

EVOLUTION FAMILIES AND THE LOEWNER EQUATION I: THE UNIT DISC

FILIPPO BRACCI, MANUEL D. CONTRERAS[†], AND SANTIAGO DÍAZ-MADRIGAL[†]

ABSTRACT. In this paper we introduce a general version of the Loewner differential equation which allows us to present a new and unified treatment of both the radial equation introduced in 1923 by K. Loewner and the chordal equation introduced in 2000 by O. Schramm. In particular, we prove that evolution families in the unit disc are in one to one correspondence with solutions to this new type of Loewner equations. Also, we give a Berkson-Porta type formula for non-autonomous weak holomorphic vector fields which generate such Loewner differential equations and study in detail geometric and dynamical properties of evolution families.

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1. INTRODUCTION

In 1923, Loewner [17] developed a machinery to “embed” a slit domain of the complex plane into a family of domains endowed with a certain order. The key idea was to represent

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such domains by means of a family (nowadays known as a *Loewner chain*) of univalent functions defined on the unit disc and satisfying a suitable differential equation. Such a machinery was then studied and extended to other types of simply connected domains by Kufarev in 1943 and Pommerenke in 1965 (see, e.g., [11], [20], and [24]).

Since the original paper of Loewner, this method has shown to be extremely useful when dealing with many different problems, especially those having some character of extremality. In fact, in 1984 de Branges used (extensions of) Loewner's theory to solve the Bieberbach conjecture.

The classical *radial Loewner equation* in the unit disc $\mathbb{D} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ is the following non-autonomous differential equation

$$(1.1) \quad \begin{cases} \dot{w} = G(w, t) & \text{for almost every } t \in [s, \infty) \\ w(s) = z \end{cases}$$

where $s \in [0, +\infty)$, $G(w, t) = -wp(w, t)$ with the function $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ measurable in t , holomorphic in z , $p(0, t) = 1$ for all $t \geq 0$ and $\operatorname{Re} p(z, t) \geq 0$. In fact Loewner himself studied the case when $p(z, t) = \frac{1+k(t)z}{1-k(t)z}$ for some continuous function $k : [0, +\infty) \rightarrow \partial\mathbb{D}$. Write $t \mapsto \varphi_{s,t}(z)$ for the solution of such a differential equation. Then for $0 \leq s \leq t < +\infty$ the maps $\varphi_{s,t}$ are holomorphic self maps of \mathbb{D} which verify the following properties:

- (1) $\varphi_{s,s} = id_{\mathbb{D}}$,
- (2) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$,
- (3) $\varphi_{s,t}(0) = 0$ and $\varphi'_{s,t}(0) = e^{s-t}$ for all $0 \leq s \leq u \leq t < +\infty$.

This family $(\varphi_{s,t})$ is an example of a more general object we introduce in this paper and that we call an *evolution family* of the unit disc (see Definition 3.1).

The hypotheses $G(0, t) \equiv 0$ and $p(0, t) \equiv 1$, which forces the evolution family $(\varphi_{s,t})$ to fix the origin and to have normalized first derivatives at 0, are strongly used in the construction of the family itself (mainly in proving semicompleteness and holomorphicity) because they allow to use distortion theorems.

Until the end of the XX century, there were only few papers where equation (1.1) was studied assuming $G(\tau, t) \equiv 0$ for some $\tau \in \partial\mathbb{D}$. We cite the pioneering works of Goryainov [12] and Goryainov and Ba [13]. After that, Schramm [25] and Lawler, Schramm and Werner [15], [16] proved the Mandelbroit conjecture using a stochastic version of this chordal Loewner equation. Also Bauer [2], Marshall and Rohde [18] and Prokhorov and Vasiliev [23] studied a similar chordal Loewner equation. In such a case, $G(w, t) = (1-w)^2 p(w, t)$ where $p(w, t) = \frac{1}{g(w)+ih(t)}$ with $g(w) = \frac{1+w}{1-w}$ and $h : [0, +\infty) \rightarrow \mathbb{R}$ continuous. Solutions to such an equation correspond to evolution families $(\varphi_{s,t})$ with boundary fixed points and are usually stated in the half plane model.

In this paper we study general evolution families of the unit disc. Our method allows to treat at the same time evolution families with inner fixed points and with no

interior fixed points. In particular, we can solve (1.1) in case of boundary fixed points *without* assuming any particular form of $G(z, t)$. More in detail, our aim is to completely characterize evolution families by means of a differential equation of type (1.1). The key observation on which our work is based, is that in all the previous studied cases, the function $w \mapsto G(w, t)$ is a semicomplete vector field for all fixed $t \geq 0$. And in fact we prove that all evolution families of the unit disc are in one-to-one correspondence with weak holomorphic vector fields which are infinitesimal generators for almost every time (see Section 2 for definitions).

More precisely, we call *Herglotz vector field of order $d \geq 1$* a function $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ which is a weak holomorphic vector field of order $d \geq 1$ (in the sense of Carathéodory's theory, see Definition 4.1) and for almost every $t \geq 0$ has the property that $z \mapsto G(z, t)$ is an infinitesimal generator. Also, an evolution family of the unit disc is said to be *of order $d \geq 1$* if $|\varphi_{s,u}(z) - \varphi_{s,t}(z)|$ is locally bounded by a non-negative function whose derivative is in L^d (see Definition 3.1).

Our main result is the following:

Theorem 1.1. *For any evolution family $(\varphi_{s,t})$ of order $d \geq 1$ in the unit disc there exists a (essentially) unique Herglotz vector field $G(z, t)$ of order d such that for all $z \in \mathbb{D}$*

$$(1.2) \quad \frac{\partial \varphi_{s,t}(z)}{\partial t} = G(\varphi_{s,t}(z), t) \quad \text{a.e. } t \in [0, +\infty).$$

Conversely, for any Herglotz vector field $G(z, t)$ of order $d \geq 1$ in the unit disc there exists a unique evolution family $(\varphi_{s,t})$ of order d such that (1.2) is satisfied.

Here *essentially unique* means that if $H(z, t)$ is another Herglotz vector field which satisfies (1.2) then $G(\cdot, t) = H(\cdot, t)$ for almost every $t \in [0, +\infty)$.

Infinitesimal generators have been characterized in several different ways. In particular, in the proof of the above theorem we use a result of [5], from which it follows that $(d\rho_{\mathbb{D}})_{(z,w)}(G(z, t), G(w, t)) \leq 0$ for all $t \geq 0$ and $z \neq w$, where $\rho_{\mathbb{D}}$ is the hyperbolic distance on \mathbb{D} . This estimate allows us to avoid considering displacement of fixed points in order to obtain suitable bounds. In fact, a version of Theorem 1.1 holds more generally on complex complete hyperbolic manifolds (see [6]).

The Berkson-Porta representation formula for infinitesimal generators implies that for a.e. fixed $t \geq 0$

$$(1.3) \quad G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t),$$

with $p(\cdot, t) : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic such that $\operatorname{Re} p(\cdot, t) \geq 0$. What is remarkable, and non-trivial, is that such a representation is essentially unique and the function $p(z, t)$, which we call a *Herglotz function of order $d \geq 1$* (see Definition 4.5), is locally in L^d in $t \geq 0$ and $[0, +\infty) \ni t \mapsto \tau(t)$ is measurable; and conversely, any vector field of the form (1.3) is a Herglotz vector field (see Theorem 4.8 for a precise statement).

There is thus an (essentially) one-to-one correspondence among evolution families $(\varphi_{s,t})$ of order $d \geq 1$, Herglotz vector fields $G(z, t)$ of order $d \geq 1$, and couples (p, τ) of Herglotz

functions $p(z, t)$ of order d and measurable functions $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$. In what follows we say that the couple (p, τ) is the *Berkson-Porta data* for $(\varphi_{s,t})$.

Going back to Loewner equations, the previous two theorems can be combined saying that the following differential equation

$$(1.4) \quad \begin{cases} \dot{w} = (w - \tau(t))(\overline{\tau(t)}w - 1)p(w, t) & \text{for a. e. } t \in [s, +\infty) \\ w(s) = z. \end{cases}$$

has a family of solutions $(\varphi_{s,t})$ which form an evolution family of order $d \geq 1$ provided $p(w, t)$ is a Herglotz function of order $d \geq 1$ and $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ is a measurable function.

We point out that equation (1.4) contains all the Loewner type equations studied so far in the literature, where in fact only Herglotz functions of order ∞ and $\tau \equiv \text{const}$ are considered. In case $\tau \equiv \text{const}$, evolution families of order $d \geq 1$ can be defined by means of weaker conditions, such as properties of regularity of first derivatives (see Theorem 7.3).

The plan of the paper is the following. In Section 2 we collect some preliminary results from iteration theory and semigroups theory. In Section 3 we deal with evolution families, proving some results about continuity in the two parameters. In Section 4 we introduce Herglotz vector fields and prove that they are semicomplete (Theorem 4.4). Then we relate Herglotz vector fields with Herglotz functions (Theorem 4.8). In Section 5 we prove Theorem 5.2 which shows that solutions of a Herglotz vector field form an evolution family (proving thus one part of Theorem 1.1). In Section 6 we prove the other part of Theorem 1.1 (see Theorem 6.2). With such a result at hand, moving from evolution families to Herglotz vector fields and Herglotz functions, we can prove some more regularity properties of evolution families with respect to the two parameters (Theorem 6.4 and Theorem 6.6). In particular, we show that

$$\frac{\partial \varphi}{\partial s}(z, s, t) = -G(z, s)\varphi'_{s,t}(z).$$

In Corollary 6.3 we show that all the elements of an evolution family must be univalent and, in Corollary 6.5, that Herglotz vector fields are (almost everywhere) characterized by their trajectories proving the essential uniqueness of the previous Theorems 1.1. In the last two sections of the paper we get back to radial and chordal Loewner equations.

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2. PRELIMINARIES

For a self-map f of \mathbb{D} , a point $p \in \partial\mathbb{D}$ is a (boundary) fixed point of f if $\angle \lim_{z \rightarrow p} f(z) = p$ (as usual, \angle -lim denotes the angular limit). If $q := \angle \lim_{z \rightarrow p} f(z) \in \partial\mathbb{D}$, then the angular limit $f'(p) := \angle \lim_{z \rightarrow p} \frac{f(z) - q}{z - p}$ exists (possibly ∞) and it is different from zero (see [22]). By the Schwarz-Pick lemma a holomorphic self-map of the unit disc, $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$, $f \neq \text{id}_{\mathbb{D}}$ may have at most one fixed point in \mathbb{D} . If such a unique fixed point in \mathbb{D} exists,

it is called the *Denjoy-Wolff point* and the sequence $\{f_n\}$ of iterates of f , defined as $f_n := f \circ f_{n-1}$, $f_1 = f$, converges to it uniformly on compacta whenever f is not a disc automorphism.

If f has no fixed points in \mathbb{D} , the Denjoy-Wolff theorem (see, e.g. [1]) guarantees the existence of a unique boundary fixed point $\tau \in \partial\mathbb{D}$ such that the sequence of iterates $\{f_n\}$ converges to τ uniformly on compacta. Such τ is again called the *Denjoy-Wolff point* of f . If $\tau \in \partial\mathbb{D}$ is the Denjoy-Wolff point of f , then $f'(\tau) \in (0, 1]$ (see [22]).

A *(one-parameter) semigroup of holomorphic functions* is a continuous semigroups-morphism $\Phi : \mathbb{R}^+ \ni t \mapsto \Phi(t) = \phi_t \in \text{Hol}(\mathbb{D}, \mathbb{D})$ between the semigroup of positive real numbers endowed with the Euclidean topology and the semigroup of holomorphic self-maps of the unit disc endowed with the topology of uniform convergence on compacta.

Given a semigroup $\Phi = (\phi_t)$, it is well-known (see [26], [3]) that there exists a *unique* holomorphic function $G : \mathbb{D} \rightarrow \mathbb{C}$, called the *vector field associated with Φ* or the *infinitesimal generator* of Φ , such that

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)) = G(z) \frac{\partial \phi_t(z)}{\partial z} \quad \text{for all } z \in \mathbb{D} \text{ and } t \geq 0.$$

By the Berkson and Porta representation formula [3], a holomorphic function $G : \mathbb{D} \rightarrow \mathbb{C}$ is the infinitesimal generator of a semigroup Φ of holomorphic self-maps of \mathbb{D} if and only if there exist $\tau \in \overline{\mathbb{D}}$ and a holomorphic function $p : \mathbb{D} \rightarrow \mathbb{C}$ with $\text{Re } p \geq 0$ such that

$$G(z) = (\tau - z)(1 - \bar{\tau}z)p(z), \quad z \in \mathbb{D}.$$

Moreover, if G is not identically zero, then such a representation is unique. In fact, the point τ is the *Denjoy-Wolff point* of all the functions of the semigroup. We denote by $\text{Gen}(\mathbb{D})$ the set of all the infinitesimal generators of semigroups of holomorphic self-maps of the unit disc. It is well-known that $\text{Gen}(\mathbb{D})$ is closed in $\text{Hol}(\mathbb{D}, \mathbb{C})$ and a real convex cone in $\text{Hol}(\mathbb{D}, \mathbb{C})$ with vertex at 0 (see, for example, [1] and [26]). A useful example of infinitesimal generator is given by $G = \varphi - \text{id}$ for $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ [26, Corollary 3.3.1]. The following two facts, related to the continuity of the so-called Heins map (see [14] and [4]) might be known but, since we do not have a reference, we sketch here their proofs.

Proposition 2.1. *Endow $\text{Gen}(\mathbb{D})$ and $\text{Hol}(\mathbb{D}, \mathbb{C})$ with the compact-open topology, and let $\underline{0}$ denote the zero function, $\underline{0}(z) = 0$ for all $z \in \mathbb{D}$. For all $F \in \text{Gen}(\mathbb{D})$, using the Berkson-Porta representation we write $F(z) = (z - \tau_F)(\bar{\tau}_F z - 1)p_F(z)$, $z \in \mathbb{D}$. Then the following two maps are continuous*

$$\begin{aligned} BP_\tau : \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\} &\rightarrow \overline{\mathbb{D}}, & \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\} \ni F &\mapsto BP_\tau(F) := \tau_F \\ BP_p : \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\} &\rightarrow \text{Hol}(\mathbb{D}, \mathbb{C}), & \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\} \ni F &\mapsto BP_p(F) := p_F. \end{aligned}$$

Proof. By the uniqueness of the Berkson-Porta representation, the maps BP_τ and BP_p are well-defined. We only prove continuity of BP_τ , the other being similar. Let $\{F_n\} \subset \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\}$ converging to $F \in \text{Gen}(\mathbb{D}) \setminus \{\underline{0}\}$. Let $\{\tau_{n_k} := BP_\tau(F_{n_k})\}$ converge to some $\alpha \in \overline{\mathbb{D}}$. Since $\{w \in \mathbb{C} : \text{Re } w > 0\}$ is hyperbolic, the family $\{p_n := BP_p(F_n) : n \in \mathbb{N}\}$

is a normal. Hence, up to subsequences, we can assume $\tau_{n_k} \rightarrow \alpha$ and $p_{n_k} \rightarrow p$ for some $p \in \text{Hol}(\mathbb{D}, \mathbb{C})$ with $\text{Re } p \geq 0$. Therefore $F(z) = (z - \alpha)(\bar{\alpha}z - 1)p(z)$, and by uniqueness of the Berkson-Porta representation, we conclude that $\alpha = \tau$ as wanted. \square

Lemma 2.2. *Let $\{G_n\}$ be a sequence in $\text{Gen}(\mathbb{D})$ such that there are two different points $z_0, z_1 \in \mathbb{D}$ and two sequences $\{u_n\}$ and $\{v_n\}$ in \mathbb{D} with $\lim_n u_n = z_0$ and $\lim_n v_n = z_1$ such that $\sup_n |G_n(u_n)| < +\infty$ and $\sup_n |G_n(v_n)| < +\infty$. Then there exists a subsequence $\{G_{n_k}\}$ converging to an infinitesimal generator $G \in \text{Gen}(\mathbb{D})$.*

Proof. By Berkson-Porta's theorem, $G_n(z) = (z - \tau_n)(\bar{\tau}_n z - 1)p_n(z)$ with $\tau_n \in \bar{\mathbb{D}}$ and $\text{Re } p_n \geq 0$. Up to subsequences, we can assume $\tau_n \rightarrow \tau \in \bar{\mathbb{D}}$. The family $\{p_n\}$ is normal, hence we have only to prove that it is not compactly divergent. Arguing by contradiction, suppose $\{p_n\}$ compactly diverges to ∞ . Since z_0 and z_1 are different, we may assume that $\tau \neq z_0$. Then we have $\lim_n |G_n(u_n)| = \lim_n |(u_n - \tau_n)(\bar{\tau}_n u_n - 1)p_n(u_n)| = |(z_0 - \tau)(\bar{\tau}z_0 - 1)| \lim_n |p_n(u_n)| = +\infty$, which contradicts $\sup_n |G_n(u_n)| < +\infty$. \square

Remark 2.3. It is worth noticing that the above lemma would not be true if we only assume that there is only one point $z_0 \in \mathbb{D}$ and one sequence (u_n) in \mathbb{D} with $\lim_n u_n = z_0$ such that $\sup_n |G_n(u_n)| < +\infty$. For example, consider the sequence of infinitesimal generators given by $G_n(z) := -nz$, for all $z \in \mathbb{D}$ (G_n is the infinitesimal generator of the semigroup $\varphi_t(z) = e^{-nt}z$).

