§1. Introduction. Let \( X^a_{ns}(N) \) denote the modular curve associated with the normalizer of a non-split Cartan group of level \( N \), where \( N \) is an arbitrary integer. The curve \( X^a_{ns}(N) \) is defined over \( \mathbb{Q} \) and the corresponding scheme over \( \mathbb{Z}[1/N] \) is smooth [1]. If \( N \) is a prime, the genus formula for \( X^a_{ns}(N) \) is given in [5, 6]. The curve \( X^a_{ns}(N) \) has genus 0 if \( N < 11 \) and \( X^a_{ns}(11) \) has genus 1. Ligozat [5] has shown that the group of \( \mathbb{Q} \)-rational points on \( X^a_{ns}(11) \) has rank 1. If the genus \( g(N) \) is greater than 1, very little is known about the \( \mathbb{Q} \)-rational points of \( X^a_{ns}(N) \). Since under simple conditions imaginary quadratic fields with class number 1 give an integral point on these curves, Serre and others have asked whether all integral points are obtained in this way [8].

In this note we determine the \( j \)-invariants of elliptic curves corresponding to points of \( X = X^a_{ns}(7) \) which are integral over \( \mathbb{Z}[1/7] \). These are points which are rational over \( \mathbb{Q} \) and do not give cusps modulo \( p \) for \( p \neq 7 \). We prove that each such point corresponds to an exceptional unit of the first kind of the field \( K = \mathbb{Q}(\cos 2\pi/7) \). Nagell [7] has shown that there are 24 such units. Half of these (those arising from the choice of the generator of \( \text{Gal}(K/\mathbb{Q}) \); the other half relate to the other generator) correspond to the integral points of \( X \). They are the values taken by a uniformizing parameter \( f \) of \( X \) at the integral points. By explicitly constructing \( f \) we are able to find a relationship between \( f \) and the modular invariant \( j \). Eight of the 12 \( \mathbb{Z}[1/7] \)-integral points correspond to elliptic curves with complex multiplication (7 of them predictably so; the exception being the point corresponding to the \( j \) invariant having the value 0). The \( j \) invariants for all the \( \mathbb{Z}[1/7] \)-integral points and the corresponding units are given in a table at the end of the paper.

A similar investigation may be made of level 9 instead of level 7. The exceptional units of that field have also been determined by Nagell [7], but all of these correspond to elliptic curves with complex multiplication.

We note that both \( N = 7 \) and \( N = 9 \) give yet another proof that a 10th imaginary quadratic field with \( h = 1 \) does not exist, since such a field would give an integral point on \( X \), distinct from those already found, since 7 and 3 would respectively have to be inert in the field.

The author thanks J.-P. Serre who found the connection between the integral points and Nagell's units and suggested the determination of the invariants.

§2. Units and integral points on \( X \). Let \( F \) be an algebraic number field, a unit \( E \) of \( F \) is called an exceptional unit if there exists another unit \( E_i \) such that

\[ E + E_i = 1. \]
It is well known [2] that there are at most finitely many exceptional units in any given number field \( F \). For \( F \) a cyclic cubic field Nagell [7] has called an exceptional unit of \( F \) which satisfies an equation of the form

\[
X^3 - pX^2 + (p - 3)X + 1 = 0, \tag{1}
\]

where \( p \) is a rational integer, an exceptional unit of the first kind. The discriminant of the cubic equation is

\[
(p^2 - 3p + 9)^2.
\]

If \( E \) satisfies equation (1) then \( E \) satisfies the equation

\[
X^3 + (p - 3)X^2 - pX + 1 = 0,
\]

so that, if \( E \) is an exceptional unit of the first kind, so is \( E_1 \), and \( E_1 \) corresponds to \( 3 - p \). For the field \( F = K \), Nagell proved that there are 24 such units corresponding to values of \( p \) from \((1, 2), (8, -5), (15, -12) \) and \((1262, -1259) \).

The modular curve \( X \) has 3 conjugate cusps which are defined over the field \( K \). Let \( P_1 \) be the projective line and \( \sigma \) the automorphism \( Z \to 1 - 1/Z \) of \( P_1 \) which is of order 3 and permutes \( 1, 0, \infty \) cyclically. Using \( \sigma \) and \( K/Q \) we can obtain a "twist" \( C \) of \( P_1 \). The curve \( C \) therefore has genus 0 and is defined over \( Q \). It has 3 "marked" points (corresponding to the cusps on \( C \)) rational over \( K \) which are permuted by the non-trivial automorphisms of \( K \). It is thus a model of \( X \) over \( Q \). Let \( s \) be the non-trivial automorphism of \( K/Q \) which corresponds to \( \sigma \) by its action on the marked points. We therefore have a \( K \)-isomorphism

\[
f: X \to P_1
\]

taking the cusps of \( X \) to \( 0, 1, \infty \) and such that \( f^s = 1 - f^{-1} \).

