemma 2.8 there exists $h_0 = h_0(U', T) > 0$ such that $\varphi_{t,t+h}(z) \in U$ for all $z \in U'$, $t \in [0,T]$ and $0 \leq h \leq h_0$. We let

$$G_h(z,t) := \frac{\varphi_{t,t+h}(z) - z}{h} \quad z \in U', \quad 0 < h \leq h_0, \quad t \in [0,T].$$

2'. *For every $0 < h \leq h_0$ fixed, the map $U' \ni z \mapsto G_h(z,t)$ is holomorphic for all fixed t.* It follows immediately from the very definition.

3'. *The function $[0,T] \ni t \mapsto G_h(z,t)$ is continuous.* It follows at once from Lemma 2.8.

4'. *For every compact set $K \subset\subset U'$ there exists $A_{T,K} > 0$ such that*

$$|G_h(z,t)| \leq A_{T,K}$$

for all $0 < h \leq h_0$, $z \in K$ and every $t \in [0,T]$. Indeed, by Property (4) of Definition 1.2 we have

$$|\varphi_{t,t+h}(z) - z| \leq c_U d(\varphi_{t,t+h}(z), \varphi_{t,t}(z)) \leq c_U c_{T,K} |t + h - t| = c_U c_{T,K} |h|,$$

and setting $A_{T,K} = c_U c_{T,K}$ we have the claim.

5'. *For all $t \in [0,+\infty)$ there exists a sequence $m_j(t) \to 0^+$ such that $G(z,t) = \lim_{j\to\infty} G_{m_j(t)}(z,t)$ has the property that $U' \ni z \mapsto G(z,t)$ is holomorphic for all fixed t and $[0,+\infty) \ni t \mapsto G(z,t)$ is measurable for all fixed $z \in U'$.* Let $\{K_r\}$ be a compact exhaustion of U' and let $T \in \mathbb{N} \setminus \{0\}$. Let

$$\Upsilon_T := \{f : U' \to \mathbb{C}^n : \sup_{z \in K_r} |f(z)| \leq A_{T,K_r}, r \in (0,+\infty)\},$$

where $A_{T,K_r} > 0$ is the constant given in Step 4'. By Step. 4' the family $\{G_h(z,t)\}$ is normal for every fixed $t \geq 0$, thus every accumulation point is holomorphic.

The space Υ_T is a closed bounded subset of $\mathsf{Hol}(U, \mathbb{C}^n)$. Thus it is a compact metrizable space (with respect to the topology of uniform convergence on compacta).

Let $\gamma_T : \Upsilon_T^{\mathbb{N}} \to \Upsilon_T$ be the measurable selector (see, *e.g.*, [12]). Then consider the sequence $\{G_{1/m}(z,t)\}_{m\in\mathbb{N}} \subset \Upsilon_T$ and let

$$G^T(z,t) := \gamma_T(\{G_{1/m}(z,t)\}) := \lim_{j\to\infty} G_{\frac{1}{m_j(t)}}(z,t). \tag{4.1}$$

By the very construction and definition of measurable selector, $[T-1,T) \ni t \mapsto G^T(z,t)$ is measurable for all z. Then define $G(z,t) := G^T(z,t)$ according to whether $t \in [T-1,T)$.

6'. *The map $G(z,t)$ is a weak holomorphic vector field of order ∞ over U' and satisfies* (1.2). By Step 5', since $G(z,t) \in \Upsilon_T$, it is clear that $G(z,t)$ is a weak holomorphic vector field of order ∞ on U. Now, by (4.1) and for almost every $t \in [0,+\infty)$ (and for $s \le t$ so that $\varphi_{s,\eta}(z) \in U'$ for $\eta \in [s,t]$) we have

$$\begin{aligned}\frac{\partial \varphi_{s,t}(z)}{\partial t} &= \lim_{h\to 0} \frac{\varphi_{s,t+h}(z) - \varphi_{s,t}(z)}{h} = \lim_{h\to 0} \frac{\varphi_{t,t+h}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{h} \\ &= \lim_{j\to\infty} \frac{\varphi_{t,t+h_j(t)}(\varphi_{s,t}(z)) - \varphi_{s,t}(z)}{h_j(t)} = G(\varphi_{s,t}(z),t).\end{aligned}$$

