Number Theory in Cryptography

Introduction

September 20, 2006
Universidad de los Andes

Guessing Numbers

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A **Hash Function** is a function f from A to B such that

- It is easy to compute $f(x)$ for any $x \in A$.
- For any $y \in B$, it is hard to find an $x \in A$ with $f(x) = y$.
- It is hard to find $x, x' \in A$ with $x \neq x'$ and $f(x) = f(x')$.

Caesar Cipher

VIXYVR XS VSQI

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A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
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RETURN TO ROME

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RETURN TO ROME

Breaking the code: just try all 26 shifts.

Substitution Cipher

MQWE WE B YXM QBLHGL

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
Q A Z X S W E D C V F R T G B N H Y U J M K I O L P

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THIS IS A LOT HARDER

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THIS IS A LOT HARDER

Breaking the code:

Can not try $26! = 403291461126605635584000000$ permutations...

Solution: Letter Frequencies

	English	Spanish
A	82	125
B	14	14
C	28	47
D	38	59
E	131	137
F	29	7
G	20	10
H	53	7
I	63	62
J	1	4
K	4	0
L	34	50
M	25	31

	English	Spanish
N	71	67
O	80	86
P	20	25
Q	1	9
R	68	69
S	61	79
T	105	46
U	25	39
V	9	9
W	15	0
X	2	2
Y	20	9
Z	1	5

out of 1000 letters

Viginere Cipher

HVD PZAHSQ JMLEIDRXPSG ZVZ UCH OVZZZSFUIY

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Shift the letters of the encrypted message according to the value of the letters of the secret keyword "**LLAVES**." (a= 1, b= 2, ...).

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

```
HVD PZAHSQ JMLEIDRXPSG ZVZ UCH OVZZSFUIY
LLA VESLLA VESLLAVESLL AVE SLL AVESLLAVE
THE LETTER FREQUENCIES ARE NOT PRESERVED
```