The isomorphism between \( C \) and \( X \) extends to that of their corresponding schemes over \( Z[1/7] \) since the scheme corresponding to \( X \) is smooth over \( Z[1/7] \). We prove

**Lemma 1.** Let \( x \in X \) be integral over \( Z[1/7] \) (equivalently \( x \in X(Q) \) and the \( j \)-invariant is in \( Z[1/7] \)). Put \( \varepsilon = f(x) \), where \( f: X \to P_1 \) is the function above, then \( \varepsilon \) is a unit of \( K \) and \( s(\varepsilon) = 1 - \varepsilon^{-1} \).

**Proof:** Let \( x \in X \) be a point of \( X \) which is integral over \( Z[1/7] \). Then \( \varepsilon = f(x) \) is a unit over \( Z[1/7] \). Also since the \( Q \)-rational points of \( X \) are defined as those corresponding under \( f \) to points \( y \) in \( P_1(K) \) satisfying

\[
s(y) = 1 - 1/y
\]

it suffices to prove that \( \varepsilon \) is a genuine unit of \( K \).

Let \( \rho \) be a generator of the prime ideal above 7 in \( K \). A priori we have \( \varepsilon = \rho^m u \), where \( m \in Z \) and \( u \) is a unit. Hence it suffices to show that \( m = 0 \).

If \( m > 0 \) we obtain a contradiction from the equation

\[
s(\varepsilon) = 1 - \varepsilon^{-1},
\]

since \( s(\varepsilon) \) is a conjugate of \( \varepsilon \) and so is a unit if \( \varepsilon \) is a unit. Similarly if \( m < 0 \) we obtain a contradiction. Hence we have \( m = 0 \) and \( \varepsilon = u \). So that \( \varepsilon \) is a
INTEGRAL POINTS OF A MODULAR CURVE OF LEVEL 7

unit. It is an exceptional unit since

$$e - e \cdot s(e) = 1.$$  

From the equation $s(e) = 1 - e^{-1}$ we can also deduce that it is of the first kind (see Nagell [7]).

Since $f$ is a $K$-isomorphism, the proof of the lemma shows also that any exceptional unit of the first kind $e$ of $K$ which satisfies $s(e) = 1 - e^{-1}$ corresponds to a $\mathbb{Z}[1/7]$-integral point of $X$.

§3. The function $f$ and the modular invariant $j$. To relate the function $f$ to the modular invariant $j$ we consider $X$ as a covering of $X(1)$, the $j$-line.

The covering

$$X \longrightarrow X(1)$$

is of degree 21 and is defined over $\mathbb{Q}$. If we extend scalars to $\mathbb{Q}(\sqrt{-7})$, this can be factored through a curve $Y$

$$X \overset{d_1}{\longrightarrow} Y \overset{d_2}{\longrightarrow} X(1)$$

where $Y$ is the modular curve attached to the symmetric group $S_4 \subset PSL_2(F_7)$, and $d_3, d_7$ are covering maps of degrees 3 and 7 respectively. We may identify $Y$ with the projective line over $\mathbb{Q}(\sqrt{-7})$ by a uniformizing parameter $y$ such that the map

$$Y \overset{d_7}{\longrightarrow} X(1)$$

is given by

$$j = y(y^2 + 7\lambda y + 7\lambda - 21)^3$$

where $\lambda = \frac{1}{2}(1 + \sqrt{-7})$ (see [3, p. 89] and [4, p. 752]). It should be noted that the parameters in Fricke–Klein [4] and here are related by $J = j/(2^33^2)$ and $\lambda \tau = y$.

Since the point $y = \infty$ on $Y$ corresponds to $f = 0, 1$ and $\infty$ on $X$ and $y = 0$ on $Y$ has a cubic ramification in the covering $X \rightarrow Y$, $y$ must be given by an equation

$$y = \frac{a(f - b)^3}{f(f - 1)^3},$$

for some coefficients $a$ and $b$ in the field $\mathbb{Q}(\sqrt[3]{1})$. So to determine the values of $j$ corresponding to the exceptional units it suffices to determine $a$ and $b$ explicitly. We do this by writing $f$ explicitly in terms of Klein forms $k_{(r,s)}$ where $r, s$ are integers not both congruent to 0 mod 7. Following the method described in [5, Ch. II] we obtain a function