7'. *For every polydisc $P \subset\subset U'$ and for all $0 \le s \le t$ such that there exists a polydisc $P' \subset\subset U$ for which $\varphi_{s,\eta}(P) \subseteq P'$ for all $\eta \in [s,t]$, it follows that $\varphi_{s,t} : P \to U$ is univalent.* Let $z, w \in P$ and $z \neq w$. Assume by contradiction that $\varphi_{s,t}(z) = \varphi_{s,t}(w)$. Let

$$\theta(\eta) := |\varphi_{s,\eta}(z) - \varphi_{s,\eta}(w)|, \quad \eta \in [s,t].$$

Let $\widetilde{C}_{K,t} \in L^\infty([0,t])$ be as in (2.2) (with $T = t$). Then

$$\begin{aligned}\theta(\eta) &= |\varphi_{s,\eta}(z) - \varphi_{s,t}(z) - \varphi_{s,\eta}(w) + \varphi_{s,t}(w)| = \left|\int_\eta^t (G(\varphi_{s,\xi}(w),\xi) - (G(\varphi_{s,\xi}(z),\xi) d\xi\right| \\ &\le \int_\eta^t |G(\varphi_{s,\xi}(z),\xi) - G(\varphi_{s,\xi}(w),\xi)| d\xi \le |\widetilde{C}_{K,t}|_{L^\infty} \int_\eta^t \theta(\xi) d\xi.\end{aligned}$$

Gronwall's Lemma 3.2 implies that $\theta(\eta) = 0$ for all $\eta \in [s,t]$. But $\theta(s) = |z-w| \neq 0$, reaching a contradiction.

8'. *Uniqueness of $G(z,t)$ on U'*. We are going to prove that if $H(z,t)$ is another weak holomorphic vector field over U' such that

$$\frac{\partial \varphi_{s,t}(z)}{\partial t} = H(\varphi_{s,t}(z),t) \tag{4.2}$$

for almost every $t \in [0,+\infty)$ (and for $s \le t$ so that $\varphi_{s,\eta}(z) \in U'$ for $\eta \in [s,t]$) then $H(z,t) = G(z,t)$ for all $z \in U'$ and almost every $t \in [0,+\infty)$. Let $P \subset\subset U'$ be a polydisc and let $s \approx t$ so that $\varphi_{s,\eta}(P)$ is contained in a fixed polydisc relatively compact in U' for all $\eta \in [s,t]$. Then from (1.2) and (4.2) we obtain that for almost every $t \in [0,+\infty)$ and $q \in \varphi_{s,t}(P)$

$$H(q,t) \equiv G(q,t).$$

Since $z \mapsto \varphi_{s,t}(z)$ is univalent on P by Step 7', the set $\varphi_{s,t}(P)$ is open in U' and thus by the identity principle for holomorphic maps $G(z,t) \equiv H(z,t)$ for all $z \in U'$ and almost every $t \in [0,+\infty)$.

Now we have a way to define the vector field solving (1.2) on each relatively compact open set of any local chart in M, and such a vector field is unique for almost every time. Since M is countable at infinity by the very definition of manifold, we can cover it with countable many coordinates charts in such a way that this covering is locally finite. Hence the previous construction allows to define globally $G(z,t)$ on M for almost every $t \in [0,+\infty)$ in such a way that (1.2) is satisfied.

To end up the proof, we need to show that $G(z,t)$ is a Herglotz vector field. To this aim, it is only left to show that $(dk_M)_{(z,w)}(G(z,t), G(w,t)) \leq 0$ for a.e. $t \in [0,+\infty)$, $z \neq w$. The map $M \ni z \mapsto \varphi_{s,t}(z)$ is holomorphic, thus by Property (2) of Definition 1.2

$$\begin{aligned} k_M(\varphi_{s,t+h}(z), \varphi_{s,t+h}(w)) &\leq k_M(\varphi_{t,t+h}(\varphi_{s,t}(z)), \varphi_{t,t+h}(\varphi_{s,t}(w))) \\ &\leq k_M(\varphi_{s,t}(z), \varphi_{s,t}(w)) \end{aligned} \tag{4.3}$$

for all $h \geq 0$.