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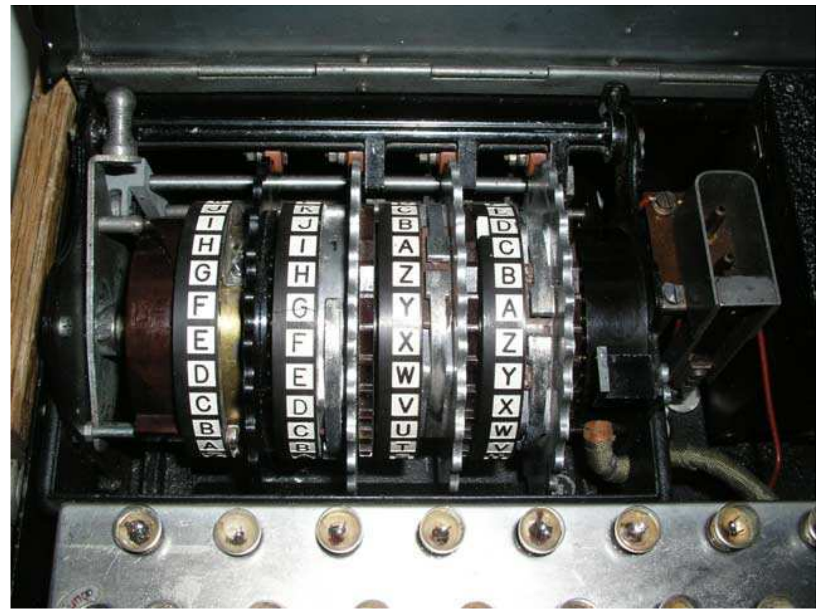
HVD PZAHSQ JMLEIDRXPSG ZVZ UCH OVZZSFUIY
LLA VESLLA VESLLAVESLL AVE SLL AVESLLAVE
THE LETTER FREQUENCIES ARE NOT PRESERVED
                 EN   ES    E N     ES
```

Repeated bigrams stay **repeated bigrams**
if their distance is a multiple of the length of the key.

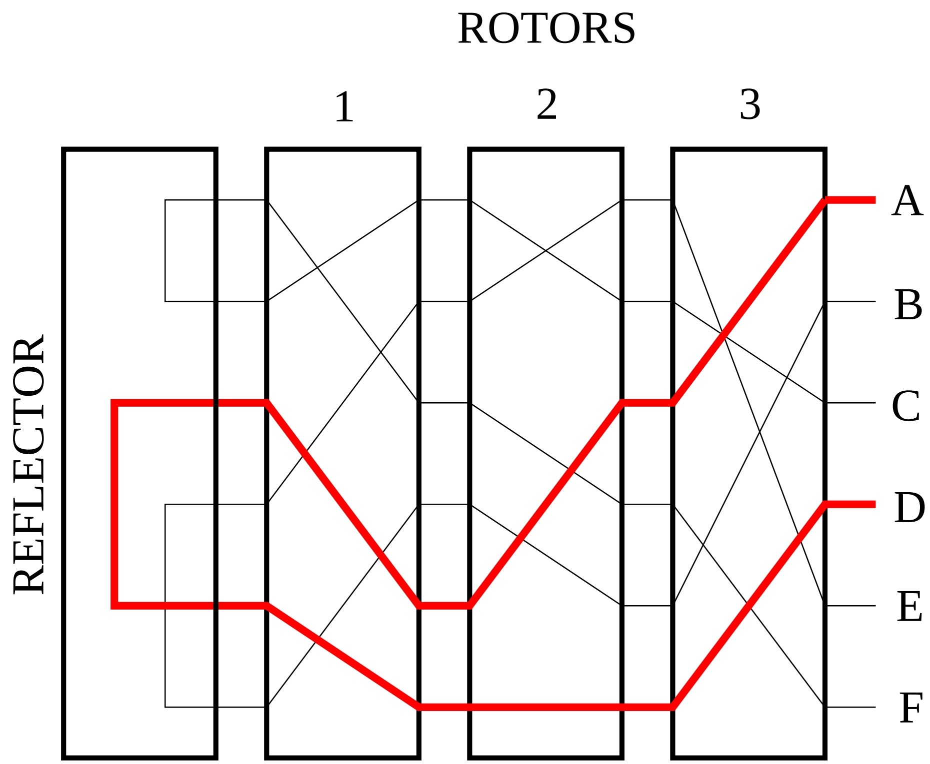
Security

All these ciphers are **breakable**
once the enemy knows
the type of encryption.

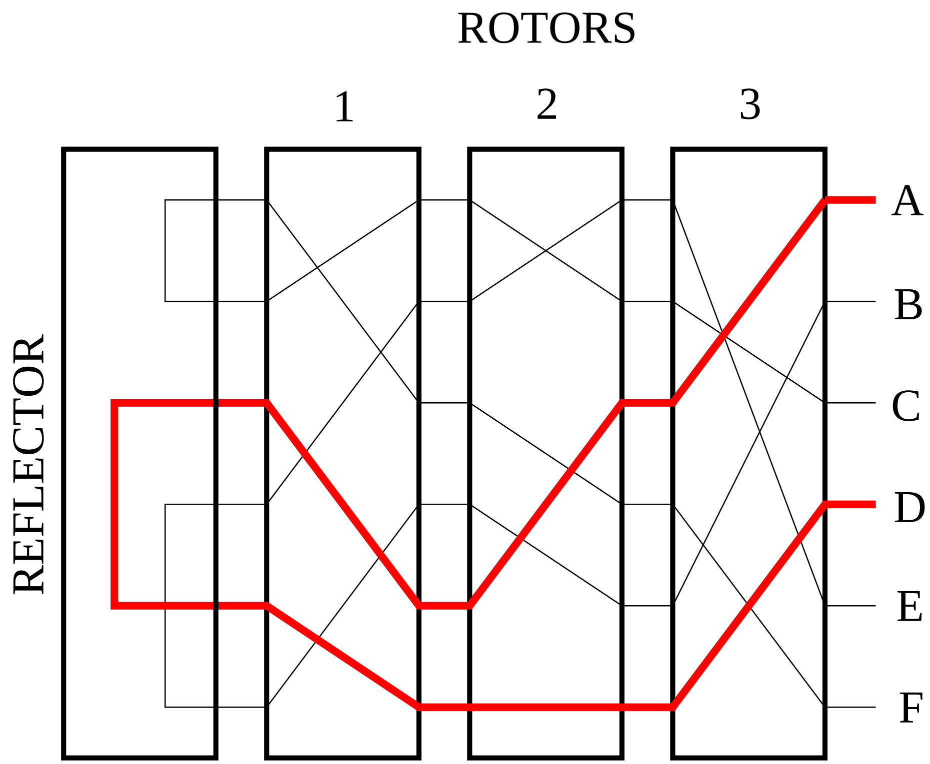
Enigma



A German WW-II encryption machine, broken by the allies



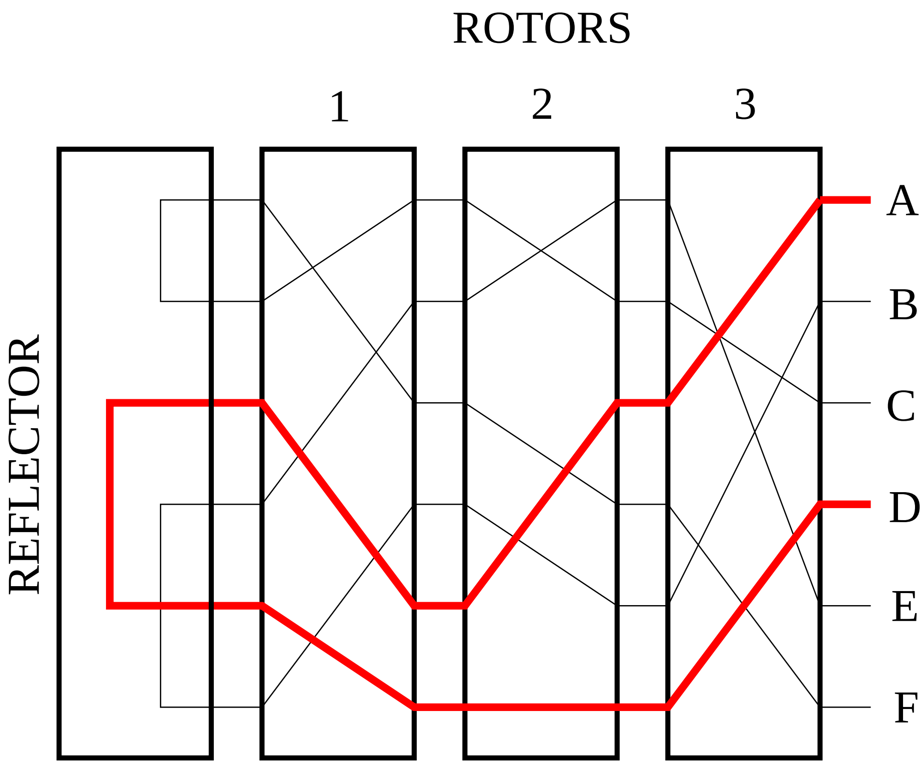
Period of 26^3 substitutions



Period of 26^3 substitutions

Weaknesses:

- Permutations are involutions
- Letter x does **not** map to x
- Rotors can be stolen
- Book of initial settings too