In what follows we will use this auxiliary lemma whose simple proof based on the Cauchy formula is omitted:

Lemma 2.4. *Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be holomorphic. Then, for all $0 < r < 1$ and $|z|, |w| \leq r$,*

$$|f(z) - f(w)| \leq 4|z - w| \sup |f(\zeta)|$$

where the supremum is taken over $|\zeta| \leq (1 + r)/2$.

3. EVOLUTION FAMILIES IN THE UNIT DISC

Definition 3.1. A family $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ (also denoted $\varphi(z, s, t)$) of holomorphic self-maps of the unit disc is an *evolution family of order d* with $d \in [1, +\infty]$ (in short, an *L^d -evolution family*) if

EF1. $\varphi_{s,s} = id_{\mathbb{D}}$,

EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$,

EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that for all $0 \leq s \leq u \leq t \leq T$

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi.$$

Remark 3.2. Clearly, by the very definition, if $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ is an evolution family of order d then it is also an evolution family of order d' for all $1 \leq d' \leq d$.

Example 3.3. Let $d \geq 1$. Let $\lambda : [0, +\infty) \rightarrow \mathbb{R}^+$ be an absolutely continuous increasing function such that $\dot{\lambda} \in L^d_{\text{loc}}([0, +\infty), \mathbb{R})$ but $\dot{\lambda} \notin L^k_{\text{loc}}([0, +\infty), \mathbb{R})$ for any $k > d$. Then $\varphi_{s,t}(z) := \exp(\lambda(s) - \lambda(t))z$ is an evolution family of order d which is not of order k for any $k > d$.

Example 3.4. Let (ϕ_t) be a semigroup of holomorphic self-maps of \mathbb{D} . Let $\varphi_{s,t} := \phi_{t-s}$ for $0 \leq s \leq t < +\infty$. From $|\phi_{u-s}(z) - \phi_{t-s}(z)| = \left| \int_{u-s}^{t-s} \frac{\partial \phi_\xi(z)}{\partial \xi} d\xi \right|$ it follows easily that $(\varphi_{s,t})$ is an evolution family of order ∞ .

Throughout the paper, we denote by $\rho_{\mathbb{D}}(z, w)$ the hyperbolic distance in the unit disc between two points $z, w \in \mathbb{D}$.

Proposition 3.5. *Let $d \geq 1$ and let $(\varphi_{s,t})$ be an evolution family of order d in the unit disc. The map $(s, t) \mapsto \varphi_{s,t} \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is jointly continuous. Namely, given a compact set $K \subset \mathbb{D}$ and two sequences $\{s_n\}, \{t_n\}$ in $[0, +\infty)$, with $0 \leq s_n \leq t_n$, $s_n \rightarrow s$, and $t_n \rightarrow t$, then $\lim_{n \rightarrow \infty} \varphi_{s_n, t_n} = \varphi_{s,t}$ uniformly on K .*

Proof. Let $\{s_n\}, \{t_n\}$ be two sequences in $[0, +\infty)$ with $0 \leq s_n \leq t_n$, $s_n \rightarrow s$ and $t_n \rightarrow t$. Since the set $\{\varphi_{u,v} : 0 \leq u \leq v\}$ is bounded in $\text{Hol}(\mathbb{D}, \mathbb{C})$, by Vitali's theorem, it is enough to show that $\lim_{n \rightarrow \infty} \varphi_{s_n, t_n}(z) = \varphi_{s,t}(z)$ for all fixed z in the unit disc. Fix $z \in \mathbb{D}$. We may (and we do) assume that the sequences $\{s_n\}$ and $\{t_n\}$ are in one of the following three cases:

Case I: $s_n \leq t_n \leq s$ for all n .

Case II: $s \leq s_n$ for all n ;

Case III: $s_n \leq s \leq t_n$ for all n ;

In case I, we have that $s = t$. Therefore, using EF3, we take the corresponding function $k_{z,t} \in L^d([0, T], \mathbb{R})$ and

$$\begin{aligned} |\varphi_{s_n, t_n}(z) - \varphi_{s,t}(z)| &= |\varphi_{s_n, t_n}(z) - z| = |\varphi_{s_n, t_n}(z) - \varphi_{s_n, s_n}(z)| \\ &\leq \int_{s_n}^{t_n} k_{z,t}(\xi) d\xi \xrightarrow{n} 0, \end{aligned}$$

where the last limit is zero because the measure of the interval $[s_n, t_n]$ tends to zero as n goes to ∞ .

If we are in case II, then

$$\begin{aligned} \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z), \varphi_{s,t}(z)) &\leq \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z), \varphi_{s, t_n}(z)) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s,t}(z)) \\ &= \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z), \varphi_{s_n, t_n}(\varphi_{s, s_n}(z))) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s,t}(z)) \\ &\leq \rho_{\mathbb{D}}(z, \varphi_{s, s_n}(z)) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s,t}(z)), \end{aligned}$$

while in case III,

$$\begin{aligned} \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z), \varphi_{s, t}(z)) &\leq \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z), \varphi_{s, t_n}(z)) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s, t}(z)) \\ &= \rho_{\mathbb{D}}(\varphi_{s, t_n}(\varphi_{s_n, s}(z)), \varphi_{s, t_n}(z)) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s, t}(z)) \\ &\leq \rho_{\mathbb{D}}(\varphi_{s_n, s}(z), z) + \rho_{\mathbb{D}}(\varphi_{s, t_n}(z), \varphi_{s, t}(z)). \end{aligned}$$

Therefore, since $\varphi_{s, t}(z)$ and z belong to \mathbb{D} , it is enough to show that the sequence $\{\varphi_{s, t_n}(z)\}$ converges to $\varphi_{s, t}(z)$ and $\{\varphi_{s_n, s}(z)\}$ (or $\{\varphi_{s, s_n}(z)\}$) converges to z as $n \rightarrow +\infty$.

Let $T > \sup_n t_n$. Let $k_{z, T} \in L^d([0, T], \mathbb{R})$ be the function given by EF3. Since $k_{z, T} \in L^1([0, T], \mathbb{R})$,

$$|\varphi_{s, t_n}(z) - \varphi_{s, t}(z)| \leq \int_t^{t_n} k_{z, T}(\xi) d\xi \xrightarrow{n \rightarrow \infty} 0,$$

if $s \leq s_n$ for all n ,

$$|\varphi_{s, s_n}(z) - z| = |\varphi_{s, s_n}(z) - \varphi_{s, s}(z)| \leq \int_s^{s_n} k_{z, T}(\xi) d\xi \xrightarrow{n \rightarrow \infty} 0,$$

while, if $s_n \leq s$ for all n ,

$$|\varphi_{s_n, s}(z) - z| = |\varphi_{s_n, s}(z) - \varphi_{s_n, s_n}(z)| \leq \int_{s_n}^s k_{z, T}(\xi) d\xi \xrightarrow{n \rightarrow \infty} 0,$$

and we are done. \square

Lemma 3.6. *Let $(\varphi_{s, t})$ be an evolution family of order $d \geq 1$ in the unit disc \mathbb{D} . Then for each $0 < T < +\infty$ and $0 < r < 1$, there exists $R = R(r, T) < 1$ such that $|\varphi_{s, t}(z)| \leq R$ for all $0 \leq s \leq t \leq T$ and $|z| \leq r$.*

Proof. Assume the result is false. Then there exist three sequences $\{z_n\}$, $\{s_n\}$, $\{t_n\}$ such that $|z_n| \leq r$, $z_n \rightarrow z_0$, $s_n, t_n \in [0, T]$, $s_n \leq t_n$, $s_n \rightarrow s_0$, $t_n \rightarrow t_0$, and $|\varphi_{s_n, t_n}(z_n)| \rightarrow 1$. Since the map φ_{s_n, t_n} is a contraction for the hyperbolic metric, we have that $\rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z_n), \varphi_{s_n, t_n}(z_0)) \leq \rho_{\mathbb{D}}(z_n, z_0) \rightarrow 0$. Then $|\varphi_{s_n, t_n}(z_0)| \rightarrow 1$. Moreover

$$\begin{aligned} \rho_{\mathbb{D}}(\varphi_{0, t_n}(z_0), \varphi_{s_n, t_n}(z_0)) &= \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(\varphi_{0, s_n}(z_0)), \varphi_{s_n, t_n}(z_0)) \\ &\leq \rho_{\mathbb{D}}(\varphi_{0, s_n}(z_0), z_0) \rightarrow \rho_{\mathbb{D}}(\varphi_{0, s}(z_0), z_0) < +\infty. \end{aligned}$$

Again this implies that $|\varphi_{0, t_n}(z_0)| \rightarrow 1$. But $\varphi_{0, t_n}(z_0) \rightarrow \varphi_{0, t}(z_0) \in \mathbb{D}$ by Proposition 2.1. A contradiction. \square

Proposition 3.7. *Let $(\varphi_{s, t})$ be an evolution family of order $d \geq 1$ in the unit disc \mathbb{D} .*

- (1) *For all $z \in \mathbb{D}$ and for all $s \geq 0$, the map $[s, \infty) \ni t \mapsto \varphi_{s, t}(z) \in \mathbb{C}$ is locally absolutely continuous, that is, for all $T > s$, the map $[s, T] \ni t \mapsto \varphi_{s, t}(z) \in \mathbb{C}$ is absolutely continuous.*
- (2) *For all $z \in \mathbb{D}$ and for all $T > 0$, the map $[0, T] \ni s \mapsto \varphi_{s, T}(z) \in \mathbb{C}$ is absolutely continuous.*

Proof. (1) It follows immediately from EF3.

(2) Fix $z \in \mathbb{D}$ and $T > 0$. By Lemma 3.6, there exists $r := R(z, T) < 1$ such that $|\varphi_{s,t}(z)| \leq r$ for all $0 \leq s \leq t \leq T$. Let $M := \sup_{|\zeta| < (1+r)/2} |\varphi_{b,T}(\zeta)|$. Hence, by Lemma 2.4, if $0 \leq a < b \leq T$,

$$|\varphi_{b,T}(z) - \varphi_{a,T}(z)| = |\varphi_{b,T}(z) - \varphi_{b,T}(\varphi_{a,b}(z))| \leq 4M|z - \varphi_{a,b}(z)|.$$

Now, let $k_{z,T} \in L^d([0, T], \mathbb{R})$ be the non-negative function given by EF3. Hence

$$|\varphi_{b,T}(z) - \varphi_{a,T}(z)| \leq 4M|\varphi_{a,a}(z) - \varphi_{a,b}(z)| \leq 4M \int_a^b k_{z,T}(\xi) d\xi.$$

Since $k_{z,T} \in L^1([0, T], \mathbb{R})$, the map $s \in [0, T] \mapsto \varphi_{s,T}(z)$ is absolutely continuous. \square

4. WEAK HOLOMORPHIC VECTOR FIELDS AND HERGLOTZ VECTOR FIELDS

Definition 4.1. Let $d \in [1, +\infty]$. A *weak holomorphic vector field of order d* on the unit disc \mathbb{D} is a function $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ with the following properties:

- WHVF1. For all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto G(z, t)$ is measurable;
- WHVF2. For all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto G(z, t)$ is holomorphic;
- WHVF3. For any compact set $K \subset \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t)| \leq k_{K,T}(t)$$

for all $z \in K$ and for almost every $t \in [0, T]$.

Lemma 4.2. Let G be a weak holomorphic vector field of order $d \geq 1$ on the unit disc \mathbb{D} . Then for any compact set $K \subset \mathbb{D}$ and for all $T > 0$, there exists a non-negative function $\widehat{k}_{K,T} \in L^d([0, T], \mathbb{R})$ such that

$$|G(z, t) - G(w, t)| \leq \widehat{k}_{K,T}(t)|z - w|$$

for all $z, w \in K$ and for almost every $t \in [0, T]$.

Proof. Fix a compact set $K \subset \mathbb{D}$ and $T > 0$. Take $0 < r < 1$ such that $K \subset \overline{\mathbb{D}}(r) := \{\zeta \in \mathbb{C} : |\zeta| \leq r\}$. Let $k_{\overline{\mathbb{D}}((r+1)/2), T} \in L^d([0, T], \mathbb{R})$ be the function given by WHVF3. Hence, using Lemma 2.4, for all $z, w \in K$

$$|G(z, t) - G(w, t)| \leq 4|z - w| \sup_{|\zeta| \leq (1+r)/2} |G(\zeta, t)| \leq 4k_{\overline{\mathbb{D}}((r+1)/2), T}(t)|z - w|,$$

and we are done. \square

By the Carathéodory theory of ODE's (see, for example, [8]), it follows from the above lemma that if G is a weak holomorphic vector field on \mathbb{D} , then for any $(z, s) \in \mathbb{D} \times [0, +\infty)$, there exist a unique $I(z, s) > s$ and a function $x : [s, I(z, s)) \rightarrow \mathbb{D}$ such that

- (1) x is locally absolutely continuous in $[s, I(z, s))$, that is, x is absolutely continuous in $[s, T]$ for all $s < T < I(z, s)$;

(2) x is the solution to the following problem:

$$\begin{cases} \dot{x}(t) = G(x(t), t) \\ x(s) = z \end{cases}$$

for almost all $t \in [s, I(z, s))$ with respect to the Lebesgue measure.

(3) The interval $[s, I(z, s))$ is maximal. Namely, if $y : [s, I) \rightarrow \mathbb{D}$ is a locally absolutely continuous function satisfying

$$\begin{cases} \dot{y}(t) = G(y(t), t) \\ y(s) = z \end{cases}$$

for almost all $t \in [s, I)$, then $I \leq I(z, s)$ and $x(t) = y(t)$ on $[s, I)$.

Such a map x is known as the *positive trajectory* of the vector field G at the pair (z, s) . The number $I(z, s)$ is known as the *escaping time* for the couple (z, s) . We say that the weak holomorphic vector field is *semi-complete* if $I(z, s) = +\infty$ for all $(z, s) \in \mathbb{D} \times [0, +\infty)$.

Definition 4.3. Let $G(z, t)$ be a weak holomorphic vector field of order $d \in [1, +\infty)$ on the unit disc \mathbb{D} . We say that G is a (*generalized*) *Herglotz vector field* (of order d) if for almost every $t \in [0, +\infty)$ it follows $G(\cdot, t) \in \text{Gen}(\mathbb{D})$. Namely, for almost every fixed $t \in [0, +\infty)$ it follows $G(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t)$ for some $\tau(t) \in \overline{\mathbb{D}}$ and holomorphic function $p(\cdot, t) : \mathbb{D} \rightarrow \mathbb{C}$ such that $\text{Re } p(z, t) \geq 0$.

