1. Introduction.

The finite simple groups have recently been classified. A short version of this result, announced in the 1980's, is the following.

Theorem 1.1. *Any finite simple group is isomorphic to one of the following groups:*

(a) a cylic group of prime order;
(b) an alternating group A_n with $n \geq 5$;
(c) a simple group of Lie type;
(d) one of the 26 sporadic simple groups.

Many mathematicians contributed to this classification [0]. The proofs cover thousands of pages, published in numerous different journals. One of the first main steps in the proof is the celebrated Feit-Thompson theorem (1961).

Theorem 1.2. *Every finite group of odd order is solvable.*

Although the proof of this theorem is much shorter than the classification of all finite simple groups (CFSG), it still takes a formidable 250 pages and occupies an entire issue of the Pacific Journal of Mathematics [0]. Very few people can be said to have read it in detail.

The proof of Theorem 1.2 is by induction. It suffices to prove that an odd order group, all of whose proper subgroups are solvable, is itself solvable. In particular, one may assume that all non-identity elements have solvable centralizers. In the odd order paper Feit and Thompson proceed by analyzing the structure of the centralizers inside a hypothetical minimal counterexample to their theorem.

Their arguments extend those in an earlier paper by themselves and Marshall Hall jr., in which they had proved a weaker version of the odd order theorem. They had, more precisely, showed that every odd order *CN-group* is solvable [0]. Here a CN-group is a group all of whose non-identity elements have nilpotent rather than solvable centralizers. The proof is difficult, but much shorter than the proof of the odd order theorem. It follows the strategy of the proof by of the following theorem.

Theorem 1.3. *Every CA-group of odd order is solvable.*

Here a CA-group is a group all of whose non-identity elements have *abelian* centralizers. Suzuki's paper appeared in 1957 and is only 20 pages long [0]. It is accessible to the non-expert and is worth reading. In fact, Thompson [0] wrote in 1984

Suzuki's CA-theorem is a marvel of cunning ...

In the same paper Thompson writes

> *In order to have a genuinely satisfying proof of the odd order theorem, it is necessary, it seems to me, not to assume this [Suzuki's] theorem. Once one accepts this theorem as a step in a general proof, one seems irresistibly drawn along the path which was followed. To my colleagues who have grumbled about the tortuous proofs in the classification of finite simple groups, I have a ready answer: find another proof of Suzuki's theorem.*

In a 2013 blog, Terence Tao in some sense takes up Thompson's suggestion and presents Suzuki's proof [0]. Tao goes back one step further and singles out the role played in the proof by a famous 1901 result of Frobenius [0].

Theorem 1.4. *Let G be a finite group and let $H \subset G$ be a subgroup with the property that for every $x \in G - H$, the intersection $H \cap xHx^{-1}$ is trivial. Then the complement of*

$$\bigcup_{x \in G} xHx^{-1} - \{1\},$$

is a subgroup $K \subset G$.

Typical examples of groups having subgroups satisfying the conditions of Theorem 1.4 are the following matrix groups.

Example 1.5. Let $\mathbf{F}_q$ be a finite field with q elements and let G be the group of matrices

$$G = \{\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} : b \in \mathbf{F}_q \text{ and } d \in \mathbf{F}_q^*\}.$$

It acts via fractional linear transformations on the projective line $\mathbf{P}_1(\mathbf{F}_q)$. All elements of G fix the point at infinity. Let H be the subgroup of elements of G that fix a given point in $\mathbf{A}^1(\mathbf{F}_q)$. Then H satisfies the conditions of Theorem 1.4. The subgroup K is precisely the subgroup of matrices *without* any fixed points in $\mathbf{A}^1(\mathbf{F}_q)$. It is the normal subgroup of matrices with $d = 1$.

Frobenius' result was one of the first results obtained with character theory. As a matter of fact, the proofs of the results above all heavily exploit character theory. Tao writes:

> *It seems to me that the above four theorems (Frobenius, Suzuki, Feit-Thompson, and CFSG) provide a ladder of sorts (with exponentially increasing complexity at each step) to the full classification, and that any new approach to the classification might first begin by revisiting the earlier theorems on this ladder and finding new proofs of these results first (in particular, if one had a robust proof of Suzuki's theorem that also gave non-trivial control on "almost CA-groups" —whatever that means— then this might lead to a new route to classifying the finite simple groups of Lie type and bounded rank). But even for the simplest two results on this ladder —Frobenius and Suzuki— it seems remarkably difficult to find any proof that is not essentially the character-based proof.*

This note is the result of my efforts to read and understand the proof presented by Terence Tao in his blog. It contains an exposition of Suzuki's theorem.

2. Inducing characters

Frobenius' 1901 theorem is an ingredient in Suzuki's proof of Theorem 1.3. In this section we prove it. We make use of standard character theory. In view of the applications to Suzuki's theorem, the presentation is slightly more general than strictly necessary for the proof of Frobenius' theorem.