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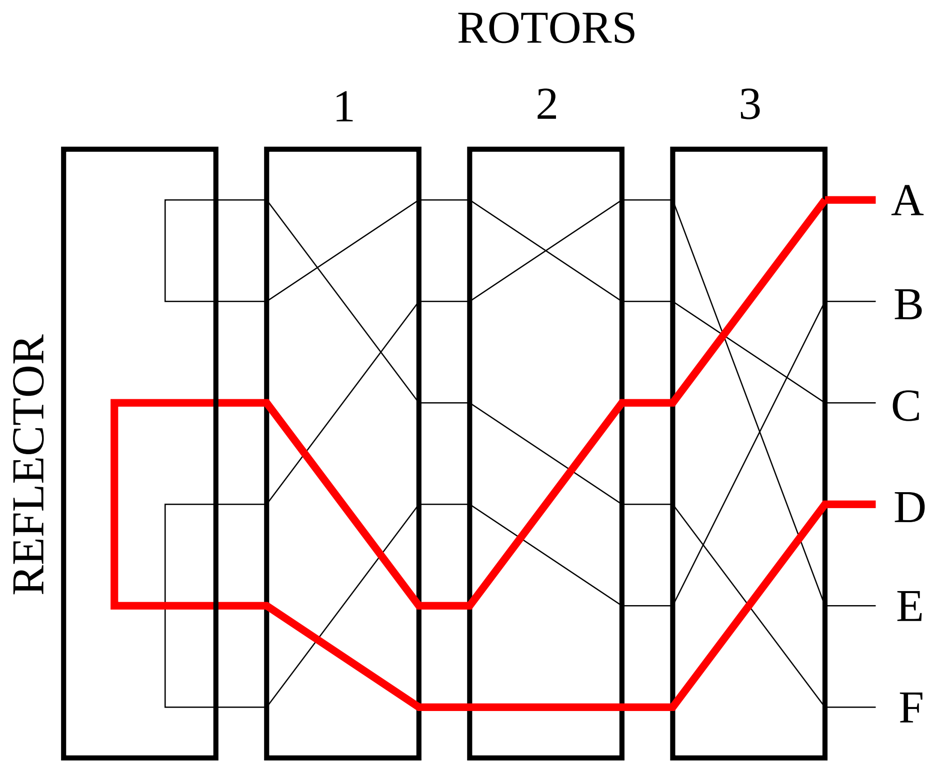
- User errors:
  - repeated initial 3 letters
  - nonrandom initial 3 letters
  - test message with only T's



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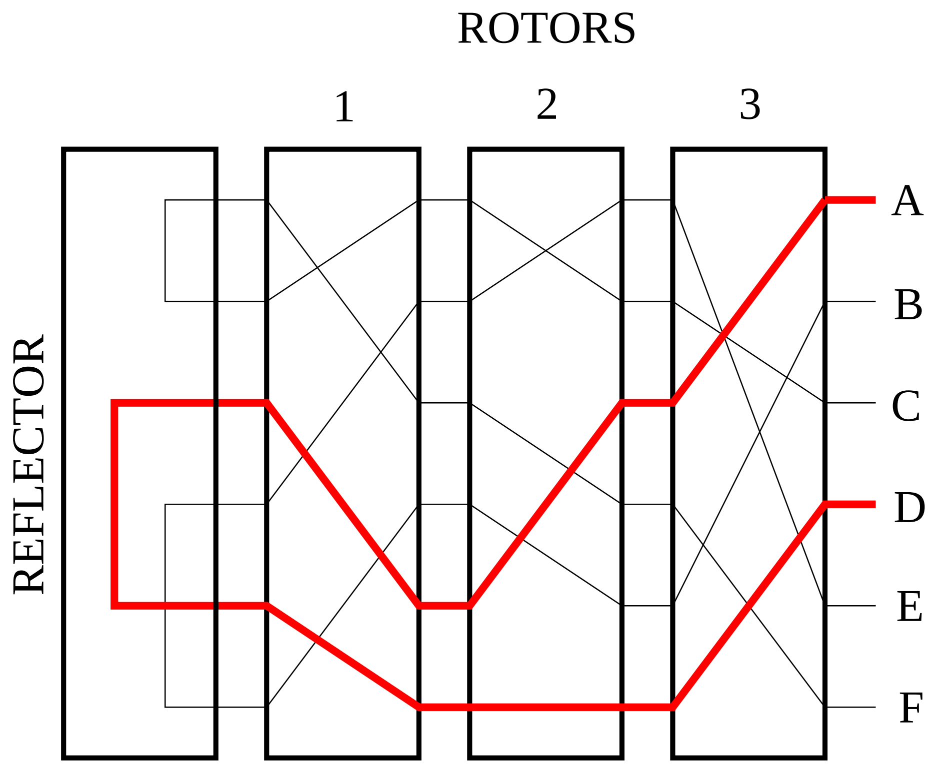

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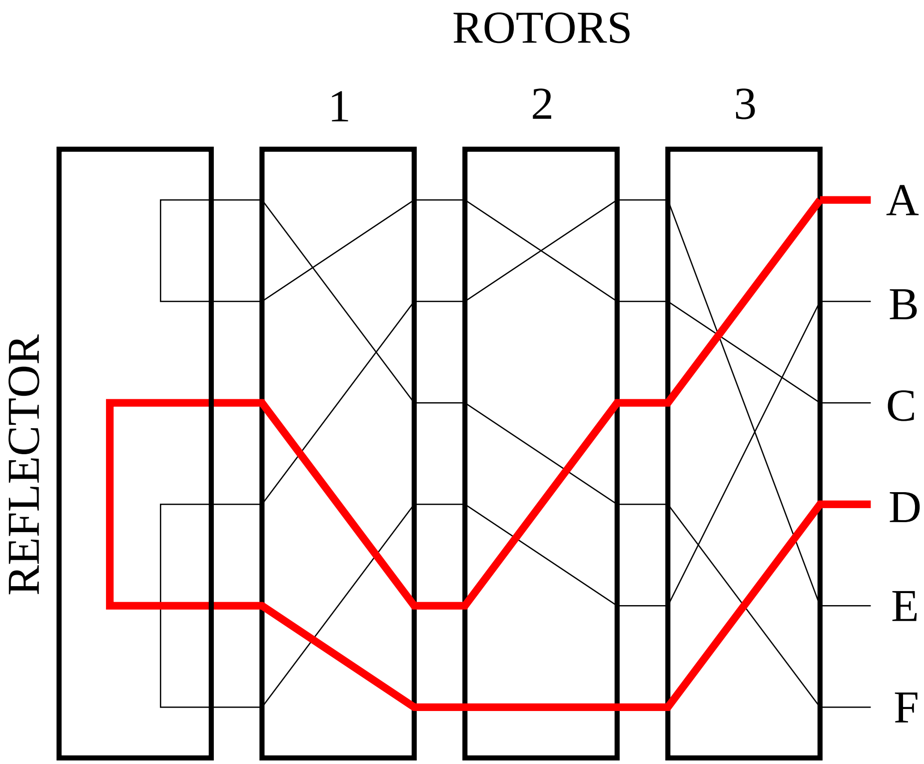

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At the end of the war all messages could be deciphered in 2 days.

The Germans were still confident about ENIGMA.

Lesson learned

A crypto system should be safe even if

- the enemy knows your encryption algorithm
- the enemy knows lots of plain texts together with their encryptions
  (no chosen plain text attacks)

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Solution