In Theorem 4.8 we prove the non-trivial fact that the dependence of $p(z, t)$ and $\tau(t)$ on t is measurable and $p(z, t)$ has some L^d -type bound. But we first examine semicompleteness:

Theorem 4.4. *Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz vector field of order $d \in [1, +\infty)$. Then G is semicomplete.*

Proof. Denote by $\phi_{s,z}$ the positive trajectory associated with the Cauchy problem

$$\begin{cases} \dot{x}(t) = G(x(t), t) \\ x(s) = z, \end{cases}$$

and let $I(z, s)$ be the corresponding escaping time. We have to show that

$$(4.1) \quad I(z, s) = +\infty \quad \text{for all } z \in \mathbb{D} \text{ and } s \geq 0.$$

In order to prove (4.1), we first show that for all $z, w \in \mathbb{D}$ and $0 \leq s \leq t < \min\{I(z, s), I(w, s)\}$

$$(4.2) \quad \rho_{\mathbb{D}}(\phi_{s,z}(t), \phi_{s,w}(t)) \leq \rho_{\mathbb{D}}(z, w).$$

Assume that $I(z, s) \leq I(w, s)$ and consider the function $h(t) := \rho_{\mathbb{D}}(\phi_{s,z}(t), \phi_{s,w}(t))$ defined for $t \in [s, I(z, s))$. By Caratheodory's ODE's theory, such a function is absolutely continuous and differentiable almost everywhere. On the other hand, by definition of Herglotz vector field, fixed a point t , the map $\mathbb{D} \ni z \mapsto G(z, t)$ is the infinitesimal generator of a semigroup of holomorphic self-maps of the unit disc for almost every $t \in [0, +\infty)$.

Therefore $(d\rho_{\mathbb{D}})_{(\phi_{s,z}(t), \phi_{s,w}(t))} (G(\phi_{s,z}(t), t), G(\phi_{s,w}(t), t)) \leq 0$ for almost every $t \in [0, +\infty)$ by [5, Thm. 0.2]. Hence, for almost every $t \geq 0$,

$$\begin{aligned} \dot{h}(t) &= (d\rho_{\mathbb{D}})_{(\phi_{s,z}(t), \phi_{s,w}(t))} \left(\dot{\phi}_{s,z}(t), \dot{\phi}_{s,w}(t) \right) \\ &= (d\rho_{\mathbb{D}})_{(\phi_{s,z}(t), \phi_{s,w}(t))} (G(\phi_{s,z}(t), t), G(\phi_{s,w}(t), t)) \leq 0. \end{aligned}$$

Thus, $h(t) \leq h(s)$ for all $t \in [s, I(z, s))$, proving (4.2).

Now, we prove that

$$(4.3) \quad I(z, s) = I(w, s) \quad \text{for all } z, w \in \mathbb{D} \text{ and } s \geq 0.$$

Fix $s \geq 0$. Suppose that there are two points z, w such that $I(z, s) < I(w, s)$. Thus, letting $t \rightarrow I(z, s)$, we have $\phi_{s,z}(t) \rightarrow \partial\mathbb{D}$, while $\phi_{s,w}(t)$ stays compact inside \mathbb{D} . In particular, $\rho_{\mathbb{D}}(\phi_{s,z}(t), \phi_{s,w}(t)) \rightarrow \infty$, which contradicts (4.2). Thus $I(z, s) \geq I(w, s)$. Swapping the role of z, w in the previous argument, we have (4.3).

Next, let $I = I(0, 0)$. We prove that

$$(4.4) \quad I(0, s) = I \quad \text{for all } s < I.$$

Let $s < I$ and $z = \phi_{0,0}(s) \in \mathbb{D}$. Take $s < t < \min\{I, I(0, s)\}$. By (4.3), $I(z, s) = I(0, s)$ so $\phi_{s,z}(t)$ is well-defined and belongs to \mathbb{D} . Therefore, by uniqueness of solutions of ODE's it follows that $\phi_{s,z}(t) = \phi_{s,\phi_{0,0}(s)}(t) = \phi_{0,0}(t)$. Moreover, by (4.2),

$$\rho_{\mathbb{D}}(\phi_{s,0}(t), \phi_{0,0}(t)) = \rho_{\mathbb{D}}(\phi_{s,0}(t), \phi_{s,z}(t)) \leq \rho_{\mathbb{D}}(0, z) = \rho_{\mathbb{D}}(0, \phi_{0,0}(s)).$$

From this, arguing as in the proof of (4.3), equation (4.4) follows.

Finally we prove that there exists $\delta > 0$ such that

$$(4.5) \quad I(0, s) \geq s + \delta \quad \text{for all } s \in [0, I).$$

Fix $0 < r < 1$ and let $K = \overline{\{|\zeta| \leq r\}}$. Let $k_{K,I+2}, \widehat{k}_{K,I+2} \in L^d([0, I+2], \mathbb{R}^+)$ be the functions given by WHVF3 and Lemma 4.2. By the absolute continuity of integrals, there exists $0 < \delta < 1$ such that for all $s \in [0, I+1]$ it holds $\int_s^{s+\delta} k_{K,I+2}(t) dt \leq r$ and $\int_s^{s+\delta} \widehat{k}_{K,I+2}(t) dt \leq r$. Moreover, if $f : [s, s+\delta] \rightarrow K$ is measurable, then $[s, s+\delta] \ni \xi \mapsto G(f(\xi), \xi)$ is integrable and

$$\left| \int_s^t G(f(\xi), \xi) d\xi \right| \leq \int_s^t |G(f(\xi), \xi)| d\xi \leq \int_s^t k_{K,I+2}(\xi) d\xi \leq r.$$

Therefore, for $s \in [0, I+1]$, we can define by induction

$$(4.6) \quad \begin{cases} x_{s,0}(t) := 0 \\ x_{s,n}(t) := \int_s^t G(x_{s,n-1}(\xi), \xi) d\xi \end{cases}$$

for $t \in [s, s + \delta]$ and $n \in \mathbb{N}$. Now, since $|x_{s,n}(t)| \leq r$, we have

$$\begin{aligned} |x_{s,n}(t) - x_{s,n-1}(t)| &\leq \int_s^t |G(x_{s,n-1}(\xi), \xi) - G(x_{s,n-2}(\xi), \xi)| d\xi \\ &\leq \int_s^t \widehat{k}_{K,I+2}(\xi) |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)| d\xi \\ &\leq \max_{\xi \in [s, s+\delta]} |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)| \int_s^t \widehat{k}_{K,I+2}(\xi) d\xi \\ &\leq r \max_{\xi \in [s, s+\delta]} |x_{s,n-1}(\xi) - x_{s,n-2}(\xi)|. \end{aligned}$$

From this inequality, we deduce that $\{x_{s,n}\}$ is a Cauchy sequence in the Banach space $C([s, s + \delta])$ of continuous complex functions from $[s, s + \delta]$, endowed with the supremum norm. Therefore, it converges uniformly on $[s, s + \delta]$ to a function $x \in C([s, s + \delta])$. Since $|G(x_{s,n-1}(\tau), \tau)| \leq k_{K,I+2}(\tau)$, the Lebesgue dominated convergence theorem implies $x(t) = \int_s^t G(x(\xi), \xi) d\xi$ for all $t \in [s, s + \delta]$. Therefore, $\phi_{s,0} = x$ on $[s, s + \delta]$, which proves that $I(0, s) \geq s + \delta$, proving (4.5).

Equation (4.1) follows immediately from (4.3), (4.4) and (4.5), and we are done. \square

As we will see, Herglotz vector fields in the unit disc can be decomposed by means of Herglotz functions (this is the reason for the name). We begin by giving the following definition:

Definition 4.5. Let $d \in [1, +\infty]$. A *Herglotz function of order d* is a function $p : \mathbb{D} \times [0, +\infty) \mapsto \mathbb{C}$ with the following properties:

- HF1. For all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto p(z, t) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$;
- HF2. For all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto p(z, t) \in \mathbb{C}$ is holomorphic;
- HF3. For all $z \in \mathbb{D}$ and for all $t \in [0, +\infty)$, we have $\operatorname{Re} p(z, t) \geq 0$.

Proposition 4.6. Let $d \in [1, \infty]$. A function $p : \mathbb{D} \times [0, +\infty) \mapsto \mathbb{C}$ is a Herglotz function of order d if and only if it satisfies HF2, HF3 and the following two statements:

- (1) for all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto p(z, t) \in \mathbb{C}$ is measurable;
- (2) there exists $z_0 \in \mathbb{D}$ such that the function $[0, +\infty) \ni t \mapsto p(z_0, t) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$.

Proof. We have to prove that if p satisfies HF2, HF3 and (1) and (2), then it satisfies HF1. Let $z \in \mathbb{D}$. Fix $t \geq 0$. Since $\mathbb{D} \ni w \mapsto p(w, t) \in \mathbb{C}$ is holomorphic, by [20, pages 39-40],

$$|p(z, t)| \leq \frac{1 + |z|}{1 - |z|} |p(0, t)| \leq \frac{1 + |z|}{1 - |z|} \frac{1 + |z_0|}{1 - |z_0|} |p(z_0, t)|.$$

Now, since the function $[0, +\infty) \ni t \mapsto p(z_0, t)$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$ and $[0, +\infty) \ni t \mapsto p(z, t)$ is measurable, the above inequality implies that the function $[0, +\infty) \ni t \mapsto p(z, t)$ also belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$. \square

We are going to show that there is essentially a one-to-one correspondence between Herglotz vector fields and Berkson-Porta data. To this aim we need a lemma:

Lemma 4.7. *Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a function such that*

- (1) *For all $t \geq 0$ the map $\mathbb{D} \ni z \mapsto G(z, t)$ is holomorphic.*
- (2) *For all $z \in \mathbb{D}$ the map $[0, +\infty) \ni t \mapsto G(z, t)$ is measurable.*

Then the map $[0, +\infty) \ni t \mapsto G(\cdot, t) \in \text{Hol}(\mathbb{D}, \mathbb{C})$, from the set $[0, +\infty)$ endowed with the Lebesgue measure to the Fréchet space $\text{Hol}(\mathbb{D}, \mathbb{C})$, is measurable.

Proof. Since $\text{Hol}(\mathbb{D}, \mathbb{C})$ is a metrizable and separable topological space, it is enough to show that, given $f \in \text{Hol}(\mathbb{D}, \mathbb{C})$ and $\epsilon > 0$, the set $\{t \in [0, +\infty) : d_H(G(\cdot, t), f) < \epsilon\}$ is measurable; where here $d_H(\cdot, \cdot)$ denotes the Fréchet distance in $\text{Hol}(\mathbb{D}, \mathbb{C})$.

Fix $t \geq 0$. Since $G(\cdot, t)$ is holomorphic in \mathbb{D} there exists a sequence $\{g_n(t)\} \subset \mathbb{C}$ such that

$$G(z, t) = \sum_{n=0}^{\infty} g_n(t) z^n.$$

The functions $[0, +\infty) \ni t \mapsto g_n(t)$ are measurable. Indeed, $g_0(t) = G(0, t)$ is measurable by hypothesis (2). By induction, assume that $g_k(t)$ is measurable for $k = 0, \dots, n$. Then

$$\begin{aligned} g_{n+1}(t) &= \frac{G^{(n+1)}(0, t)}{(n+1)!} = \lim_{h \rightarrow 0} \frac{1}{h^{n+1}} [G(h, t) - \sum_{k=0}^n \frac{G^{(k)}(0, t)}{k!} h^k] \\ &= \lim_{m \rightarrow \infty} m^{n+1} [G(\frac{1}{m}, t) - \sum_{k=0}^n g_k(t) (\frac{1}{m})^k], \end{aligned}$$

which proves that $t \mapsto g_{n+1}(t)$ is measurable, concluding the induction.

Let $G_m(z, t) := \sum_{n=0}^m g_n(t) z^n$. Since the map $\mathbb{C}^m \ni (a_0, \dots, a_m) \mapsto \sum_{n=0}^m a_n z^n \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is continuous, and $t \mapsto g_n(t)$ is measurable for all $n \in \mathbb{N}$, then $[0, +\infty) \ni t \mapsto G_m(\cdot, t) \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is measurable for all $m \in \mathbb{N}$. Moreover, $\{G_m(\cdot, t)\}$ converges to $G(\cdot, t)$ in $\text{Hol}(\mathbb{D}, \mathbb{C})$. Therefore, $d_H(G(\cdot, t), f) = \lim_{m \rightarrow \infty} d_H(G_m(\cdot, t), f)$, and hence it follows easily that

$$\begin{aligned} &\{t \in [0, +\infty) : d_H(G(\cdot, t), f) < \epsilon\} \\ &= \bigcup_{p=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{t \in [0, +\infty) : d_H(G_m(\cdot, t), f) \leq \epsilon(1 - \frac{1}{p+1})\}. \end{aligned}$$

Since $\{G_m(\cdot, t)\}$ are measurable, $\{t \in [0, +\infty) : d_H(G_m(\cdot, t), f) \leq \epsilon(1 - \frac{1}{p+1})\}$ is measurable and this proves the result. \square

Theorem 4.8. *Let $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ be a measurable function and let $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz function of order $d \in [1, +\infty)$. Then the map $G_{\tau, p} : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ given by*

$$(4.7) \quad G_{\tau, p}(z, t) = (z - \tau(t))(\overline{\tau(t)}z - 1)p(z, t),$$

for all $z \in \mathbb{D}$ and for all $t \in [0, +\infty)$, is a Herglotz vector field of order d on the unit disc.

Conversely, if $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ is a Herglotz vector field of order $d \in [1, +\infty)$ on the unit disc, then there exist a measurable function $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ and a Herglotz function $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ of order d such that $G(z, t) = G_{\tau, p}(z, t)$ for almost every $t \in [0, +\infty)$ and all $z \in \mathbb{D}$ (here $G_{\tau, p}$ is given by (4.7)).

Moreover, if $\tilde{\tau} : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ is another measurable function and $\tilde{p} : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ is another Herglotz function of order d such that $G = G_{\tilde{\tau}, \tilde{p}}$ for almost every $t \in [0, +\infty)$ then $p(z, t) = \tilde{p}(z, t)$ for almost every $t \in [0, +\infty)$ and all $z \in \mathbb{D}$ and $\tau(t) = \tilde{\tau}(t)$ for almost all $t \in [0, +\infty)$ such that $G(\cdot, t) \not\equiv 0$.

Proof. Assume (τ, p) be given. By the Berkson-Porta representation formula, for each fixed $t \in [0, +\infty)$ the function $\mathbb{D} \ni z \mapsto G_{\tau, p}(z, t)$ is an infinitesimal generator. Thus we need to prove that $G_{\tau, p}$ is a weak holomorphic vector field of order d on \mathbb{D} .

On the one hand, it is clear that for all $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto G_{\tau, p}(z, t)$ is measurable and that for all $t \in [0, +\infty)$, the function $\mathbb{D} \ni z \mapsto G_{\tau, p}(z, t)$ is holomorphic. That is, $G_{\tau, p}$ satisfies WHVF1 and WHVF2. On the other hand, fix a compact set $K \subset \mathbb{D}$ and $T > 0$. Let $0 < r < 1$ be such that $K \subset \mathbb{D}(r) = \{\zeta \in \mathbb{D} : |\zeta| < r\}$. Fix $z \in K$ and $t \in [0, T]$. By [20, pages 39-40],

$$\begin{aligned} |G_{\tau, p}(z, t)| &= |(z - \tau(t))(\overline{\tau(t)}z - 1)||p(z, t)| \leq 4|p(z, t)| \\ &\leq 4\frac{1 + |z|}{1 - |z|}|p(0, t)| \leq 4\frac{1 + r}{1 - r}|p(0, t)|. \end{aligned}$$

Since the function $[0, +\infty) \ni t \mapsto p(0, t)$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$, writing $k_{K, T}(t) = 4\frac{1+r}{1-r}|p(0, t)|$ we conclude that $G_{\tau, p}$ satisfies WHVF3 and it is a weak holomorphic vector field of order d .

Conversely, let G be a Herglotz vector field. Hence $z \mapsto G(z, s)$ belongs to $\text{Gen}(\mathbb{D})$, for almost every $s \in [0, +\infty)$. Therefore, by the Berkson-Porta representation formula, we can find $\alpha_s \in \overline{\mathbb{D}}$ and $p_s \in \text{Hol}(\mathbb{D}, \mathbb{C})$ with $\text{Re } p_s \geq 0$ such that $G(z, s) = (z - \alpha_s)(\overline{\alpha_s}z - 1)p_s(z)$, for all $z \in \mathbb{D}$ and almost every $s \in [0, +\infty)$. By WHVF1 for each fixed $z \in \mathbb{D}$ the function $[0, +\infty) \ni t \mapsto G(z, t)$ is measurable. By Lemma 4.7, the map $\Psi : [0, +\infty) \ni t \mapsto G(\cdot, t) \in \text{Hol}(\mathbb{D}, \mathbb{C})$ from the set $[0, +\infty)$ endowed with the Lebesgue measure to the Fréchet space $\text{Hol}(\mathbb{D}, \mathbb{C})$, is measurable.