In general, for a finite group G, we write $\mathbf{C}[G]$ for the group ring of G and $\mathbf{C}(G)$ for its sub-$\mathbf{C}$-algebra of class functions $G \longrightarrow \mathbf{C}$. On $\mathbf{C}[G]$ we have the usual Hermitian product given by

$$\langle f, g\rangle \;=\; \frac{1}{\#G}\sum_{x\in G} f(x)\overline{g(x)}.$$

The *characters* of the irreducible representations of G form an orthormal basis of $\mathbf{C}(G)$. We denote their $\mathbf{Z}$-span by $\chi(G)$. Let $\chi(G)^0$ denote the subgroup of elements $f \in \chi(G)$ for which $f(1) = 0$. We have the inclusions

$$\chi(G)^0 \;\subset\; \chi(G) \;\subset\; \mathbf{C}(G) \;\subset\; \mathbf{C}[G].$$

Let $H \subset G$ be a subgroup. Then any class function of G can be *restricted* to H. Restriction is a $\mathbf{C}$-linear map $\mathbf{C}(G) \longrightarrow \mathbf{C}(H)$. Conversely, a class function f of H can be induced to a class function $\operatorname{Ind} f$ on G as follows. For $z \in G$ we put

$$\operatorname{Ind} f(z) = \frac{1}{\#H}\sum_{x\in G} f(xzx^{-1}),$$

with the convention that $f(y) = 0$ whenever $y \notin H$. Induction is a $\mathbf{C}$-linear map from $\mathbf{C}(H)$ to $\mathbf{C}(G)$. It maps $\chi(H)$ to $\chi(G)$ and $\chi(H)^0$ to $\chi(G)^0$.

In this section we study the induction homomorphism $\operatorname{Ind} : \chi(H) \longrightarrow \chi(G)$ in the following special situation. We assume that

any two conjugates of H either coincide or have trivial intersection with one another. $(*)$

We write N for the normalizer of H in G and W for the "Weyl group" N/H. Let $w = \#W$. The group W acts on class functions $f \in \mathbf{C}(H)$ by $f^s(z) = f(szs^{-1})$ for $s \in W$ and $z \in H$. It follows easily that for every $s \in W$ we have $\operatorname{Ind} f = \operatorname{Ind} f^s$.

Proposition 2.1. *Suppose G is a finite group and $H \subset G$ is a subgroup satisfying $(*)$. Let $f \in \mathbf{C}(H)$. Then we have for $z \in G$*

$$\operatorname{Ind} f(z) \;=\; \begin{cases} [G:H]f(1), & \text{if } z = 1;\\ 0, & \text{if } z \notin \cup_{x\in G} xHx^{-1};\\ \sum_{s\in W} f^s(y), & \text{if } z = xyx^{-1} \text{ for some } x \in G \text{ and } y \in H - \{1\}.\end{cases}$$

This is an application of the induction formula given above. We leave the straightforward computation to the reader.

Proposition 2.2. *Suppose G is a finite group and $H \subset G$ is a subgroup satisfying $(*)$. Let $f, g \in \chi(H)^0$. Then we have*

$$\langle \mathrm{Ind}\, f, \mathrm{Ind}\, g\rangle_G \;=\; \frac{1}{w}\langle \sum_{s\in W} f^s, \sum_{s\in W} g^s\rangle_H.$$

This follows from Proposition 2.1.

Corollary 2.3. *Let χ be a non-trivial irreducible character of H of dimension d. Then*

$$\|\mathrm{Ind}\,(\chi - d)\|_G^2 \;=\; 1 + wd^2.$$

and

$$\langle \mathrm{Ind}\,(\chi - d), 1\rangle_G \;=\; -d.$$

In particular, $\mathrm{Ind}\,(\chi - d) + d$ *is orthogonal to* 1 *in* $\chi(G)$.

Proof. Note that $\chi - d$ is in $\chi(H)^0$. By Proposition 2.2 we have

$$\begin{aligned} \|\mathrm{Ind}\,(\chi - d)\|_G^2 &= \frac{1}{w}\langle \sum_{s\in W} \chi^s - wd, \sum_{s\in W} \chi^s - wd\rangle_H, \\ &= \frac{1}{w}\langle \sum_{s\in W} \chi^s, \sum_{s\in W} \chi^s\rangle_H + wd^2, \\ &= 1 + wd^2. \end{aligned}$$

For the second statement we apply Frobenius reciprocity:

$$\langle \mathrm{Ind}\,(\chi - d), 1\rangle_G = \langle \chi - d, 1\rangle_H = 0 - d = -d.$$

Theorem 1.4. *(Frobenius 1901). Let G be a finite group and let $H \subset G$ be a subgroup with the property that for every $x \in G - H$, the intersection $H \cap xHx^{-1}$ is trivial. Then the complement of*

$$\bigcup_{x\in G} xHx^{-1} - \{1\},$$

is a subgroup $K \subset G$.

Proof. The condition on H implies that $N_G H = H$ and hence $w = 1$. Let χ be an irreducible non-trivial character of H. Let d be its degree. To χ we associate the element of $\chi(G)$ given by

$$\chi^* \;=\; \mathrm{Ind}\,(\chi - d) + d.$$

By Corollary 2.3, we have $\|\mathrm{Ind}\,(\chi - d)\|_G^2 = 1 + d^2$. Since χ^* is orthogonal to 1, the Pythagorean Theorem implies that $\|\chi^*\|^2 = (d^2 + 1) - d^2 = 1$. Since $\chi^*(1) = \chi(1) = d$ is positive, χ^* is an irreducible non-trivial character of G.