- Use a public algorithm with a secret key.

Data Encryption Standard (DES, 1974)

Xor:

$\oplus$	0	1
0	0	1
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key	0 1 1 0 1 0 0 1 0 0 0 1 0 0 1 0 $\oplus$
encryption	1 1 0 0 1 1 0 0 0 1 0 1 1 0 1 1

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encryption $\oplus$ key $=$ message

message $\oplus$ encryption $=$ key **!DANGER!**

Data Encryption Standard (DES, 1974)

- Pick a secret shared key of 64 bits.
- Divide the message in blocks of 64 bits.
- Encrypting one block consists of a combination of
  repeated $\oplus$ with parts of the key, permutations,
  breaking up in subblocks, and small functions by table.

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System uses a **secret shared key**

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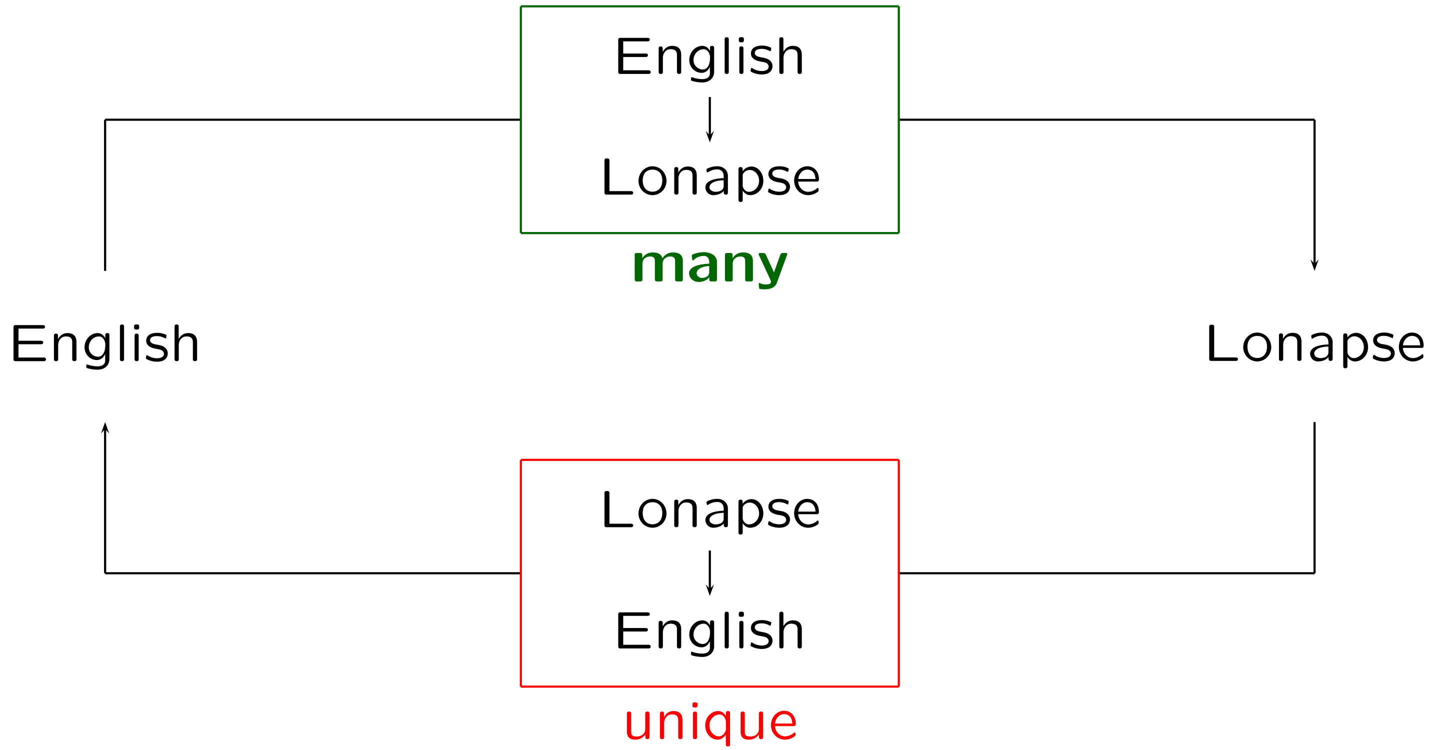