Note that if $G(\cdot, s) \equiv 0$ then necessarily $p_s(\cdot) \equiv 0$ and in such a case α_s can take any value. We set $\alpha_s = 0$ if $G(\cdot, s) \equiv 0$. Let $E := \{s \in [0, +\infty) : G(\cdot, s) \equiv 0\}$ that is, $s \in E$ if and only if $G(\cdot, s) \equiv 0$. Note that since $E = \Psi^{-1}(\{0\})$, the set E is a measurable subset of $[0, +\infty)$. Hence, $\alpha_s = 0$ for $s \in E$ and $\alpha_s = BP_\tau \circ \Psi(s)$ for $s \in [0, +\infty) \setminus E$. Since E is measurable, Ψ is measurable, BP_τ is continuous by Proposition 2.1 and $\text{Gen}(\mathbb{D})$ is a closed subset of $\text{Hol}(\mathbb{D}, \mathbb{C})$, it follows that α_s is a measurable mapping from $[0, +\infty)$ into $\overline{\mathbb{D}}$.

Similarly, being $p_s(z) \equiv 0$ for $s \in E$ and $p_s(z) = BP_p \circ \Psi(s)$ for $s \in [0, +\infty) \setminus E$ and being BP_p continuous by Proposition 2.1 we deduce that p_s is a measurable map from $[0, +\infty)$ into $\text{Hol}(\mathbb{D}, \mathbb{C})$.

We are left to check that p_s is a Herglotz function of order d . By Proposition 4.6 this is equivalent to show that there exists a point $z_0 \in \mathbb{D}$ such that the mapping $[0, +\infty) \ni s \mapsto p_s(0) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$.

Let $A := \{s \in [0, +\infty) : |\alpha_s| \geq \frac{1}{2}\}$. Since $A = \alpha_s^{-1}(\mathbb{D} \setminus \mathbb{D}(1/2))$, we see that A is a Lebesgue measurable subset of $[0, +\infty)$. Moreover, when $s \in A$ clearly $\alpha_s \neq 0$ and

$$|p_s(0)| = \frac{|G(0, s)|}{|\alpha_s|} \leq 2|G(0, s)|.$$

Hence $[0, +\infty) \ni s \mapsto \chi_A(s)p_s(0) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$, where $\chi_A(s) = 1$ for $s \in A$ and $\chi_A(s) = 0$ otherwise.

Moreover, by the very definition of A , when $s \in [0, +\infty) \setminus A$,

$$\begin{aligned} |G(3/4, s)| &= |3/4 - \alpha_s| |\overline{\alpha_s} 3/4 - 1| |p_s(3/4)| \\ &\geq \left(\frac{3}{4} - \frac{1}{2}\right) \left(1 - \frac{3}{4}\right) |p_s(3/4)| = \frac{1}{16} |p_s(3/4)|. \end{aligned}$$

Hence $[0, +\infty) \ni s \mapsto \chi_{[0, +\infty) \setminus A}(s)p_s(3/4) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$.

By the distortion theorem for Carathéodory functions [20, pages 39-40], for every $s \in [0, +\infty)$,

$$|p_s(0)| \leq \frac{1 + 3/4}{1 - 3/4} |p_s(3/4)| = 7|p_s(3/4)|.$$

Therefore, $[0, +\infty) \ni s \mapsto \chi_{[0, +\infty) \setminus A}(s)p_s(0) \in \mathbb{C}$ belongs to $L_{loc}^d([0, +\infty), \mathbb{C})$. Thus $p_s(0) \in L_{loc}^d([0, +\infty), \mathbb{C})$, and we are done.

The statement about uniqueness follows at once from the uniqueness of the Berkson-Porta representation formula. \square

The representation of Herglotz vector fields by means of Herglotz functions given by Theorem 4.8 will turn out to be a very powerful tool, because it allows to use distortion theorems for Carathéodory's function, a tool which is not available in higher dimensions (see [6]).

5. FROM HERGLOTZ VECTOR FIELDS TO EVOLUTION FAMILIES

For the sake of clearness, we begin by recalling the well-known Gronwall's Lemma as needed for our aims.

Lemma 5.1. *Let $\theta : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $k \in L^1([a, b], \mathbb{R})$ non-negative. If there exists $C \geq 0$ such that for all $t \in [a, b]$*

$$\theta(t) \leq C + \int_a^t \theta(\xi)k(\xi)d\xi \quad (\text{resp.}, \theta(t) \leq C + \int_t^b \theta(\xi)k(\xi)d\xi),$$

then

$$\theta(t) \leq C \exp \left(\int_a^t k(\xi) d\xi \right) \quad (\text{resp.}, \theta(t) \leq C \exp \left(\int_t^b k(\xi) d\xi \right)).$$

Theorem 5.2. *Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz vector field of order $d \in [1, +\infty)$. For all $s \geq 0$ and $z \in \mathbb{D}$, let $\phi_{s,z}$ be the solution of the problem*

$$\begin{cases} \dot{x}(t) = G(x(t), t) \text{ for a. e. } t \in [s, +\infty) \\ x(s) = z. \end{cases}$$

Let $\varphi_{s,t}(z) := \phi_{s,z}(t)$ for all $0 \leq s \leq t < +\infty$ and for all $z \in \mathbb{D}$. Then $(\varphi_{s,t})$ is an evolution family in the unit disc of order d .

Proof. By Theorem 4.4, the Herglotz vector field G is a semi-complete weak holomorphic vector field on the unit disc. Therefore, the value $\phi_{s,z}(t)$ is well-defined for all $0 \leq s \leq t < +\infty$ and for all $z \in \mathbb{D}$. Moreover, by uniqueness of solutions of ODE's, it follows that $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ for all $0 \leq s \leq u \leq t < +\infty$. Thus EF1, EF2 hold, and we are left to prove EF3 and the holomorphicity of $\varphi_{s,t}$.

We prove that $\varphi_{s,t} : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic for all $0 \leq s \leq u \leq t < +\infty$. First, we claim that for each $0 < T < +\infty$ and $0 < r < 1$, there exists $R = R(r, T) < 1$ such that

$$(5.1) \quad |\varphi_{s,t}(z)| \leq R$$

for all $0 \leq s \leq t \leq T$ and $|z| \leq r$.

Seeking for a contradiction, assume (5.1) is not true. Then there exist three sequences (z_n) , (s_n) , and (t_n) such that $|z_n| \leq r$, $z_n \rightarrow z_0$, $s_n, t_n \in [0, T]$, $s_n \leq t_n$, $s_n \rightarrow s_0$, $t_n \rightarrow t_0$, and $|\varphi_{s_n, t_n}(z_n)| \rightarrow 1$. Since the map φ_{s_n, t_n} is a contraction for the hyperbolic metric (see the proof of Theorem 4.8), we have that $\rho_{\mathbb{D}}(\varphi_{s_n, t_n}(z_n), \varphi_{s_n, t_n}(z_0)) \leq \rho_{\mathbb{D}}(z_n, z_0) \rightarrow 0$. Then $|\varphi_{s_n, t_n}(z_0)| \rightarrow 1$. The map $t \mapsto \varphi_{0,t}(z_0)$ is continuous (because ϕ_{0,z_0} is a positive trajectory of the semi-complete vector field G). Moreover,

$$\begin{aligned} \rho_{\mathbb{D}}(\varphi_{0, t_n}(z_0), \varphi_{s_n, t_n}(z_0)) &= \rho_{\mathbb{D}}(\varphi_{s_n, t_n}(\varphi_{0, s_n}(z_0)), \varphi_{s_n, t_n}(z_0)) \\ &\leq \rho_{\mathbb{D}}(\varphi_{0, s_n}(z_0), z_0) \rightarrow \rho_{\mathbb{D}}(\varphi_{0, s_0}(z_0), z_0) < +\infty. \end{aligned}$$

Again this implies that $|\varphi_{0, t_n}(z_0)| \rightarrow 1$. But $\varphi_{0, t_n}(z_0) \rightarrow \varphi_{0, t_0}(z_0) \in \mathbb{D}$. A contradiction. Hence (5.1) holds.

Fix $s < t$ and $z \in \mathbb{D}$. Let $|z| < r < 1$, $T > t$ and let $R = R(r, T)$ be given in (5.1). Write $\widehat{R} = (R + 1)/2$. Let $k_{R,T}, \widehat{k}_{R,T} \in L^d([0, T], \mathbb{R}^+)$ be the functions given by WHVF3 and by Lemma 4.2.

The map $u \mapsto G'(z, u)$ is clearly measurable. Thus, the function $t \mapsto G'(\varphi_{s,t}(z), t)$ is also measurable. Therefore,

$$|G'(\varphi_{s,u}(z), u)| = \frac{1}{2\pi} \left| \int_{C(0, \widehat{R})^+} \frac{G(\xi, u)}{\xi - \varphi_{s,u}(z)} d\xi \right| \leq k_{R,T}(u) \frac{2R}{1-R}.$$

Hence, the map $u \mapsto G'(\varphi_{s,u}(z), u)$ belongs to $L^d([s, T], \mathbb{R})$. Once we know that this function is integrable, we claim that

$$\lim_{h \rightarrow 0} \frac{\varphi_{s,u}(z+h) - \varphi_{s,u}(z)}{h} = \exp \left(\int_s^t G'(\varphi_{s,u}(z), u) du \right).$$

To simplify the notation we write $H(u) := \exp \left(- \int_s^u G'_{\tau,p}(\varphi_{s,\xi}(z), \xi) d\xi \right)$. Moreover, for $|h| < R - |z|$ we define $\theta(u) := |\varphi_{s,u}(z+h) - \varphi_{s,u}(z)|$ and $f_h(u) := \frac{\varphi_{s,u}(z+h) - \varphi_{s,u}(z)}{h}$. We have

$$\begin{aligned} \theta(u) &= \left| h + \int_s^u G(\varphi_{s,\xi}(z+h), \xi) d\xi - \int_s^u G(\varphi_{s,\xi}(z), \xi) d\xi \right| \\ &\leq |h| + \int_s^u \theta(\xi) \widehat{k}_{R,T}(\xi) d\xi = \theta(s) + \int_s^u \theta(\xi) \widehat{k}_{R,T}(\xi) d\xi. \end{aligned}$$

Lemma 5.1 implies that

$$\theta(u) \leq \theta(s) \exp \left(\int_s^u \widehat{k}_{R,T}(\xi) d\xi \right) = |h| \exp \left(\int_s^u \widehat{k}_{R,T}(\xi) d\xi \right).$$

That is, $|f_h(u)| \leq \exp \left(\int_s^u \widehat{k}_{R,T}(\xi) d\xi \right)$. In a similar way, for all $0 \leq s \leq v \leq u \leq t$, we have that

$$\theta(v) \leq \theta(u) + \left| \int_v^u G(\varphi_{s,\xi}(z+h), \xi) d\xi - \int_v^u G(\varphi_{s,\xi}(z), \xi) d\xi \right| \leq \theta(u) + \int_v^u \theta(\xi) \widehat{k}_{R,T}(\xi) d\xi.$$

Using again Lemma 5.1, we have that

$$|\varphi_{s,v}(z+h) - \varphi_{s,v}(z)| \leq |\varphi_{s,u}(z+h) - \varphi_{s,u}(z)| \exp \left(\int_v^u \widehat{k}_{R,T}(\xi) d\xi \right) \text{ for all } s \leq v \leq u \leq t.$$

In particular,

$$(5.2) \quad |h| \leq |\varphi_{s,u}(z+h) - \varphi_{s,u}(z)| \exp \left(\int_s^t \widehat{k}_{R,T}(\xi) d\xi \right).$$

This means that if $h \neq 0$, then $\varphi_{s,u}(z+h) \neq \varphi_{s,u}(z)$ for all $u \in [s, t]$. Fix $h > 0$. Then there is a set $A = A(h)$ of zero measure such that for all $u \in [s, T] \setminus A(h)$, we have

$$\begin{aligned} f'_h(u) &= \frac{\dot{\varphi}_{s,u}(z+h) - \dot{\varphi}_{s,u}(z)}{h} \\ &= \frac{G(\varphi_{s,u}(z+h), u) - G(\varphi_{s,u}(z), u)}{h} = G'(\varphi_{s,u}(z), u) f_h(u) + L_h(u) \end{aligned}$$

where

$$L_h(u) = f_h(u) \left[\frac{G(\varphi_{s,u}(z+h), u) - G_{\tau,p}(\varphi_{s,u}(z), u)}{\varphi_{s,u}(z+h) - \varphi_{s,u}(z)} - G'_{\tau,p}(\varphi_{s,u}(z), u) \right].$$

Note $L_h(u)$ is well-defined because $\varphi_{s,u}(z+h) \neq \varphi_{s,u}(z)$. Then, by the very definition of H , it holds

$$\frac{d(H(u)f_h(u))}{du} = H(u)f'_h(u) - H(u)G'(\varphi_{s,u}(z), u)f_h(u) = H(u)L_h(u) \quad \text{a.e. on } u \in [s, t].$$

Integrating on $u \in [s, t]$, we obtain $H(t)f_h(t) - 1 = \int_s^t H(u)L_h(u)du$. Moreover

$$|H(u)L_h(u)| \leq \left| \exp\left(-\int_s^u G'(\varphi_{s,\xi}(z), \xi)d\xi\right) \right| \exp\left(\int_s^u \widehat{k}_{R,T}(\xi)d\xi\right) \left[\widehat{k}_{R,T}(u) + k_{R,T}(u) \frac{2R}{r-1} \right].$$

Since this bound does not depend on h and

$$\lim_{h \rightarrow 0} \frac{G(\varphi_{s,u}(z+h), u) - G(\varphi_{s,u}(z), u)}{\varphi_{s,u}(z+h) - \varphi_{s,u}(z)} = G'(\varphi_{s,u}(z), u),$$

by the Lebesgue dominated convergence theorem, we have $\lim_{h \rightarrow 0} \int_s^t H(u)L_h(u)du = 0$. Therefore,

$$\lim_{h \rightarrow 0} \frac{\varphi_{s,u}(z+h) - \varphi_{s,u}(z)}{h} = \frac{1}{H(t)},$$

proving that $\varphi_{s,u}(z)$ is holomorphic for all $0 \leq s \leq t < +\infty$.

To end up the proof we need to check property EF3. Let $0 \leq s \leq u \leq t \leq T$, $z \in \mathbb{D}$ and let $R = R(T, |z|)$ be the number given by (5.1). Then

$$|\varphi_{s,u}(z) - \varphi_{s,v}(z)| = \left| \int_u^v \dot{\varphi}_{s,\xi}(z)d\xi \right| = \left| \int_u^v G(\varphi_{s,\xi}(z), \xi)d\xi \right| \leq \int_u^v k_{R,T}(\xi)d\xi.$$

Note that this also implies that if G is of order d for some $d \in [1, +\infty]$, then $(\varphi_{s,t})$ is also of order d . \square

6. FROM EVOLUTION FAMILIES TO HERGLOTZ VECTOR FIELDS

In this section we prove the converse of Theorem 5.2. Part of the proof relies on the following result on measurable selections:

Theorem 6.1. [7, Theorem III.30, page 80] *Let (Ω, Σ, μ) be a positive σ -finite complete measure space, $[X, d]$ a separable and complete metric space and Γ a multifunction from Ω to the subsets of X . Assume that:*

(i) *For every $\omega \in \Omega$, $\Gamma(\omega)$ is a closed non-empty subset of X .*

(ii) *For every $x \in X$ and every $r > 0$, $\{\omega \in \Omega : \Gamma(\omega) \cap B(x, r) \neq \emptyset\} \in \Sigma$. (As usual, $B(x, r)$ denotes the open unit ball in X with center x and radius r).*

Then Γ admits a measurable selector $\sigma : \Omega \rightarrow X$; namely, for every $\omega \in \Omega$, we have $\sigma(\omega) \in \Gamma(\omega)$ and the inverse image by σ of any borelian in X belongs to Σ .