We claim that $K = \cap_\chi \ker \chi^*$, where χ runs over the irreducible non-trivial characters of H. Indeed, let $\chi \neq 1$ be an irreducible character of H. Since $\mathrm{Ind}(\chi - d)$ vanishes on K, we have $\chi^*(x) = d$ for every $x \in K$. This shows that $K \subset \ker \chi^*$. On the other hand, if $x \notin K$, then x is a non-trivial element of gHg^{-1} for some $g \in G$. Therefore there is an irreducible character $\chi \neq 1$ of H for which $\chi(g^{-1}xg)$ is not equal to $d = \chi(1)$. This means that $\chi^*(x) \neq d$, so that $x \notin \ker \chi^*$, as required.

The following proposition is used in the next section.

Proposition 2.4. *Let χ, χ' be non-trivial irreducible characters of H of the same dimension. If they are in the same W-orbit, then* $\mathrm{Ind}\,\chi = \mathrm{Ind}\,\chi'$. *If not, then we have*

$$\|\mathrm{Ind}\,(\chi - \chi')\|_G^2 \;=\; 2.$$

Proof. Note that $\chi - \chi'$ is in $\chi(H)^0$. By Proposition 2.2 we have

$$\|\mathrm{Ind}\,(\chi - \chi')\|_G^2 = \frac{1}{w}\langle \sum_{s\in W}(\chi^s - \chi'^s), \sum_{s\in W}(\chi^s - \chi'^s)\rangle_H.$$

Since χ and χ' are not conjugate by W and since the characters χ^s, $s \in W$ are distinct, orthogonality of characters implies that this is equal to

$$\frac{1}{w}\langle \sum_{s\in W}\chi^s, \sum_{s\in W}\chi^s\rangle_H + \frac{1}{w}\langle \sum_{s\in W}\chi'^s, \sum_{s\in W}\chi'^s\rangle_H = 1 + 1 = 2,$$

as required.

3. CA-groups.

In this section we describe the basic properties of CA-groups. This part of Suzuki's proof is the so-called *local analysis.* It is easy for CA-groups, but becomes much harder for CN-groups and in the relevant parts of the Feit-Thompson paper.

Definition. A CA-group is a finite group with the property that the centralizers of its non-trivial elements are abelian subgroups.

For a centralizer A in a CA-group we let $N_G A$ denote its normalizer in G and $W_A = N_G A/A$ its "Weyl group". It acts on A by conjugation. By t_A we denote the number of W_A-orbits in $A - \{1\}$. By $S(A)$ we denote the set $\cup_{x\in G} xAx^{-1} - \{1\}$. By $\mathcal{A}$ we denote the set of centralizers A in G up to conjugacy. A CA-group is abelian if and only if $\#\mathcal{A} = 1$.

Theorem 3.1. *Let G be a CA-group. Then*

(a) *The maximal abelian subgroups of G are precisely the centralizers of the non-trivial elements of G.*

(b) *Any two distinct maximal abelian subgroups have trivial intersection. In particular, any maximal abelian subgroup $A \subset G$ has property $(*)$ of section 2.*

(c) *Let A be a maximal abelian subgroup. Then each non-trivial element of W_A acts without fixed points on $A - \{1\}$. The number of W_A-orbits t_A is $(\#A - 1)/\#W_A$. The cardinality of $S(A)$ is $[G : A]t_A = (\#A - 1)[G : N_G A]$.*

(d) *The sets $S(A)$ for $A \in \mathcal{A}$ form a partition of $G - \{1\}$. We have*

$$\#G - 1 \;=\; \sum_{A\in\mathcal{A}} [G : A]t_A.$$

(e) *The number of conjugacy classes of G is*

$$1 + \sum_{A\in\mathcal{A}} t_A.$$

(f) The cardinalities of non-conjugate maximal subgroups of G are coprime. We have

$$\#G = \prod_{A \in \mathcal{A}} \#A.$$

Proof. (a) Let $1 \neq x \in G$ and let A be a maximal abelian subgroup that contains it. Then A is contained in the centralizer $C(x)$ of x. (b) Let A, A' be two maximal abelian subgroups and let $x \in A \cap A'$. Then $A, A' \subset C(x)$. Since $C(x)$ is abelian, all three groups must be equal. (c) Suppose $x \in N_G A$ commutes with a non-trivial element $a \in A$. Then x and A are in the centralizer $C(a)$ of a. Since A is maximal, we have $A = C(a)$ and hence $x \in A$. Parts (d) and (e) are obvious. For (f), let A, A' be maximal abelian subgroups and let p be a prime number dividing both $\#A$ and $\#A'$. Since p-groups have non-trivial centers, the p-Sylow subgroups of CA-groups are abelian. Therefore both A and A' contain a p-Sylow subgroup. This means that for some $g \in G$ the subgroups gAg^{-1} and A' contain the same p-Sylow subgroup. Therefore (b) implies $gAg^{-1} = A'$ and (f) follows. slower.