$$f = \mu \frac{k_{(1,0)}k_{(0,1)}k_{(3,2)}k_{(2,3)}k_{(2,5)}k_{(5,3)}k_{(5,9)}k_{(9,2)}}{k_{(1,1)}k_{(2,1)}k_{(1,2)}k_{(1,5)}k_{(1,6)}k_{(3,9)}k_{(0,3)}k_{(5,1)}}$$

where $\mu = \xi^{-2}(1 - \xi^2)(1 - \xi)/(1 - \xi^2)^2$ and $\xi = \exp(2\pi i/7)$. The function $f$
INTEGRAL POINTS OF A MODULAR CURVE OF LEVEL 7

takes 0, 1 and \(\infty\) respectively at the cusps of \(X\) and is normalized so that expansion of \(y\) at the cusp, where \(f\) has a pole of order 1, is

\[\xi^a q^{-\frac{1}{7}} - 3\lambda + \text{(terms with positive powers of } q)\].

The constant \(\alpha\) satisfies \(0 \leq \alpha \leq 6\) and reflects the ambiguity of \(y\). Expressing \(y\) as a function of \(f\) gives values \(a\) and \(b\) depending on \(\alpha\). The only value of \(\alpha\) which gives \(y\) lying in \(\mathbb{Q}(\sqrt{-7})\) for the exceptional units is 4. It yields the values \(a = 4u^{-1}\) and \(b = 1 + \xi + \xi^4\). From these we obtain the following table.

<table>
<thead>
<tr>
<th>(f)</th>
<th>(y)</th>
<th>(p)</th>
<th>(j)</th>
<th>Discriminant (d) and conductor (l) of the order corresponding to (j) for CM cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-(\xi + \xi^6))</td>
<td>(-(\lambda)^6)</td>
<td>1</td>
<td>2^3</td>
<td>(d = -4, \ l = 1)</td>
</tr>
<tr>
<td>(-(\xi + \xi^2))</td>
<td>-1</td>
<td>1</td>
<td>2^5</td>
<td>(d = -8, \ l = 1)</td>
</tr>
<tr>
<td>(-(\xi + \xi^7))</td>
<td>(-(\lambda)^3)</td>
<td>1</td>
<td>2^{15}</td>
<td>(d = -43, \ l = 1)</td>
</tr>
<tr>
<td>(-(\xi + \xi^3)(\xi^2 + \xi^4)^3)</td>
<td>((3 - \lambda)^3)</td>
<td>15</td>
<td>2^{18}3^5</td>
<td>(d = -67, \ l = 1)</td>
</tr>
<tr>
<td>((\xi^2 + \xi^3)^3(\xi + \xi^4)^3)</td>
<td>(-(\lambda)^3)</td>
<td>15</td>
<td>2^{13}3^511^3</td>
<td>(d = -3, \ l = 1)</td>
</tr>
<tr>
<td>(-(\xi + \xi^3)(\xi^2 + \xi^4)^2)</td>
<td>(-15\lambda)</td>
<td>-5</td>
<td>2^{25}7^5</td>
<td>Non-CM case</td>
</tr>
<tr>
<td>(-(\xi + \xi^3)^3(\xi^2 + \xi^4)^2)</td>
<td>(1 - 15\lambda)</td>
<td>-5</td>
<td>2^{16}3^5</td>
<td>Non-CM case</td>
</tr>
<tr>
<td>(-(\xi + \xi^3)^3(\xi^2 + \xi^4)^2)</td>
<td>(-1(1 + 5\lambda))</td>
<td>-5</td>
<td>2^{25}7^5</td>
<td>Non-CM case</td>
</tr>
<tr>
<td>(440 + 244(\xi^2 + \xi^4) - 305(\xi^3 + \xi^4)(-1 + 5\lambda)^3)</td>
<td>(-1259)</td>
<td>-1259</td>
<td>2^{13}3^5</td>
<td>Non-CM case</td>
</tr>
<tr>
<td>(135 + 305(\xi^2 + \xi^4) + 549(\xi^3 + \xi^4)(5 + \lambda)^3)</td>
<td>(-1259)</td>
<td>-1259</td>
<td>2^817^3</td>
<td>Non-CM case</td>
</tr>
<tr>
<td>(-684 - 549(\xi^2 + \xi^4) - 244(\xi^3 + \xi^4)(-13 + 4\lambda)^3)</td>
<td>(-1259)</td>
<td>-1259</td>
<td>2^911^3</td>
<td>Non-CM case</td>
</tr>
</tbody>
</table>

**References**


Dr. M. A. Kenku,
Department of Mathematics,
University of Lagos,
Akoka, Lagos, Nigeria.

14K07: ALGEBRAIC GEOMETRY; Special ground fields, arithmetic problems; Elliptic curves. Received on the 23rd of January, 1984.