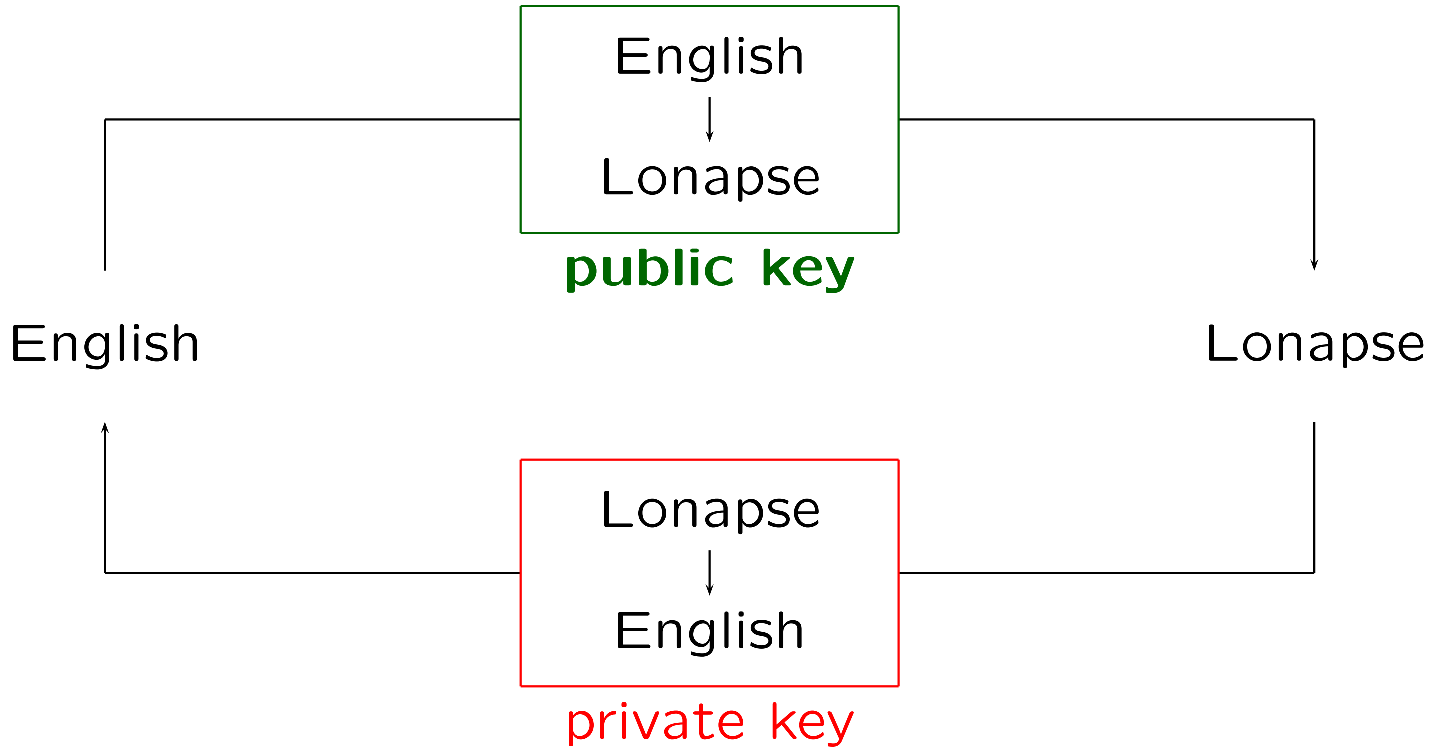
Problem: How do you prove a cryptography system is "secure"?

Public Keys


English
Lonapse
many
English
Lonapse
Lonapse
English
unique

Public Keys



English
Lonapse
public key
English
Lonapse
Lonapse
English
private key

Public Keys

$$
\begin{tikzcd}
M_E & M_L \arrow[l, "\text{L2E}"'] & \textbf{B} \\
M_E \arrow[r, "\text{E2L}"] & M_L \arrow[u] & \textbf{A}
\end{tikzcd}
$$

encrypting, sending,
and decrypting
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English and Lonapse have same words!

Public Keys

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M_E \arrow[r, "\text{E2L}"] & M_L \arrow[u] & \text{A} & ?M_E? & M_{NL} \arrow[l, "\text{E2L}"']
\end{tikzcd}$$

encrypting, sending,
and decrypting
a message

signing, sending,
and **checking the signature**
of a message