Finite abelian groups are of course CA-groups. If A, B are finite abelian groups and B acts on A, then the semidirect product $A \rtimes B$ is a CA-group if every $b \in B - \{1\}$ acts without fixed points on $A - \{1\}$. Example 1.5 is a special instance. None of these groups are simple. But simple CA-groups, albeit of even order, do exist. Indeed, for every 2-power q the group $\mathrm{SL}_2(\mathbf{F}_q)$ is a CA-groups. For $q = 2$ and $q = 4$ it is isomorphic to S_3 and A_5 respectively. For $q \geq 4$ it is a simple group.

Example 3.2. Let $q \geq 4$ be a power of 2 and let $G = \mathrm{SL}_2(\mathbf{F}_q)$. Then $\#\mathcal{A} = 3$. In other words, up to conjugacy there are three maximal abelian subgroups. Indeed, by Theorem 3.1 (a), they are the centralizers of three different types of elements of $g \in G$. If the characteristic polynomial f of g has two distinct eigenvalues in $\mathbf{F}_q$, then its centralizer is conjugate to the *split Cartan subgroup*

$$A = \{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbf{F}_q^*\}.$$

We have $\#A = q - 1$. Since $[N_G A : A] = 2$, the Weyl group has order 2 and we have $t_A = (q-2)/2$. If f has two conjugate eigenvalues in $\mathbf{F}_{q^2} - \mathbf{F}_q$, then the centralizer of g is conjugate to the *non-split Cartan subgroup*

$$A = \{\begin{pmatrix} a & b\lambda \\ b & a+b \end{pmatrix} : a, b \in \mathbf{F}_q \text{ with } a^2 + ab + \lambda b^2 = 1\}.$$

Here λ is an element in $\mathbf{F}_q$ whose trace to $\mathbf{F}_2$ is 1. In this case A is cyclic of order $q+1$. Its Weyl group has order 2, so that $t_A = q/2$. Finally, if $g \neq 1$ and f has a double eigenvalue, then the centralizer of g is conjugate to the *unipotent subgroup*

$$A = \{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbf{F}_q\}.$$

It is isomorphic to the additive group of $\mathbf{F}_q$. Its normalizer is the subgroup of upper triangular matrices. So the Weyl group has order $q-1$. It follows that W_A-acts transively on $A - \{1\}$, so that $t_A = 1$ in this case. Part (e) of Theorem 3.1 says that there are $(q-2)/2 + q/2 + 1 + 1 = q + 1$ conjugacy classes while part (f) confirms the well known formula $\#\mathrm{SL}_2(\mathbf{F}_q) = (q-1)q(q+1)$.

4. Character theory of CA-groups.

In this section we show that the irreducible non-trivial characters of CA-groups can in a precise sense be obtained from the inductions of the 1-dimensional characters of its maximal abelian subgroups. In this section we describe a one-to-one correspondence due to Brauer [0] and Suzuki [0] between the non-trivial irreducible characters of CA-groups and the non-trivial characters of its maximal abelian subgroups.

Let G be a finite group and let A be an abelian subgroup of G, having the property of section 2:

any two conjugates of A either coincide or have trivial intersection with one another. $(*)$

Note that CA-groups satisfiy this condition. The Weyl group W_A naturally acts on the set of non-trivial 1-dimensional characters of A. We denote the set of orbits by Γ_A. We have $\#\Gamma_A = t_A$.

Proposition 4.1. *(Brauer-Suzuki). Let $A \subset G$ be as above.*

(a) If $\#\Gamma_A > 2$, there exist a unique sign $\epsilon \in \{\pm 1\}$ and for each $\chi \in \Gamma_A$ a unique irreducible character χ^* of G such that $\mathrm{Ind}(\chi - \chi') = \epsilon(\chi^* - {\chi'}^*)$ for all $\chi, \chi' \in \Gamma_A$.

(b) When $\#\Gamma_A = 2$, the same is true, except that there is an ambiguity: one can switch χ and χ' and change the sign of ϵ.

Proof. For every pair of irreducible characters χ, χ' of the same dimension, the difference $\chi - \chi'$ is in $\chi(H)^0$. Corollary 2.4 implies then that $\|\mathrm{Ind}(\chi - \chi')\|^2 = 2$. Therefore there are for every pair of irreducible characters χ, χ' of H two distinct irreducible characters ψ, ψ' of G such that $\mathrm{Ind}(\chi - \chi') = \pm(\psi - \psi')$. The set $\{\psi, \psi'\}$ is uniquely determined by χ and χ'. When $\#\Gamma_A = 2$, we put $\chi^* = \psi$ and ${\chi'}^* = \psi'$ or vice versa. The sign ϵ is determined by this choice.