Now we are going to prove the main result of this section:

Theorem 6.2. *Let $(\varphi_{s,t})$ be an evolution family of order d in the unit disc. Then there exists a Herglotz vector field G which has positive trajectories $(\varphi_{s,t})$; namely, for any $(z, s) \in \mathbb{D} \times [0, +\infty)$, the positive trajectory of the vector field G with initial data (z, s) is exactly $[s, +\infty) \ni t \rightarrow \varphi_{s,t}(z)$.*

Proof. The proof of this theorem is rather long and has three main parts which will be exposed separately. In short: (a) construction of a candidate function $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ verifying that $G(\cdot, s) \in \text{Gen}(\mathbb{D})$, for all $s \geq 0$; (b) checking that G is a weak holomorphic vector field; (c) verification of the assertion of the theorem.

Part (a): We are going to apply Theorem 6.1 to a suitably chosen $\Gamma : [0, +\infty) \rightarrow 2^{\text{Hol}(\mathbb{D}, \mathbb{C})}$, where the set $[0, +\infty)$ is endowed with the Lebesgue measure and $\text{Hol}(\mathbb{D}, \mathbb{C})$ has its natural structure of Fréchet space.

Fix $z \in \mathbb{D}$ and $T > 0$. Let $k := k_{z,T} \in L^d([0, T+1], \mathbb{R})$ be the non-negative function given by EF3. We extend k to all of \mathbb{R} by setting zero outside the interval $[0, T+1]$. Then for $0 \leq s \leq T$ and every $n \in \mathbb{N}$

$$n \left| \varphi\left(z, s, s + \frac{1}{n}\right) - \varphi(z, s, s) \right| \leq n \int_s^{s+1/n} k(\xi) d\xi \leq \text{Max}_k(s),$$

where

$$\text{Max}_k(s) := \sup \left\{ \frac{1}{|I|} \int_I k(\xi) d\xi : I \text{ is a closed interval of the real line and } s \in I \right\}$$

is the so-called maximal function associated with k . Since $k \in L^1(\mathbb{R}, \mathbb{R})$, by Hardy-Littlewood maximal theorem there exists a subset $N(T, z) \subset [0, +\infty)$ of zero measure such that $\text{Max}_k(s) < +\infty$ for every $s \in [0, T] \setminus N(T, z)$. Let $N(z) := \cup_{m \in \mathbb{N}} N(m, z)$. Then for all $s \in [0, +\infty) \setminus N(z)$

$$(6.1) \quad \sup_n \left| n\left(\varphi\left(z, s, s + \frac{1}{n}\right) - z\right) \right| < +\infty.$$

Let $M := N(0) \cup N(1/2)$. Clearly, M is a subset of $[0, +\infty)$ of zero measure. We let

$$\Gamma : [0, +\infty) \rightarrow 2^{\text{Hol}(\mathbb{D}, \mathbb{C})}, \quad s \mapsto \Gamma(s) = \begin{cases} \text{ac}(g_{n,s}) & s \notin M, \\ \{\text{id}\} & s \in M, \end{cases}$$

where $g_{n,s} := n(\varphi_{s,s+1/n} - \text{id}) \in \text{Hol}(\mathbb{D}, \mathbb{C})$ and $\text{ac}(g_{n,s})$ denotes the accumulation points of the sequence $\{g_{n,s}\}_n$ in the metric space $\text{Hol}(\mathbb{D}, \mathbb{C})$. The multifunction Γ is well-defined and, since $\text{Hol}(\mathbb{D}, \mathbb{C})$ is a metric space, $\Gamma(s)$ is a closed subset of $\text{Hol}(\mathbb{D}, \mathbb{C})$ for every $s \geq 0$.

Next step is to prove that $\Gamma(s)$ is non-empty for all $s \geq 0$. This is true by definition if $s \in M$. Thus, fix $s \in [0, +\infty) \setminus M$. As recalled in Section 2, $\varphi_{s,s+1/n} - \text{id}$ belongs to $\text{Gen}(\mathbb{D})$ for all $n \in \mathbb{N}$. Moreover, $\text{Gen}(\mathbb{D})$ is a real cone in $\text{Hol}(\mathbb{D}, \mathbb{C})$, thus $\{g_{n,s}\}$ is a sequence in $\text{Gen}(\mathbb{D})$. By the definition of M , $\max\{\sup_n |g_{n,s}(0)|, \sup_n |g_{n,s}(1/2)|\} < +\infty$. Hence, we can apply Lemma 2.2 and conclude that the sequence $\{g_{n,s}\}$ has accumulation points in $\text{Hol}(\mathbb{D}, \mathbb{C})$, so that $\Gamma(s)$ is not empty. Thus Γ satisfies hypothesis (i) of Theorem 6.1.

In order to check condition (ii) in Theorem 6.1 for Γ , we fix $f \in \text{Hol}(\mathbb{D}, \mathbb{C})$ and $r > 0$. Since M has zero measure, we have only to prove that

$$A_{f,r} = \{s \in [0, +\infty) \setminus M : \exists g \in \text{ac}(g_{n,s}) \text{ with } d_H(f, g) < r\}$$

is Lebesgue measurable, where d_H is the canonical Fréchet distance defining the topology of $\text{Hol}(\mathbb{D}, \mathbb{C})$. Bearing in mind Lemma 2.2 and the argument above, we see that

$$A_{f,r} := \bigcup_{l=2}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{ s \in [0, +\infty) \setminus M : d_H(f, g_{k,s}) < r \left(1 - \frac{1}{l}\right) \right\}.$$

Hence, it is enough to prove that, for every $k \in \mathbb{N}$, every $s \geq 0$ and every $r^* > 0$, the subset $B_{k,f,r^*} := \{s \in [0, +\infty) \setminus M : d_H(f, g_{k,s}) < r^*\}$ is Lebesgue measurable. Since the functions $[0, +\infty) \ni s \mapsto \varphi_{s,s+1/k} \in \text{Hol}(\mathbb{D}, \mathbb{C})$ are continuous (see Proposition 3.5), then $[0, +\infty) \ni s \mapsto P_k(s) := g_{k,s} \in \text{Hol}(\mathbb{D}, \mathbb{C})$ is also continuous for every $k \in \mathbb{N}$. Therefore, the inverse image by P_k of $B(f, r^*)$ (the open ball in $\text{Hol}(\mathbb{D}, \mathbb{C})$ with center f and radius r^*) is an open subset of $[0, +\infty)$. Since $B_{k,f,r^*} = P_k^{-1}(B(f, r^*)) \setminus M$, the set B_{k,f,r^*} is Lebesgue measurable.

Therefore, the multifunction Γ satisfies the hypotheses of Theorem 6.1. Thus there exists a measurable selector $\sigma : [0, +\infty) \rightarrow \text{Hol}(\mathbb{D}, \mathbb{C})$ for Γ . We define $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ by $G(z, s) := \sigma[s](z)$, for $z \in \mathbb{D}$ and $s \geq 0$. Hence, for every $s \in [0, +\infty) \setminus M$, there exists a strictly increasing sequence $\{n_k(s)\}$ of natural numbers such that, for all $z \in \mathbb{D}$,

$$(6.2) \quad G(z, s) := \lim_{k \rightarrow \infty} n_k(s)(\varphi(z, s, s + 1/n_k(s)) - z)$$

and the convergence is uniform on compacta of \mathbb{D} . In particular, because $\text{Gen}(\mathbb{D})$ is a closed subset of $\text{Hol}(\mathbb{D}, \mathbb{C})$ (see [1, Consequence of Theorem 1.4.14] or [26, p.76]), we see that $z \mapsto G(z, s)$ belongs to $\text{Gen}(\mathbb{D})$, for every $s \in [0, +\infty) \setminus M$. Moreover, since $z \mapsto G(z, s) = z$, for every $s \in M$, we deduce that $z \mapsto G(z, s)$ belongs to $\text{Gen}(\mathbb{D})$, for every $s \in [0, +\infty)$.

Part (b): According to Definition 4.1, we have to check WHVF1, WHVF2 and WHVF3. Fixing $z \in \mathbb{D}$, we see that by definition, $[0, +\infty) \ni s \mapsto G(z, s) \in \mathbb{C}$ is the composition of the measurable selector σ and the continuous functional of $\text{Hol}(\mathbb{D}, \mathbb{C})$ given by evaluation at z . Thus, WHVF1 holds. Also, WHVF2 holds trivially.

To prove property WHVF3, we argue as follows. Fix $z \in \mathbb{D}$ and $T > 0$. By EF3, there exists $k_z \in L^d([0, T+1], \mathbb{R})$ non-negative such that

$$\left| \varphi(z, t, t + \frac{1}{n}) - \varphi(z, t, t) \right| \leq \int_t^{t+1/n} k_z(\xi) d\xi,$$

for $0 \leq t \leq T$ and every $n \in \mathbb{N}$. The map $s \mapsto \int_s^{s+1/n} k_z(\xi) d\xi$ is differentiable with derivative $k(s)$ in $[0, T]$ outside a set $N_0(z, T)$ of zero measure. Let $N(z, T) := M \cup$

$N_0(z, T)$. Then for every $s \in [0, T] \setminus N(z, T)$,

$$|n_k(s)(\varphi(z, s, s + 1/n_k(s)) - z)| \leq n_k(s) \int_s^{s+1/n_k(s)} k_z(\xi) d\xi.$$

Taking limit in k , by (6.2), we conclude that

$$(6.3) \quad |G(z, s)| \leq k_z(s)$$

for almost every $s \in [0, T]$.

Now fix $r \in (0, 1)$ and $T > 0$. By Part (a), we know that $z \mapsto G(z, s)$ belongs to $\text{Gen}(\mathbb{D})$, for every $s \in [0, +\infty)$. By [26, Section 3.5], there exist $a_s \in \mathbb{C}$ and $q_s \in \text{Hol}(\mathbb{D}, \mathbb{C})$ with $\text{Re } q_s \geq 0$ and $G(z, s) = a_s - \overline{a_s}z^2 - zq_s(z)$, for $z \in \mathbb{D}$, $s \geq 0$. Since $G(0, s) = a_s$, equation (6.3) provides a function $k_0 \in L^d([0, T], \mathbb{R})$ such that $|a_s - \overline{a_s}z^2| \leq 2k_0(t)$, for $s \in [0, T]$ and $|z| \leq r$. Again by (6.3), we can find another function $k_{1/2} \in L^d([0, T], \mathbb{R})$ such that $|G(1/2, s)| \leq k_{1/2}(s)$, for $s \in [0, T]$. Therefore, for $s \in [0, T]$, we have

$$|q_s(1/2)| \leq 2|a_s - \frac{1}{4}\overline{a_s}| + 2|G(1/2, s)| \leq 3k_0(s) + 2k_{1/2}(s).$$

Since $s \mapsto G(z, s)$ is measurable for all fixed $z \in \mathbb{D}$, it follows that both maps $s \mapsto a_s$ and $s \mapsto q_s(1/2)$ belong to $L^d([0, T], \mathbb{C})$. Now, the distortion theorem for Carathéodory functions [20, pages 39-40] shows that, when $|z| \leq r$ and $s \in [0, T]$

$$\begin{aligned} |G(z, s)| &\leq 2k_0(s) + |q_s(z)| \leq 2k_0(s) + \frac{1 + |z|}{1 - |z|} |q_s(0)| \\ &\leq 2k_0(s) + \frac{1 + |z|}{1 - |z|} \frac{1 + 1/2}{1 - 1/2} |q_s(1/2)| \leq 2k_0(s) + \frac{1 + r}{1 - r} (9k_0(s) + 6k_{1/2}(s)), \end{aligned}$$

showing WHVF3.

Part (c): We have to prove that, given $(z, s) \in \mathbb{D} \times [0, +\infty)$, the positive trajectory of the weak holomorphic vector field G with initial data (z, s) is $[s, +\infty) \ni t \rightarrow \varphi(z, s, t)$. Recall that by Proposition 3.7, this function is absolutely continuous in $[s, +\infty)$ and $\varphi(z, s, s) = z$. Thus we have only to show that for almost every $t \in (s, +\infty)$

$$\frac{\partial \varphi}{\partial t}(z, s, t) := \lim_{h \rightarrow 0} \frac{\varphi(z, s, t + h) - \varphi(z, s, t)}{h} = G(\varphi(z, s, t), t).$$

Let us fix $z \in \mathbb{D}$ and $s \geq 0$. Let $N_1(z, s) \subset [s, +\infty)$ be a set of zero measure such that $[s, +\infty) \ni t \rightarrow \varphi(z, s, t)$ is differentiable for every $t \in (s, +\infty) \setminus N_1(z, s)$. Let M be the set of zero measure defined in Part (a). Then, for every $t \in (s, +\infty) \setminus (N_1(z, s) \cup M)$,

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(z, s, t) &= \lim_k \frac{\varphi(z, s, t + 1/n_k(t)) - \varphi(z, s, t)}{1/n_k(t)} \\ &= \lim_k n_k(t) (\varphi(\varphi(z, s, t), t, t + 1/n_k(t)) - \varphi(z, s, t)) \\ &= G(\varphi(z, s, t), t), \end{aligned}$$

and we are done. \square

As a consequence of the previous results we have the following interesting fact¹:

Corollary 6.3. *Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ in the unit disc. Then, every $\varphi_{s,t}$ is univalent.*

Proof. By Theorem 6.2 the elements of the evolution family $(\varphi_{s,t})$ are trajectories of a weak holomorphic vector fields. By inequality (5.2) they are univalent in the unit disc. \square

Theorem 6.4. *Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ in the unit disc.*

- (1) *For every $s \geq 0$, there exists a set $M(s) \subset [s, +\infty)$ (not depending on z) of zero measure such that, for every $t \in (s, +\infty) \setminus M(s)$, the function*

$$\mathbb{D} \ni z \mapsto \frac{\partial \varphi}{\partial t}(z, s, t) = \lim_{h \rightarrow 0} \frac{\varphi_{s,t+h}(z) - \varphi_{s,t}(z)}{h} \in \mathbb{C}$$

is a well-defined holomorphic function on \mathbb{D} and the limit is uniform on compacta.

- (2) *Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz vector field whose positive trajectories are $(\varphi_{s,t})$. Fix $s \geq 0$. Then there exists a set $M(s) \subset [s, +\infty)$ (not depending on z) of zero measure such that, for every $t \in (s, +\infty) \setminus M(s)$ and every $z \in \mathbb{D}$*

$$(6.4) \quad \frac{\partial \varphi}{\partial t}(z, s, t) = G(\varphi_{s,t}(z), t).$$

Proof. (1) Fix $s \geq 0$. By Proposition 3.7 the map $[s, +\infty) \ni t \mapsto \varphi(z, s, t) \in \mathbb{C}$ is absolutely continuous in $[s, +\infty)$ for all fixed $z \in \mathbb{D}$. Thus there exists a set of zero measure $N(z, s) \subset [s, +\infty)$ such that, for every $t \in (s, +\infty) \setminus N(z, s)$ the following limit exists

$$D_{s,t}(z) = \frac{\partial \varphi}{\partial t}(z, s, t) = \lim_{h \rightarrow 0} \frac{\varphi(z, s, t+h) - \varphi(z, s, t)}{h}.$$

Now, define $N(s) := \bigcup_{n=1}^{\infty} N(\frac{1}{n+1}, s)$. The set $N(s)$ has zero measure and it is independent of z . We are going to show that $\lim_{h \rightarrow 0} (\varphi(z, s, t+h) - \varphi(z, s, t))/h$ exists for all $t \in (s, +\infty) \setminus N(s)$ uniformly on compacta of \mathbb{D} .

First of all we show that for every $t \in (s, +\infty) \setminus N(s)$, the family $\mathcal{F}_{s,t} := \{\frac{1}{h}(\varphi_{s,t+h} - \varphi_{s,t}) : 0 < |h| < t - s\}$ is relatively compact in $\text{Hol}(\mathbb{D}, \mathbb{C})$. To this aim there are two cases to be considered, namely, $0 < h < t - s$ and $s - t < h < 0$. The proofs being similar, we only consider the first case. Since $h > 0$ and by EF2, $\mathcal{F}_{s,t} = \{f_h \circ \varphi_{s,t} : 0 < h < t - s\}$, where $f_h := \frac{1}{h}(\varphi_{t,t+h} - \text{id}) \in \text{Hol}(\mathbb{D}, \mathbb{C})$. Since $\varphi_{s,t}$ is holomorphic in \mathbb{D} , and by Montel's theorem, we only need to check that $\mathcal{F}_{s,t}^* := \{f_h : 0 < h < t - s\}$ is a bounded subset of $\text{Hol}(\mathbb{D}, \mathbb{C})$. Assume this is not the case. Then there exists a sequence $\{f_n\}$ (with $f_n := f_{h_n}$) in $\mathcal{F}_{s,t}^*$ and $r \in (0, 1)$ such that

$$(6.5) \quad \lim_{n \rightarrow \infty} \max \{|f_n(z)| : |z| \leq r\} = +\infty.$$

¹With some work one can even prove this result directly, without using Theorem 6.2.

