In this note we show that the only non-commutative simple group of order ≤ 100 is the alternating group A_5 of order 60.

Lemma 1. *Let G be a non-commutative simple group of order n. Let p be a prime dividing n and let m be the number of p-Sylow subgroups. Then G is isomorphic to a subgroup of S_m. In particular, n divides $m!$.*

Proof. Since G is simple but not commutative, we have $m > 1$. The action by conjugation of G on its p-Sylow subgroups leads to a homomorphism $i : G \longrightarrow S_m$. Since the action of G is transitive, the image of i is a non-trivial subgroup of S_m. It follows that i is injective and the Lemma follows.

Proposition 2. *Let G be a group of order n. In the following cases G admits a proper normal subgroup:*
(a) $n = 2m$ with $m > 1$ odd;
(b) $n = p^k$ where p is prime and $k \geq 2$;
(c) $n = pq$ where p and q are primes;
(d) $n = p^2q$ where p and q are primes;
(e) $n = 8p$ with p prime.

Proof. (a) Let $x \in G$ be an element of order 2. The permutation $G \longrightarrow G$ induced by left multiplication by x is a product of m disjoint 2-cycles. Therefore it is odd. The kernel of the surjective homomorphism

$$G \longrightarrow S(G) \xrightarrow{\text{sign}} \{\pm 1\}$$

is a normal subgroup of index 2 of G.
(b) This is clear when G is commutative. In the other cases the center of G is a proper normal subgroup.
(c) If $p = q$ this follows from (b). In the other cases, either the p-Sylow subgroup or the q-Sylow subgroup is unique and hence normal. Indeed, if not, we have $p \geq q + 1$ as well as $q \geq p + 1$, which is impossible.
(d) If $p = q$ this follows from (b). In the other cases, either the p-Sylow subgroup or the q-Sylow subgroup is unique and hence normal. Indeed, the number of q-Sylow subgroups is 1, p or p^2. If it is p^2, then the cardinality of the complement of the subset of elements of order q is $p^2q - p^2(q-1) = p^2$. This shows that there can be only one p-Sylow subgroup. If the number of q-Sylow subgroups is p and the p-Sylow subgroup is not unique, we have $p \geq q + 1$ as well as $q \geq p + 1$, which is impossible.
(e) If $p = 2$ the result follows from (b). If $p > 2$ and the p-Sylow subgroup of G is unique, it is normal and we are done. Suppose it is not unique. Then their number m divides 8. Since $m \equiv 1 \pmod{p}$, this implies that $p = 3$ or 7. If $p = 7$, there are eight 7-Sylow subgroups and the complement of the subset of elements of order 7 is $56 - 8(7-1) = 8$, showing that the 2-Sylow subgroup is normal. If $p = 3$, there are four 3-Sylow subgroups. If G were simple, then Lemma 1 shows $G \cong S_4$, a contradiction.

Proposition 3. *Any non-commutative simple group of order ≤ 100 has order 60.*

Proof. Let G be a non-commutative simple group of order ≤ 100. Since groups of prime order are commutative, Proposition 2 implies easily that $n \in \{36, 48, 60, 72, 84, 80, 96, 100\}$.

If $n = 84$, the 7-Sylow subgroup of G is normal and if $n = 100$ the 5-Sylow subgroup of G is normal. This follows from Sylow's theorems. If $n = 36$ or 72, there are exactly four 3-Sylow subgroups. If $n = 48$ or 96, there are precisely three 2-Sylow subgroups. If $n = 80$ there are five 2-Sylow subgroups. In each case this contradicts Lemma 1.

Proposition 4. *A simple group of order 60 is isomorphic to A_5.*

Proof. Let G be a simple group of order 60. Consider the 2-Sylow subgroups of G. They have order 4 and their number m divides 15. Since G is simple, Lemma 1 implies that m is not equal to 1 or 3. If $m = 5$, Lemma 1 provides us with an injective homomorphism $G \longrightarrow S_5$.

Suppose that $m = 15$. Since G admits six 5-Sylow subgroups, the complement of the set of elements of order 5 has $60 - 6(5 - 1) = 36$ elements. Therefore there are at least two 2-Sylow subgroups P and P' with non trivial intersection. Let $1 \neq x \in P \cap P'$ and let H be the centralizer of x. Since P' centralizes x, the group P is a proper subgroup of H. It follows that $\#H \geq 12$ and hence $k = [G : H] \leq 5$. Translation by G of the cosets of H induces a non-trivial and hence injective homomorphism $G \longrightarrow S_k$. It follows that $k = 5$.

So both when $m = 5$ and $m = 15$, we obtain an injective homomorphism $G \longrightarrow S_5$. Since A_5 does not admit a subgroup of index 2, we have $G \cong A_5$ as required.