When $\#\Gamma_A > 2$, then for every $\chi \in \Gamma_A$, there is precisely one irreducible character ψ of G that "appears" in $\mathrm{Ind}(\chi - \chi')$ for every $\chi' \in \Gamma_A$. We leave this exercise in combinatorics to the reader. By definition $\chi^* = \psi$. Since induction is linear, it is easy to see that $\mathrm{Ind}(\chi - \chi') = \epsilon(\chi^* - {\chi'}^*)$ for some sign ϵ, that does not depend on the characters χ, χ'.

This proves the proposition.

Proposition 4.2. *Let $A \subset G$ as in Proposition 4.1. Then the characters χ^* associated to the characters $\chi \in \Gamma_A$ agree outside $S(A)$ and are integer valued on $G - S(A)$.*

Proof. Proposition 2.1 implies that for any two characters χ, χ' of A, the support of $\mathrm{Ind}(\chi - \chi')$ is contained in $S(A) = \cup_{x \in G} xAx^{-1} - \{1\}$. Therefore the characters χ^* associated to $\chi \in \Gamma_A$ all agree outside the set $S(A)$. This proves the first statement. Since induction commutes with Galois action, $\chi^* = \sigma(\chi^*) = \sigma(\chi)^*$ outside $S(A)$ for every σ in the absolute Galois group of $\mathbf{Q}$. Since the sum over σ of $\sigma(\chi^*)$ has values in $\mathbf{Q}$, the same is true for the values of χ^* on $S(A)$. From the fact that traces of representations are algebraic integers, it follows that χ^* has values in $\mathbf{Z}$ on $G - S(A)$, as required. Write better.

Proposition 4.3. *Let G be a CA-group. Suppose that each maximal abelian subgroup A has the property that there are at least two W_A-orbits of non-trivial characters . Then every non-trivial irreducible character of G is of the form χ^* for some irreducible character χ of some maximal abelian subgroup A.*

Proof. The condition means that $t_A > 1$ for every maximal subgroup A of G. Therefore Proposition 4.1 applies to *every* A. By Corollary 2.4, the t_A distinct W_A-conjugacy classes of characters χ of A give rise to t_A distinct irreducible characters χ^* of G. Since the irreducible characters χ^* associated to different $A \in \mathcal{A}$ have disjoint supports, they are orthogonal. Therefore the set of irreducible characters χ^* associated to some maximal abelian subgroup A of G has cardinality t_A. Since the number of non-identity conjugacy classes of G is $\sum_{A\in\mathcal{A}} t_A$, Theorem 3.1 (e) implies that *every* non-trivial irreducible character of G is of the form χ^* for some character χ of a maximal abelian subgroup $A \subset G$. This proves the proposition.

The following formula plays a role in the proof of Suzuki's theorem. We introduce some notation. For a maximal abelian subgroup A of an CA-group G, we let Γ_A denote the set of non-trivial characters of A up to conjugacy by G and we put

$$X_A^* = \sum_{\chi\in\Gamma_A} \chi^*.$$

Proposition 4.4. *Let G be a CA-group with the property that $t_A > 1$ for all maximal subgroups A. Let A_0 be a maximal abelian subgroup and let ϕ be a non-trivial character of A_0. Then the character* $\mathrm{Ind}(\phi - 1)$ *has the following Fourier expansion.*

$$\mathrm{Ind}(\phi - 1) = -1 + \epsilon\phi^* + \sum_{A\in\mathcal{A}} c_A X_A^*,$$

for certain coefficients $c_A \in \mathbf{Z}$ and some $\epsilon = \pm 1$.

Proof. By Proposition 4.3 the non-trivial irreducible characters of G are of the form χ^* as in Proposition 4.1. By Corollary 2.3 we have $\langle \mathrm{Ind}(\phi - 1), 1\rangle_G = -1$. For any non-trivial character $\chi \neq \phi$ of A_0 we have the "Fourier expansion"

$$\begin{aligned}
\langle \mathrm{Ind}(\phi - 1), \chi^* - \phi^*\rangle &= \frac{\epsilon}{w}\langle \sum_s (\phi - 1)^s, \sum_s (\chi - \phi)^s\rangle_{A_0}, \\
&= -\frac{\epsilon}{w}\langle \sum_s \phi^s, \sum_s \phi^s\rangle_{A_0}, \\
&= -\epsilon,
\end{aligned}$$

for some $\epsilon \in \{\pm 1\}$. It follows that the χ^*-Fourier coefficients of $\mathrm{Ind}(\phi - 1)$ are all equal to the one of ϕ^* minus ϵ. Similarly, if χ and χ' are characters of a maximal abelian subgroup $A \neq A_0$, then

$$\langle \mathrm{Ind}(\phi - 1), \chi^* - {\chi'}^*\rangle = 0,$$

because the support of $\chi^* - {\chi'}^*$ is $S(A)$, which is disjoint from the support $S(A_0)$ of $\mathrm{Ind}(\phi - 1)$. It follows that the χ^* and ${\chi'}^*$-Fourier coefficients are equal to some constant c_A independent of the character.

This proves the proposition.