Since the sequence $\{\varphi_{t,t+h_n} - \text{id}\}_n$ belongs to $H^\infty(\mathbb{D})$, we may assume that $\lim_n h_n = 0$. Moreover, letting $z_1 := \varphi(1/2, s, t)$ and $z_2 := \varphi(1/3, s, t)$, then $z_1 \neq z_2$ because $\varphi_{s,t}$ is univalent (see Corollary 6.3). Hence, since $h_n > 0$ and $t \notin N(1/2, s)$

$$\begin{aligned} D_{s,t}(1/2) &= \lim_n \frac{\varphi(1/2, s, t + h_n) - \varphi(1/2, s, t)}{h_n} \\ &= \lim_n \frac{\varphi(\varphi(1/2, s, t), t, t + h_n) - \varphi(1/2, s, t)}{h_n} \\ &= \lim_n \frac{\varphi(z_1, t, t + h_n) - z_1}{h_n} = \lim_n f_n(z_1). \end{aligned}$$

Similarly, one can check the existence of $\lim_n f_n(z_2)$. Now, note that $\mathcal{F}_{s,t}^* \subset \text{Gen}(\mathbb{D})$ since $h > 0$. Therefore, we can apply Lemma 2.2 to the sequence $\{f_n\}$ and the two points z_0, z_1 , contradicting (6.5).

Thus the family $\mathcal{F}_{s,t}$ is relatively compact in $\text{Hol}(\mathbb{D}, \mathbb{C})$. Let ψ, ϕ be two of its limits. By the definition of $N(s)$, $D_{s,t}(\frac{1}{m+1}) = \psi(\frac{1}{m+1}) = \phi(\frac{1}{m+1})$, for every $m \in \mathbb{N}$. But $\{\frac{1}{m+1}\}$ is a sequence accumulating at 0, hence by the identity principle $\psi = \phi$. This shows that $\lim_{h \rightarrow 0} \frac{\varphi(z, s, t+h) - \varphi(z, s, t)}{h}$ exists, for all $t \in (s, +\infty) \setminus N(s)$ uniformly on compacta of \mathbb{D} , ending the proof of (1).

(2) Fix $s \geq 0$. By part (1), there exists a set $N_0(s) \subset [s, +\infty)$ of zero measure (not depending on z) such that, for every $t \in (s, +\infty) \setminus N_0(s)$, the function $\mathbb{D} \ni z \mapsto \frac{\partial \varphi}{\partial t}(z, s, t)$ is a well defined holomorphic function on \mathbb{D} . Let $\{z_n\}$ be any sequence converging to 0. Then for all $n \in \mathbb{N}$ there exists a set of zero measure $N(z_n, s)$ such that

$$(6.6) \quad \frac{\partial \varphi}{\partial t}(z_n, s, t) = G(\varphi(z_n, s, t), t)$$

for all $t \in [s, +\infty) \setminus N(z_n, s)$. Let $N(s) = N_0(s) \cup \bigcup_n N(z_n, s)$. Then $N(s)$ has measure zero and for all $t \in [s, +\infty) \setminus N(s)$ equation (6.6) holds. By the identity principle for holomorphic maps, the two holomorphic functions $\frac{\partial \varphi}{\partial t}(\cdot, s, t)$ and $G(\varphi(\cdot, s, t), t)$ are then equal on \mathbb{D} , proving (2). \square

Corollary 6.5. *If G, \tilde{G} are Herglotz vector fields with the same positive trajectories then $G(z, t) = \tilde{G}(z, t)$ for almost every $t \in [0, +\infty)$ and all $z \in \mathbb{D}$.*

Proof. By Theorem 5.2 the positive trajectories of G and \tilde{G} are evolution families of the unit disc. In particular they are univalent by Corollary 6.3. The claim follows then from (6.4). \square

The next result studies the dependence of evolution families with respect to the “ s variable”.

Theorem 6.6. *Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ in the unit disc.*

- (1) For every $t > 0$, there exists a set $N(t) \subset [0, t]$ (not depending on z) of zero measure such that, for every $s \in (0, t) \setminus N(t)$, the function

$$\mathbb{D} \ni z \mapsto \frac{\partial \varphi}{\partial s}(z, s, t) := \lim_{h \rightarrow 0} \frac{\varphi_{s+h, t}(z) - \varphi_{s, t}(z)}{h} \in \mathbb{C}$$

is a well-defined holomorphic function on \mathbb{D} and the limit is uniform on compacta.

- (2) Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz vector field whose positive trajectories are $(\varphi_{s, t})$. Fix $t > 0$. Then, there exists a set $N(t) \subset [0, t]$ (not depending on z) of zero measure such that, for every $s \in (0, t) \setminus N(t)$ and every $z \in \mathbb{D}$

$$(6.7) \quad \frac{\partial \varphi}{\partial s}(z, s, t) = -G(z, s)\varphi'_{s, t}(z).$$

Proof. (1) Fix $t > 0$. By Proposition 3.7 the map $[0, t] \ni s \mapsto \varphi(z, s, t) \in \mathbb{C}$ is absolutely continuous in $[0, t]$, for all fixed $z \in \mathbb{D}$. Thus there exists a set of zero measure $N_1(z, t) \subset [0, t]$ such that, for every $s \in (0, t) \setminus N_1(z, t)$, the following limit exists

$$D_{s, t}(z) = \frac{\partial \varphi}{\partial s}(z, s, t) = \lim_{h \rightarrow 0} \frac{\varphi(z, s+h, t) - \varphi(z, s, t)}{h}.$$

Moreover, let $\{r_n\} \subset (0, 1)$ be a sequence converging to 1. For $R > 0$, let $\mathbb{D}(R) = \{\zeta \in \mathbb{C} : |\zeta| < R\}$. By Lemma 3.6, for all $n \in \mathbb{N}$ there exists $R_n := R(r_n, t) \in (0, 1)$ such that $A_n := \{\varphi(z, u, v) : |z| \leq r_n, 0 \leq u \leq v \leq t+1\} \subset \mathbb{D}(R_n)$.

Let $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ be a Herglotz vector field whose positive trajectories are $(\varphi_{s, t})$ (see Theorem 6.2). Let $k_n := k_{R_n, t} \in L^d([0, t+1], \mathbb{R})$ be the non negative function given by property WHVF3 in Definition 4.1. There exists a set $N_2(n, t) \subset [0, t]$ of zero measure such that $k_n(s) = \lim_{h \rightarrow 0} \frac{1}{h} \int_s^{s+h} k_n(\eta) d\eta$ for every $s \in (0, t) \setminus N_2(n, t)$. Let us define

$$N(t) := \left(\bigcup_{n=1}^{\infty} N_1\left(\frac{1}{n+1}, t\right) \right) \cup \left(\bigcup_{n=1}^{\infty} N_2(n, t) \right).$$

Obviously, $N(t)$ is a subset of $[0, t]$ of zero measure, independent of z . We prove that $\lim_{h \rightarrow 0} \frac{\varphi(z, s+h, t) - \varphi(z, s, t)}{h}$ exists uniformly on compacta of \mathbb{D} for all $s \in (0, t) \setminus N(t)$.

First of all we show that for every $s \in (0, t) \setminus N(t)$ the family $\mathcal{F}_{s, t} := \{F_h := \frac{1}{h}(\varphi_{s+h, t} - \varphi_{s, t}) : 0 < h < t-s \text{ or } -s < h < 0\}$ is relatively compact in $\text{Hol}(\mathbb{D}, \mathbb{C})$. To this aim there are two cases to be considered, namely $0 < h < t-s$ and $-s < h < 0$. We only consider the first case, the proof of the second being similar.

Fix $r \in (0, 1)$. Let $n \in \mathbb{N}$ be such that $r_n > r$, and let $\rho_n \in (0, 1)$ be such that $\rho_n > R_n$. Set $z_h := \varphi(z, s, s+h)$. Then, for every $|z| \leq r$, the point $z_h \in A_n$ and

$$\begin{aligned} |F_h(z)| &= \left| \frac{1}{h} (\varphi(z, s+h, t) - \varphi(\varphi(z, s, s+h), s+h, t)) \right| \\ &= \frac{1}{h} \left| \frac{1}{2\pi i} \int_{C^+(0, \rho_n)} \varphi(\xi, s+h, t) \left(\frac{1}{\xi-z} - \frac{1}{\xi-z_h} \right) d\xi \right| \leq \frac{1}{h} \rho_n \frac{|z-z_h|}{(\rho_n-r_n)(\rho_n-R_n)}. \end{aligned}$$

Setting $C := C(r, t) = \frac{\rho_n}{(\rho_n - r_n)(\rho_n - R_n)} > 0$ and recalling the definition of A_n , we have that there exists $\tilde{C} > 0$ such that

$$\begin{aligned} |F_h(z)| &\leq C \frac{1}{h} |\varphi(z, s, s+h) - z| = C \frac{1}{h} \left| \int_s^{s+h} G(\varphi(z, s, \xi), \xi) d\xi \right| \\ &\leq C \frac{1}{h} \int_s^{s+h} k_n(\xi) d\xi \leq \tilde{C} < +\infty, \end{aligned}$$

where the last inequality follows from $s \notin N_2(n, t)$. Hence, $\sup\{|F_h(z)| : |z| \leq r, 0 < h < t - s\} < +\infty$ as wanted.

Now, arguing as in the last part of the proof of part (1) of Theorem 6.4 we can see that $\lim_{h \rightarrow 0} (\varphi(z, s+h, t) - \varphi(z, s, t))/h$ exists for all $s \in (0, t) \setminus N(t)$ uniformly on compacta of \mathbb{D} , concluding the proof of (1).

(2) Fix $t > 0$. Let $N_1 \subset [0, +\infty)$ be the set of zero measure given by Theorem 6.4.(1) such that $\frac{\partial \varphi}{\partial t}(z, 0, u) = G(\varphi(z, 0, u), u)$ for all $u \in (0, +\infty) \setminus N_1$ and for all $z \in \mathbb{D}$. Let $N_2 := N_2(t) \subset [0, t]$ be the set of zero measure prescribed by part (1) of this theorem.

Let $N := N_1 \cup N_2$. Differentiating with respect to u the identity $\varphi(z, 0, t) = \varphi(\varphi(z, 0, u), u, t)$, for $z \in \mathbb{D}$ and $u \in (0, t) \setminus N$ and taking into account that the $\varphi_{0,u}$'s are univalent, we get the result. \square

Now we are going to show that the τ_s appearing in the Berkson-Porta type decomposition formula for Herglotz vector fields are related to Denjoy-Wolff points of the elements of the associated evolution families as in the classical Berkson-Porta formula for semigroups:

Theorem 6.7. *Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ in the unit disc, let $G(z, t)$ be the Herglotz vector field of order $d \geq 1$ which solves (1.2) and let $G(z, s) = (z - \tau_s)(\bar{\tau}_s z - 1)p(z, s)$ be its Berkson-Porta type decomposition (1.3). Let $Z := \{s \in [0, +\infty) : G(\cdot, s) \not\equiv 0\}$. Then for almost every $s \in Z$ there exists a decreasing sequence $\{t_n(s)\}$ converging to s such that $\varphi_{s,t_n(s)} \not\equiv \text{id}_{\mathbb{D}}$ and, denoting by $\tau(s, n)$ the Denjoy-Wolff point of $\varphi_{s,t_n(s)}$, it holds $\tau_s = \lim_{n \rightarrow \infty} \tau(s, n)$.*

Proof. By (6.2) there exists a set of zero measure $M \subset [0, +\infty)$, such that for every $s \in (0, +\infty) \setminus M$ there exists a strictly increasing sequence of natural numbers $\{n_k(s)\}$ such that, defining $f_k(z, s) := n_k(s)(\varphi(z, s, s + 1/n_k(s)) - z)$, it follows that $G(z, s)$ is the uniform limit on compacta of \mathbb{D} of the sequence $\{f_k(z, s)\}$. Note that by the classical Berkson-Porta formula, $G(\cdot, s) \in \text{Gen}(\mathbb{D})$ for $s \geq 0$ fixed and also $f_k(\cdot, s) \in \text{Gen}(\mathbb{D})$ for $s \geq 0$ fixed and all $k \geq 0$ (see Section 2).

Fix $s \in Z \setminus M$. Therefore there exists $m(s) \in \mathbb{N}$ such that $\varphi(\cdot, s, s + 1/n_k(s)) \not\equiv \text{id}_{\mathbb{D}}$ and $f_k(\cdot, s) \not\equiv 0$ for $k \geq m(s)$. We show that such a sequence $\{s + 1/n_k(s)\}_{k \geq m(s)}$ (which we relabel $\{t_n(s)\}$) is the one we are looking for. Let $\tau(s, k)$ be the Denjoy-Wolff point of $\varphi_{s, s+1/n_k(s)}$. We claim that $BP_{\tau}(f_k(\cdot, s)) = \tau(s, k)$ for all k . Once this is proved the result follows at once from Proposition 2.1.

In case $\tau(s, k) \in \mathbb{D}$ then clearly $f_k(\tau(s, k), s) = 0$ and hence $BP_\tau(f_k(\cdot, s)) = \tau(s, k)$, as wanted. In case $\tau(s, k) \in \partial\mathbb{D}$, then $\angle \lim_{z \rightarrow \tau(s, k)} f_k(z, s) = 0$, so $\tau(s, k)$ is a boundary critical point for the generator $f_k(\cdot, s)$ (see [9] for further details about critical points). Since $\tau(s, k) \in \partial\mathbb{D}$ is the Denjoy-Wolff point of $\varphi_{s, s+1/n_k(s)}$, it follows $\angle \lim_{z \rightarrow \tau(s, k)} \varphi'(z, s, s + 1/n_k(s)) \in (0, 1]$. Hence

$$\angle \lim_{z \rightarrow \tau(s, k)} f'_k(z, s) = n_k(s) \left(\angle \lim_{z \rightarrow \tau(s, k)} \varphi'(z, s, s + 1/n_k(s)) - 1 \right) \in [0, +\infty).$$

According to [9], this implies that $BP_\tau(f_k(\cdot, s)) = \tau(s, k)$, as needed. \square

7. EVOLUTION FAMILIES WITH A COMMON FIXED POINT

Theorem 7.1. *Let $(\varphi_{s,t})$ be an evolution family of order $d \geq 1$ of the unit disc with Berkson-Porta data (p, τ) . Suppose that $\tau(t) \equiv \tau \in \overline{\mathbb{D}}$ is constant. Then there exists a unique locally absolutely continuous function $\lambda : [0, +\infty) \rightarrow \mathbb{C}$ with $\lambda' \in L^d_{loc}([0, +\infty), \mathbb{C})$, $\lambda(0) = 0$ and $\operatorname{Re} \lambda(t) \geq \operatorname{Re} \lambda(s) \geq 0$ for all $0 \leq s \leq t < +\infty$ such that $\varphi'_{s,t}(\tau) = \exp(\lambda(s) - \lambda(t))$ for all $s \leq t$. Moreover, if $\tau \in \mathbb{D}$, then*

$$\lambda(t) = (1 - |\tau|^2) \int_0^t p(\tau, \xi) d\xi \quad \text{for all } t \geq 0,$$

while, if $\tau \in \partial\mathbb{D}$, then

$$\lambda(t) = \int_0^t \left(\angle \lim_{z \rightarrow \tau} \frac{2|\tau - z|^2 p(z, \xi)}{1 - |z|^2} \right) d\xi \quad \text{for all } t \geq 0.$$

Proof. Case $\tau \in \mathbb{D}$. Up to conjugation with a fixed automorphism mapping τ to 0, we can assume $\tau = 0$. We write $\lambda(t) = \int_0^t p(0, \xi) d\xi$ for all $t \geq 0$. Fixed $s < t$ we have only to prove that $\varphi'_{s,t}(0) = \exp(\lambda(s) - \lambda(t))$.