Let $Z(s)$ be the zero measure set such that $[s,+\infty) \setminus Z(s) \ni t \mapsto \varphi_{s,t}(z)$ and $[s,+\infty) \setminus Z(s) \ni t \mapsto \varphi_{s,t}(w)$ are differentiable. Let $Z := \cup_{s \in \mathbb{Q}} Z(s)$. Then Z has zero measure. Let $t_0 \in [0,+\infty) \setminus Z$. By (4.3) and (1.2) (and taking into account that $\varphi_{s,t}(z) \neq \varphi_{s,t}(w)$ as we will show in Proposition 5.1), we have

$$\begin{aligned} 0 &\geq \lim_{h \to 0} \frac{k_M(\varphi_{s,t_0+h}(z), \varphi_{s,t_0+h}(w)) - k_M(\varphi_{s,t_0}(z), \varphi_{s,t_0}(w))}{h} \\ &= (dk_M)_{(\varphi_{s,t_0}(z), \varphi_{s,t_0}(w))}(G(\varphi_{s,t_0}(z), t_0), G(\varphi_{s,t_0}(w), t_0)), \end{aligned}$$

which holds for every $s \in \mathbb{Q}$ such that $s \leq t_0$. Taking the limit for $\mathbb{Q} \ni s \to t_0$ by Lemma 2.8 we have the result. $\square$

Remark 4.2. If $M = \mathbb{D}$ the unit disc in $\mathbb{C}$, in [4] we proved that Proposition 4.1 holds also for evolution families of order $d \geq 1$ (and not just $d = \infty$). This was done by using the Berkson-Porta representation formula for infinitesimal generators, which is missing in higher dimensions.

Remark 4.3. Dropping the hypothesis on the regularity of k_M in Proposition 4.1, it follows from the above proof that given an evolution family $(\varphi_{s,t})$ on M then there exists a weak holomorphic vector field G of order ∞ which verifies (1.2). It does not seem to be clear how to obtain that G is Herglotz, not even in the distributional sense.

5. Univalence of evolution families

Proposition 5.1. *Let $(\varphi_{s,t})$ be an evolution family on a taut manifold M. Then for every $0 \leq s \leq t < +\infty$ the map $M \ni z \mapsto \varphi_{s,t}(z) \in M$ is univalent.*

Proof. Fix $0 \leq s \leq t < +\infty$. We show that $\varphi_{s,t}$ is injective on M. By contradiction, assume $\varphi_{s,t}(z_0) = \varphi_{s,t}(z_1)$ with $z_0 \neq z_1$. Let $t_0 \in [s,t]$ be the smallest number such that $\varphi_{s,t_0}(z_0) = \varphi_{s,t_0}(z_1)$. Since $\varphi_{s,s} = \mathsf{id}_M$, Lemma 2.8 implies $t_0 > s$. Let $z_2 = \varphi_{s,t_0}(z_0)$. Then for all $u \in (s,t_0)$

$$\varphi_{u,t_0}(\varphi_{s,u}(z_0)) = \varphi_{s,t_0}(z_0) = z_2 = \varphi_{s,t_0}(z_1) = \varphi_{u,t_0}(\varphi_{s,u}(z_1)). \tag{5.1}$$

Let U be a local chart around z_2 and let P be a relatively compact polydisc in U. By Lemma 2.8 we can choose $s < u < t_0$ such that both $\varphi_{s,u}(z_0)$ and $\varphi_{s,u}(z_1)$ are contained in P and $\varphi_{u,\eta}(P)$ is contained in a relatively compact polydisc in U for all $\eta \in [u,t_0]$. Looking at the proof of Proposition 4.1 (Steps 1' to 7') it follows that φ_{u,t_0} is injective on P. But $\varphi_{s,u}(z_0) \neq \varphi_{s,u}(z_1)$ by definition of t_0, hence we get a contradiction with (5.1). $\square$

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