5. Character theory of $\mathrm{SL}_2(\mathbf{F}_q)$.

This section plays no role in the proof of Suzuki's theorem. We merely illustrate the results of the previous sections in the case of the CA-groups $G = \mathrm{SL}_2(\mathbf{F}_q)$. So, q is a power of 2. The character table of the group $\mathrm{SL}_2(\mathbf{F}_q)$ is as follows.

Table 5.1.

$\mathrm{SL}_2(\mathbf{F}_q)$	Id	$S(U)$	$S(T)$	$S(T')$
1	1	1	1	1
σ	q	0	1	-1
$\mathrm{Ind}\,\psi$	$q+1$	1	$\zeta+\zeta^{-1}$	0
$\mathrm{Ind}\,\lambda - \mathrm{Ind}\,\theta$	$q-1$	-1	0	$-(\xi+\xi^{-1})$

The columns correspond to the conjugacy classes of $\mathrm{SL}_2(\mathbf{F}_q)$. We write U, T and T' for a unipotent subgroup, a split-Cartan subgroup and a non-split Cartan subgroup respectively. The first column lists the identity element, the second the conjugacy class of unipotent matrices. There are q^2-1 unipotent matrices, and they are all conjugate. The third column stands for the $(q-2)/2$ conjugacy classes of matrices whose characteristic polynomials have two distinct zeroes in $\mathbf{F}_q$. Each conjugacy class contains $q(q+1)$ matrices. The fourth column stands for the $q/2$ conjugacy classes of matrices whose characteristic polynomials are irreducible over $\mathbf{F}_q$. Each conjugacy class contains $q(q-1)$ matrices. We have

$$1 + (q^2-1) + (q/2-1)\cdot q(q+1) + q/2\cdot q(q-1) = q(q^2-1) = \#\mathrm{SL}_2(\mathbf{F}_q).$$

There are precisely $q+1$ conjugacy classes. When $q = 2$, the third column is missing.

The rows correspond to the irreducible characters of $\mathrm{SL}_2(\mathbf{F}_q)$. The first row contains the character 1. The second row is the *Steinberg character* σ. This is the character of the representation given by the natural action by fractional linear transformations of $\mathrm{SL}_2(\mathbf{F}_q)$ on the p-dimensional vector space $\{f : \mathbf{P}_1(\mathbf{F}_q) \longrightarrow \mathbf{C} : \sum_{P\in\mathbf{P}_1(\mathbf{F}_q)} f(P) = 0\}$.

The third row contains the $(q-2)/2$ distinct inductions of the $q-2$ non-trivial characters ψ of a Borel subgroup B. Here $\zeta = \psi(t)$, where t denotes a fixed generator the Cartan subgroup $T \subset B$. It is a non-trivial $(q-1)$-th root of unity. The relation with Propositions 4.1 and 4.2 is as follows. The non-trivial characters χ of T lift uniquely to characters ψ of the Borel subgroup. One computes that $\mathrm{Ind}_T^G\chi = \mathrm{Ind}_U^G\lambda + \mathrm{Ind}_B^G\psi$. Here λ is any non-trivial character of U. We put $\chi^* = \mathrm{Ind}_B^G\psi$. It is an irreducible character. It follows that for any two characters χ, χ' of T we have $\mathrm{Ind}(\chi-\chi') = \chi^* - \chi'^*$, confirming Proposition 4.1. We see that the characters in the third row of the character table agree outside the set $S(T)$ and are $\mathbf{Z}$-valued on $G - S(T)$, confirming Proposition 4.2.

In the fourth row the so-called cuspidal representations are listed. They correspond to the $q/2$ distinct inductions of the q non-trivial characters θ of the non-split Cartan subgroup T'. Here $\xi = \theta(t)$, where t denotes a fixed generator of T'. It is a non-trivial $(q+1)$-th root of unity. The irreducible character θ^* corresponding to a non-trivial character θ of T' is given by $\mathrm{Ind}_U^G\,\lambda - \mathrm{Ind}_{T'}^G\,\theta$, where λ is any non-trivial character of U. For any two characters θ, θ' of T' we have $\mathrm{Ind}(\theta-\theta') = -(\theta^* - \theta'^*)$, confirming Proposition 4.1.

Proposition 4.2 follows from the information in the fourth row of the character table. Note that the characters in this row agree outside the set $S(T')$ and are $\mathbf{Z}$-valued there.

This accounts for $1+1+(q-2)/2+q/2=q+1$ irreducible characters. When $q=2$, the fourth row is missing.

Propositions 4.1 and 4.2 do not apply to the maximal abelian subgroup U. Since the normalizer of U is the Borel subgroup, the Weyl group acts transitively on the non-trivial characters of U. Therefore the character $\mathrm{Ind}_U^G \lambda$ does not depend on λ. In fact, $\mathrm{Ind}_U^G \lambda$ is the sum of all non-trivial irreducible characters of G. Indeed, its dimension is equal to $q+(q+1)(q-2)/2+(q-1)q/2=q^2-1$, which is the index of U.