Claim 1. *For all $z \in \mathbb{D}$ and for all $0 \leq s < t$, the function $[s, t] \ni \xi \mapsto p(\varphi_{s,\xi}(z), \xi)$ belongs to $L^d([s, t], \mathbb{C})$. Moreover,*

$$(7.1) \quad \varphi_{s,t}(z) = z \exp \left(- \int_s^t p(\varphi_{s,\xi}(z), \xi) d\xi \right).$$

Assuming the claim, since $\varphi'_{s,t}(0) = \lim_{z \rightarrow 0} \frac{\varphi_{s,t}(z)}{z}$, by (7.1) we are left to prove that

$$(7.2) \quad \lim_{z \rightarrow 0} \int_s^t p(\varphi_{s,\xi}(z), \xi) d\xi = \int_s^t p(0, \xi) d\xi.$$

Fix ξ . If $\operatorname{Re} p(z, \xi) = 0$ for some $z \in \mathbb{D}$ then $p(\cdot, \xi) \equiv ia_\xi$ for some $a_\xi \in \mathbb{R}$. If $\operatorname{Re} p(\cdot, \xi) > 0$, the holomorphic map $\mathbb{D} \ni z \mapsto \frac{p(z, \xi) - p(0, \xi)}{p(z, \xi) + p(0, \xi)}$ sends the unit disc into itself and fixes 0. Then

$$\begin{aligned} |p(z, \xi) - p(0, \xi)| &\leq |z| \left| p(z, \xi) + \overline{p(0, \xi)} \right| \leq |z| |p(z, \xi)| + |z| |p(0, \xi)| \\ &\leq |z| \frac{1 + |z|}{1 - |z|} |p(0, \xi)| + |z| |p(0, \xi)| = \frac{2|z|}{1 - |z|} |p(0, \xi)|, \end{aligned}$$

where in the last inequality we have used [20, pages 39-40]. Therefore,

$$|p(\varphi_{s, \xi}(z), \xi) - p(0, \xi)| \leq \frac{2|\varphi_{s, \xi}(z)|}{1 - |\varphi_{s, \xi}(z)|} |p(0, \xi)| \leq \frac{2|z|}{1 - |z|} |p(0, \xi)|.$$

Since the function $[s, t] \ni \xi \mapsto p(0, \xi)$ belongs to $L^d([s, t], \mathbb{C})$, we have

$$\left| \int_s^t p(\varphi_{s, \xi}(z), \xi) d\xi - \int_s^t p(0, \xi) d\xi \right| \leq \frac{2|z|}{1 - |z|} \int_s^t |p(0, \xi)| d\xi$$

and (7.2) follows from the dominated convergence theorem.

Case $\tau \in \partial\mathbb{D}$. Conjugating with the Cayley transform $T_\tau : z \mapsto \frac{\tau+z}{\tau-z}$, we define a family $\phi_{s, t} := T_\tau \circ \varphi_{s, t} \circ T_\tau^{-1}$ of holomorphic self maps of $\mathbb{H} := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. For $w \in \mathbb{H}$ and $t \in [0, +\infty)$ we let $P(w, t) := 2p(T_\tau^{-1}(w), t)$.

Claim 2. *For all $w \in \mathbb{H}$ and for all $0 \leq s < t$ the function $[s, t] \ni \xi \mapsto \frac{\operatorname{Re} P(\phi_{s, \xi}(w), \xi)}{\operatorname{Re} \phi_{s, \xi}(w)} \in L^d([s, t], \mathbb{C})$. Moreover,*

$$(7.3) \quad \operatorname{Re} \phi_{s, t}(w) = \operatorname{Re} w \exp \left(\int_s^t \frac{\operatorname{Re} P(\phi_{s, \xi}(w), \xi)}{\operatorname{Re} \phi_{s, \xi}(w)} d\xi \right) \quad \text{for all } w \in \mathbb{H}$$

and ∞ is the Denjoy-Wolff point of $\phi_{s, t}$.

We assume that $\operatorname{Re} p(w, \xi) > 0$ for all w and for all $\xi \geq 0$ (leaving to the reader the obvious modifications in case $\operatorname{Re} p(\cdot, \xi) \equiv a_\xi i$, $a_\xi \in \mathbb{R}$ for some ξ). By the Julia-Wolff-Carathéodory theorem (see, e.g., [1]), the number

$$\widehat{\lambda}(\xi) := \inf \left\{ \frac{\operatorname{Re} P(w, \xi)}{\operatorname{Re} w} : w \in \mathbb{H} \right\} = \angle \lim_{w \rightarrow \infty} \frac{P(w, \xi)}{w} = \angle \lim_{w \rightarrow \infty} \frac{\operatorname{Re} P(w, \xi)}{\operatorname{Re} w}$$

is well-defined for all ξ . Moreover, the function $\xi \in [0, +\infty) \mapsto \widehat{\lambda}(\xi)$ is measurable since $\widehat{\lambda}(\xi) = \lim_{n \rightarrow \infty} \frac{P(n, \xi)}{n}$ and $\xi \mapsto P(w, \xi)$ is measurable for all $w \in \mathbb{H}$. In addition, since $[0, +\infty) \ni \xi \mapsto \operatorname{Re} P(1, \xi)$ belongs to $L^d_{loc}([0, +\infty))$ and $0 \leq \widehat{\lambda}(\xi) \leq \operatorname{Re} P(1, \xi)$, we conclude that the function $[0, +\infty) \ni \xi \mapsto \widehat{\lambda}(\xi)$ also belongs to $L^d_{loc}([0, +\infty))$. By (7.3)

$$\varphi'_{s, t}(\tau)^{-1} = \lim_{n \rightarrow +\infty} \frac{\operatorname{Re} \phi_{s, t}(n)}{n} = \exp \left(\lim_{n \rightarrow +\infty} \int_s^t \frac{\operatorname{Re} P(\phi_{s, \xi}(n), \xi)}{\operatorname{Re} \phi_{s, \xi}(n)} d\xi \right).$$

Now, for all fixed $n \in \mathbb{N}$

$$\int_s^t \widehat{\lambda}(\xi) d\xi = \int_s^t \inf \left\{ \frac{\operatorname{Re} P(w, \xi)}{\operatorname{Re} w} : w \in \mathbb{H} \right\} d\xi \leq \int_s^t \frac{\operatorname{Re} P(\phi_{s,\xi}(n), \xi)}{\operatorname{Re} \phi_{s,\xi}(n)} d\xi.$$

Thus, $\int_s^t \widehat{\lambda}(\xi) d\xi \leq \lim_{n \rightarrow +\infty} \int_s^t \frac{\operatorname{Re} P(\phi_{s,\xi}(n), \xi)}{\operatorname{Re} \phi_{s,\xi}(n)} d\xi$. Moreover, since ∞ is the Denjoy-Wolff point of the function $\phi_{s,t}$ we have that $\operatorname{Re} \phi_{s,t}(n) \geq n$ and, by [21, (3.2)] and [20, pages 39-40] we have

$$\frac{\operatorname{Re} P(\phi_{s,\xi}(n), \xi)}{\operatorname{Re} \phi_{s,\xi}(n)} \leq \frac{\operatorname{Re} P(n, \xi)}{n} \leq 4 \operatorname{Re} P(1, \xi).$$

Then, the sequence of measurable functions $[s, t] \ni \xi \mapsto \frac{\operatorname{Re} P(n, \xi)}{n}$ is uniformly bounded by a $L^d_{loc}([0, +\infty))$ -function and converges pointwise to $\widehat{\lambda}$. Thus, the dominated convergence theorem shows that

$$\lim_n \int_s^t \frac{\operatorname{Re} P(\phi_{s,\xi}(n), \xi)}{\operatorname{Re} \phi_{s,\xi}(n)} d\xi \leq \lim_n \int_s^t \frac{\operatorname{Re} P(n, \xi)}{n} d\xi = \int_s^t \left(\lim_n \frac{\operatorname{Re} P(n, \xi)}{n} \right) d\xi = \int_s^t \widehat{\lambda}(\xi) d\xi.$$

Summing up, we have $\varphi'_{s,t}(\tau) = \exp\left(-\int_s^t \widehat{\lambda}(\xi) d\xi\right)$ as wanted. Now, we are left to prove Claim 1 and Claim 2.

Proof of Claim 1. Fix z and $0 \leq s < t < +\infty$. Since the function $\xi \mapsto p(z, \xi)$ is measurable and $\xi \mapsto \varphi_{s,\xi}(z)$ is continuous by Proposition 3.5, it follows that the function $[s, t] \ni \xi \mapsto p(\varphi_{s,\xi}(z), \xi)$ is measurable. Moreover, for all ξ , by [20, pages 39-40], we have

$$|p(\varphi_{s,\xi}(z), \xi)| \leq \frac{1 + |\varphi_{s,\xi}(z)|}{1 - |\varphi_{s,\xi}(z)|} |p(0, \xi)| \leq \frac{2}{1 - |z|} |p(0, \xi)|.$$

Therefore, the map $[s, t] \ni \xi \mapsto p(\varphi_{s,\xi}(z), \xi) \in L^d([s, t], \mathbb{C})$. Hence, the function $\phi(u) := z \exp\left(-\int_s^u p(\varphi_{s,\xi}(z), \xi) d\xi\right)$ is absolutely continuous in $[s, t]$ and $\phi'(u) = -\phi(u)p(\varphi_{s,u}(z), u)$. Assume that $z \neq 0$. By (6.4) and (4.7), recalling that $\tau \equiv 0$, it follows $\frac{\partial \varphi_{s,u}(z)}{\partial u} = -\varphi_{s,u}(z)p(\varphi_{s,u}(z), u)$ almost everywhere, thus $\frac{\partial}{\partial u} \left(\frac{\phi(u)}{\varphi_{s,u}(z)} \right) \equiv 0$ for almost every $u \in [s, t]$ (notice that since $\varphi_{s,u}$ is univalent then $\varphi_{s,u}^{-1}(0) = 0$ and the above quotient is well-defined for $z \neq 0$). Therefore, there exists c such that $\varphi_{s,u}(z) = c\phi(u)$ for all u . But, $\varphi_{s,s}(z) = z$ and $\phi(s) = z$. Hence, $c = 1$ and the claim is proved.

Proof of Claim 2. A direct computation shows that equation (6.4) translates to \mathbb{H} in

$$(7.4) \quad \frac{\partial \phi_{s,t}(w)}{\partial t} = P(\phi_{s,t}(w), t),$$

which holds for almost every t and every $w \in \mathbb{H}$.

Fix w and $0 \leq s < t < +\infty$. Since the function $\xi \mapsto P(w, \xi)$ is measurable and $\xi \mapsto \phi_{s,\xi}(z)$ is continuous by Proposition 3.5, the function $[s, t] \ni \xi \mapsto P(\phi_{s,\xi}(z), \xi)$ is

measurable. Moreover by the distortion theorem for Carathéodory functions (see [21]), for each $\xi \in [s, t]$,

$$(7.5) \quad \frac{\operatorname{Re} P(\phi_{s,\xi}(w), \xi)}{\operatorname{Re} \phi_{s,\xi}(w)} \leq \frac{|\phi_{s,\xi}(w) + 1|^2}{(\operatorname{Re} \phi_{s,\xi}(w))^2} \operatorname{Re} P(1, \xi).$$

Fix a compact set K in \mathbb{H} . By Lemma 3.6, the set $\{\phi_{s,\xi}(w) : w \in K, \xi \in [s, t]\}$ is compact in \mathbb{H} . Since the function $[s, t] \ni \xi \mapsto \operatorname{Re} P(1, \xi) \in L^d([s, t], \mathbb{C})$, equation (7.5) shows that $[s, t] \ni \xi \mapsto \frac{\operatorname{Re} P(\phi_{s,\xi}(w), \xi)}{\operatorname{Re} \phi_{s,\xi}(w)}$ belongs to $L^d([s, t], \mathbb{C})$.

Hence, the function $\phi(u) := \operatorname{Re} w \exp\left(\int_s^u \frac{\operatorname{Re} P(\phi_{s,\xi}(w), \xi)}{\operatorname{Re} \phi_{s,\xi}(w)} d\xi\right)$ is absolutely continuous in $[s, t]$ and $\phi'(u) = \phi(u) \frac{\operatorname{Re} P(\phi_{s,u}(w), u)}{\operatorname{Re} \phi_{s,u}(w)}$ for almost every $u \in [s, t]$. By (7.5), $\frac{\partial}{\partial u} \left(\frac{\phi(u)}{\operatorname{Re} \phi_{s,u}(w)}\right) \equiv 0$ for almost every $u \in [s, t]$. That is there exists c such that $\operatorname{Re} \phi_{s,u}(w) = c\phi(u)$ for all u . But, $\operatorname{Re} \phi_{s,s}(w) = \operatorname{Re} w = \phi(s)$. That is, $c = 1$ and the claim is proved. \square

Corollary 7.2. *Let $(\varphi_{s,t})$ be an evolution family of the unit disc with Berkson-Porta data (p, τ) . Suppose τ is constant. Then for all $0 \leq s \leq t < +\infty$ either τ is the Denjoy-Wolff point of $(\varphi_{s,t})$ or $\varphi_{s,t} = \operatorname{id}$.*

Proof. It follows at once from Claim 1 and Claim 2 in the proof of Theorem 7.1. \square

Our next result shows that a nice behavior of the derivative at the common fixed point τ allows us to replace the topological property EF3 in the definition of evolution family by a much weaker hypothesis. In order to understand the naturality of our hypothesis on the first derivative at the Denjoy-Wolff point, we remark that if $(\varphi_{s,t})$ is an evolution family on the unit disc with common Denjoy-Wolff point $\tau \in \mathbb{D}$, then by univalence, $\varphi'_{0,t}(\tau) \neq 0$ for all $t \geq 0$. If $\tau \in \partial\mathbb{D}$ then the same (as angular limit) is true by the classical Julia lemma (see, e.g., [1]).

Theorem 7.3. *Let $(\varphi_{s,t})$ be a family of holomorphic self-maps of the unit disc having a common Denjoy-Wolff point $\tau \in \overline{\mathbb{D}}$. Assume that $(\varphi_{s,t})$ satisfies EF1 and EF2 and $\varphi'_{0,t}(\tau) \neq 0$ for all $t \geq 0$. Then the following are equivalent:*

- (1) $(\varphi_{s,t})$ is an evolution family of order $d \geq 1$.
- (2) The following properties are satisfied:
 - 2.1 the map $[0, +\infty) \ni t \mapsto \mu(t) := \varphi'_{0,t}(\tau)$ is absolutely continuous and $\mu' \in L^d_{loc}([0, +\infty), \mathbb{R})$,
 - 2.2 If $\tau \in \partial\mathbb{D}$, there exists a point $z_0 \in \mathbb{D}$ such that for all $T > 0$ there exists a non-negative function $k_{z_0,T} \in L^d([0, T], \mathbb{R})$ such that for all $0 \leq s \leq u \leq t \leq T$

$$|\varphi_{s,u}(z_0) - \varphi_{s,t}(z_0)| \leq \int_u^t k_{z_0,T}(\xi) d\xi.$$

Proof. By Theorem 7.1 and the definition of evolution family, (1) implies (2).

Conversely, assume (2). Again we have to split up the inner and boundary cases.