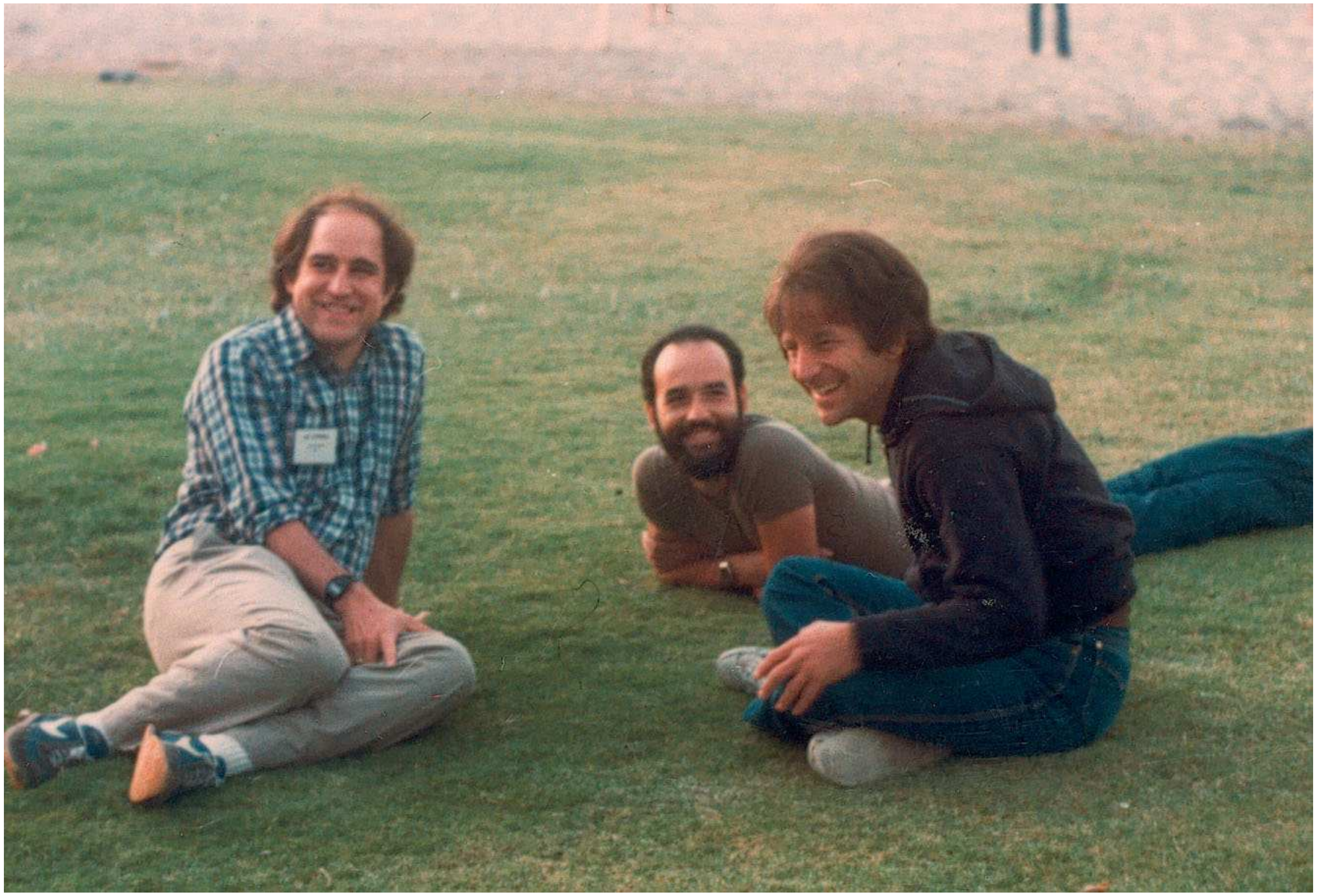
English and Lonapse have same words!

Public Keys (RSA)

RSA (Rivest, Shamir, Adleman):

An $n >> 0$, a **public** key e, and a private key d,

such that $x^{de} \equiv x \mod n$ for all x.



Public Keys (RSA)

$0 < M < n$
$x^{de} \equiv x \mod n$

$$
\begin{tikzcd}
M \equiv (M^e)^d & M^e \arrow[l] & \mathbf{B} & M \arrow[r] & M^d \arrow[d] \\
M \arrow[r] & M^e \arrow[u] & \mathbf{A} & M \stackrel{?}{\equiv} (M^d)^e & M^d \arrow[l]
\end{tikzcd}
$$

encrypting, sending,
and decrypting
a message M

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Public Keys (RSA)

Security of this system is based on our inability to take e-th roots.

A factorization of n allows one to compute d from e.

It is believed that finding d is as hard as factorizing n.

So breaking this system would be as hard as factorizing n.