The groups $\mathrm{SL}_2(\mathbf{F}_q)$ do not satisfy its conditions, but Proposition 4.3 is *almost* true. Indeed, it is easy to see that the characters χ^* that occur in Proposition 4.1 for A a split or non-split Cartan subgroup are precisely the non-trivial characters of G except for the character σ of the Steinberg representation. Proposition 4.4 does not apply either, but since we know all characters of G anyway, we can still compute a similar Fourier expansion of $\mathrm{Ind}(\phi-1)$ for a non-trivial character irriducibile ϕ of a maximal abelian subgroup A. When A is a Cartan subgroup, Corollary 2.3 implies that $\|\mathrm{Ind}(\phi-1)\|^2=1+w_A=3$. One finds that

$$\mathrm{Ind}(\phi-1)=\begin{cases}-1+\phi^*-\sigma, & \text{when } A \text{ is split;}\\ -1-\phi^*+\sigma, & \text{when } A \text{ is non-split.}\end{cases}$$

We have $w=q-1$ for the subgroup U and hence $\|\mathrm{Ind}(\phi-1)\|^2=q$. In this case one can show that

$$\mathrm{Ind}(\phi-1)=-1-X_T^*+X_{T'}^*.$$

6. Suzuki's Theorem.

In this section we prove Suzuki's theorem.

Theorem 6.1. *There does not exist a non-abelian simple CA-group of odd order.*

Proof. Let G be a simple non-commutative CA-group of odd order and let A be a maximal abelian subgroup. If the Weyl group W_A is trivial, G satisfies the conditions of Frobenius' Theorem and therefore cannot be simple. Therefore we may assume that $\#W_A\geq 2$. Since $\#W_A$ divides $\#G$, we have $\#W_A\geq 3$. Similarly, since $\#A-1$ is even, while $\#W_A$ is odd, the number t_A of W_A-orbits is even and hence at least 2.

By Theorem 3.1 (d) we have

$$1-\frac{1}{\#G}\;=\;\sum_{A\in\mathcal{A}}\frac{t_A}{\#A}. \tag{1}$$

The proof proceeds by providing upper bounds for the terms on the right hand side of this equality. We'll succeed in finding a bound that is smaller than 1, which leads to an upper bound on $\#G$. Then we take care of the finitely many remaining cases by hand.

Fix a maximal abelian subgroup A_0 of G and a non-trivial character ϕ of A_0. By Propositions 4.3 and 4.4 the character $\mathrm{Ind}(\phi-1)$ has the following Fourier expansion

$$\mathrm{Ind}(\phi-1)\;=\;-1+\epsilon\phi^*+\sum_{A\in\mathcal{A}}c_AX_A^*, \tag{2}$$

with coefficients $c_A \in \mathbf{Z}$ and $\epsilon = \pm 1$.

In view of this expansion, we split the sum in equation (1) into three pieces:

$$1 - \frac{1}{\#G} = \frac{t_0}{\#A_0} + \sum_{A \neq A_0,\, c_A \neq 0} \frac{t_A}{\#A} + \sum_{A \neq A_0,\, c_A = 0} \frac{t_A}{\#A}.$$

For brevity we write $w_0 = \#W_{A_0}$, $t_0 = t_{A_0}$, $\Gamma_0 = \Gamma_{A_0}$, $c_0 = c_{A_0}$ and $X_0^* = X_{A_0}^*$.

The following two lemmas provide estimates for the sums appearing in this formula.

Lemma 6.2. *We have* $\|\mathrm{Ind}(\phi - 1)\|^2 = w_0 + 1$ *and*

$$\sum_{A \neq A_0,\, c_A \neq 0} \frac{t_A}{\#A} \leq \frac{w_0 - 1}{\min \#A}. \tag{3}$$

Here the minimum is taken over all $A \neq A_0$.

Proof. The first equality follows from Corollary 2.3. By orthogonality of irreducible characters, formula (2) gives us therefore

$$w_0 + 1 = 1 + (c_0 + \epsilon)^2 - c_0^2 + \sum_{A \in \mathcal{A}} c_A^2 t_A.$$

Since $t_0 \geq 2$, the contribution $(c_0 - \epsilon)^2 - c_0^2 + c_0^2 t_0$ from the Fourier coefficients of the irreducible characters associated to A_0 is at least 1. Therefore we find

$$\sum_{A \neq A_0,\, c_A \neq 0} t_A \leq \sum_{A \neq A_0, c_A \neq 0} c_A^2 t_A \leq w_0 - 1,$$

which easily implies inequality. This proves the lemma.

Lemma 6.3. *We have* $\|X_0^*\|^2 = t_0$ *and*

$$\sum_{A \neq A_0,\, c_A = 0} \frac{t_A}{\#A} \leq \frac{1}{t_0}. \tag{4}$$

Proof. The equality holds by definition. For the inequality, consider a maximal abelian subgroup $A \neq A_0$ with $c_A = 0$. Then Propositions 4.3 and 4.4 imply that $\mathrm{Ind}(\phi - 1)$ vanishes on $S(A)$. By Propoposition 4.2, the irreducible characters χ^* that come from any maximal abelian subgroup $A' \neq A$ all agree on $S(A)$ and have integral values there. Since $t_{A'}$ is even, $X_{A'}^*$ takes even integral values on $S(A)$. From (2) we see that *on the set* $S(A)$ we have

$$0 = -1 + \epsilon\phi^* + \sum_{A' \in \mathcal{A}} c_{A'} X_{A'}^*.$$

It follows that on each $S(A)$ with $A \neq A, A_0$ for which $c_A = 0$ the character ϕ^* takes *odd* integral values and hence X_0^* takes integral values that are odd multiples of t_0. This implies

$$t_0 = \|X_0^*\|^2 \geq \frac{1}{\#G} \sum_{A \neq A_0,\, c_A = 0} t_0^2 \# S(A),$$

from which the lemma easily follows.