Firstly, suppose that $\tau \in \mathbb{D}$. Up to conjugation, we may assume $\tau = 0$. Fix $0 < T < +\infty$. Since $\mu(t) \neq 0$ for all t , there exist two absolutely continuous functions $a, b : [0, +\infty) \rightarrow \mathbb{R}$ such that $\mu(t) = e^{a(t)+ib(t)}$ for all t and $a', b' \in L^1_{loc}([0, +\infty), \mathbb{R})$. By the chain rule for derivatives, $\varphi'_{s,t}(\tau) = \mu(t)/\mu(s)$ for all $0 \leq s \leq t < +\infty$ and since $|\varphi'_{s,t}(0)| = |\mu(t)/\mu(s)| = e^{a(t)-a(s)} \leq 1$, the map a is decreasing. Let $h_{s,t}(z) := \frac{\varphi_{s,t}(z)}{e^{i(b(t)-b(s))z}}$ for $z \in \mathbb{D} \setminus \{0\}$ and $h_{s,t}(0) := e^{a(t)-a(s)}$ for all $0 \leq s \leq t$. The map $h_{s,t}$ is holomorphic and $\operatorname{Re}(1 - h_{s,t}) \geq 0$. Therefore, by [20, pages 39-40],

$$\begin{aligned} |1 - h_{s,t}(z)| &\leq \frac{1 + |z|}{1 - |z|} |1 - h_{s,t}(0)| = \frac{1 + |z|}{1 - |z|} (1 - e^{a(t)-a(s)}) \\ &= \frac{1 + |z|}{1 - |z|} e^{-a(s)} (e^{a(s)} - e^{a(t)}) \leq \frac{1 + |z|}{1 - |z|} e^{-a(T)} (e^{a(s)} - e^{a(t)}) \end{aligned}$$

whenever $z \in \mathbb{D}$ and $0 \leq s \leq t \leq T$. Now, fix $z \in \mathbb{D}$ and $0 \leq s \leq u \leq t \leq T$. On the one hand, if $\varphi_{s,u}(z) = 0$, then $|\varphi_{s,u}(z) - \varphi_{s,t}(z)| = |\varphi_{u,t}(\varphi_{s,u}(z))| = |\varphi_{u,t}(0)| = 0$. On the other hand, if $w := \varphi_{s,u}(z) \neq 0$, then

$$\begin{aligned} |\varphi_{s,u}(z) - \varphi_{s,t}(z)| &= |w - \varphi_{u,t}(w)| \leq \left| 1 - \frac{\varphi_{u,t}(w)}{w} \right| \\ &\leq |1 - h_{u,t}(w)| + \left| \frac{\varphi_{u,t}(w)}{w} \right| \left| \frac{1}{e^{i(b(t)-b(u))}} - 1 \right| \\ &\leq \frac{1 + |w|}{1 - |w|} e^{-a(T)} (e^{a(u)} - e^{a(t)}) + |e^{ib(t)} - e^{ib(u)}| \\ &\leq \frac{1 + |z|}{1 - |z|} e^{-a(T)} (e^{a(u)} - e^{a(t)}) + |e^{ib(t)} - e^{ib(u)}|. \end{aligned}$$

In any case, we have that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \frac{1 + |z|}{1 - |z|} e^{-a(T)} ((e^{a(u)} - e^{a(t)}) + |e^{ib(t)} - e^{ib(u)}|).$$

Then, the function

$$k_{z,T}(\xi) := \frac{1 + |z|}{1 - |z|} e^{-a(T)} \frac{d}{d\xi} (e^{a(\xi)} + |e^{ib(\xi)}|)$$

belongs to $L^d([0, T], \mathbb{C})$ and

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi,$$

proving the result in this case.

Next, suppose that $\tau \in \partial\mathbb{D}$. Again by the chain rule for angular derivatives, $\varphi'_{s,t}(\tau) = \mu(t)/\mu(s)$ for all $0 \leq s \leq t < +\infty$ and, since $\varphi'_{s,t}(\tau) \in (0, 1]$, it follows that $0 < \mu(t) \leq \mu(s) \leq 1$ for all $0 \leq s \leq t$. Without loss of generality we assume $z_0 = 0$. Again, we

move to the right half-plane by means of the Cayley transform T_τ given by $T_\tau(z) = \frac{\tau+z}{\tau-z}$ and we let $(\phi_{s,t})$ be the family of holomorphic self-maps of the right half-plane given by $\phi_{s,t} := T_\tau \circ \varphi_{s,t} \circ T_\tau^{-1}$ for all $0 \leq s \leq t < +\infty$. The Denjoy-Wolff point of $\phi_{s,t}$ is ∞ with multiplier $1/\varphi'_{s,t}(\tau) = \mu(s)/\mu(t)$. Thus the function $\operatorname{Re} [\phi_{s,t}(w) - \frac{\mu(s)}{\mu(t)}w] \geq 0$ for all $w \in \mathbb{H}$. Then, by [20, pages 39-40],

$$\left| \phi_{s,t}(w) - \frac{\mu(s)}{\mu(t)}w \right| \leq \frac{|w+1| + |w-1|}{|w+1| - |w-1|} \left| \phi_{s,t}(1) - \frac{\mu(s)}{\mu(t)} \right|.$$

Therefore,

$$|w - \phi_{s,t}(w)| \leq \left| 1 - \frac{\mu(s)}{\mu(t)} \right| |w| + \frac{|w+1| + |w-1|}{|w+1| - |w-1|} \left| \phi_{s,t}(1) - \frac{\mu(s)}{\mu(t)} \right|.$$

Fix $0 < T < +\infty$. By hypothesis and arguing as in the proofs of Proposition 3.5 and Lemma 3.6, there is a number R such that $|\varphi_{s,t}(0)| \leq R$ for all $0 \leq s \leq t \leq T$ and then the set $\{\phi_{s,t}(1) : 0 \leq s \leq t \leq T\}$ is a compact subset of the right half-plane.

Now fix $w \in \mathbb{H}$. If $0 \leq s \leq t \leq T$, then $\rho_{\mathbb{H}}(\phi_{s,t}(w), \phi_{s,t}(1)) \leq \rho_{\mathbb{H}}(w, 1)$. Hence there is a compact set K in \mathbb{H} such that $\phi_{s,t}(w) \in K$ for all $0 \leq s \leq t \leq T$. Therefore there exists $M > 0$ such that

$$\max_{0 \leq s \leq t \leq T} \left\{ |\phi_{s,t}(w)|, \frac{|\phi_{s,t}(w) + 1| + |\phi_{s,t}(w) - 1|}{|\phi_{s,t}(w) + 1| - |\phi_{s,t}(w) - 1|} \right\} \leq M.$$

Fix $0 \leq s \leq u \leq t \leq T$. Write $v = \phi_{s,u}(w)$. Then

$$\begin{aligned} |\phi_{s,u}(w) - \phi_{s,t}(w)| &= |v - \phi_{u,t}(v)| \leq \left| 1 - \frac{\mu(u)}{\mu(t)} \right| |v| + \frac{|v+1| + |v-1|}{|v+1| - |v-1|} \left| \phi_{u,t}(1) - \frac{\mu(u)}{\mu(t)} \right| \\ &\leq M \left(2 \left| 1 - \frac{\mu(u)}{\mu(t)} \right| + |\phi_{u,t}(1) - 1| \right) \\ &\leq M \left(\frac{2}{\mu(T)} |\mu(t) - \mu(u)| + \frac{|\varphi_{u,t}(0)|}{1 - |\varphi_{u,t}(0)|} \right) \\ &\leq M \left(\frac{2}{\mu(T)} |\mu(t) - \mu(u)| + \frac{1}{1-R} |\varphi_{u,t}(0)| \right). \end{aligned}$$

From this inequality, one can easily finish the proof arguing as in the previous case. \square

Remark 7.4. We point out that hypothesis 2.2 is not needed in case $\tau \in \mathbb{D}$. While, in case $\tau \in \partial\mathbb{D}$ hypothesis 2.2 in Theorem 7.3 is necessary: the family $\varphi_{s,t}(z) = T^{-1}(T(z) + ic(s) - ic(t))$ with $c : [0, +\infty) \rightarrow \mathbb{R}$ continuous but not absolutely continuous and $T(z) = \frac{1+z}{1-z}$ satisfies EF1, EF2 and 2.1 (being $\mu \equiv 1$) but it is not an evolution family according to our definition.

Classically, evolution families that comes out from Loewner types equations are those $(\varphi_{s,t})$ with a common fixed point 0 and such that $\varphi'_{0,t}(0) = e^{-t}$ (see, e.g., [17], [19], [20],

and [24]). The above result shows that EF3 is not necessary in the classical case: it follows automatically from the normalization hypothesis on the first derivative at 0.

We end up this section with a technical result which better relates the “classical evolution families” in Loewner’s theory with the definition introduced in this paper.

Proposition 7.5. *Let $(\varphi_{s,t})$ be an evolution family on the unit disc. Then for all $r < 1$ and for all $T < +\infty$, the set of functions $\{[0, T] \ni t \mapsto \varphi_{0,t}(z) \in \mathbb{D} : |z| \leq r\}$ is uniformly absolutely continuous.*

Conversely, assume $(\varphi_{s,t})$ is a family of holomorphic self-maps of the unit disc which satisfies EF1 and EF2 and has a common Denjoy-Wolff point $\tau \in \mathbb{D}$. Assume moreover that $\varphi'_{0,t}(\tau) \neq 0$ for all $t \geq 0$. If for all $r < 1$ and for all $T < +\infty$, the set of functions $\{[0, T] \ni t \mapsto \varphi_{0,t}(z) \in \mathbb{D} : |z| \leq r\}$ is uniformly absolutely continuous then $(\varphi_{s,t})$ is an evolution family.

Proof. By Lemma 3.6, there exists $R > 0$ such that $|\varphi_{s,t}(z)| \leq R$ for all $0 \leq s \leq t \leq T$ and $|z| \leq r$. Let $G(z, t)$ be the Herglotz vector field which solves (1.2), and let $k_{R,T} \in L^1([0, T], \mathbb{R})$ be the function given by WHVF3. Then

$$\begin{aligned} \sup_{|z| \leq r} \sum_{k=1}^n |\varphi_{0,b_k}(z) - \varphi_{0,a_k}(z)| &= \sup_{|z| \leq r} \sum_{k=1}^n \left| \int_{a_k}^{b_k} \frac{\partial \varphi_{0,\xi}(z)}{\partial \xi} d\xi \right| \\ &\leq \sup_{|z| \leq r} \sum_{k=1}^n \int_{a_k}^{b_k} |G(\varphi_{0,\xi}(z), \xi)| d\xi \leq \sum_{k=1}^n \int_{a_k}^{b_k} k_{R,T}(\xi) d\xi. \end{aligned}$$

Conversely, assuming $\tau = 0$, it is not difficult to see that the absolute continuity and Cauchy formula imply that $[0, +\infty) \ni t \mapsto \varphi'_{0,t}(\tau)$ is absolutely continuous. Hence the result follows from Theorem 7.3. \square

8. EVOLUTION FAMILIES IN THE HALF PLANE

In this section, we examine evolution families with a common Denjoy-Wolff point at the boundary of \mathbb{D} . By means of a conjugation with a Cayley map, we can transfer everything to the right half-plane $\mathbb{H} := \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$. As a matter of notation, we say that a family $(\phi_{s,t})$ of holomorphic self-maps of \mathbb{H} is an *evolution family of order $d \geq 1$* if there exists a biholomorphic map $T : \mathbb{H} \rightarrow \mathbb{D}$ such that $(T \circ \phi_{s,t} \circ T^{-1})$ is an evolution family of order d in \mathbb{D} . Similar definition are given for Herglotz vector fields and Herglotz functions.

Translating Theorems 6.2 and 5.2 to the right half-plane, we can state the following result

Theorem 8.1. *Let $(\phi_{s,t})$ be a family of holomorphic self-maps of the right half-plane \mathbb{H} . Then $(\phi_{s,t})$ is an evolution family of order $d \geq 1$ in the right half-plane with ∞ as common boundary fixed point if and only if there exists a Herglotz function $P(w, t)$ of order d in the right half-plane such that, given $s \geq 0$, there exists a set $M = M(s) \subset [s, +\infty)$ (not*

depending on w) of zero measure such that, for every $t \in (s, +\infty) \setminus M$ and every $w \in \mathbb{H}$, it holds that

$$\frac{\partial \phi_{s,t}(w)}{\partial t} = P(\phi_{s,t}(w), t).$$

Let P be a Herglotz function of order $d \geq 1$ in the right half-plane. For all $s \geq 0$ and $w \in \mathbb{H}$, let $\psi_{s,w}$ be the solution of the problem

$$\begin{cases} \dot{w}(t) = P(w(t), t) & \text{for a. e. } t \in [s, +\infty) \\ w(s) = w. \end{cases}$$

Then defining $\phi_{s,t}(w) := \psi_{s,w}(t)$ for all $0 \leq s \leq t < +\infty$ and for all $w \in \mathbb{H}$, the family $(\phi_{s,t})$ is an evolution family of order d in the right half-plane with ∞ as common boundary fixed point.

From the very beginning of this century, there have been many authors interested on a very particular case of Herglotz functions in the right half-plane (see, [25], [18], [23]). Namely, let $h : [0, +\infty) \rightarrow i\mathbb{R}$ be a measurable function (in fact, those papers always assume that h is continuous). Then $P(w, t) = \frac{1}{w+h(t)}$ is a Herglotz function in \mathbb{H} of order ∞ since $|P(w, t)| \leq \frac{1}{\operatorname{Re} w}$ for all $w \in \mathbb{H}$ and $t \geq 0$. Moreover, it is clear that $\angle \lim_{w \rightarrow \infty} \frac{P(w, \xi)}{w} = 0$ for all ξ . Therefore, by Theorem 8.1, if $(\phi_{s,t})$ is the evolution family in the right half-plane with Herglotz function P , then all the functions $\phi_{s,t}$ are parabolic, that is, $\angle \lim_{w \rightarrow \infty} \frac{\phi_{s,t}(w)}{w} = 1$. By Claim 2 in the proof of Theorem 7.1, we have that for all $w \in \mathbb{H}$ and for all $0 \leq s < t$, the function $[s, t] \ni \xi \mapsto \frac{P(\phi_{s,\xi}(w), \xi)}{\phi_{s,\xi}(w)}$ belongs to $L^\infty([s, t], \mathbb{C})$ and

$$\phi_{s,t}(w) = w \exp \left(\int_s^t \frac{P(\phi_{s,\xi}(w), \xi)}{\phi_{s,\xi}(w)} d\xi \right) \quad \text{for all } w \in \mathbb{H}.$$

Now, write $k(w) = \int_s^t \frac{P(\phi_{s,\xi}(w), \xi)}{\phi_{s,\xi}(w)} d\xi$ for all $w \in \mathbb{H}$. Since $\phi_{s,\xi}$ is a parabolic function with Denjoy-Wolff point ∞ , if $w \in \mathbb{H} \cap \mathbb{R}$, then $|\phi_{s,\xi}(w)| \geq \operatorname{Re} \phi_{s,\xi}(w) \geq w$. Therefore, $\left| w^2 \frac{P(\phi_{s,\xi}(w), \xi)}{\phi_{s,\xi}(w)} \right| \leq 1$ for all ξ and $\lim_{w \in \mathbb{R}, w \rightarrow +\infty} w^2 \frac{P(\phi_{s,\xi}(w), \xi)}{\phi_{s,\xi}(w)} = 1$. Then, by the dominated converge theorem, we have

$$t - s = \lim_{w \in \mathbb{R}, w \rightarrow +\infty} w^2 k(w) = \lim_{w \in \mathbb{R}, w \rightarrow +\infty} w^2 (e^{k(w)} - 1) = \lim_{w \in \mathbb{R}, w \rightarrow +\infty} w(\phi_{s,t}(w) - w).$$

Thus, by Lehto-Virtanen theorem, $\angle \lim_{w \rightarrow \infty} (\phi_{s,t}(w) - w) = 0$ and $\angle \lim_{w \rightarrow \infty} w(\phi_{s,t}(w) - w) = t - s$. That is, $\phi_{s,t}(w) = w + \frac{t-s}{w} + \gamma_{s,t}(w)$ where $\angle \lim_{w \rightarrow \infty} w\gamma_{s,t}(w) = 0$ and the functions of the evolution family satisfies the so called hydrodynamic normalization. Following the terminology introduced by the last two authors and Pommerenke in [10], this means that if $(\varphi_{s,t})$ is the corresponding evolution family in the unit disc with fixed point τ , then there exist the second and third angular derivatives of $\varphi_{s,t}$ at τ and, in fact, $\varphi_{s,t}''(\tau) = 0$ and $\varphi_{s,t}'''(\tau) = \frac{3}{2}(s-t)\bar{\tau}^2$.

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DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA “TOR VERGATA”, VIA DELLA RICERCA SCIENTIFICA 1, 00133, ROMA, ITALY.

E-mail address: `fbracci@mat.uniroma2.it`

CAMINO DE LOS DESCUBRIMIENTOS, S/N, DEPARTAMENTO DE MATEMÁTICA APLICADA II, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS, UNIVERSIDAD DE SEVILLA, SEVILLA, 41092, SPAIN.

E-mail address: `contreras@us.es`

E-mail address: `madrigoal@us.es`