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Advantages:

compact, use in smart cards

both encryption and signing

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Disadvantages:

Computationally intensive

only small messages

man-in-the-middle attack
(weakness of public keys)

RSA only encripts small messages

For **signing**, you can just sign a hash-function of the message instead.

$$\begin{tikzcd}
\text{B} & & M \arrow[rr] & & {[M, H(M)^{d}]} \arrow[d] \\
\text{A} & H(M) \stackrel{?}{\equiv} (H(M)^{d})^{e} & & & {[M,\ H(M)^{d}]} \arrow[lll]
\end{tikzcd}$$

signing, sending,
and **checking the signature**
of a message

RSA only encripts small messages

For **encryption**, one can use public-key systems to agree on a shared secret key for a more efficient encryption algorithm (like **triple-DES**).

A certain way of doing this is called **PGP** (Pretty Good Privacy)

Public key systems and the man-in-the-middle attack

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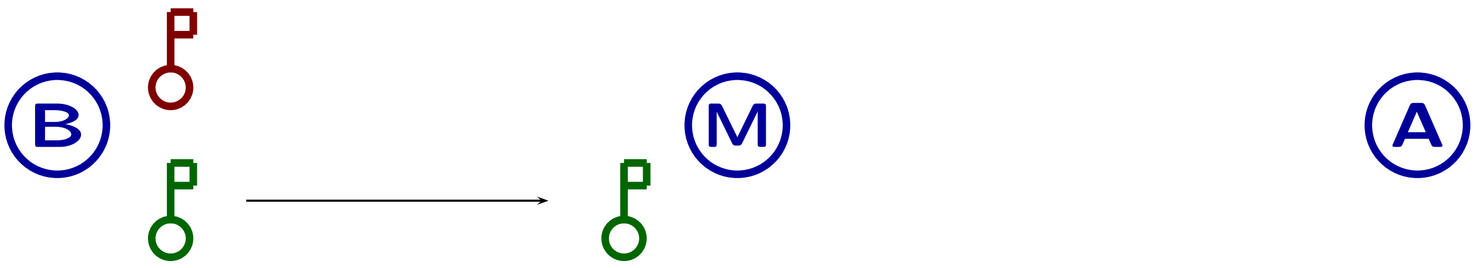

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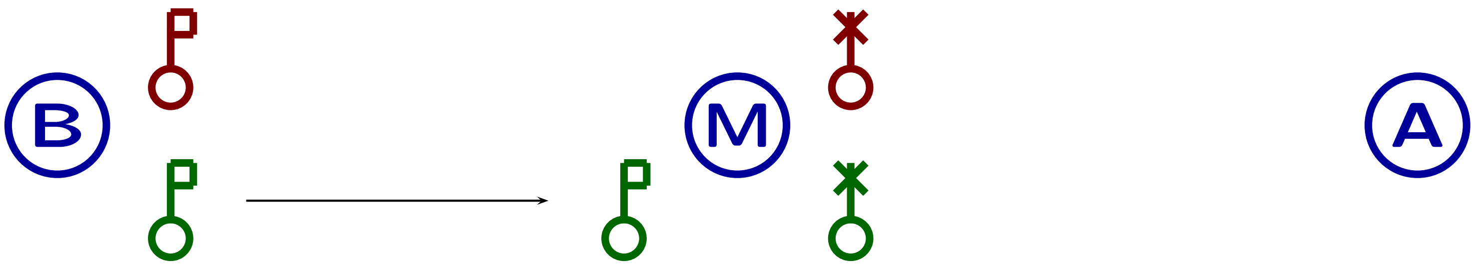

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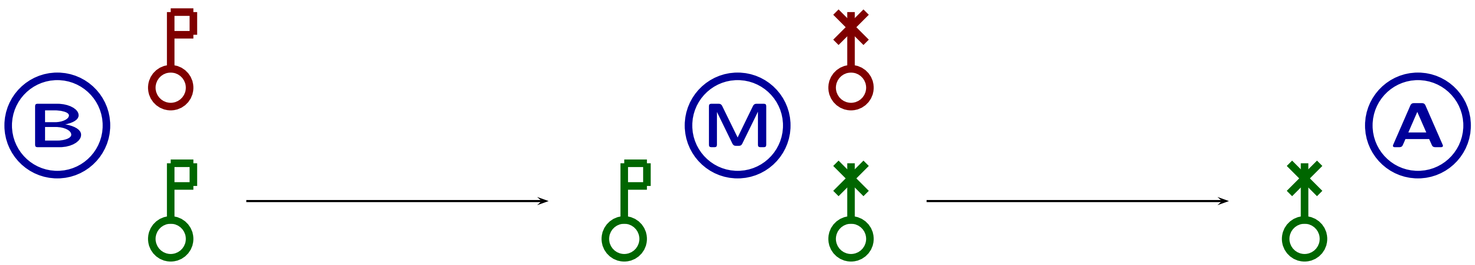

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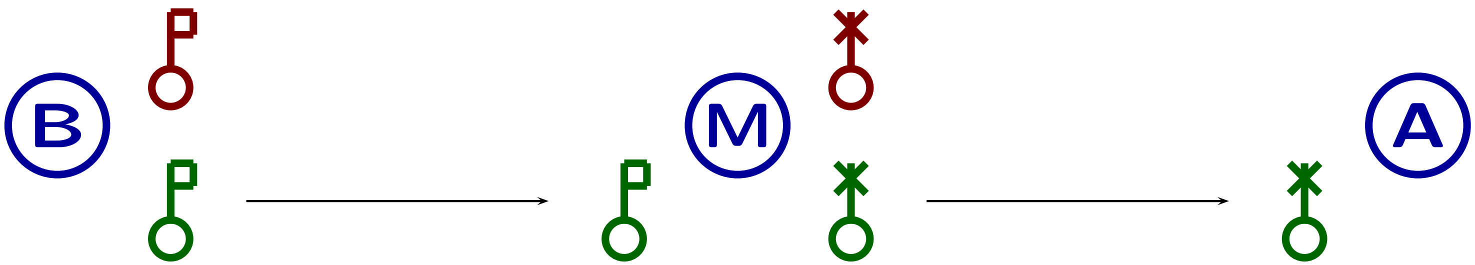

Solution: A trusted third party
(online companies that garantee you are you
by checking your credit card info)

Important

- Factorizing integers

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- Discrete logarithms (tomorrow)

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- Factorizing integers
- Discrete logarithms (tomorrow)
- Coffee (now)