From (1), (3) and (4) we obtain the estimate:

$$1-\frac{1}{\#G} \leq \underbrace{\frac{t_0}{\#A_0}}_{<\frac{1}{3}}+\underbrace{\frac{w_0-1}{\min\#A}}_{<\frac{1}{2}}+\underbrace{\frac{1}{t_0}}_{\leq\frac{1}{2}} \qquad (4).$$

From now on we assume that A_0 is a maximal abelian subgroup for which $w_0 = w_{A_0}$ is *minimal*. Then we see that this estimate is already pretty good. Since $\#A_0 = 1 + w_0t_0$, the first term is less than $1/w_0 \leq 1/3$. Since $\#A \geq 1 + 2w_0$ for each A, the second term is at most $w_0/(1 + 2w_0) < 1/2$. Since $t_0 \leq 1/2$, the third are also at most $1/2$. In order to get an estimate of the right hand side that is smaller than 1, we proceed a little bit more carefully.

First we show that $t_0 = 2$. Indeed, since $\#G \geq \#N_GA_0 = w_0\#A_0$ and since $t_0/\#A_0 = (\#A_0 - 1)/w_0\#A_0$, equation (4) implies

$$1 \leq \frac{1}{w_0}+\frac{w_0-1}{\min\#A}+\frac{1}{t_0}. \qquad (5)$$

Since $\#A \geq 2w_0$ for every A and since $w\_0 \geq 3$, this leads to

$$1 \leq \frac{1}{w_0}+\frac{w_0-1}{2w_0}+\frac{1}{t_0}=\frac{1}{2w_0}+\frac{1}{2}+\frac{1}{t_0}<\frac{2}{3}+\frac{1}{t_0},$$

implying $t_0 < 3$ and hence $t_0 = 2$.

Let $A \neq A_0$ be a maximal subgroup of minimal cardinality. If $t_A = 2$, the fact that non-conjugate maximal subgroups have coprime order, implies that $w_A \neq w_0$. By minimality of w_0 we have $w_A \geq w_0 + 2$ and hence $\#A = 1 + t_Aw_A \geq 5 + 2w_0$. If $t_A \neq 2$, then $t_A \geq 4$ and we also have $\#A \geq 1 + 4w_0 \geq 5 + 2w_0$. Substititung this in (5) gives

$$1 \leq \frac{1}{w_0}+\frac{w_0-1}{5+2w_0}+\frac{1}{2}.$$

This implies that $w_0 \leq 10/3$ and hence $w_0 = 3$. Substituting $t_0 = 2$, $w_0 = 3$ and $\min\#A \geq 5 + 2w_0 = 11$ in inequality (4) finally gives

$$1-\frac{1}{\#G} \leq \frac{2}{7}+\frac{2}{11}+\frac{1}{2}.$$

We conclude that $\#G < 31$, which easily implies that G is solvable, as required.

Finally we prove Suzuki's Theorem by reducing to the simple case.

Theorem 1.3. *Every CA-group of odd order is solvable.*

Proof. Let G be an odd order CA-group. By Theorem 6.1 it cannot be simple. Let N be a proper normal subgroup. By induction N is solvable, so the last step N' in its derived series is abelian. Let A be the centralizer of N' in G. If $A = G$, the fact that G is an

CA-group implies that G is abelian and we are done. So we may assume that A is a proper subgroup. Since it is a CA-group, it is by induction, solvable. The group A is a normal subgroup of G and we claim that G/A is a CA-group as well. By induction G/A is then solvable and we are done.

It remains to prove that G/A is a CA-group. Let $x \in G - A$ and let $y, z \in G$ centralize x modulo A. This means that the commutators $[x, y]$ and $[x, z]$ are in A. Since A is an abelian normal subgroup, the map $f : A \longrightarrow A$ given by $f(a) = [x, a]$ is an endomorphism of A. Since $x \notin A$ and A is the centralizer of C, the homomorphism f is injective. Therefore it is surjective. Since $[x, y] \in A$, there exists $u \in A$ such that $[x, u] = [x, y]$. But then $[x, u^{-1}y] = xu^{-1}yx^{-1}y^{-1}u = xu^{-1}x^{-1}[x, y]u = xu^{-1}x^{-1}[x, u]u = 1$. In other words, $u^{-1}y$ commutes with x. Similarly, there is $v \in A$ such that $v^{-1}z$ commutes with x. Since G is a CA-group, it follows that $u^{-1}y$ and $v^{-1}z$ commute. But then y and z commute modulo A. This proves the theorem.

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